KTH ROYAL INSTITUTE OF TECHNOLOGY

SF2822 Applied Nonlinear Optimization

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Project Assignment 1C

Group 1C4

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Abstract

This project addresses a nonlinear optimization problem faced by the petrochemical company Oljeblandarna in planning the production and refining of crude oil over a three-week period. The objective is to maximize expected profit by optimizing purchases, conversions, and sales under an uncertain supply of crude oil and nonlinear storage losses. The problem is first modeled as a deterministic nonlinear program incorporating random supply modeled via a normal distribution, followed by an extension with nonlinear storage effects. The solution process involves mathematical modeling, implementation in GAMS, and interpretation of the resulting production strategy. The report analyzes the efficiency of the proposed plan and explores the potential for global optimality.

Contents

1 Introduction

1.1 Background

Planning and optimization play a crucial role in the petrochemical industry, particularly when managing complex production processes involving multiple stages, limited resources, and uncertain supply. This project considers a real-world inspired scenario where a company must develop a production and purchasing plan over a fixed planning horizon. The company converts crude oil into refined products through a multi-stage process, making decisions that influence profitability under operational and market constraints.

The problem combines elements from classical blending and production planning models. Still, it introduces realistic complexities, such as uncertain input availability and nonlinear losses during storage, that require a nonlinear optimization approach. These aspects reflect actual challenges faced in industry and offer an opportunity to apply advanced optimization methods to support better decision-making.

1.2 Problem description

This project focuses on determining an optimal production plan over multiple time periods that maximizes expected profit. This involves decisions on how much of each type of crude oil to purchase, how to allocate it across processing stages, and how to manage intermediate and final products through storage and sales, all while operating under a set of constraints.

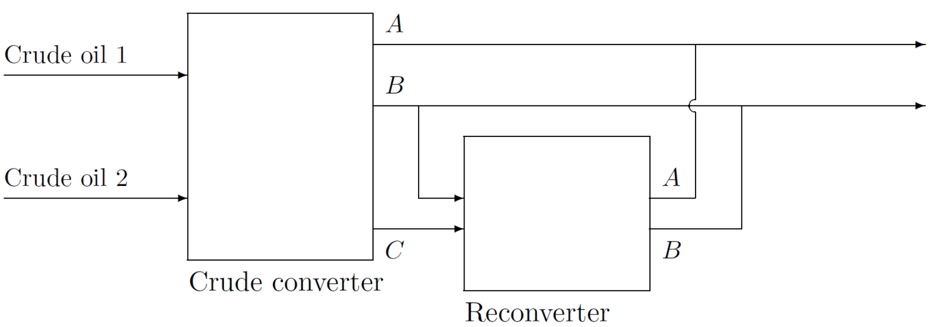

Figure 1: Simplified schematic of process.

A simplified schematic of the overall process is shown in Figure 1. It illustrates the main flow from crude oil inputs through the conversion and reconversion stages, highlighting some of the decision points, for example, the ones regarding processing and selling of intermediate products.

Crude oil is transformed into intermediate products through an initial conversion stage. Some of these products can be sold directly, while others may be refined further or stored for future use. Each processing stage and storage decision comes with operational costs and capacity limitations. Market constraints restrict how much of each product can be sold at different times, and prices vary depending on the product and timing.

One of the main complexities is the uncertain availability of one crude oil type, which is modeled as a random variable with a known probability distribution. This uncertainty requires decisions to be made in advance for the entire planning horizon, introducing a stochastic component into the optimization problem. Additionally, one of the intermediate products deteriorates in quantity during storage, introducing a nonlinear effect that must be accurately represented in the model.

To address these challenges, the problem is formulated as a nonlinear programming (NLP) model. The nonlinearities arise from both the expected value calculations under uncertain supply and the storage loss function, which depends quadratically on the amount stored. This mathematical formulation allows for a more realistic and effective optimization of the production plan, capturing the dynamic and uncertain nature of the real-world system.

2 Problem formulation

2.1 Definition of variables and parameters

There are different variables concerning our problem. It is obvious that we have to choose the values of the four variables for each week, and the state of the system is fully determined. More specifically, if for every week, we choose the values of purchased oil 1 and 2, the unit of product B to flow through the reconverter, and the amount of product B to store (at the end of each week), we can fully determine other values of the process. Although for a more precise definition of the problem and achieving all the values automatically from the GAMS, we also introduce some dummy variables. It is obvious that the value of these dummy variables could be determined manually using the constraints and main variables. The graphical representation of the main variables of our problem is presented in Figure 2.

- x_{1t}: Crude oil 1 purchased — Variable
- x_{2t}: Crude oil 2 purchased — Variable
- q_{At}: Product A sold — Dummy
- q_{Bt}: Product B sold directly each week — Dummy
- s_{Bt}: Product B stored at the end of week t — Variable
- r_{Bt}: Product B sent to the reconverter — Variable
- r_{Ct}: Product C sent to the reconverter — Dummy
- S_t: Supply of crude oil 1 at weak t

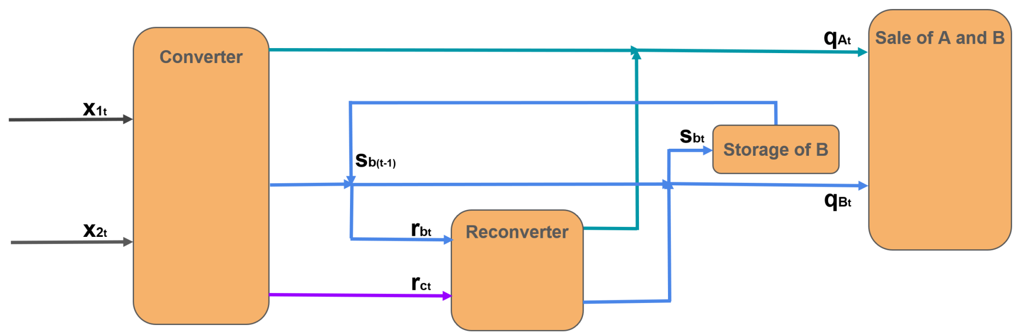

Figure 2: Graphical representation of the variables.

2.2 Objective function definition

Without supply uncertainty, the optimization problem can be modeled as a linear problem as follow:

$$\max \sum_{t=1}^{3} [1000q_{At} + 740q_{Bt} - (500x_{1t} + 600x_{2t} + 100(x_{1t} + x_{2t}) + 80(r_{Bt} + r_{Ct}) + 20s_{Bt})] \tag{1}$$

This objective function consists of two parts:

- Income due to the sale of product A and B: $1000q_{At} + 740q_{Bt}$
- Operational Costs: $500x_{1t} + 600x_{2t} + 100(x_{1t} + x_{2t}) + 80(r_{Bt} + r_{Ct}) + 20s_{Bt}$

But in reality, there are different types of uncertainty. One of the most crucial is the uncertainty in supply. Based on the problem definition, the supply of crude oil 1 is modeled as a stochastic process, where the weekly supply is treated as a normally distributed random variable with a mean of 300 units and a standard deviation of 20 units.

