Optimal Traffic Flow

Project Assignment 2A, Group 4

SF2822 Applied Nonlinear Optimization

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Nomenclature

I Set of nodes (zones) in the network.

A Set of directed arcs (i, j) representing road connections.

q_{ij} Binary parameter indicating whether arc (i, j) exists: $q_{ij} = 1$ if $(i, j) \in A$, 0 otherwise.

t_{ij} Free-flow travel time [min] on arc (i, j).

u_{ij} Capacity [people/hour] of arc (i, j).

ϵ Small constant used to avoid division by zero in the congestion function.

o, d Origin and destination nodes for a given trip.

p_{od} Number of people traveling from origin o to destination d.

x_{odij} Number of people from o to d using arc (i, j).

f_{ij} Total flow [people/hour] on arc (i, j), aggregated over all O-D pairs.

$T_{ij}(f_{ij})$ Congestion-adjusted travel time [min] on arc (i, j).

a_{ij} Congestion sensitivity parameter on arc (i, j).

$G = (I, A)$ Directed graph representing the traffic network.

Abstract

This project addresses the problem of modeling traffic in an urban region consisting of a central downtown area and several surrounding suburban areas. The region is represented as a directed graph, where each zone corresponds to a node, and the roads between the zones are modeled as arcs. Travel demand is primarily from suburban commuters heading to downtown, although inter-suburban travel is also considered. Each arc is associated with a free-flow travel time, reflecting congestion-free travel time. The objective is to model and analyze the distribution of traffic flows in this network, taking into account the network topology and travel demand, and to evaluate strategies for reducing traffic congestion through capacity improvements in the roads. Using optimization techniques, we aim to provide insight into the most effective road improvements and validate model predictions through the case study referred to Stockholm inner region.

Problem Description

Urban commuters do not choose their routes in isolation; every driver added to a busy link impairs the speed of all others. Capturing this feedback loop is fundamental to solving the traffic-assignment problem in optimization. When the objective is to minimize the total travel time experienced by all travellers, it is known as the System-Optimal Traffic Assignment (SOTA) problem, first cast as a convex programme by Beckmann, McGuire & Winsten in the 1950s.[1]

Study area

Our case study focuses on a 14-node abstraction of the Stockholm inner region 1a: node 0 represents the downtown (Norrmalm), and nodes 1-13 represent major suburban districts. Arcs are the principal radial or beltway corridors 1b; their free-flow travel times and practical lane capacities are summarized in 2 (these are real values for the network. In parts of the project for evaluation of different scenarios, other values might be used). Furthermore, based on the population and total number of cars in each node (Table2), some random destinations are assigned to the population of each node based on (Table 3)

