Homework 7 Solution

Problem 1. *Describe a randomized algorithm which, for a given graph $G = (V, E)$, finds a partition $V = V_1 \cup V_2$ that satisfies*

$$|E(V_1, V_2)| \geq \frac{1}{2}. \max |E(U_1, U_2)|$$

where the maximum is over all partitions $V = U_1 \cup U_2$.

Assume we have a classifier $C : V \to \{-1, 1\}$ where for each $v \in V$, $C(v)$ is a symmetric Bernoulli random variable independent of other nodes. Consider the random variable $1_{C(u)C(v)=-1}$ for each $u, v \in V$. The size of the cut given by our algorithm would be

$$\frac{1}{2} \sum_{(u,v) \in E} 1_{C(u)C(v)=-1}$$

for which the expected value is $\frac{2|E|}{4} = \frac{|E|}{2}$.
Since the expected value is $\frac{|E|}{2}$, our algorithm can find a cut of size at least $\frac{|E|}{2}$.
Note that the size of a cut cannot exceed $|E|$, which implies that

$$|E(V_1, V_2)| \geq \frac{1}{2}. \max |E(U_1, U_2)|$$

for all partitions $V = U_1 \cup U_2$. □

Problem 2. *Let $K(x, y)$ and, $M(x, y)$ be kernels. Show that all of the following are kernels, too:*

(a) $aK(x, y) + bM(x, y)$, *where $a, b > 0$ are constants.*

(b) $K(x, y)^p$ *where $p \in \mathbb{N}$*

(c) $P(K(x, y))$ *where P is a polynomial with nonnegative coefficients.*

(d) $K(x, y)f(x)f(y)$, *where $f : \mathbb{R}^d \to \mathbb{R}$ is a function.*

(a) We know that the matrices related to K and M are symmetric positive semidefinite, which makes the matrices of aK and bM symmetric positive semidefinite since a and b are positive. Finally, the sum of two symmetric positive semidefinite matrices is a symmetric positive semidefinite matrix.

(b) If the matrix of K is symmetric positive semidefinite, then K^p is also symmetric positive semidefinite for $p \in \mathbb{N}$

(c) The polynomial can be written as a sum of aK^r for some $r \in \mathbb{N}$ and $a > 0$, which is symmetric and positive semidefinite as a result of part (b), and their sum is also symmetric and positive semidefinite as a result of part (a).

(d) Let $K'(x, y) = f(x)f(y)$. It is easy to check that K' is a PD kernel, and we also know that the multiplication of two PD kernels is a PD kernel.

$\square$

Problem 3. *Let u, v be unit vectors in $\mathbb{R}^d$, and g be a standard normal random vector in $\mathbb{R}^d$, i.e. $g \sim N(0, I_d)$. Prove the following identities.*

(a) $\mathbb{E}\langle u, g\rangle\langle v, g\rangle = \langle u, v\rangle$

(b) $\mathbb{E}\langle u, g\rangle \mathrm{sign}(\langle v, g\rangle) = \sqrt{\frac{2}{\pi}}\langle u, v\rangle$

(c) Consider the random variable $X_u = \langle u, g\rangle - \sqrt{\frac{\pi}{2}}\mathrm{sign}(\langle u, g\rangle)$ and similarly for X_v. Deduce from (a) and (b) that

$$\frac{\pi}{2}\mathbb{E}\mathrm{sign}(\langle u, g\rangle)\mathrm{sign}(\langle v, g\rangle) = \langle u, v\rangle + \mathbb{E}[X_u X_v] \quad (1)$$

(a) $\mathbb{E}\langle u, g\rangle\langle v, g\rangle = \sum_{i,j=1}^n u_i v_j \mathbb{E}[g_i g_j] = \sum_{i=1}^n u_i v_i = \langle u, v\rangle$

(b) Using rotation invariance, let $u = v = (1, 0, \dots, 0)$, then:

$$\mathbb{E}\langle u, g\rangle \mathrm{sign}(\langle v, g\rangle) = \mathbb{E}g_1 \mathrm{sign}(g_1) = \sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-\frac{x^2}{2}} \mathrm{d}x = \sqrt{\frac{2}{\pi}}$$

Further,$\langle u, v\rangle = 1$, so:

$$\mathbb{E}\langle u, g\rangle \mathrm{sign}(\langle v, g\rangle) = \sqrt{\frac{2}{\pi}} \langle u, v\rangle$$

(c) This simply results from expanding $\mathbb{E}X_u X_v$, using the linearity of expected value and replacing with the relations derived in part (a) and (b).

□

Problem 4. *Let $A = [a_{ij}]_{i,j=1}^n$ be a symmetric, positive semi-definite matrix, and let $u_1, \dots, u_n$ be unit vectors in $\mathbb{R}^d$. Perform the randomized rounding of the vectors u_i, i.e. let $x_i = \mathrm{sign}(\langle g, u_i\rangle)$, where $g \sim \mathcal{N}(0, I_d)$. Using identity (1), show that:*

$$\mathbb{E}\left[\frac{\pi}{2} \sum_{i,j=1}^n a_{ij} x_i x_j\right] \geq \sum_{i,j=1}^n a_{ij} \langle u_i, u_j\rangle$$

$$\mathbb{E}\left[\frac{\pi}{2} \sum_{i,j=1}^n a_{ij} x_i x_j\right] = \sum_{i,j=1}^n a_{ij} \frac{\pi}{2} \mathbb{E}[x_i x_j] = \sum_{i,j=1}^n a_{ij} \langle u_i, u_j\rangle + \sum_{i,j=1}^n a_{ij} \mathbb{E}[X_{u_i} X_{u_j}]$$

Note that the right summand in the last equality is non-negative due to A being positive semi-definite:

$$\sum_{i,j=1}^n a_{ij} \mathbb{E}[X_{u_i} X_{u_j}] = \mathbb{E}X^\top A X \geq 0$$

where $X = [X_{u_1}, \dots, X_{u_n}]^\top$.
This implies the inequality given in the question. □