Furthermore, due to technical reasons the decision on purchased quality and production plan for all three weeks has to be determined before the time period in hand. But, in the case of a possible shortage of Crude oil 1, Oljeblandarna may purchase additional quantities of Crude oil 1 at the price of $700kr/unit$. Considering all of this, we can incorporate the expected cost of crude oil 1 into our objective function.

$$\max \sum_{t=1}^{3} [1000q_{At} + 740q_{Bt} - ([Cost_{crude1,t}] + 600x_{2t} + 100(x_{1t} + x_{2t}) + 80(r_{Bt} + r_{Ct}) + 20s_{Bt})] \tag{2}$$

The cost each week for Crude 1 can be calculated as follows:

$$\underbrace{500 \min\{x_{1t},\ S_t\}}_{\text{regular supply at 500 kr}} + \underbrace{700\,(x_{1t} - S_t)^+}_{\text{spot purchase if } S_t < x_{1t}}$$

This can be further simplified:

$$500x_{1t} + (700 - 500)\,(x_{1t} - S_t)^+ = 500x_{1t} + 200\,(x_{1t} - S_t)^+$$

Because S_t is a random variable, the cost is random. Our objective now is to maximize the expected profit, so we need the expected cost:

$$\mathbb{E}[\text{Crude 1 cost}] = 500\,x_{1t} + 200\,\mathbb{E}\left[(x_{1t} - S_t)^+\right]$$

In this way, the final objective will be:

$$\max \sum_{t=1}^{3} \Big[1000q_{At} + 740q_{Bt} - \big(500\,x_{1t} + 200\,\mathbb{E}[(x_{1t} - S_t)^+] \\ +600x_{2t} + 100(x_{1t} + x_{2t}) + 80(r_{Bt} + r_{Ct}) + 20s_{Bt}\big)\Big] \tag{3}$$

The expected term in the objective is the loss function that can be evaluated using PDF and CDF of the normalized variable z:

$$z = \frac{x - \mu}{\sigma}$$

with $S_t \sim \mathcal{N}(\mu_t, \sigma_t^2)$:

$$\mathbb{E}[(x - S)^+] = (x - \mu)\,\Phi\,(z) + \sigma\,\phi\,(z) \tag{4}$$

where:

- $\phi(z)$ is the standard normal PDF:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z^2\right) \tag{5}$$

- $\Phi(z)$ is the standard normal CDF:

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} t^2\right) dt = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{z(t)}{\sqrt{2}}\right)\right) \tag{6}$$

2.3 Constraints

In our model, the constraints are designed to reflect the operational and physical limitations of the production system while ensuring feasibility across all three time periods. Each week, the total amount of crude oil processed through the converter is limited by a maximum capacity, which represents the technical throughput the system can handle. Additionally, the amount of each crude type purchased is bounded by upper limits, simulating the maximum availability or procurement agreements for those raw materials.

The model also respects downstream constraints related to product transformation and flow. The quantities of final product A and by-product B generated from crude processing are governed by fixed yield coefficients. Product A can be sold up to a maximum weekly cap, representing demand or contractual limitations. Product B, which can either be sold, stored, or reprocessed, must be carefully tracked over time. This requires a balance equation that ensures all produced and carried-forward inventory of B is either sold, reconverted, or stored, with a final condition that no inventory of B remains at the end of the planning horizon. Finally, the reconverter is subject to its own weekly capacity constraint, which restricts how much of B and C can be reprocessed in each period. Together, these constraints enforce a realistic and manageable flow of materials through the system while capturing intertemporal dependencies. Our constraints are graphically presented in Figure 3 and mathematically formulated as follow:

$$x_{1t} \leq 300, \quad x_{2t} \leq 300 \tag{7}$$

$$x_{1t} + x_{2t} \leq 500 \tag{8}$$

$$q_{At} = 0.5x_{1t} + 0.7x_{2t} + 0.9r_{Bt} + 0.4r_{Ct}, q_{At} \leq 250 \tag{9}$$

$$r_{Ct} = 0.2x_{1t} + 0.1x_{2t} \tag{10}$$

$$0.3x_{1t} + 0.2x_{2t} + s_{B(t-1)} - r_{Bt} + 0.1r_{Bt} + 0.6r_{Ct} = q_{Bt} + s_{Bt} \tag{11}$$

$$r_{Bt} + r_{Ct} \leq 300 \tag{12}$$

$$s_{B0} = s_{B3} = 0 \tag{13}$$

$$x_{1t},\ x_{2t},\ s_{Bt},\ r_{Bt} \geq 0 \tag{14}$$

$$q_{B1} \leq 30, \quad q_{B2} \leq 130, \quad q_{B3} \leq 130 \tag{15}$$

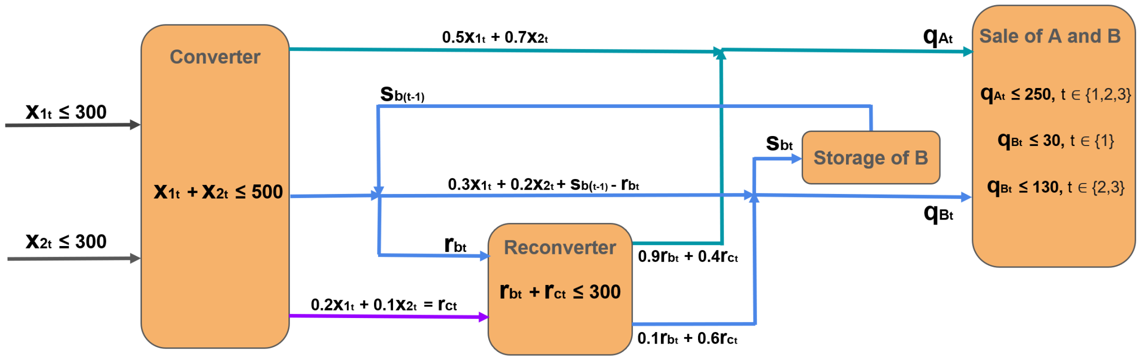

Figure 3: Graphical representation of the constraints.

2.4 Advanced formulation

In this new formulation of our problem, not all of the product B stored can be used in the following week, unlike in the initial formulation. This is due to the fact that some of the oil becomes waste during storage. Therefore, a nonlinear dependency in storage must be included in the model. This results in a modification of the previous constraint (11) and also requires the addition of a new constraint that limits the maximum quantity of product B that can be stored. The new constraints are:

$$0.3x_{1t} + 0.2x_{2t} + s_{B(t-1)} - 0.01s_{B(t-1)}^2 - r_{Bt} + 0.1r_{Bt} + 0.6r_{Ct} = q_{Bt} + s_{Bt} \tag{16}$$

$$s_{Bt} \leq 100 \tag{17}$$

All constraints of the new problem formulation are presented in Figure 4.

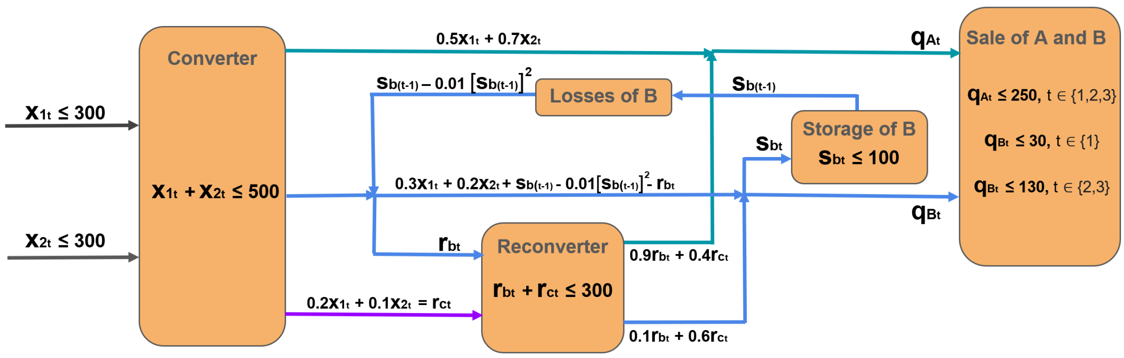

Figure 4: Graphical representation of the constraints of the new problem formulation.