(a) Distinct areas

(b) Travel Network

Figure 1: Stockholm Regions

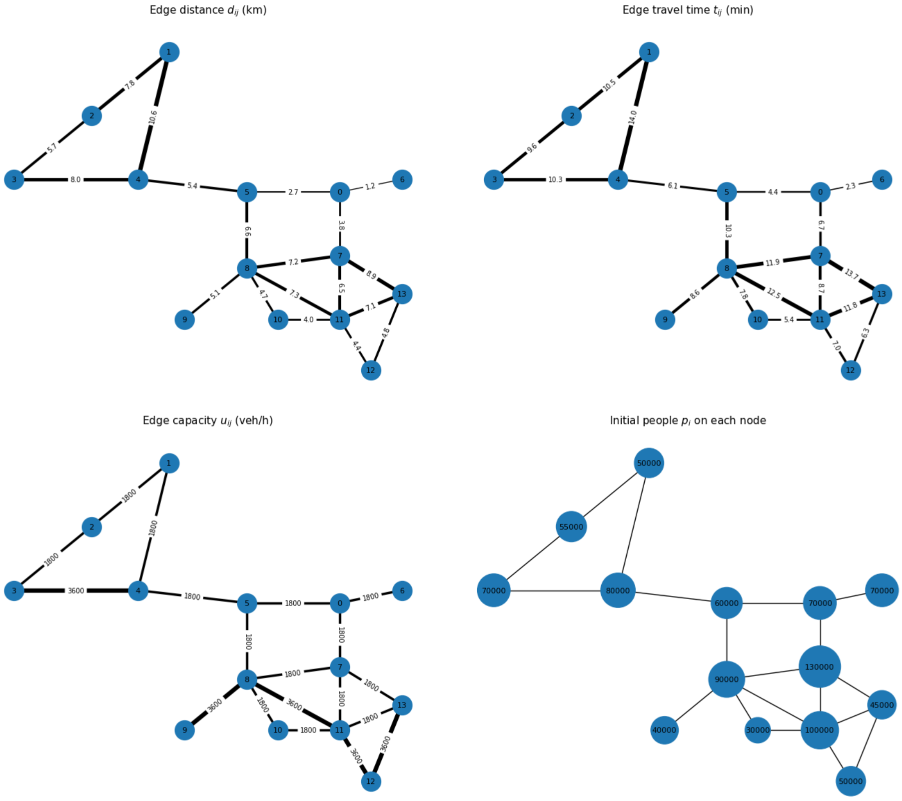

Figure 2: Network parameters

The importance of congestion modelling

If link travel times were fixed, routing would be a simple linear-programming exercise that rarely matches field observations. In reality, empirical studies show that travel time rises sharply as volume approaches capacity; ignoring that effect can understate delay by 30-50% on saturated links in European cities.

The Davidson volume-delay function[2] models how travel time on a road link rises as traffic flow approaches its practical capacity. It augments the free-flow time T_0 with a hyperbolic delay term:

$$T = T_0 + \frac{av}{c - v}, \quad 0 \leq v < c \tag{1}$$

where v is the link flow (veh h^{-1}), c the capacity (veh h^{-1}), and a a single positive parameter. The denominator gives the curve a vertical asymptote at $v = c$, reflecting the abrupt breakdown empirically observed when demand reaches capacity. Davidson (1966) derived the expression from queueing considerations, and subsequent calibration by Akçelik (1991) showed that choosing $a \approx 0.25\,T_0$ makes travel time roughly double at a volume-to-capacity ratio of 0.8.[3]

In summary, the project addresses a canonical optimization problem— System-Optimal Traffic Assignment— augmented with the Davidson congestion law, whose asymptotic behavior makes it well suited to capacity-constrained urban networks

Mathematical Formulation

Fee Flow Formulation

We consider a transportation region consisting of a central downtown area and several surrounding suburban zones. We model a transportation region consisting of a central downtown area and several surrounding suburban zones as a directed graph $G = (I, A)$, where I is the set of nodes, each representing a zone, and A is the set of directed arcs indicating possible travel between zones. Each road connection is bidirectional, so for every arc $(i, j) \in A$, the reverse arc (j, i) is also included in the graph.

The majority of daily travel originates from the suburbs and is directed toward the downtown area, although inter-suburban travel also occurs and must be accounted for. Each arc (i, j) is associated with a baseline travel time t_{ij}, which assumes ideal conditions with no traffic congestion. These travel times are considered known.

The objective of the model is to determine how people should be routed through the network to minimize the total travel time for all trips, given a known pattern of demand between origin-destination.

$$\min \sum_{(o,d,i,j)} x_{odij} \cdot t_{ij} \tag{2}$$

Here, x_{odij} the number of people traveling from origin o to destination d using the arc from node i to node j. To ensure the solution is feasible and respects both the network structure and flow balance, the model includes the constraints described below.

Travelers can only use arcs that actually exist in the network, which is described as

$$x_{odij} \leq a_{ij} \cdot p_{od} \tag{3}$$

where $q_{ij} \in \{0, 1\}$ indicates whether arc (i, j) exists, and p_{od} is the number of people traveling from o to d.

At the origin node o, the total outflow must equal the number of people traveling from that origin

$$\sum_j x_{odoj} - \sum_j x_{odjo} = p_{od} \tag{4}$$

At the destination node d, the total inflow must equal the number of people arriving.

$$\sum_j x_{oddj} - \sum_j x_{odjd} = -p_{od} \tag{5}$$

At all intermediate nodes $i \neq o, d$, flow is conserved

$$\sum_j x_{odij} - \sum_j x_{odji} = 0 \tag{6}$$

Accordingly, the optimization model is defined as

$$
\begin{aligned}
\text{minimize} \quad & \sum_{(o,d,i,j)} x_{odij} \cdot t_{ij} && && (7) \\
\text{subject to} \quad & x_{odij} \leq q_{ij} \cdot p_{od} && \forall o, d, i, j && (8) \\
& \sum_{j:(i,j)\in A} x_{odij} - \sum_{j:(j,i)\in A} x_{odij} = p_{od} && \text{if } i = o && (9) \\
& \sum_{j:(i,j)\in A} x_{odij} - \sum_{j:(j,i)\in A} x_{odij} = -p_{od} && \text{if } i = d && (10) \\
& \sum_{j:(i,j)\in A} x_{odij} - \sum_{j:(j,i)\in A} x_{odij} = 0 && \text{if } i \neq o, d && (11) \\
& x_{odij} \geq 0 && \forall o, d, i, j && (12)
\end{aligned}
$$

Since congestion is not considered, the travel times t_{ij} are fixed, and all roads are assumed to have unlimited capacity. As a result, the objective reduces to computing the shortest path for each traveler. The total travel time is obtained by summing the minimum travel times for all origin-destination pairs, making the problem equivalent to solving multiple independent shortest-path problems.

