Homework 8 Solution

Problem 1.

(a) Compute the operator and Frobenius norms of the identity matrix.

(b) Compute the operator and Frobenius norms of the matrix whose all entries = 1.

(a) $\|I\|_F = n$,$\|I\| = 1$

(b) $\|A\|_F = n^2$,$\|A\| = n$

Problem 2. *Let A be an $n \times n$ symmetric matrix.*

(a) Show that
$$\|A\| = \max_{i=1,\dots,n} |\lambda_i(A)|$$
where $\lambda_i(A)$ denote the eigenvalues of A.

(b) Show that $\|A\| = \max_{x \in S^{n-1}} |x^\top Ax|$

(c) Show by example that the formula in (b) may fail for non-symmetric matrices.

(a) We can write the spectral decomposition of A since A is self-adjoint:
$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^\top$$
$$AA^\top = \sum_{i=1}^{n} \lambda_i^2 u_i u_i^\top$$
which implies that $\sigma_i = |\lambda_i|$, so:
$$\|A\| = \max_{i=1,\dots,n} |\lambda_i(A)|$$

(b) We can rewrite the objective as $\langle x, Ax \rangle$. Knowing that A is symmetric, we have a orthonormal basis of eigenvectors of A due to the real spectral theorem. We can then write $x = \sum_{i=1}^{n} \langle x, u_i \rangle u_i$ where u_i's form the aforementioned basis. Thus, we can write the objective as:

$$\begin{aligned}
|\langle A \sum_{i=1}^{n} \langle x, u_i \rangle u_i, \sum_{i=1}^{n} \langle x, u_i \rangle u_i \rangle| &= |\langle \sum_{i=1}^{n} \lambda_i \langle x, u_i \rangle u_i, \sum_{i=1}^{n} \langle x, u_i \rangle u_i \rangle| \\
&= |\sum_{i=1}^{n} \langle x, u_i \rangle^2 \lambda_i| \\
&\leq |\lambda_{max}| | \sum_{i=1}^{n} \langle x, u_i \rangle^2 | \\
&= |\lambda_{max}|
\end{aligned}$$

So we have:

$$\max_{x \in S^{n-1}} x^\top A x = \max_{i=1,\dots,n} |\lambda_i(A)| = \|A\|$$

where the second equality is the result of part (a).

(c) Consider the matrix:

$$A = \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix}$$

we have:

$$AA^\top = \begin{bmatrix} 64 & 0 \\ 0 & 4 \end{bmatrix}$$

so $\|A\| = 8$.

Now consider a unit vector $u = [u_1, u_2]^\top$, then $u^\top A u = 10 u_1 u_2$, we know that $u_1^2 + u_2^2 = 1$, so $u_1 u_2 \leq \frac{1}{2}$ (this is easy to check using Lagrangian dual), which implies that the relation given in part (b) would give us 5 as the answer, which is not equal to $\|A\| = 8$.

$\square$

Problem 3. *Let X_1, X_2 be independent $N(0,1)$ random variables. Show that $Y_1 = (X_1 + X_2)/\sqrt{2}$ and $Y_2 = (X_1 - X_2)/\sqrt{2}$ are independent $N(0,1)$ random variables.*

Let $X = (X_1, X_2) \sim \mathcal{N}(0, I)$ and $Y = (Y_1, Y_2)$, then $Y = AX$, where:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

we know that $Cov(Y) = AA^\top = I$, which implies that Y_1 and Y_2 are jointly normal independent random variables with variance 1.
It is also easy to check that Y_1 and Y_2 have mean 0 using the linearity of expected value.
Note that 0 covariance does not imply independence in general, but it does in the case where the variables are jointly normal, which holds in our case.

Problem 4. *Let A be an $n \times n$ symmetric matrix, $\varepsilon \in (0, 1/2)$, and N be an ε-net of the unit sphere S^{n-1}. Show that*

$$\|A\| \leq \frac{1}{1 - 2\varepsilon} \cdot \max_{x \in \mathcal{N}} |x^\top A x|$$

Assume that $\|A\| = x^\top A x$ for some $x \in S^{n-1}$. There exists some $y \in \mathcal{N}$ s.t. $\|x - y\|_2 \leq \varepsilon$. We have:

$$\begin{aligned}
|x^\top Ax| - |y^\top Ay| &\leq |x^\top Ax - y^\top Ay| \\
&= |x^\top Ax - x^\top Ay + x^\top Ay - y^\top Ay| \\
&\leq |x^\top Ax - x^\top Ay| + |x^\top Ay - y^\top Ay| \\
&= |\langle x, A(x-y)\rangle| + |\langle x - y, Ay\rangle| \\
&\leq 2\|A\|\varepsilon \\
&\Rightarrow \|A\| - 2\varepsilon\|A\| \leq |y^\top Ay| \\
&\Rightarrow \|A\| \leq \frac{1}{1-2\varepsilon} \cdot \max_{x \in \mathcal{N}} |x^\top Ax|
\end{aligned}$$