2.5 Mathematical formulation of the NPL models

Taking into consideration the objective function and all the constraints required for the basic and advanced NPL models discussed in the previous sections, it is now possible to fully define the respective problems:

$$
(P_{\text{Bas}}) \left[\begin{array}{ll}
\max & \sum_{t=1}^{3} \Big[1000q_{At} + 740q_{Bt} - \big(500\, x_{1t} + 200\, \mathbb{E}[(x_{1t} - S_t)^+] + 600x_{2t} + \\
& +100(x_{1t} + x_{2t}) + 80(r_{Bt} + r_{Ct}) + 20s_{Bt}\big)\Big] \\
& \\
\text{s.t} & 0 \le x_{1t} \le 300, \quad \forall t \in \{1,2,3\} \\
& 0 \le x_{2t} \le 300, \quad \forall t \in \{1,2,3\} \\
& s_{Bt} \ge 0, \quad t \in \{1,2\} \\
& s_{Bt} = 0, \quad t \in \{0,3\} \\
& r_{Bt} \ge 0, \quad \forall t \in \{1,2,3\} \\
& q_{At} \le 250, \quad \forall t \in \{1,2,3\} \\
& q_{Bt} \le 30, \quad t = 1 \\
& q_{Bt} \le 130, \quad t \in \{2,3\} \\
& x_{1t} + x_{2t} \le 500, \quad \forall t \in \{1,2,3\} \\
& q_{At} = 0.5x_{1t} + 0.7x_{2t} + 0.9r_{Bt} + 0.4r_{Ct}, \quad \forall t \in \{1,2,3\} \\
& r_{Ct} = 0.2x_{1t} + 0.1x_{2t}, \quad \forall t \in \{1,2,3\} \\
& 0.3x_{1t} + 0.2x_{2t} + s_{B(t-1)} - r_{Bt} + 0.1r_{Bt} + 0.6r_{Ct} = q_{Bt} + s_{Bt}, \\
& \quad \forall t \in \{1,2,3\} \\
& r_{Bt} + r_{Ct} \le 300, \quad \forall t \in \{1,2,3\}
\end{array} \right] \tag{18}
$$

$$
(P_{\text{Adv}}) \left[\begin{array}{ll}
\max & \sum_{t=1}^{3} \Big[1000q_{At} + 740q_{Bt} - \big(500\, x_{1t} + 200\, \mathbb{E}[(x_{1t} - S_t)^+] + 600x_{2t} + \\
& +100(x_{1t} + x_{2t}) + 80(r_{Bt} + r_{Ct}) + 20s_{Bt}\big)\Big] \\
& \\
\text{s.t} & 0 \le x_{1t} \le 300, \quad \forall t \in \{1,2,3\} \\
& 0 \le x_{2t} \le 300, \quad \forall t \in \{1,2,3\} \\
& 0 \le s_{Bt} \le 100, \quad t \in \{1,2\} \\
& s_{Bt} = 0, \quad t \in \{0,3\} \\
& r_{Bt} \ge 0, \quad \forall t \in \{1,2,3\} \\
& q_{At} \le 250, \quad \forall t \in \{1,2,3\} \\
& q_{Bt} \le 30, \quad t = 1 \\
& q_{Bt} \le 130, \quad t \in \{2,3\} \\
& x_{1t} + x_{2t} \le 500, \quad \forall t \in \{1,2,3\} \\
& q_{At} = 0.5x_{1t} + 0.7x_{2t} + 0.9r_{Bt} + 0.4r_{Ct}, \quad \forall t \in \{1,2,3\} \\
& r_{Ct} = 0.2x_{1t} + 0.1x_{2t}, \quad \forall t \in \{1,2,3\} \\
& 0.3x_{1t} + 0.2x_{2t} + s_{B(t-1)} - 0.01s_{B(t-1)}^2 - r_{Bt} + 0.1r_{Bt} + 0.6r_{Ct} = \\
& \quad = q_{Bt} + s_{Bt}, \quad \forall t \in \{1,2,3\} \\
& r_{Bt} + r_{Ct} \le 300, \quad \forall t \in \{1,2,3\}
\end{array} \right] \tag{19}
$$

3 Results and analysis

3.1 Basic NLP model

Using the nonlinear programming method to solve the expected profit model results in an optimal profit of 294834.915 kr, and the corresponding optimal purchase and sales plan as shown in Tables 1 and 2 and in Figure 5.

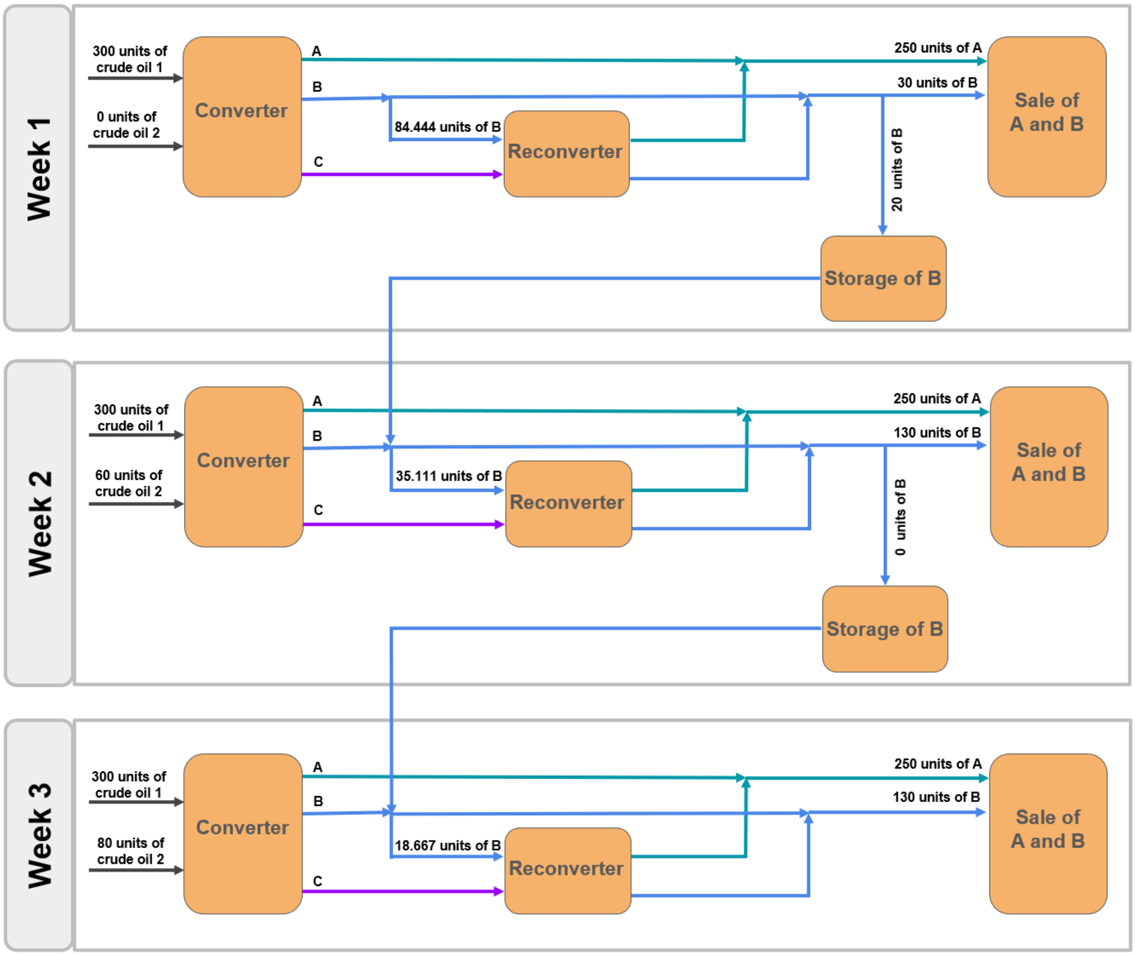

Figure 5: Optimal production flow.