This problem is affine and convex and a global optimum can therefor easily be obtained.

Congestion Formulation

To realistically model traffic dynamics, congestion effects are explicitly incorporated into the objective function. This is achieved using a modified version of Davidson's function, which adjusts the travel time on each arc based on its total flow:

$$T_{ij}(f_{ij}) = t_{ij} + \frac{a_{ij} \cdot 60 \cdot f_{ij}}{u_{ij} - f_{ij} + \epsilon} \tag{13}$$

Here, t_{ij} denotes the free-flow travel time on arc (i, j), u_{ij} is the maximum capacity of the arc, and a_{ij} is a parameter that determines the sensitivity of congestion.

The total flow on arc (i, j), denoted by f_{ij}, is defined as the sum of all individual origin-destination flows that use that arc:

$$f_{ij} = \sum_{(o,d)} x_{odij} \tag{14}$$

As f_{ij} approaches the arc's capacity u_{ij}, the denominator shrinks and $T_{ij}(f_{ij})$ increases rapidly. When flows are low, the travel time closely approximates the free-flow time. The flow f_{ij} is not allowed to be greater than u_{ij} and the small constant ϵ is included to avoid division by zero when f_{ij} is very close to u_{ij}.

Accordingly, the optimization model is defined as

$$
\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in A} \left(t_{ij} + \frac{a_{ij} \cdot 60 \cdot f_{ij}}{u_{ij} - f_{ij} + \epsilon} \right) \cdot f_{ij} && && (15)\\
\text{subject to} \quad & f_{ij} = \sum_{(o,d)} x_{odij} && \forall (i,j) \in A && (16)\\
& \sum_{j:(i,j)\in A} x_{odij} - \sum_{j:(j,i)\in A} x_{odij} = p_{od} && \text{if } i = o && (17)\\
& \sum_{j:(i,j)\in A} x_{odij} - \sum_{j:(j,i)\in A} x_{odij} = -p_{od} && \text{if } i = d && (18)\\
& \sum_{j:(i,j)\in A} x_{odij} - \sum_{j:(j,i)\in A} x_{odij} = 0 && \text{if } i \neq o, d && (19)\\
& 0 \leq f_{ij} \leq u_{ij} && \forall (i,j) \in A && (20)\\
& x_{odij} \geq 0 && \forall o, d, i, j && (21)
\end{aligned}
$$

Although the objective function is nonlinear, a closer analysis reveals that the overall optimization problem is convex given the structure of the constraints. This implies that any locally optimal solution is also globally optimal, and thus the global optimum can be reliably found.

Results and Analysis

In this chapter, we examine the performance of the transportation network under different modeling assumptions and analyze the resulting flow patterns.

As shown in Table 1, the maximum flow occurs on arc (4,5) in both the congestion-aware and congestion-free models. Given that the minimum arc capacity is 1800, it is not surprising that this high-flow arc remains unchanged. However, the overall flow distribution does shift when congestion is taken into account. For instance, arcs such as (11,8), (11,12), and (11,13), which were unused in the initial model, begin to carry traffic in the congestion-aware formulation. This indicates that congestion encourages the use of alternative routes to alleviate overloaded paths, while in the case of free flow the trivial choice is to take the shortest possible pass.

One could argue from the results that arcs with lower travel times are more heavily used. However, it is not clear from the results how the travel time of each arc influences the paths that are chosen. This may be due to the similarity in travel times across different paths, as well as the limited number of available paths between arcs. Further analysis of the model could investigate how travel time impacts the objective function, the resulting paths, and the load distribution on the arcs.

The results presented in Appendix B could serve as a valuable foundation for planning a bus route. By analyzing travel times, path usage, and arc loads, transit planners can identify the most efficient routes, optimize stop placements, and improve overall service reliability.

