

# STATISTICAL PATTERN RECOGNITION

## ASSIGNMENT 1

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### Abstract

This is an introductory assignment to the world of *Statistics* and *Probability* in the context of *Pattern Recognition*. We'll introduce some key concepts like *Probability Distribution Function*, *Cumulative Distribution Function*, *Probability Density Function*, *Probability Mass Function*, *Joint Probability Density Function*, *Joint Cumulative Density Function*, *Marginal Density* & more details as the probabilistic point of view. Furthermore, we'll review the concepts of *Expected Value*, *Variance*, *Standard Deviation*, *Covariance & Correlation of Random Variables*(e.g. *Random Vectors*), *Univariate & Multivariate Gaussian Distribution*, *Total Probability & Bayes Theorem*, *Geometric & Mahalanobis Distances*, *Central Limit Theorem*, *Independence & Correlation* as the statistics point of view. Also, a principal concept called *Linear Transformation* is discussed. The relationship between these fields is far more important than each separately.

**Keywords.** *PDF, PMF, JPDF, JPMF, CDF, JCDF, Covariance Matrix, Correlation Coefficient, Correlation, Variance, Expected Vector, Gaussian Distribution, Marginal Probability, Linear Transformation, Eigenvector, Eigenvalue, Rank.*

## 1 Expectation & Variance

A random variable  $X$  has  $E(X) = -4$  and  $E(X^2) = 30$ . Let  $Y = -3X + 7$ . Compute the following.

- (a)  $V(X)$
- (b)  $V(Y)$
- (c)  $E((X + 5)^2)$
- (d)  $E(Y^2)$

## Solution

The main equation to calculate the *Variance* of a random variable  $X$  is given in 1.1.

$$V(X) = E[(X - E[X])^2] \quad (1.1)$$

Expanding the equation 1.1, we'll have the equation 1.2 using simple calculus.

$$\begin{aligned} V(X) &= E[X^2 + E[X]^2 - 2XE[X]] \\ V(X) &= E[X^2] + E[X]^2 - 2E[X]^2 \\ V(X) &= E[X^2] - E[X]^2 \end{aligned} \quad (1.2)$$

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(a) The equation 1.2 can be conducted directly to compute the *Variance*. Replacing the values from the problem description we get the following as result.

$$\begin{aligned} V(X) &= E[X^2] - E[X]^2 \\ V(X) &= 30 - 16 = 14 \end{aligned}$$

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(b) It is important to mention the *Linearity* of *Expectation* operator as formally described in 1.3.

$$E[aX + b] = aE[X] + b \quad (1.3)$$

Using property 1.3, we can write the  $V(Y)$  as

$$\begin{aligned} V(Y) &= V(-3X + 7) = E[(-3X + 7)^2] - E[-3X + 7]^2 \\ V(Y) &= E[9X^2 + 49 - 42X] - E[-3X + 7]^2 \\ V(Y) &= 9E[X^2] + E[49] - 42E[X] - 9E[X]^2 - E[49] \\ V(Y) &= 9 * 30 + 49 - 42 * (-4) - 9 * 30 - 49 \\ V(Y) &= 168 \end{aligned}$$

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(c) Expanding the internals of the expectation, we'll get the following.

$$\begin{aligned} E[(X + 5)^2] &= E[X^2 + 10X + 25] \\ E[(X + 5)^2] &= E[X^2] + 10E[X] + E[25] \\ E[(X + 5)^2] &= 30 + 10 * (-4) + 25 \\ E[(X + 5)^2] &= 15 \end{aligned}$$

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(d) Same as above, we'll use 1.2 to get the following.

$$\begin{aligned} E[Y^2] &= E[(3X + 7)^2] \\ E[Y^2] &= E[9X^2 + 49 - 42X] \\ E[Y^2] &= 487 \end{aligned}$$

## 2 Eigenvector & Eigenvalue

(a) Compute eigenvalues and eigenvectors of  $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 9 & -5 \end{bmatrix}$  and compare your results with Matlab outputs.

