Statistical Pattern Recognition
Assignment 3

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Abstract

In this paper, we'll review the *parametric* techniques to estimate the *unknown* parameters of data distributions. We'll use, *MLE* and *Bayesian* estimation for *parameter estimation*. Also, we'll delve into the *non-parametric* techniques to estimate the unknown *density* of data distribution. We'll use *Kernel Density Estimation* methods such as *Parzen Windows* and other techniques such as *Histogram* and *k-NN* density estimation.

Keywords. *Parameter Estimation, Density Estimation, Non-parametric Methods, Parametric Methods, Kernel Density Estimation, Maximum Likelihood Estimation, Bayesian Estimation, Histogram Density Estimation, K-NN Density Estimation.*

1 General Maximum Likelihood Estimation

Let x_k, $k = 1, 2, ..., N$ denote independent training samples from one of the following densities. Obtain the Maximum Likelihood estimate of θ in each case.

(a) $f(x_k;\theta) = \frac{x_k}{\theta^2}\exp(\frac{-x_k^2}{2\theta^2})$ where $x_k \geq 0$ and $\theta \geq 0$

(b) $f(x_k;\theta) = \sqrt{\theta}x_k^{\sqrt{\theta}-1}$ where $0 \leq x_k \leq 1$ and $\theta \geq 0$

Solution

(a) Substituting the given density inside the *MLE* equation yields the following results.

$$\hat{\theta} = \arg\max_{\theta}\{P(D|\theta)\} = \arg\max_{\theta}\{\sum_{k=1}^{n}\ln P(x_k|\theta)\} \tag{1.1}$$

$$\hat{\theta} = \arg\max_{\theta}\{\sum_{k=1}^{n}\ln\frac{x_k}{\theta^2}\exp(\frac{-x_k^2}{2\theta^2})\}$$

$$\nabla_{\theta} l(\theta) = 0$$

where $l(\theta)$ is $\sum_{k=1}^{n} \ln \frac{x_k}{\theta^2} \exp(\frac{-x_k^2}{2\theta^2})$ in this case. Performing the gradient on the given equation yields the following results.

$$\sum_{k=1}^{n} (\frac{-2}{\theta} + \frac{x_k^2}{\theta^3}) = 0$$

The simplified estimate of unknown θ is given below.

$$\hat{\theta} = \sqrt{\frac{\sum_{k=1}^{n} x_k^2}{2N}}$$

(b) Substituting the given density in the Maximum Likelihood method yields the following result.

$$\hat{\theta} = \arg\max_{\theta} \{\sum_{k=1}^{n} \ln \sqrt{\theta} x_k^{\sqrt{\theta}-1}\} \tag{1.2}$$

we can obtain the estimate for the unknown θ:

$$\nabla_{\theta} l(\theta) = 0$$

where $l(\theta)$ is $\sum_{k=1}^{n} \ln \sqrt{\theta} x_k^{\sqrt{\theta}-1}$ in this case. Performing the gradient on the given equation yields the following results.

$$\frac{n}{2\theta} + \frac{1}{2\sqrt{\theta} \sum_{k=1}^{n} \ln x_k} = 0$$

multiplying the whole equation by θ results in the following equation:

$$\hat{\theta} = \frac{n^2}{(\sum_{k=1}^{n} \ln x_k)^2}$$

2 Uniform Maximum Likelihood Estimation

Let x have a uniform density

$$f_x(x|\theta) \sim U(0, \theta) = \begin{cases} \frac{1}{\theta} & 0 \leq x \leq 0 \\ 0 & otherwise \end{cases}$$

(a) Suppose that n samples $D = x_1, x_2, ..., x_n$ are drawn independently according to $f_x(x|\theta)$. Show that the maximum likelihood estimate for θ is $max[D]$.

(b) Suppose that n = 5 points are drawn from the distribution and the maximum value of which happens to be $max x_k = 0.6$. Plot the likelihood $f_x(D|\theta)$ in the range $0 \leq \theta \leq 1$. Explain in words why you do not need to know the values of the other four points.

Solution

(a) Substituting the uniform density function in the Maximum Likelihood method yields the following results.

$$\hat{\theta} = \arg\max_{\theta} \{\sum_{k=1}^{n} \ln \frac{1}{\theta}\} \tag{2.1}$$

This equation can be written as

$$\hat{\theta} = \arg\max_{\theta} \{l(\theta)\}$$

where $l(\theta) = \sum_{k=1}^{n} \ln \frac{1}{\theta}$. Performing a gradient on $l(\theta)$ would give us the Maximum Likelihood Estimate of θ.

$$\sum_{k=1}^{n} \frac{-1}{\theta} = 0 \rightarrow \frac{n}{\theta} = 0$$

thus

$$\hat{\theta} \rightarrow \inf$$

Since $\hat{\theta} \rightarrow \inf$ and $\hat{\theta} \epsilon \{x_1, x_2, ..., x_n\}$ we'll have:

$$\hat{\theta} = max[D]$$

(b) Since $\hat{\theta} = max[D]$ we can simply plot the diagram as following.