The scatter plot 3 shows that most links lie close to the 45-degree line, indicating modest re-routing. But a handful deviate strongly: arc $8 \to 11$ and, to a lesser degree, $2 \to 3$ and $3 \to 4$ pick up substantial traffic once congestion is introduced, whereas the arc $1 \to 4$ and $0 \to 5$ lose flow as vehicles divert to less direct but now faster routes. The network map 4 confirms the same story. After congestion, the thick red edges shift from the original radial corridor toward the square formed by nodes 7, 8 and 11, showing that drivers reroute onto the inner grid when the downtown approaches become slower. (This is because longer passes are now faster than the saturated radial. As a result, some flow shifts there until the costs are balanced again.)

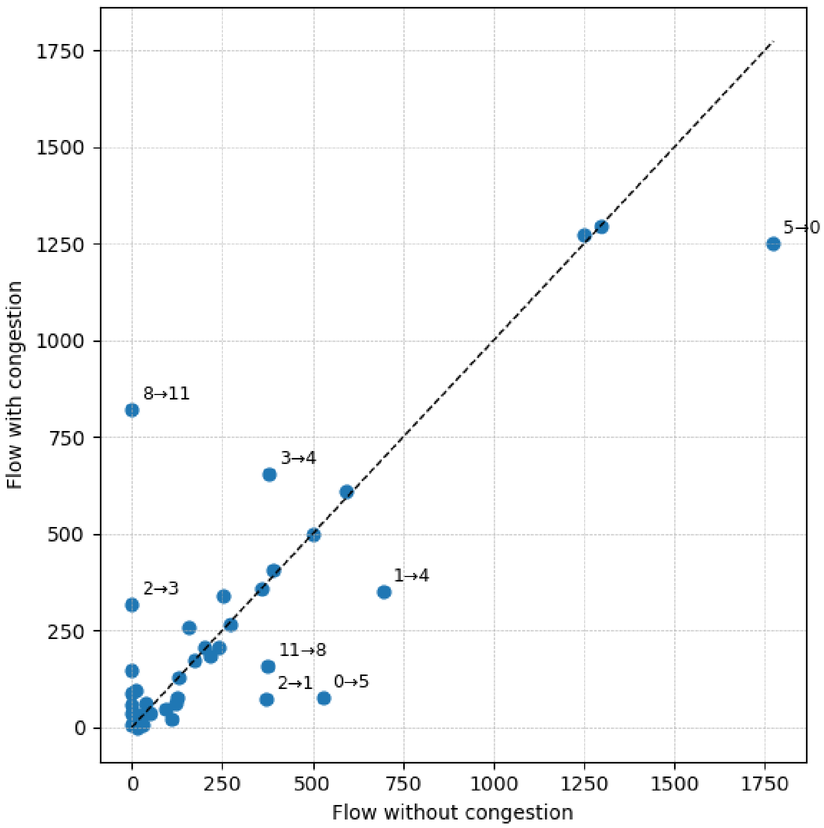

Figure 3: Arc flows: with vs without congestion

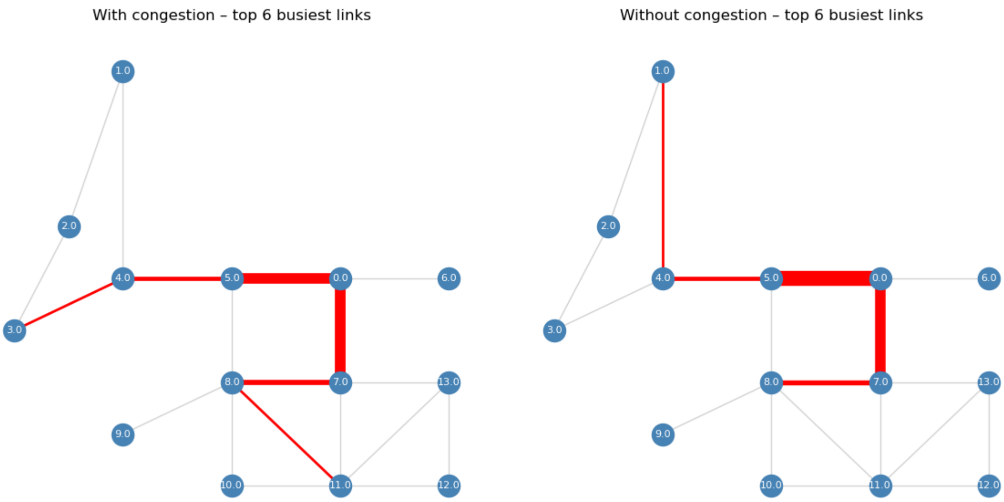

Figure 4: Arc flows: Networks Comparison

From	To	Flow (NLP)	Flow (LP)	Travel Time
0	5	77	528.11	13
0	6	20	20	13
0	7	47.11	94.44	10
1	2	32.87	20	14
1	4	349.49	695.22	30
2	1	72.38	370.11	14
2	3	317.73	0	21
3	2	207.13	200	21
3	4	655.84	378.11	15
4	1	182.87	218	30
4	3	206.13	239	15
4	5	**1296.44**	**1296.44**	21
5	0	1250.39	1773.44	13
5	4	500	500	21
5	8	338.16	253.22	22
6	0	359.11	359.11	13
7	0	1274.61	1250	10
7	8	147	0	6
7	11	22.22	111.1	20
7	13	0	15	16
8	5	405	392	22
8	7	609.26	591.44	6
8	9	130	130	20
8	10	36	49	13
8	11	82.77	0	13
9	8	173.11	173.11	20
10	8	95.41	10	13
10	11	60.45	121.11	27
11	7	259.25	156	20
11	8	160.24	377	13
11	10	37.76	0	27
11	12	5	0	19
11	13	87.97	0	12
12	11	63.25	37.11	19
12	13	78.13	125	21
13	7	266.1	272.11	16
13	11	57.84	0	12
13	12	5.27	31	21

Table 1: Flow comparison between Free flow and Congestion models

Importance of continues modeling

When we model the problem with a continuous variable x_{ij}, the objective and constraints become convex, and the whole problem becomes a convex nonlinear program. Convexity can guarantee a unique global optimum that can be found in polynomial time (for example, using the interior point method).

But if we model the problem as an integer network problem, even with congestion-free conditions and a relatively small network, the problem becomes MINLP and NP-hard to solve[4]. In this case we should use methods like branch and bound to solve the problem.

Global Optimality

Free Flow model