(b) A  $2 \times 2$  matrix  $A$  has  $\lambda_1 = 2$  and  $\lambda_2 = 5$ , with corresponding eigenvectors  $V_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$  and  $V_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ . Find  $A$ .

### Solution

(a) First of all, we'll find the *eigenvalues* using the *characteristic equation* given in 2.1.

$$|A - \lambda I| = 0 \quad (2.1)$$

Using this equation, all of the diagonal components of matrix  $A$  will be decremented by a  $\lambda$  term. The determinant of the resulting matrix will be as following.

$$A - \lambda I = \begin{bmatrix} 4 - \lambda & 0 & 0 \\ 0 & 2 - \lambda & 2 \\ 0 & 9 & -5 - \lambda \end{bmatrix}$$

$$|A - \lambda I| = (4 - \lambda)(\lambda^2 + 3\lambda - 28)$$

This will result in the following two roots for the  $\lambda$  term.

$$\boxed{\lambda_1 = 4} \quad \boxed{\lambda_2 = -7}$$

These are the *eigenvalues* of the given matrix  $A$ . We'll continue using the *Gaussian Elimination* technique to compute the *eigenvectors* of matrix  $A$ . According to the equation 2.2 we are looking for all possible vectors that can be substitute with vector  $X$ .

$$|A - \lambda I|X = 0 \quad (2.2)$$

Thus, for each *eigenvalue* determined in the previous computations, we'll find the proper *eigenvector* by converting the equation 2.2 to a *Row Echelon Form* and solving the resulting linear system by *Back Substitution*. Using the computed *eigenvalues*, we'll have 2.3 and 2.4

$$A - 4I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 9 & -9 \end{bmatrix} \quad (2.3)$$

$$A + 7I = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 9 & 2 \\ 0 & 9 & 2 \end{bmatrix} \quad (2.4)$$

Now we can find the proper  $X$  for each *eigenvalue*, using *augmented* version of 2.3 and 2.4. We'll have 2.5 and 2.6 as result.

$$E_1 = (A - 4I \mid 0) = \left[ \begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 9 & -9 & 0 \end{array} \right] \quad (2.5)$$

$$E_2 = (A + 7I \mid 0) = \left[ \begin{array}{ccc|c} 11 & 0 & 0 & 0 \\ 0 & 9 & 2 & 0 \\ 0 & 9 & 2 & 0 \end{array} \right] \quad (2.6)$$

The above matrices will be converted to the *Row Echelon Form* below using *Row Operations*.

$$E_1 = \left[ \begin{array}{ccc|c} 0 & 9 & -9 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 9 & -9 & 0 \end{array} \right]$$

$$E_1 = \left[ \begin{array}{ccc|c} 0 & 9 & -9 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$E_1 = \left[ \begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$E_1 = \left[ \begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The resulting equation for  $E_1$  will be as following.

$$(0)(X_1) + (1)(X_2) - (1)(X_3) = 0$$

Thus, we'll have the following *eigenvector* and *eigenvalue*.

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ X_2 \\ X_2 \end{bmatrix}$$

$$X = X_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda = 4$$

Calculating the *eigenvector* using the same procedure, we'll get the following results.

$$E_2 = (A + 7I \mid 0) = \left[ \begin{array}{ccc|c} 11 & 0 & 0 & 0 \\ 0 & 9 & 2 & 0 \\ 0 & 9 & 2 & 0 \end{array} \right] \quad (2.7)$$

Using *Row Operations* we'll have:

$$E_2 = \left[ \begin{array}{ccc|c} 11 & 0 & 0 & 0 \\ 0 & 9 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Since  $E_2$  is in the *Row Echelon Form*, we can infer the equation for  $E_2$  as:

$$(11)(X_1) + (1)(X_2) = 0$$

For the *eigenvector* we'll have:

$$X = X_2 \begin{bmatrix} \frac{-1}{11} \\ 1 \\ 0 \end{bmatrix}, \quad \lambda = -7$$

(b) Since  $A$  is a  $2 \times 2$  matrix, we have:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Given *eigenvectors* and *eigenvalues* in the problem description, we can conduct the *Characteristic Equation* presented in 2.2 and get the following results.

$$(A - 2I)X = \begin{bmatrix} a - 2 & b \\ c & d - 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(A - 5I)X = \begin{bmatrix} a - 5 & b \\ c & d - 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Simply solving these equations will give us the following results.

$$\boxed{a = 2} \quad \boxed{b = 3} \quad \boxed{c = 0} \quad \boxed{d = 5}$$

Finally, we'll rewrite the matrix  $A$  replacing the new parameters in it.