In the case of crude oil 1, even though we are meeting the upper bound of the purchase, the marginal values (Lagrange multipliers) related to this bound is equal to zero this is mainly because of the fact that the real constraint, as we will explain, is the boundary on sales, especially for product A.

Crude Type	Week 1	Week 2	Week 3
1	300.000	300.000	300.000
2	0	60.000	80.000

Table 1: Units of crude purchased.

The bar chart of Figure 6 illustrates the weekly sales of product A and B over the three-week planning horizon, revealing a consistent strategy centered on maximizing A sales while dynamically managing B through a combination of sales and storage due to

Product	Week 1	Week 2	Week 3
Product A	250.000	250.000	250.000
Product B	30.000	130.000	130.000

Table 2: Units of product A and B sold.

constraints. Product A is sold at its weekly upper limit of 250 units across all periods, reflecting its high profitability and the binding nature of the sales cap constraint. In contrast, the upper limit for product B differs from the first week to the subsequent weeks. This means we should find the optimal plan with respect to production and storage of product B relative to this constraint.

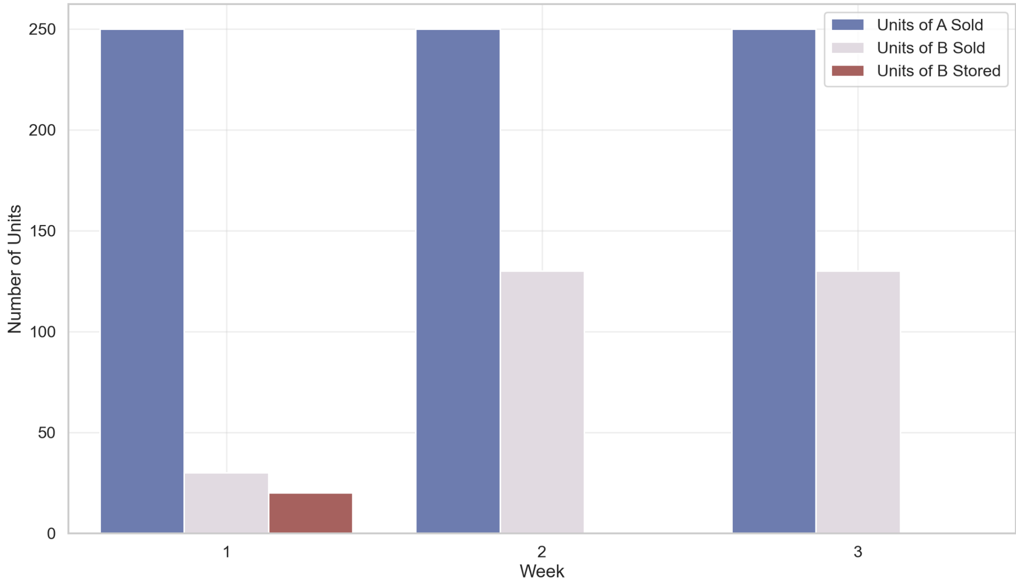

Figure 6: Sales and storage during the three-week period.

Furthermore, Figure 7 gives more insights about the process and possible refinements. From this plot, it can be seen that the company never uses the maximum capacity of the converter. After careful investigation of the NLP model, we concluded that this is mainly due to the limitation on the weekly sales of product A. By finding new markets for this product or organizing a storage facility similar to the one for product B, the company will be able to use the full capacity of the converter.

Another insightful source of information is the flow to and usage of the reconverter. The reconverter never works at its maximum capacity, as can be seen in Figure 8. The proportion of product B sent to the reconverter is 2.5 times higher than week two and more than 4 times higher than that of week three. This has multiple reasons, such as the strict limit on the sales of product B in week one and the fact that the company is not allowed to store the product at the end of week 3.

-	Week 1	Week 2	Week 3
Product B Sent to Reconverter	84.444	35.111	18.667

Table 3: Units of product B sent to reconverter.

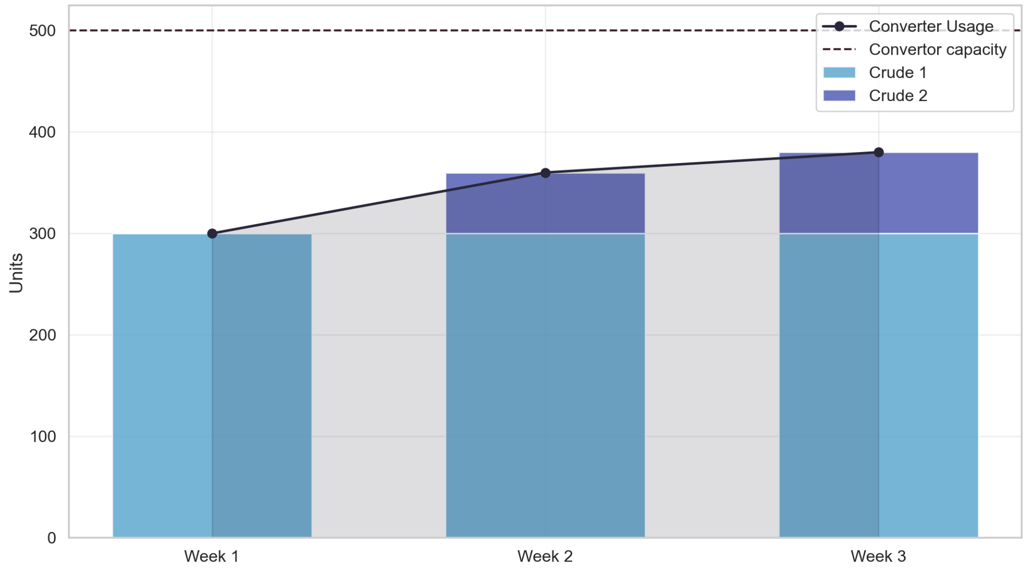

Figure 7: Purchase plan and converter usage.