In the case of free flow, because cost coefficients t_{ij} are fixed numbers and every constraint is linear, the whole model is an LP model, whose decision vector is the set of link flows x_{odij}. The LP model minimizes a linear objective over a polyhedral feasible region. This problem is convex, and basic results for convex optimization state that any feasible point that satisfies the complementary slackness condition is a global minimizer to the LP problem (in fact, any local optimum is globally optimal).
Congestion model
Let

$$p_{ij}(f) = f\left(t_{ij} + \frac{a_{ij}f}{u_{ij} - f + \varepsilon}\right), \quad a_{ij} > 0, \quad 0 \le f < u_{ij}, \tag{22}$$

be the cost contribution (single term in the objective function) for each arc that appears in the objective. ($t_{ij} > 0$ is the free-flow time, $u_{ij} > 0$ the capacity, and $\varepsilon > 0$ the small safeguarding constant used in your formulation.)
We can show that $p_{ij}(f)$ is strictly convex on $[0, u_{ij})$ by taking the first and second derivative with respect to flow:

$$p'_{ij}(f) = t_{ij} + \frac{a_{ij}f}{u_{ij} - f + \varepsilon} + \frac{a_{ij}(u_{ij} + \varepsilon)f}{(u_{ij} - f + \varepsilon)^2} > 0 \tag{23}$$

$$p''_{ij}(f) = \frac{2a_{ij}(u_{ij} + \varepsilon)(u_{ij} - f + \varepsilon) + a_{ij}f(2u_{ij} + \varepsilon)}{(u_{ij} - f + \varepsilon)^3} > 0 \quad \text{for } 0 \le f < u_{ij} \tag{24}$$

because every term in the numerator and denominator is positive. We can conclude p_{ij} is strictly convex.
The objective (15) is

$$P(\mathbf{f}) = \sum_{(i,j)} p_{ij}(f_{ij}),$$

a non-negative sum of strictly convex functions, therefore strictly convex.
Constraints (16)–(21) are *affine*:

- equality constraints (flow balance),
- upper/lower bounds $0 \le f_{ij} \le u_{ij}$ and $x_{od}^{ij} \ge 0$.

The feasible set is thus a non-empty, closed, convex polyhedron

$$\mathcal{F} \subset \mathbb{R}^{|\mathcal{A}| + |\mathcal{A}||\mathcal{K}|}.$$

Because P is strictly convex and $\mathcal{F}$ is convex, the optimization problem is a convex nonlinear program (CNP). Fundamental results of convex analysis give: Any feasible point that satisfies the Karush–Kuhn–Tucker conditions is a global minimizer and, owing to strict convexity, is unique.
sufficiency of KKT Conditions
Let $(f^\star, x^\star, \lambda^\star, \mu^\star)$ satisfy the KKT system derived from (15)–(21). Since:

1. P is continuously differentiable and convex,

2. All constraints are affine

The KKT conditions are both necessary and sufficient for optimality [5].

