Homework 6 Solution

Problem 1. *Prove the following:*

(a) *Let $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ be independent symmetric Bernoulli random vectors in $\mathbb{R}^n$, i.e. all coordinates X_i and Y_i areindependent random variables that take values ±1 with probability 1/2. Show that:*

$$\mathbb{P}\{|\langle X, Y\rangle| \geq 0.001n\} \leq 2exp(-cn)$$

(b) *Deduce that the angle $\angle(X, Y)$ between the vectors X and Y satisfies*

$$\mathbb{P}\{\left|\angle(X, Y) - \frac{\pi}{2}\right| \geq 0.01\pi\} \leq 2exp(-c_1 n)$$

(c) *Prove that for every dimension $n > C$, there exist $N \geq \frac{1}{2}exp(cn)$ vectors $v_1, \dots, v_N$ in $\mathbb{R}^n$ so that all pairwise angles between these vectors are between $89°$ and $91°$.*

(a) for every i, X_iY_i is a symmetric Bernoulli random variable, and the inner product can be written as a sum of independent symmetric Bernoulli random variables.Applying Hoeffding's inequality:

$$\mathbb{P}\{\left|\sum_{i=1}^{n} X_iY_i\right| \geq 0.001n\} \leq 2exp(-5.10^{-7}n)$$

.

(b) Denoting the angle with θ, $\cos\theta = \frac{\langle X,Y\rangle}{n}$. Further

$$\begin{aligned}\mathbb{P}\{\left|\theta - \frac{\pi}{2}\right| \geq 0.01\pi\} &= \mathbb{P}\{|\cos\theta| \geq \sin(0.01\pi)\}\\ &= \mathbb{P}\{|\langle X, Y\rangle| \geq n\sin(0.01\pi)\}\\ &\leq 2exp(-c_1 n) \;\; (Hoeffding's)\end{aligned}$$

(c) Denoting the angle between every pair of vectors as θ_{ij}.

$$\mathbb{P}\{\left|\theta_{ij} - \frac{\pi}{2}\right| \geq \frac{\pi}{180}\} \leq 2exp(c'n)$$

Thus the probability of existing a pair with an angle not lying in the given interval is bounded above by $2exp(-c'n)N^2$ using union bound. We want at least $\frac{exp(cn)}{2}$ such vectors. Setting an upper bound of 0.01, we want

$$exp(n(2c - c')) \leq 2.10^{-2}$$

picking $c < \frac{c'}{2}$, the inequality holds for $n > C$ for some C since the exponential function is not bounded.

□

Problem 2. *Prove the following:*

(a) *Let z be a fixed unit vector in $\mathbb{R}^d$. Show that the $X = Gz$ is a random vector in $\mathbb{R}^n$ whose all coordinates X_j are independent random variables, which satisfy*

$$\mathbb{E}X_j = 0, \quad Var(X_j) = 1, \quad \|X_j\|_{\psi_2} \leq C_1$$

(b) *Prove a thin-shell inequality for $X = Gz$:*

$$\mathbb{P}\{0.99\sqrt{n} \leq \|X\|_2 \leq 1.01\sqrt{n}\} \geq 1 - 2exp(-cn)$$

(c) *Let $x_1, \ldots, x_N$ be any set of fixed vectors in $\mathbb{R}^d$. Let G be an $n \times d$ Bernoulli random matrix, and set $T = \frac{G}{\sqrt{n}}$. Prove that if $n = ClogN$, then the map T approximately preserves the pair-wise geometry of the data, namely that following event holds with positive probability:*

$$0.99\|x_i - x_j\|_2 \leq \|Tx_i - Tx_j\|_2 \leq 1.01\|x_i - x_j\|_2 \quad \textit{for all } i, j \in [N]$$

(a) Denoting the ith row of G with G_i

$$\mathbb{E}X_j = \mathbb{E}G_j z = \sum_{i=1}^{n} \mathbb{E}G_{ji} z_i = 0$$

$$Var(X_j) = \mathbb{E}X_j^2 = \sum_{k,i=1}^{n} \mathbb{E}[G_{ji}G_{jk}]z_i z_k = \sum_{i=1}^{n} z_i^2 = 1 \ \ (unit)$$

For the third one, we know that X_j is a sum of independent symmetric Bernoulli random variables with coefficients z_i, thus we can apply Hoeffding's inequality, which yields

$$\mathbb{P}\{|G_j z| \geq t\} \leq 2exp(\frac{-t^2}{2\|z\|_2^2}) = 2exp(\frac{-t^2}{2})$$

which implies that $\|X_j\|_{\psi_2} \leq \sqrt{2}$

(b)

$$\mathbb{E}\|X\|_2^2 = \sum_{i=1}^{n} \mathbb{E}X_j^2 = n$$

$$\mathbb{P}\{\big|\|X\|_2^2 - n\big| \geq 0.01n\} = \mathbb{P}\left\{\left|\sum_{i=1}^{n}(X_i^2 - 1)\right| \geq 0.01n\right\}$$

As shown before, each X_i is subgaussian, which makes $X_i^2 - 1$ sub-exponential for each i.