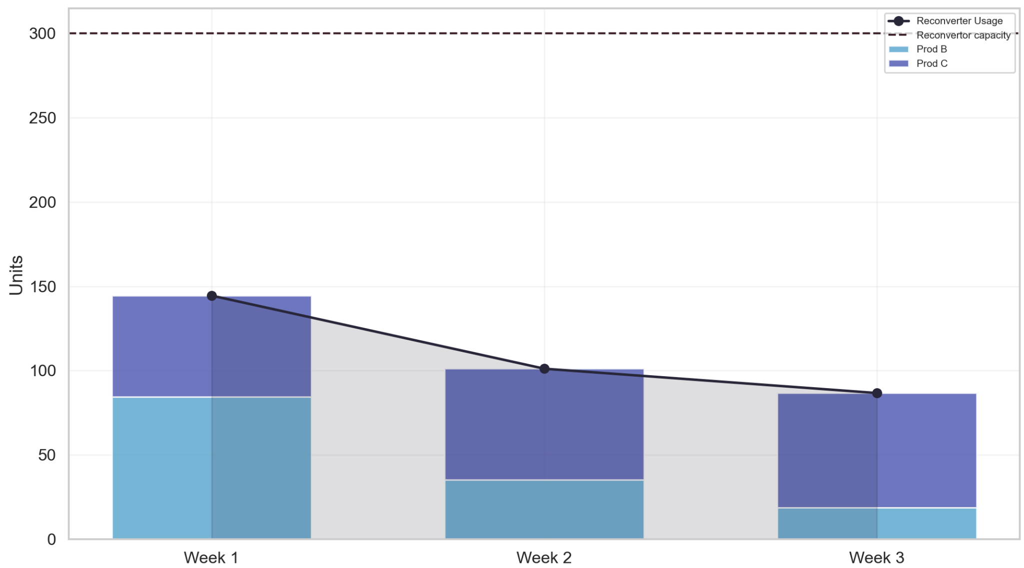

Figure 8: Usage of reconverter.

Investigating the marginal values (Lagrange multipliers) related to some constraints may help use to give better feedback to the company about their operation. The most value comes from further sale of product A. If the company can change the upper limit for the sales of product A, then the best choice is to do it for the first week, which will add 288.8889 kr per unit in revenue. Also in the second and third week, this value is equal to 268.8889 kr per unit of increase. Increasing the sale of product B also adds value to the operation; the highest profit comes from an increase in sales in the first week (117.7778 kr/unit), which has the smallest bound between the three weeks. furthermore.

Furthermore, it is optimal not to use the storage unit at the end of week 2, because the Lagrange multiplier related to this variable is equal to the negative of the cost of storage per unit of product B.

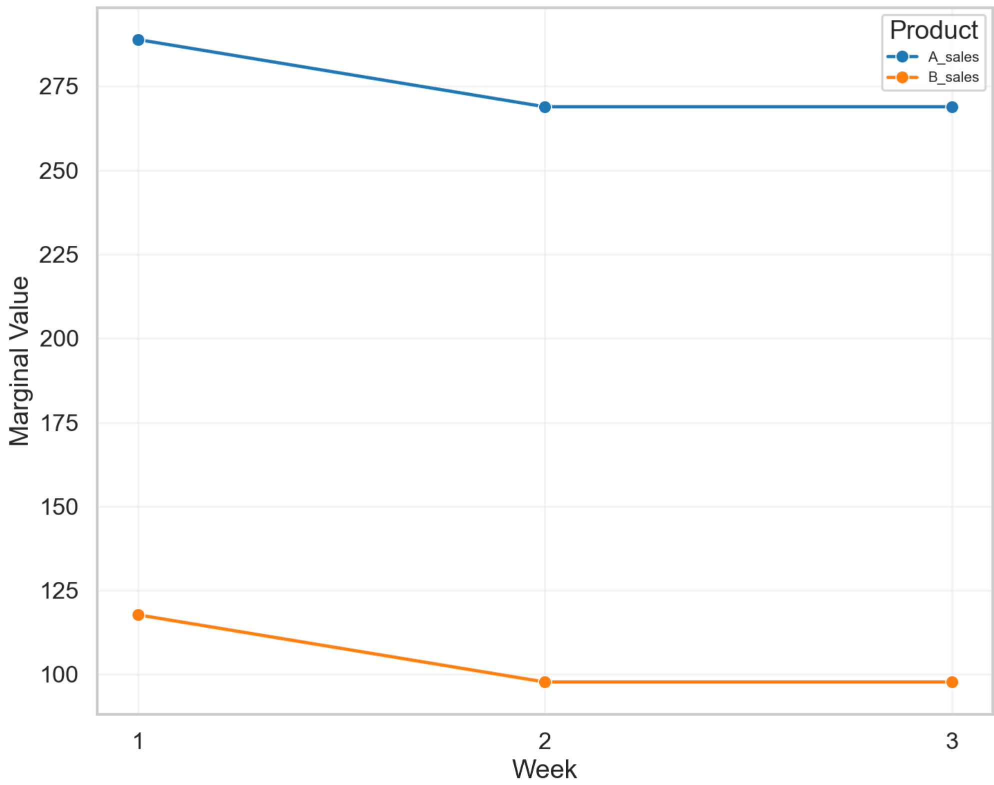

Figure 9: Marginal values related to constraints.

3.1.1 Local and global optimality

The objective function for the basic model is of the form:

$$\text{Profit}(\boldsymbol{x}) = \underbrace{\text{Revenue}}_{\text{linear in } \boldsymbol{qi}} - \underbrace{\left[\text{linear cost} + 200\,\mathbb{E}\left((x-\mu)^+\right)\right]}_{\text{convex in } \boldsymbol{decision\ var.}}.$$

Firstly, $\mathbb{E}\left((x-\mu)^+\right)$ is a convex function of $\boldsymbol{x}$. The reason is the fact that $(x-S)^+$ is a convex function in x and taking its expectation preserves convexity.

Furthermore, in closed form for $S \sim \mathcal{N}(\mu, \sigma^2)$, we have:

$$\mathbb{E}[(x-S)^+] = (x-\mu)\,\Phi\left(\frac{x-\mu}{\sigma}\right) + \sigma\,\phi\left(\frac{x-\mu}{\sigma}\right)$$

which is well-known to be convex in x.

Hence, Cost $=$ (linear terms) $+\,200 \times \mathbb{E}[(x-\mu)^+]$ is a convex function with respect to our decision variables (and the negative of it is concave). Also, Revenue is a linear function of our decision variables.

With regard to all of these, we can conclude that profit (objective function) is concave with respect to our decision variables.

Concerning the constraints, it is obvious that all of them are linear, so the feasible region is a convex polyhedron.

As a result, the NL program tries to maximize a concave function over a convex polyhedron so the local optimum is also **Globally optimal**.

(Here, it should be mentioned that the nonlinear constraints defined in GAMS software for the basic part are used to define auxiliary variables for the expected shortfall. These equations do not restrict the feasible set.)

3.2 Advanced NLP model

Using the same nonlinear programming method to solve the expected profit model results in an optimal profit for this new model of 292266.026 kr, and the corresponding optimal plan is shown in Figure 10.

