Notes on a Simple Epidemic Model

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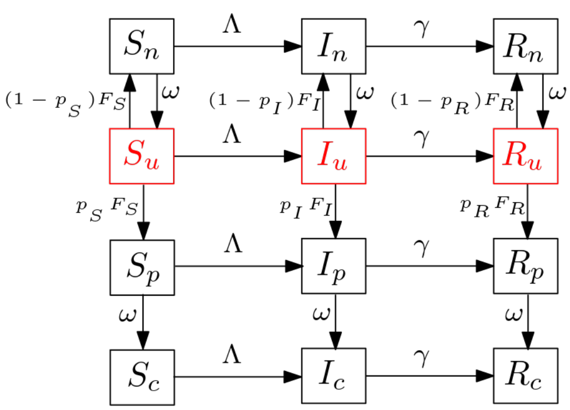
1 Method

1.1 model and parameters



The model is

$$
\begin{aligned}
dS_u/dt &= -\Lambda S_u - F_S S_u + \omega S_n, &(1)\\
dS_n/dt &= -\Lambda S_n + (1-p_S)F_S S_u - \omega S_n, &(2)\\
dS_p/dt &= -\Lambda S_p + p_S F_S S_u - \omega S_p, &(3)\\
dS_c/dt &= -\Lambda S_c + \omega S_p, &(4)\\
dI_u/dt &= \Lambda S_u - F_I I_u + \omega I_n - \gamma I_u, &(5)\\
dI_n/dt &= \Lambda S_n + (1-p_I)F_I I_u - \omega I_n - \gamma I_n, &(6)\\
dI_p/dt &= \Lambda S_p + p_I F_I I_u - \omega I_p - \gamma I_p, &(7)\\
dI_c/dt &= \Lambda S_c + \omega I_p - \gamma I_c, &(8)\\
dR_u/dt &= \gamma I_u - F_R R_u + \omega R_n, &(9)\\
dR_n/dt &= \gamma I_n + (1-p_R)F_R R_u - \omega R_n, &(10)\\
dR_p/dt &= \gamma I_p + p_R F_R R_u - \omega R_p, &(11)\\
dR_c/dt &= \gamma I_c + \omega R_p, &(12)\\
dN/dt &= \omega(S_n + I_n + R_n), &(13)\\
dP/dt &= \omega(I_p + R_p), &(14)
\end{aligned}
$$

Symbol	Description	Unit	Value
N_0	Total population size	people	10^6
ω	Rate of onward flow from the awaiting to reported or untested compartments	1/day	-
γ	Recovery rate	1/day	1/3
ρ	Per capita testing intensity	1/day	0.01
η_w	Relative probability of transmission for isolated awaiting individuals	-	-
η_c	Relative probability of transmission for isolated confirmed individuals	-	-
Λ	Force of infection	1/day	-
p_S	Probability of false positive for susceptible	-	0
p_I	Probability of being infected and tested positive	-	1
p_R	Probability of being recovered and tested positive	-	0.5
W_S, W_I, W_R	Relative testing weight	-	Random testing: $W_S = W_I = W_R = 1$, Non-random testing: $W_S = 0.3, W_I = W_R = 1$

Table 1: The underlying parameters of model, Eqs. (1) to (12).

2 Results

2.1 On the calculation of $\mathcal{R}_0$

- **DFE** is given by solving the following system [1]

$$S_u + S_n = N_0$$
$$F_S S_u - \omega S_n = 0$$

- **DFE:**

$$S_u = (1 - \tfrac{\rho}{\omega}) N_0,$$
$$S_n = \tfrac{\rho}{\omega} N_0,$$
$$I_j = R_j = 0 \quad \text{for all} \quad j.$$

- At the DFE, $F_I^* = \frac{\rho}{(1-\rho/\omega)} W_I/W_S$ also note that $\partial F_I^*/\partial \rho > 0$.

$$F = \beta/N_0 \begin{bmatrix} S_u & \eta_w S_u & \eta_w S_u & \eta_c S_u \\ S_n & \eta_w S_n & \eta_w S_n & \eta_c S_n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{15}$$

$$V = \begin{bmatrix} F_I + \gamma & -\omega & 0 & 0 \\ -(1-p_I)F_I & \omega + \gamma & 0 & 0 \\ -p_I F_I & 0 & \omega + \gamma & 0 \\ 0 & 0 & -\omega & \gamma \end{bmatrix}, \text{thus} \tag{16}$$

$$V^{-1} = \begin{bmatrix} \frac{\omega+\gamma}{\omega\gamma+\gamma F_I+\gamma^2+\omega F_I p_I} & \frac{\omega}{\omega\gamma+\gamma F_I+\gamma^2+\omega F_I p_I} & 0 & 0 \\ \frac{(1-p_I)F_I}{\omega\gamma+\gamma F_I+\gamma^2+\omega F_I p_I} & \frac{F_I+\gamma}{\omega\gamma+\gamma F_I+\gamma^2+\omega F_I p_I} & 0 & 0 \\ \frac{p_I F_I}{\omega\gamma+\gamma F_I+\gamma^2+\omega F_I p_I} & \frac{\omega p_I F_I}{(\omega\gamma+\gamma F_I+\gamma^2+\omega F_I p_I)(\omega+\gamma)} & \frac{1}{\omega+\gamma} & 0 \\ \frac{\omega F_I p_I}{(\omega\gamma+\gamma F_I+\gamma^2+\omega F_I p_I)\gamma} & \frac{\omega^2 F_I p_I}{(\omega\gamma+\gamma F_I+\gamma^2+\omega F_I p_I)(\omega+\gamma)\gamma} & \frac{\omega}{(\omega+\gamma)\gamma} & \frac{1}{\gamma} \end{bmatrix}. \tag{17}$$