Thus, any solution returned by an NLP solver that reports KKT optimality—for example IPOPT or SNOPT—is guaranteed to be globally optimal for our formulation.

Capacity modifications

To analyze any change in the capacity value of the different roads included in the problem, we should check the values of the obtained marginals in the constraint (20) that controls that the maximum flow of people moving between nodes must be less than the capacity in the roads.

At first, given the values of people used to solve the congestion model, we have obtained that every marginal is cero, which suggested that no change in capacities was needed. However, when we tried to increase these values, considering that the number of people who want to go downtown (node 0) is higher, we have obtained some information from the marginals.

The capacity in arcs (4,5), (5,0), (8,5) and (7,0) should be increased, and we will get a decrease in the objective function of one unit per unit increase in the capacity of these arcs.

Increasing the capacity of these arcs makes sense since, all these arcs are the only paths that most of the nodes have to reach downtown, and also, arcs (5,0) and (7,0) are the ones that go directly to downtown. In addition, we can observe that the capacity value of these arcs is also low (1800 cars), as we can see in Figure 2.

To check also how close our predictions were with the reality, we have checked the traffic situation in real time in the morning, when most people go to the city center, in services like [6]. Thanks to this service we can observe that, as predicted with our model, an increase in road capacity is needed, given by arcs (4,5), (5,0) and (8,5), and a better approximation of the data used would have given us more accurate results with the reality.

Model improvement

The model used and analyzed in this report aims to minimize the total travel time. However, this approach can result in some individuals experiencing very short travel times while others face significantly longer ones. To ensure a more balanced outcome where all travelers have a reasonable travel time, the objective could instead focus on minimizing the longest individual travel time. There is also other properties that could be minimized such as minimizing total CO_2 emissions or fuel consumption across the network. It would also be valuable to further analyze how the capacity u_{ij} and the congestion sensitivity parameter a_{ij} influence overall travel time. Such analysis can provide useful insights for urban planning and inform future decisions regarding road construction and infrastructure development.

To more accurately reflect real traffic conditions, additional factors, such as road pricing and the time of the day should also be considered.

Conclusion

A Linear Programming (LP) model was formulated to solve the non congested version of the urban transportation problem, which aims to minimize the total travel time between areas in an urban network. To account for congestion and bypass capacity constraints, a Nonlinear Programming (NLP) model was developed by incorporating nonlinear terms into the objective function.

Both models were implemented in GAMS. For the congested NLP model, we used the solver SNOPT, which applies the Sequential Quadratic Programming (SQP) method to efficiently handle nonlinear objectives with equality and inequality constraints.

The LP model provided a solution for an ideal case under perfect flow conditions, while the congested model introduced more realistic traffic dynamics. In the congested problem, we can see from the results that there will be a distribution in the arcs of the number of people traveling between the nodes according to the capacity values. This was translated into an increase in the total travel time in the network, compared to the case without congestion, where people traveling from one node to another are allowed to take the most direct paths without considering limitations in capacity.

This modeling approach provides a deeper insight into traffic behavior and helps to identify potential bottlenecks in the network, while at the same time a dual problem analysis allows us to know the improvement obtained by changing the road capacity at the points more affected by congestion.

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Appendix A: Data Tables

Index	Suberb	Total population	Amount of cars
0	Norrmalm	70 000	25 000
1	Rinkeby	50 000	18 000
2	Spånga	55 000	19 800
3	Hässelby	70 000	25 800
4	Bromma	80 000	28 800
5	Kungsholmen	60 000	21 600
6	Östermalm	70 000	25 200
7	Södermalm	130 000	46 800
8	Liljeholmen	90 000	32 400
9	Skärholmen	40 000	14 400
10	Älvsjö	30 000	10 800
11	Enskede	100 000	36 000
12	Farsta	50 000	18 000
13	Skarpnäck	45 000	16 200
Total amount		**1 000 000**	**388 800**

Table 2: Statistical values of regions

Nodes	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	52000	0	2000	1000	2000	900	0	1000	700	1000	0	1111	0	0
2	53000	2000	0	0	2000	800	0	1000	600	1000	0	1111	0	0
3	55000	3000	2000	0	2000	600	0	2000	600	1000	0	1111	0	0
4	56000	2000	2000	900	0	1000	800	2000	700	1000	0	1111	500	0
5	54000	2000	2000	600	2000	0	0	2000	600	1000	800	1111	0	0
6	47000	2000	2000	800	2000	900	0	2000	0	1000	600	1111	0	0
7	60000	3000	2000	1000	3000	700	0	0	600	2000	800	1111	0	0
8	53000	2000	2000	700	2000	800	600	2000	0	1000	800	1111	0	0
9	18000	2000	1000	0	2000	600	0	1000	0	0	600	1111	0	0
10	14000	1000	1000	0	1000	0	0	1000	0	1000	0	1111	0	0
11	54000	2000	2000	900	2000	700	600	2000	800	1000	700	0	0	0
12	14000	1000	1000	0	1000	500	0	1000	0	1000	0	1111	0	0
13	18000	1000	1000	0	1000	600	0	1000	0	1000	600	1111	0	0