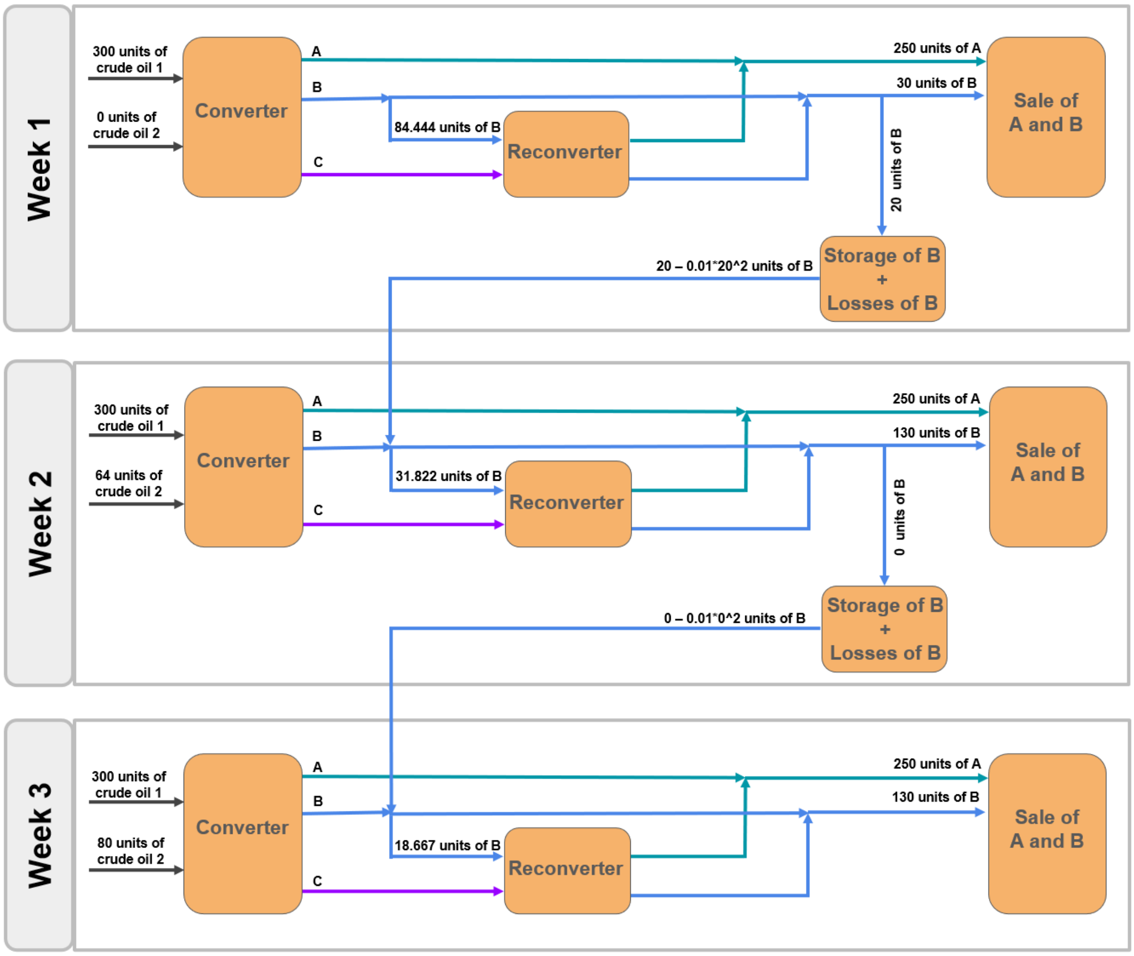

Figure 10: Optimal production flow.

Compared to the basic model, the profit in this case is slightly lower by approximately 2568.89 kr. While the overall purchase and sales patterns remain similar, the slight changes, particularly in the purchase of crude oil 2 in week 2 (from 60.000 to 64.000), as can be seen in Table 4, reflect the model's adaptation to the updated constraints or assumptions.

The optimal plan related to the sales of the products A and B is presented in the bar chart of Figure 11 and in Table 5.

As in the basic model, the sales of product A consistently reach the upper limit of 250 units per week, indicating its high profitability and the binding nature of the sales cap across all three weeks.

Crude Type	Week 1	Week 2	Week 3
1	300.000	300.000	300.000
2	0	64.000	80.000

Table 4: Units of crude purchased.

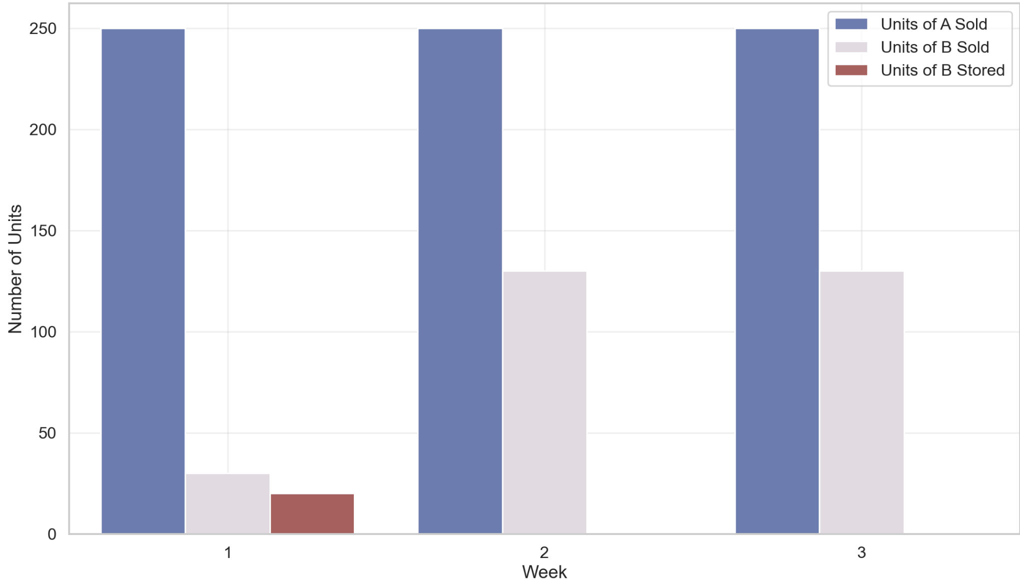

Figure 11: Sales and storage during the three-week period.

Product	Week 1	Week 2	Week 3
Product A	250.000	250.000	250.000
Product B	30.000	130.000	130.000

Table 5: Units of product A and B sold.

For product B, the sales follow the same weekly pattern as in the first model: 30 units in week 1 and 130 units in both week 2 and Week 3. Interestingly, the optimal strategy also involves storing 20 units of product B from week 1 to week 2, just as in the basic model. However, unlike in the first case, where all stored product is eventually used, the second model leads to a situation where not all of the stored product is utilized. Despite this potential for profit loss due to unused inventory, the model still identifies the same storage decision as optimal.

Similarly to the results obtained for the basic problem, in this case, the full capacity of both the converter and reconverter is never utilized, due to the constraint on the maximum units of product A that can be sold weekly, as shown in Figures 12 and 13.

Analyzing Table 6, the quantities of product B to be sent to the reconverter are essentially the same, except for the quantity in week 2, where less product B should be sent to the reconverter. Specifically, the amount decreases from 35.111 units in the optimal solution of the first formulation to 31.882 units in this new formulation.