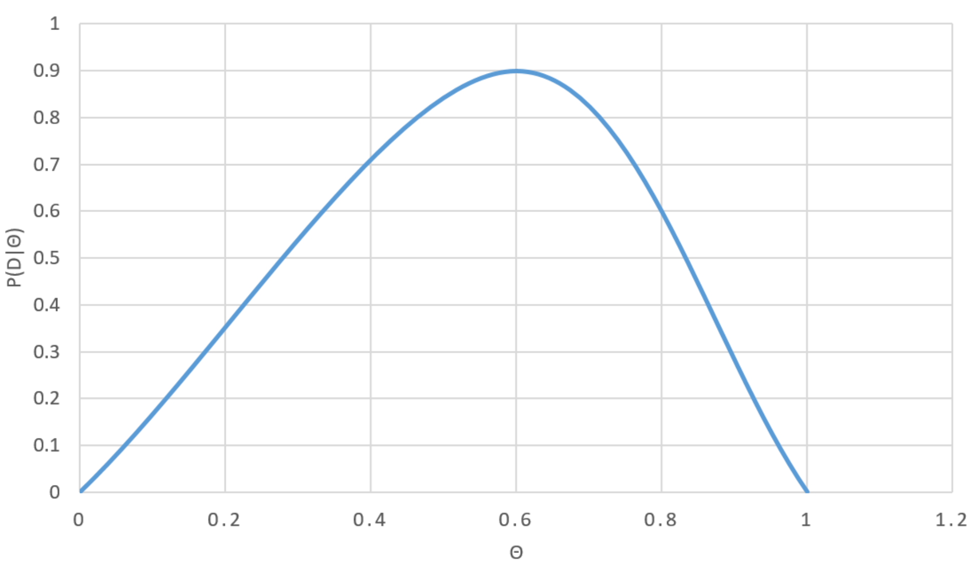

Figure 2.1: Maximum Likelihood Estimation of θ.

The other four points wouldn't have maximized the likelihood $P(D|\theta)$.

3 Density Estimation; Histogram, Parzen Windows and K-NN Method

In this section, we'll be implementing top non-parametric methods for density estimation. We've implemented the 1-D and 2-D Scenario of density estimation in the *src* folder. The experiment results are provided here.

Histogram Density Estimation

Core Definition

In this method, we'll divide the range of available samples to multiple bins. Then we'll count the number of samples in each bin. Let k_i be the number of sample in ith bin and V the size of bins($V = h^d$ where h is the size of the bin in each dimension). The density for the ith bin can be estimated using the following formula. n is the number of total samples.

$$\hat{p}_{(x)} = \frac{k}{n * V} \tag{3.1}$$

Python Implementation

Full implementation with guiding comments can be found in *src* folder. Note that in this implementation i've not used the internal bindings for kernel density estimation of Sklearn.

- (1-Dimensional) — ($\mu = 5$) — ($var = 3$) — ($bin = 2$) — ($|D| = 100$)

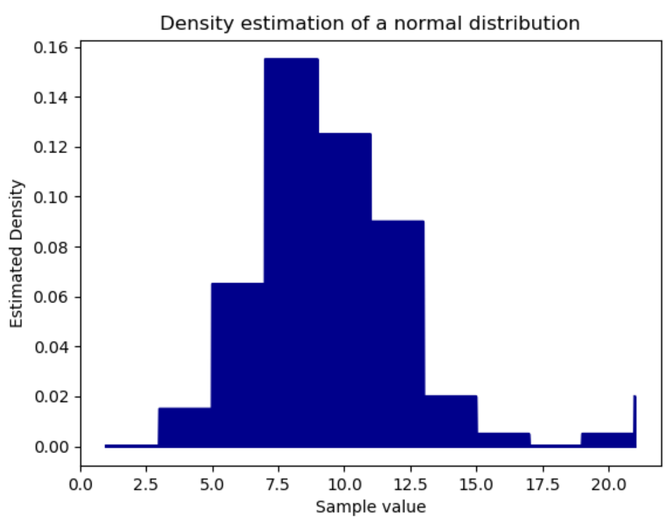

Figure 3.1: 1-D Histogram Density Estimation.

- (2-Dimensional) — ($\mu = \begin{bmatrix} 8 & 8 \end{bmatrix}$) — ($var = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$) — ($bin = 1$) — ($|D| = 5000$)

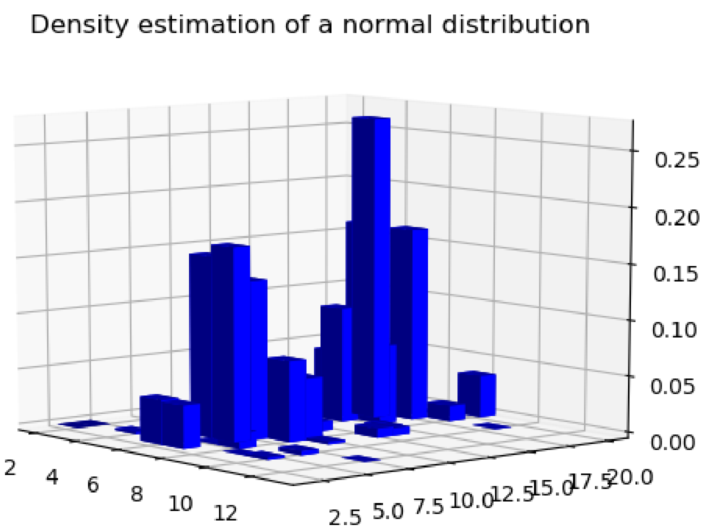

Figure 3.2: 2-D Histogram Density Estimation.

Density Estimation with Parzen Windows