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$$

### 3 Probability Density Function

Let

$$f(x, y) = \begin{cases} c(x + y^2) & \text{if } 0 < x < 1 ; 0 < y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find each of the following.

- (a)  $c$
- (b)  $f_X(x)$
- (c)  $f_{X|Y}(y)$
- (d) Are  $X$  and  $Y$  independent?
- (e) What is the probability  $Pr(X < \frac{1}{2} | Y = \frac{1}{2})$ ?

#### Solution

(a) The *Joint Probability Distribution* is given above. It is obvious that the total probability is always equal to 1; Hence we can integrate over the variables in the *JPDF* and that will give us the *Total Probability* which is equal to 1.

$$\int_0^1 \int_0^1 f(x, y) \, dx \, dy = 1$$

Replacing the given equation we'll have the following.

$$\int_0^1 \int_0^1 c(x + y^2) \, dx \, dy = 1$$

$$\int_0^1 \left( \frac{c}{2} + cy^2 \right) dy = 1$$

$$\frac{c}{2} + \frac{c}{3} = 1$$

$$\boxed{c = \frac{6}{5}}$$

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(b) *Marginal Density* can easily be computed by integrating over the variable of not interest.

$$f_X(x) = \int_0^1 cx + cy^2 \, dy$$

$$f_X(x) = \int_0^1 \frac{6x}{5} + \frac{6y^2}{5} dy$$

$$\boxed{f_X(x) = \frac{6x+2}{5}}$$

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(c) While having the relation for the *Joint Probability* and the ease of finding *Marginal Density* of each of the variables, we can simply find the conditional probability using the *Bayes* theorem.

$$f_{X|Y}(x) = \frac{f(x,y)}{g(y)} = \frac{\text{joint density of } x \text{ \& } y}{\text{marginal density of } y} \quad (3.1)$$

The  $g(y)$  can be found by integrating the *JPDF* over the variable  $X$ .

$$g(y) = \int_0^1 \frac{6}{5}(x + y^2) dx$$

$$g(y) = \frac{6y^2}{5} + \frac{3}{2}$$

Finally, the  $f_{X|Y}(x)$  will be:

$$f_{X|Y}(x) = \frac{\frac{6}{5}(x + y^2)}{\frac{6y^2}{5} + \frac{3}{2}}$$

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(d) In the case the  $X$  and  $Y$  are independent, the conditional probability  $f_{X|Y}(x)$  is equal to  $f_X(x)$ . In the previous section, we calculated the  $f_{X|Y}(x)$  using *Bayes* theorem. We can simply find  $f_X(x)$  from the *JPDF* by integrating over  $Y$ . If the results are equal:

$$f_{X|Y}(x) = f_X(x)$$

we say that  $X$  and  $Y$  are independent. Computing the integral of *JPDF* over  $Y$  we get:

$$f_X(x) = \frac{6x+2}{5}$$

So the variables  $X$  and  $Y$  are not *independent*.

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(e) Given the conditional density introduced in 3.1, we'll have:

$$Pr(X < \frac{1}{2} \mid Y = \frac{1}{2}) = \frac{f(X < \frac{1}{2}, Y = \frac{1}{2})}{g(\frac{1}{2})}$$

$$Pr(X < \frac{1}{2} \mid Y = \frac{1}{2}) = \frac{\frac{3}{36}}{\frac{3}{20}}$$

$$Pr(X < \frac{1}{2} \mid Y = \frac{1}{2}) = \frac{1}{6}$$