Table 3: Flow of people between nodes

Appendix B: Result Tables

Origin	Destination	Demand	Path	People on Path
1	0	260	$1 \to 4 \to 5 \to 0$	152
			$1 \to 4 \to 5 \to 8 \to 7 \to 0$	108
1	2	20	$1 \to 2$	20
1	3	20	$1 \to 2 \to 3$	20
1	4	20	$1 \to 4$	20
1	5	9	$1 \to 4 \to 5$	9
1	7	10	$1 \to 4 \to 5 \to 8 \to 7$	10
1	8	7	$1 \to 4 \to 5 \to 8$	7
1	9	10	$1 \to 4 \to 5 \to 8 \to 9$	10
1	11	11	$1 \to 4 \to 5 \to 8 \to 11$	11
2	0	265	$2 \to 3 \to 4 \to 5 \to 0$	265
2	1	20	$2 \to 1$	20
2	3	20	$2 \to 3$	20
2	4	20	$2 \to 3 \to 4$	20
2	5	8	$2 \to 3 \to 4 \to 5$	8
2	7	10	$2 \to 3 \to 4 \to 5 \to 8 \to 7$	10
2	8	6	$2 \to 3 \to 4 \to 5 \to 8$	6
2	9	10	$2 \to 3 \to 4 \to 5 \to 8 \to 9$	10
2	11	11	$2 \to 3 \to 4 \to 5 \to 8 \to 11$	11

Table 4: Origin–destination paths and the number of people traveling on each path.

Origin	Destination	Demand	Path	People on Path
3	0	275	$3 \to 4 \to 5 \to 0$	275
3	1	30	$3 \to 2 \to 1$	30
3	2	20	$3 \to 2$	20
3	4	20	$3 \to 4$	20
3	5	6	$3 \to 4 \to 5$	6
3	7	20	$3 \to 4 \to 5 \to 8 \to 7$	20
3	8	6	$3 \to 4 \to 5 \to 8$	6
3	9	10	$3 \to 4 \to 5 \to 8 \to 9$	10
3	11	11	$3 \to 4 \to 5 \to 8 \to 11$	11
4	0	280	$4 \to 5 \to 0$	280
4	1	20	$4 \to 1$	20
4	2	20	$4 \to 3 \to 2$	20
4	3	9	$4 \to 3$	9
4	5	10	$4 \to 5$	10
4	6	8	$4 \to 5 \to 0 \to 6$	8
4	7	20	$4 \to 5 \to 8 \to 7$	20
4	8	7	$4 \to 5 \to 8$	7
4	9	10	$4 \to 5 \to 8 \to 9$	10
4	11	11	$4 \to 5 \to 8 \to 11$	11
4	12	5	$4 \to 5 \to 8 \to 11 \to 12$	5

Table 5: Origin–destination paths and the number of people traveling on each path.

Origin	Destination	Demand	Path	People on Path
5	0	270	$5 \to 0$	270
5	1	20	$5 \to 4 \to 1$	20
5	2	20	$5 \to 4 \to 3 \to 2$	20
5	3	6	$5 \to 4 \to 3$	6
5	4	20	$5 \to 4$	20
5	7	20	$5 \to 8 \to 7$	20
5	8	6	$5 \to 8$	6
5	9	10	$5 \to 8 \to 9$	10
5	10	8	$5 \to 8 \to 10$	8
5	11	11	$5 \to 8 \to 11$	11
6	0	235	$6 \to 0$	235
6	1	20	$6 \to 0 \to 5 \to 4 \to 1$	20
6	2	20	$6 \to 0 \to 5 \to 4 \to 3 \to 2$	20
6	3	8	$6 \to 0 \to 5 \to 4 \to 3$	8
6	4	20	$6 \to 0 \to 5 \to 4$	20
6	5	9	$6 \to 0 \to 5$	9
6	7	20	$6 \to 0 \to 7$	20
6	9	10	$6 \to 0 \to 7 \to 8 \to 9$	10
6	10	6	$6 \to 0 \to 7 \to 8 \to 10$	6
6	11	11	$6 \to 0 \to 7 \to 11$	11
7	0	300	$7 \to 0$	300
7	1	30	$7 \to 8 \to 5 \to 4 \to 1$	30
7	2	20	$7 \to 8 \to 5 \to 4 \to 3 \to 2$	20
7	3	10	$7 \to 8 \to 5 \to 4 \to 3$	10
7	4	30	$7 \to 8 \to 5 \to 4$	30
7	5	7	$7 \to 8 \to 5$	7
7	8	6	$7 \to 8$	6
7	9	20	$7 \to 8 \to 9$	20
7	10	8	$7 \to 8 \to 10$	8
7	11	11	$7 \to 11$	11

