Homework 5 Solutions

Problem 1. *Prove The following:*

(a) $\sigma(X) \leq C_1\|X - \mathbb{E}X\|_{\psi_2}$ *for any subgaussian random variable* X

(b) $\|X - \mathbb{E}X\|_{\psi_2} \leq C_2\sigma(X)$ *for any normally distributed random variable* $X \sim N(\mu, \sigma^2)$ *and for the random variable* X *uniformly distributed on an interval* $[a, b]$

(c) *Check that the random variable* X *that has Poisson distribution (with any parameter* $\lambda > 0$*) is not subgaussian*

Solution.

(a) Taking the ℓ_2 norm of $X - \mathbb{E}X$ and using the second property, we have:

$$\sigma \leq \sqrt{2}\|X - \mathbb{E}X\|_{\psi_2}$$

(b) First, assume $X \sim N(\mu, \sigma^2)$.Then we know that $\frac{X-\mathbb{E}X}{\sigma} \sim N(0, 1)$. Using the gaussian tail, we know that:

$$\mathbb{P}\{|X - \mathbb{E}X| \geq t\} = \mathbb{P}\{\left|\frac{X - \mathbb{E}X}{\sigma}\right| \geq \frac{t}{\sigma}\} \leq 2e^{\frac{-t^2}{2\sigma^2}}$$

Which implies that $\|X - \mathbb{E}X\|_{\psi_2} \leq \sqrt{2}\sigma$ using the first property.

Now assume that X is uniformly distributed over $[a, b]$ with $\sigma = \frac{(b-a)}{\sqrt{12}}$. Applying Hoeffding's inequality gives us:

$$\mathbb{P}\{|X - \mathbb{E}X| \geq t\} \leq 2e^{\frac{-t^2}{\frac{(b-a)^2}{2}}}$$

which implies that:

$$\|X - \mathbb{E}X\|_{\psi_2} \leq \frac{b - a}{\sqrt{2}} = \sqrt{6}\sigma$$

(c) Applying the MGF method and Markov's inequality:

$$\mathbb{P}\{e^{rX} \geq e^{rk}\} \leq e^{-rk}e^{\lambda(e^r-1)}$$

optimizing for r gives us an optimal bound of the form:

$$\mathbb{P}\{X \geq k\} \leq e^{k-\lambda}(\frac{\lambda}{k})^k$$

comparing it with the upper bound given in property 1, it's easy to check that X can not be subgaussian.

$\square$

Problem 2. *Prove that any random variable X satisfies:*

$$\mathbb{E}X = \int_0^\infty \mathbb{P}\{X > t\}\mathrm{d}t - \int_{-\infty}^0 \mathbb{P}\{X < t\}\mathrm{d}t$$

Solution. Any real value x can be written as:

$$x = \int_0^\infty 1_{t<x}\,\mathrm{d}t - \int_{-\infty}^0 1_{t>x}\,\mathrm{d}t$$

so for a real-valued random variable X, we have:

$$\begin{aligned}\mathbb{E}X &= \int_0^\infty \mathbb{E}[1_{t<X}]\,\mathrm{d}t - \int_{-\infty}^0 \mathbb{E}[1_{t>X}]\,\mathrm{d}t \quad \text{(Tonelli's Theorem)}\\ &= \int_0^\infty \mathbb{P}\{X > t\}\mathrm{d}t - \int_{-\infty}^0 \mathbb{P}\{X < t\}\mathrm{d}t\end{aligned}$$

$\square$

Problem 3. *Let X be a random variable that satisfies $\mathbb{E}\exp(\lambda X) \leq \exp(K^2\lambda^2)$ for all $\lambda \in \mathbb{R}$. Prove that $\mathbb{E}X = 0$.*

Solution. Applying Jensen's inequality, we have:

$$e^{\lambda \mathbb{E}X} \leq \mathbb{E}e^{\lambda X} \leq e^{K^2\lambda^2}$$

Taking the log yields:

$$\lambda \mathbb{E}X \leq K^2\lambda^2$$

Denoting $\mathbb{E}X = \mu$, We have two cases:

1. $\mu \geq 0$:

   Consider a sequence $\lambda_n \to 0^+$.We have:

   $$\mu \leq K^2\lambda_n$$

   Taking the limit as $n \to +\infty$ yields $\mu \leq 0$ and therefore $\mu = 0$.

2. $\mu \leq 0$

   Consider a sequence $\lambda_n \to 0^-$. We have:

   $$\mu \geq K^2\lambda_n$$

   Note that the inequality changes when dividing by λ_n since $\lambda_n \leq 0$ for all $n \in \mathbb{N}$.

   Taking the limit as $n \to +\infty$ yields $\mu \geq 0$ and therefore $\mu = 0$.

□

Problem 4. *let $X_1, \ldots, X_n$ be sub-gaussian random variables (not necessarily independent), and assume that $\|X_i\|_{\psi_2} \leq K$ for all i. Show that:*

$$\mathbb{E}\left[\max_{i\in[n]} |X_i|\right] \leq CK\sqrt{\log n}$$

Solution.

$$\begin{aligned}
\mathbb{E}\left[\max_{i\in[n]} |X_i|\right] \leq \mathbb{E}\left(\sum_{i=1}^{n} |X_i|^p\right)^{1/p} &\leq \left(\sum_{i=1}^{n} \mathbb{E}|X_i|^p\right)^{1/p} \quad (Jensen's) \\
&\leq \left(nK^p p^{p/2}\right)^{1/p} \\
&= n^{1/p}K\sqrt{p} \qquad (1)
\end{aligned}$$

Optimizing for p:

$$\frac{n^{1/p} \log n \sqrt{p}}{p^2} = \frac{n^{1/p}}{2\sqrt{p}}$$

We have $p = 2 \log n$. Substituting in (1), we have:

$$\mathbb{E}\left[\max_{i \in [n]} |X_i|\right] \leq \sqrt{2} n^{1/2 \log n} K \sqrt{\log n}$$

for $n \geq 2$, $n^{1/2 \log n}$ is bounded above by a constant. This means that for some absolute constant C, we have:

$$\mathbb{E}\left[\max_{i \in [n]} |X_i|\right] \leq CK \sqrt{\log n}$$

As stated in the question. $\square$