The particular form of F with two rows of zeros at the bottom, simplifies G as

$$G = \begin{bmatrix} G_{11} & G_{12} \\ 0 & 0 \end{bmatrix}, \text{ where } G_{11} = C \begin{bmatrix} A\,S_u & B\,S_u \\ A\,S_n & B\,S_n \end{bmatrix}. \tag{18}$$

[1] Note when $I_u = R_u = 0$, $F_S = \frac{\rho N_0}{W_S S_u}$.

The Disease-Free Equilibrium (DFE) for the SIR model, Eqs. (1) to (12), is given by solving the coupled system including $S_u + S_n = N_0$ and $F_S S_u - \omega S_n = 0$. The DFE is

$$S_n^* = \frac{\rho}{\omega} N_0, \; S_u^* = N_0 - S_n^*, \text{and} I_j = R_j = 0 \text{ for all j.} \tag{19}$$

The basic reproduction number, $\mathcal{R}_0$, was calculated by using the next generation matrix method developed by Van den Driessche and Watmough (2002). $\mathcal{R}_0$ is

$$\mathcal{R}_0 = (A \times S_u^* + B \times S_n^*) \times C, \tag{20}$$

where

$$\begin{aligned} A =& \gamma(\omega+\gamma) + (\gamma\eta_w + \omega\eta_c p_I)F_I, \\ B =& \big(\omega + (F_I+\gamma)\eta_w\big)\gamma + \frac{(\eta_w\gamma + \eta_c\omega)\omega p_I F_I}{\omega+\gamma}, \\ C =& \frac{\beta/\gamma}{N_0(\gamma(\omega+\gamma) + F_I(\gamma+\omega p_I))}. \end{aligned}$$

Note that the block matrix G_{12} does not influence $\mathcal{R}_0$ defined as the spectral radius of G. All matters here are the eigenvalues of G_{11}, which are 0 and $\mathcal{R}_0$ (20).

2.2 Sensitivity of $\mathcal{R}_0$ with respect to the underlying parameters

note 1. The tricky one is the $\partial\mathcal{R}_0/\partial\rho$; Following JD comment and some analysis, we can prove that analytically.

Jonathan's comment: "There should be a logical route to a clear proof about ρ. The total amount of time spent in I_x when starting from a given starting point should not depend on ρ (or anything but γ). And this should be reflected in the column sums of V^{-1}. Since both of the η's are < 1, all we should then need to prove is that the (nonzero) values on the first row of V^{-1} decrease with ρ to show that the elements of FV^{-1} decrease with ρ, which presumably shows that $\mathcal{R}_0$ decreases as well. This needs to be filled ink but should work."

note 2. Notice that matrix F is a rank one matrix, thus FV^{-1} is of a rank one. This is why FV^{-1} has only one non-zero eigenvalue. Specifically, F is a rank one means that it can be written as matrix product of a column vector and a row vector as follows

$$F = \beta/N_0 \begin{bmatrix} S_u \\ S_n \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1\eta_w\eta_w\eta_c \end{bmatrix}.$$

Proposition 1. Let f and g be smooth real functions such that (i) $f(x) \leq g(x)$ for all $x \in (x_0, x_0+\epsilon)$ and (ii) $f(x_0) = g(x_0)$, then the definition of the derivative yields

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0+h) - g(x_0)}{h} \leq \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} = g'(x_0).$$

Based on the note 2, Also notice that FV^{-1} is of rank one matrix since it can be written as matrix product of a column vector and a row vector. Thus, the next generation matrix $G = FV^{-1}$ has only one non-zero eigenvalue given by the trace G [2].

Thus,

$$\text{trace}(G) = \text{trace}(G_{11}) \leq \frac{\beta}{\gamma}\frac{1}{N_0}(S_u^* + S_n^*) \;\; \forall \rho \in (0, \epsilon). \tag{21}$$

It is easy to show that $\text{trace}(G) = \frac{\beta}{\gamma}\frac{1}{N_0}(S_u^* + S_n^*) = 1/\gamma$. Thus, following the Prop.1, $\partial\mathcal{R}_0\rho \leq 0$.

References

Van den Driessche, P. and Watmough, J. (2002). Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Mathematical biosciences*, 180(1-2):29–48.

[2]see the argument in `https://www.physicsforums.com/threads/eigenvalues-of-a-rank-1-matrix.682216/`