Table 6: Origin–destination paths and the number of people traveling on each path.

Origin	Destination	Demand	Path	People on Path
8	0	265	$8 \to 7 \to 0$	265
8	1	20	$8 \to 5 \to 4 \to 1$	20
8	2	20	$8 \to 5 \to 4 \to 3 \to 2$	20
8	3	7	$8 \to 5 \to 4 \to 3$	7
8	4	20	$8 \to 5 \to 4$	20
8	5	8	$8 \to 5$	8
8	7	20	$8 \to 7$	20
8	9	10	$8 \to 9$	10
8	10	8	$8 \to 10$	8
8	11	11	$8 \to 11$	11
9	0	90	$9 \to 8 \to 7 \to 0$	90
9	1	20	$9 \to 8 \to 5 \to 4 \to 1$	20
9	2	10	$9 \to 8 \to 5 \to 4 \to 3 \to 2$	10
9	4	20	$9 \to 8 \to 5 \to 4$	20
9	5	6	$9 \to 8 \to 5$	6
9	7	10	$9 \to 8 \to 7$	10
9	10	6	$9 \to 8 \to 10$	6
9	11	11	$9 \to 8 \to 11$	11
10	0	70	$10 \to 8 \to 7 \to 0$	31
			$10 \to 11 \to 13 \to 7 \to 0$	39
10	1	10	$10 \to 8 \to 5 \to 4 \to 1$	10
10	2	10	$10 \to 8 \to 5 \to 4 \to 3 \to 2$	10
10	4	10	$10 \to 8 \to 5 \to 4$	10
10	7	10	$10 \to 11 \to 7$	10
10	9	10	$10 \to 8 \to 9$	10
10	11	11	$10 \to 11$	11

Table 7: Origin–destination paths and the number of people traveling on each path.

Origin	Destination	Demand	Path	People on Path
11	0	270	$11 \to 7 \to 0$	223
			$11 \to 13 \to 7 \to 0$	47
11	1	20	$11 \to 8 \to 5 \to 4 \to 1$	20
11	2	20	$11 \to 8 \to 5 \to 4 \to 3 \to 2$	20
11	3	9	$11 \to 8 \to 5 \to 4 \to 3$	9
11	4	20	$11 \to 8 \to 5 \to 4$	20
11	5	7	$11 \to 8 \to 5$	7
11	7	20	$11 \to 7$	20
11	8	8	$11 \to 8$	8
11	9	10	$11 \to 8 \to 9$	10
11	10	7	$11 \to 10$	7
12	0	70	$12 \to 11 \to 13 \to 7 \to 0$	68
			$12 \to 13 \to 7 \to 0$	2
12	1	10	$12 \to 11 \to 8 \to 5 \to 4 \to 1$	10
12	2	10	$12 \to 11 \to 8 \to 5 \to 4 \to 3 \to 2$	10
12	4	10	$12 \to 11 \to 8 \to 5 \to 4$	10
12	5	5	$12 \to 11 \to 8 \to 5$	5
12	7	10	$12 \to 13 \to 7$	10
12	9	10	$12 \to 11 \to 8 \to 9$	10
12	11	11	$12 \to 11$	11
13	0	90	$13 \to 7 \to 0$	90
13	1	10	$13 \to 11 \to 8 \to 5 \to 4 \to 1$	10
13	2	10	$13 \to 11 \to 10 \to 8 \to 5 \to 4 \to 3 \to 2$	10
13	4	10	$13 \to 11 \to 8 \to 5 \to 4$	10
13	5	6	$13 \to 11 \to 8 \to 5$	6
13	7	10	$13 \to 7$	10
13	9	10	$13 \to 11 \to 8 \to 9$	10
13	11	11	$13 \to 11$	11

Table 8: Origin–destination paths and the number of people traveling on each path.