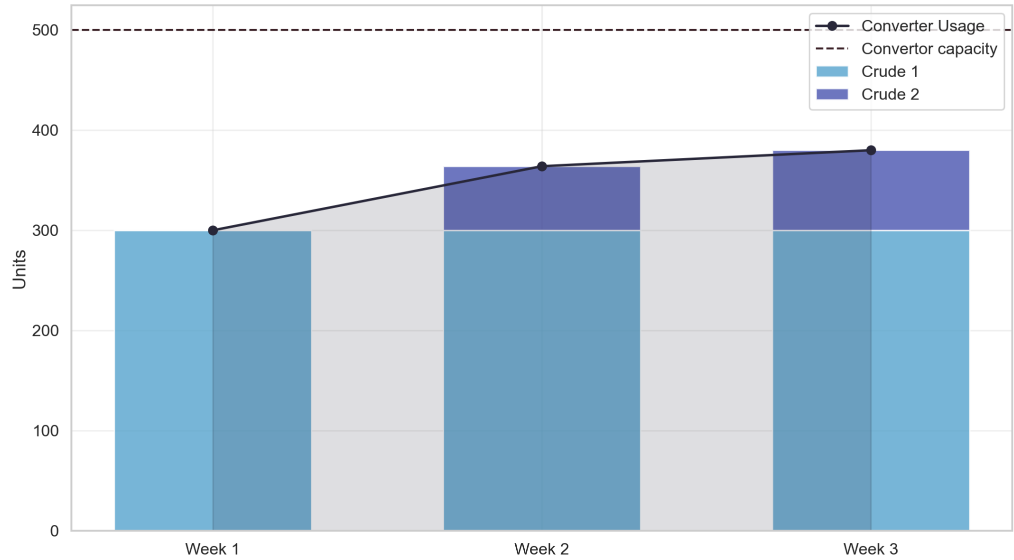

Figure 12: Purchase plan and converter usage.

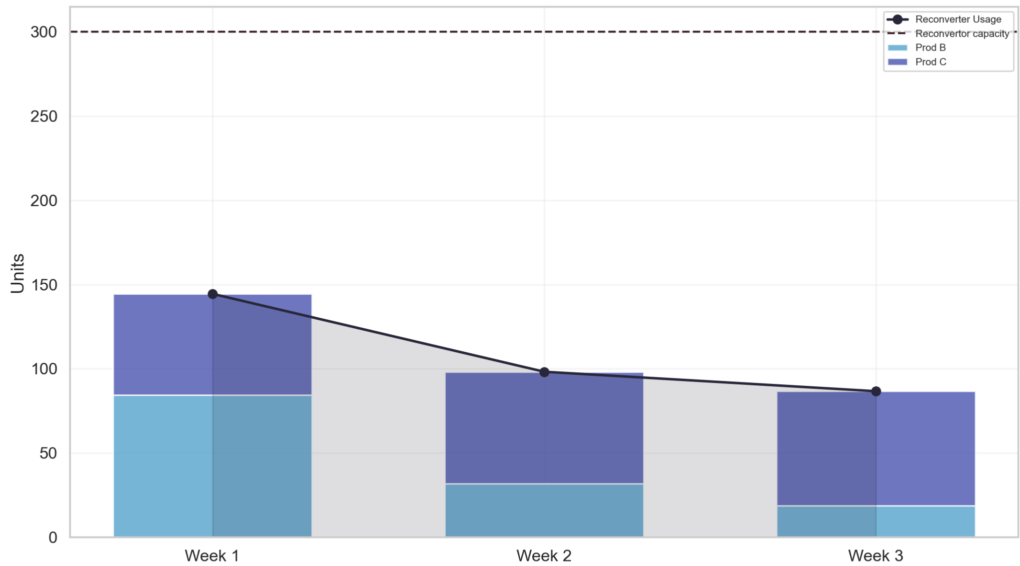

Figure 13: Usage of reconverter.

-	Week 1	Week 2	Week 3
Product B Sent to Reconverter	84.444	31.822	18.667

Table 6: Units of product B sent to reconverter.

Investigating the marginal values obtained, it can be concluded that if the company can increase the upper limit on the sales of product A, the optimal choice would be to do so in the first week. This aligns with the conclusion drawn for the basic problem and would contribute an additional 545.78 kr per unit in revenue. In the second and third weeks, increasing the sales limit of product A would still be beneficial, yielding 268.89 kr per unit of increase in both cases. Similarly, increasing the sales of product B adds value to the operation, with the highest profit achieved in the first week, 374.67 kr per additional unit sold.

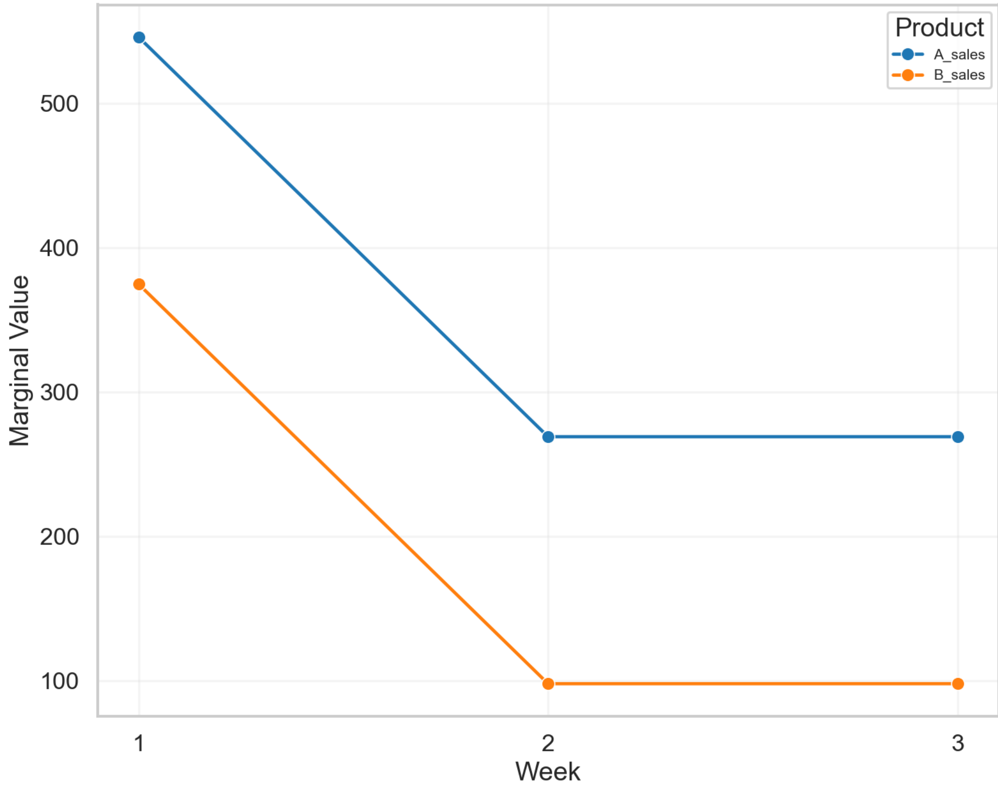

Figure 14: Marginal values related to constraints

3.2.1 Local and global optimality

In the case of an advanced model, things will get more complicated. Based on equation **??**, we introduced a new equality constraint which has the $-0.01\, s_B(t-1)^2$ term. Because $s_B(t-1)$ is a decision variable and appears squared, this constraint is not affine. So this negative quadratic term may jeopardize the convexity of the feasible set. (We should emphasize that this is an **equality** constraint with a non-affine term, so we can almost be certain that the feasible set is nonconvex)
Local solvers like IPOPT, CONPOT, or SNOPT (which we used so far in the GAMS model) will find locally optimal solutions that satisfy KKT conditions. This answer can also be globally optimal (like the case of our basic model), but it is not guaranteed in our advanced model.

To be able to scrutinize our solution furthermore, we suggest three approaches.