Applying Bernstein's inequality

$$\mathbb{P}\left\{\left|\sum_{i=1}^{n}(X_i^2 - 1)\right| \geq 0.01n\right\} \leq 2exp(-c\min(\frac{10^{-4}n^2}{\sigma^2}, \frac{0.01n}{k})) \tag{1}$$

Where

$$\sigma^2 = \sum_{i=1}^{n} \|X_i - 1\|_{\psi_1}^2 \leq nC$$

$$k = \max_{i\in[n]} \|X_i - 1\|_{\psi_1}^2 \leq C$$

since all variables are sub-exponential. Substituting in (1), we have

$$\mathbb{P}\{\big|\|X\|_2^2 - n\big| \leq 0.01n\} \geq 1 - 2exp(-c'n)$$

which implies that

$$0.99\sqrt{n} \leq \|X\|_2 \leq 1.01\sqrt{n}$$

with probability at least $1 - 2exp(-c'n)$

(c) Fix a pair x_i and x_j. Normalizing $\frac{x_i - x_j}{\|x_i - x_j\|}$, we can apply the thin-shell we found in previous section, yielding

$$\mathbb{P}\left\{0.99\sqrt{n} \leq \frac{\|Gx_i - Gx_j\|_2}{\|x_i - x_j\|_2} \leq 1.01\sqrt{n}\right\} =$$
$$\mathbb{P}\left\{0.99\|x_i - x_j\|_2 \leq \|Tx_i - Tx_j\|_2 \leq 1.01\|x_i - x_j\|_2\right\} \geq$$
$$1 - 2exp(-cn)$$

Applying union bound on all pairs, the probability of existing a pair not lying in the interval is bounded above by $2exp(2\log N - cn)$. Setting $n \geq \frac{4\log N}{c}$ would give us a high confidence.

$\square$

Problem 3.

(a) Check that the Gram matrix G of any system of vectors is positive semidefinite.

(b) Conversely, prove that any $n \times n$ positive semidefinite matrix G is a Gram matrix of some system of vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$.

(a) Assume we have a set of vectors $v_1, \ldots, v_n$ whose gram matrix is G.

$$V = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$G = \begin{bmatrix} \langle v_1, v_1 \rangle & \ldots & \langle v_1, v_n \rangle \\ \vdots & \ddots & \vdots \\ \langle v_n, v_1 \rangle & \ldots & \langle v_n, v_n \rangle \end{bmatrix} = VV^\top$$

let x be a non-zero vector.

$$x^\top Gx = x^\top VV^\top x = (V^\top x)^\top V^\top x = \langle V^\top x, V^\top x \rangle = \|V^\top x\|^2 \geq 0$$

where $\|.\|$ denotes the norm induced by the inner product.

(b) According to the real spectral theorem, we can pick an orthonormal basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ and write

$$G = V\Sigma V^\top$$

where V is the column vector $V = [v_1, \dots, v_n]$ and Σ is the diagonal matrix of eigenvalues. expanding the relation, we get $G = \sum_{i=1}^n \lambda_i v_i v_i^\top$. Setting $A = \sum_{i=1}^n \sqrt{\lambda_i} v_i v_i^\top$,

$$A^2 = \sum_{i,j=1}^n \sqrt{\lambda_i \lambda_j} v_i \langle v_i, v_j \rangle v_j^\top = \sum_{i=1}^n \lambda_i v_i v_i^\top = G$$

Note that the square root is real-valued since all eigenvalues are non-negative. Further, it's easy to check that $A^2 = AA^\top$, which is the gram matrix of the rows of A as shown in part (a). So G is the gram matrix of the rows of A.

$\square$

Problem 4. *Let θ be a random vector uniformly distributed on the unit circle $S^1 = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$. Prove that for any pair of vectors $u, v \in S^1$, we have*

$$\mathbb{E}\text{sign}(\langle u, \theta \rangle)\text{sign}(\langle v, \theta \rangle) = \frac{2}{\pi}\text{arcsin}(\langle u, v \rangle)$$

Setting $\angle(u, v) = \alpha$, we have $\alpha = \cos^{-1}(\langle u, v \rangle)$

$$\begin{aligned} \mathbb{E}\text{sign}(\langle u, \theta \rangle)\text{sign}(\langle v, \theta \rangle) &= 1 - \frac{2\alpha}{\pi} \\ &= \frac{2}{\pi}\sin^{-1}(\langle u, v \rangle) \end{aligned}$$

$\square$