Homework 1 Solution

Exercise 1. *Study whether the following kernels are positive definite:*

1. $\chi = \mathbb{N},\ K(x, x') = 2^{(x+x')}$
2. $\chi = \mathbb{R},\ K(x, x') = \cos(x + x')$
3. $\chi = \mathbb{R},\ K(x, x') = \cos(x - x')$

Assume we have a set of points $X = \{x_1, \ldots, x_n\}$.

1. We can represent the matrix of K(x,y) with respect to X as $VV^\top$ where $V = [2^{x_1}, \ldots, 2^{x_n}]^\top$, which is positive semi-definite. Further, it is also symmetric since $2^{(x+y)} = 2^{(y+x)}$. Considering these facts, K is positive semi-definite.

2. K is not a PD kernel.

   Consider $X = \{\pi, -\frac{\pi}{2}\}$. The matrix of K with respect to X is not positive semi-definite since it has eigenvalues 1 and -1.

3. For every $x_i \in X$, define $v_i = [\cos(x_i), \sin(x_i)]$, and set

$$V = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

   it is easy to check that the matrix of K with respect to X is equal to $VV^\top$, which is positive semi-definite. as for symmetry, we know that $\cos(x - y) = \cos(-(x - y)) = \cos(y - x)$. Putting the two together implies that K is a PD kernel.

□

Exercise 2. *Consider a p.d. kernel* $K : X \times X \to \mathbb{R}$ *such that* $K(x, z) \le b^2$ *for all* x, z *in* X. *Show that* $\|f\|_\infty = \sup_{x \in \chi} |f(x)| \le b$ *for any function* f *in the unit ball of the corresponding RKHS.*

Let $\mathbb{H}$ be the RKHS given by the kernel. We know that $K(x, y) = \langle K_x, K_y \rangle$. Setting $x = y$:

$$\langle K_x, K_x \rangle = \|K_x\|_{\mathbb{H}}^2 \leq b^2 \Rightarrow \|K_x\|_{\mathbb{H}} \leq b$$

for all $x \in \chi$. On the other hand, $f(x) = \langle f, K_x \rangle$ and $\|f\|_{\mathbb{H}} \leq 1$. We have:

$$\sup_{x\in\chi} |f(x)| = \sup_{x\in\chi} |\langle f, K_x \rangle| \leq \sup_{x\in\chi} \|f\|_{\mathbb{H}} \|K_x\|_{\mathbb{H}} \leq b$$

□

Exercise 3. *Consider the Gaussian kernel $K : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ such that for all pair of points x, x' in $\mathbb{R}^p$,*

$$K(x, x') = e^{-\frac{\alpha}{2}\|x-x'\|^2}$$

where $\|.\|$ is the Euclidean norm on $\mathbb{R}^p$. Call $\mathcal{H}$ the RKHS of K and consider its RKHS mapping $\varphi : \mathbb{R}^p \to \mathcal{H}$ such that $K(x, x') = \langle \varphi(x), \varphi(x') \rangle_{\mathcal{H}}$ for all x, x' in $\mathbb{R}^p$. Show that

$$\|\varphi(x) - \varphi(x')\|_{\mathcal{H}} \leq \sqrt{\alpha}\|x - x'\|$$

The mapping is called non-expansive whenever $\alpha \leq 1$.

$$\begin{aligned}
\|\varphi(x) - \varphi(y)\|_{\mathcal{H}}^2 &= \langle \varphi(x) - \varphi(y), \varphi(x) - \varphi(y) \rangle_{\mathcal{H}} \\
&= 2(1 - exp(-\frac{\alpha}{2}\|x - y\|^2)) \\
&\leq 2(1 - (1 - \frac{\alpha}{2}\|x - y\|^2)) \\
&= \alpha\|x - y\|^2
\end{aligned}$$

taking the square root would complete the proof. □