- First approach: Using global solvers
  Many global solvers such as BARON and Couenne work internally by constructing convex relaxations of the nonconvex functions (by forming their convex envelopes) and using a branch-and-bound procedure. After solving the model, these solvers will give the optimal solution and the best bound related to this solution. They also report the optimality gap, and if this optimality gap is zero, we can ensure that the

solution is globally optimal. But there are two problems with regard to this approach. Firstly, we need a commercial license to use these solvers. Secondly, there are some limitations for our case. For example, with BARON solver, we cannot use functions like errorf() and exp(). One solution to this can be to estimate the value of the pdf and cdf using series or to discretise them and solve the problem as minlp problem. We should aso ensure that model is formulated in a way that the nonconvex part is relaxable.

- Second approach: choosing random initial starting points
  The second idea is to use different initial feasible starting points and see if the final optimal solution will change. We tried this approach for multiple scenarios, but the final objective and configuration did not change (since values are the same, mentioning them here is unnecessary). This can make us more confident in our results, but it can never be enough. There might still be some initial starting point for which we can achieve a higher objective value. For this reason we also try the third approach

- Third approach: Relaxation of nonconvex constraint and replacing it with its convex envelope
  In this approach, we try to replace the nonconvex term by the convex envelope (Based on the instructions that says this model is only good for the storage capacity of up to 100, then the convex envelope is the linear function connecting the end-points (0, 0) and (100, 0))in the constraint and solve the resulting convex problem using a convex NLP solver. In this way, the objective value from the relaxed problem will be a bound on the best possible value of the original nonconvex problem. If the local optimal solution (from the original problem) is less than or equal to but very close to the optimal solution of the relaxed problem, then it is likely the global solution. Here is an explanation of how we used relaxation.
  The original constraint includes a concave quadratic term $-0.01 \cdot s_B(t-1)^2$, which introduces non-convexity. To relax the problem, we remove the non-convex term $-0.01 \cdot s_B(t-1)^2$, leaving only the linear term $s_B(t-1)$.
  The original term $s_B(t-1) - 0.01 \cdot s_B(t-1)^2$ models a storage decay where stored product B loses value over time. By removing $-0.01 \cdot s_B(t-1)^2$, the relaxed constraint ignores this decay, and the model will be the same as the basic model. This also makes sense physically. In fact our basic model is an upper bound for our advanced model.
  The optimality gap quantifies how far the original solution might be from the global optimum:
  $$\text{Gap (\%)} = \frac{P_{\text{relaxed}} - P_{\text{orig}}}{|P_{\text{relaxed}}|} \times 100.$$
  In our case this value is:
  $$\text{Gap (\%)} = 0.8713$$
  Since this value is really small it suggests the original solution is near-optimal

3.3 Basic NLP model *vs* Advanced NLP model

Comparing the Basic and Advanced NLP models shows that even small changes in how the model is built or what assumptions are used can affect the final results. The Basic model gives a slightly higher profit (294834.915 kr) than the Advanced model (292266.03

kr), even though both follow a similar plan for buying crude oil and selling products. This small difference in profit shows how important it is to design the model carefully, especially when decisions are based on narrow profit margins.

One key difference is how the two models handle storage. Both choose to store 20 units of product B from week 1 to week 2, but in the Basic model, all of it is used later, while in the Advanced model, some of it is not. This means that the Advanced model still sees value in storing the product, even if it's not all used. This highlights how having flexibility in storage and production can be valuable, especially when future demand or constraints are uncertain.

Lastly, both models agree that increasing the sales limit for product A in week 1 would give the biggest boost to profit. However, the value of this increase is much higher in the Advanced model. This shows that changes in the model can affect how valuable certain decisions look. It also reminds us that good modeling helps businesses understand where they can get the most benefit from changes in operations.

4 Potential model improvements

While the current model effectively incorporates nonlinearities and stochasticity in crude oil supply, several enhancements could significantly improve its realism, robustness, and performance. The suggestions below aim to fix some theoretical issues and make the model more useful in real-world situations.

- Handling non-negativity in stochastic supply

  One of the most important improvements concerns the treatment of the stochastic variable representing the supply of crude oil 1, $S_t \sim \mathcal{N}(300, 20^2)$. In the current formulation, the expected cost term is computed using the standard normal distribution's cumulative distribution function (CDF) and probability density function (PDF). While this approach is mathematically valid, it implicitly assumes that S_t can take any real value, including negative ones.

  In practice, however, a negative supply is physically impossible. Although the probability of a negative value is negligible with a mean of 300 and standard deviation of 20, this assumption becomes problematic if the distribution's parameters were to shift, for example, due to geopolitical instability or market fluctuations. A lower mean or higher variance would increase the probability of negative realizations, making the model's results unrealistic or misleading.

  To address this, the distribution can be modified to exclude negative values, effectively restricting $S_t \geq 0$. This adjustment preserves the essential stochastic behavior of the supply while eliminating outcomes that are not physically feasible. Although computing the expected shortage $\mathbb{E}[(x_{1t} - S_t)^+]$ under this adjusted distribution involves slightly more complex calculations, it significantly enhances the robustness and realism of the model.

- Incorporating a robust optimization

  Another way to make the model better is to switch from using average outcomes to using a more cautious approach called robust optimization. Right now, the model assumes that making decisions based on the average expected supply will work well in all situations. But in real life, it's often not the average result that matters most, it's whether the system can handle bad or unexpected situations.

  With robust optimization, the model would prepare for the worst-case scenarios instead of relying on specific guesses about how likely different outcomes are. For example, instead of focusing on reducing the average shortage, the model could try to avoid the highest possible costs within a certain range of uncertainty. This would help avoid being caught off guard by major supply problems and lead to safer, more cautious planning.

- Modeling nonlinear or time-varying demand

  The current model uses fixed upper limits for the weekly sales of products A and B. While this simplifies the formulation, it does not reflect the dynamic nature of real market demand, which often varies with time and price. Introducing nonlinear or time-dependent demand functions could significantly improve the model's realism and decision-making power.

  For instance, the sale price or maximum allowable sales could be modeled as a decreasing function of quantity sold, to simulate price elasticity, or vary week-by-week, to simulate seasonal fluctuations. This would allow the model to better align production and storage strategies with actual market conditions and potentially uncover higher-profit operational plans.

5 Conclusion

In this project, we developed and solved two nonlinear programming models to help a petrochemical company plan its operations under uncertain supply conditions. The goal was to find the best way to purchase crude oil, produce, store, and sell products in order to maximize profit over a three-week period. We started with a basic model that included uncertainty in crude oil supply, and then improved it by adding nonlinear storage losses to make it more realistic.

The results showed that both models led to similar production plans, but the basic model gave a slightly higher profit. This was expected, as the advanced model included more realistic conditions, which made it a bit more restrictive. In both models, product A was always sold at its upper limit due to its high profitability, while product B had to be managed more carefully because of varying sales caps and storage issues. The storage of product B showed how planning for future use can be helpful, even if not all stored units are used later.

We also looked into how changing certain limits, like allowing more sales of product A in week one, could lead to higher profits. This helped us understand which decisions have the most impact and where the company could improve its operations. The analysis of marginal values supported these findings and pointed out where small changes could bring big benefits.

Overall, the project showed how nonlinear optimization can be a powerful tool for solving complex planning problems in industry. It also highlighted the importance of careful modeling and testing different assumptions. By using more advanced features like robust optimization or time-based demand, the models could become even more useful for real-life decision-making.