Quiz 1 Solution

Problem 1. *For each given kernel, check whether they are positive definite*

1. $\chi = (-1, 1), K(x, y) = \frac{1}{1-xy}$
2. $\chi = \mathbb{N}, K(x, y) = 2^{xy}$
3. $\chi = \mathbb{R}_+, K(x, y) = \log(1 + xy)$
4. $\chi = \mathbb{R}, K(x, y) = \exp(-|x - y|^2)$
5. $\chi = \mathbb{R}_+, K(x, y) = \max(x, y)$
6. $\chi = \mathbb{R}_+, K(x, y) = \min(x, y)$
7. $\chi = \mathbb{R}_+, K(x, y) = \frac{\min(x,y)}{\max(x,y)}$
8. $\chi = \mathbb{N}, K(x, y) = \gcd(x, y)$
9. $\chi = \mathbb{N}, K(x, y) = \mathrm{lcm}(x, y)$
10. $\chi = \mathbb{N}, K(x, y) = \frac{\gcd(\mathrm{x,y})}{lcm(x,y)}$

1. Set $f(x, y) = xy$. it is easy to see that f is a PD kernel. Set $K_n(x, y) = \sum_{r=0}^{n} f(x, y)^r$. Since $|f(x, y)| < 1, \lim\limits_{n\to\infty} K_n(x, y) = K(x, y)$. We also know that each K_n is a PD kernel (using the basic facts about PD kernels), which makes the limit i.e. K a PD kernel.

2. As we know, for any PD kernel K',$e^{K'}$ is also a PD kernel. Again, setting $f(x, y) = xy$, we have $2^{f(x,y)} = e^{\ln(2)f(x,y)}$. We know that $\ln(2) > 0$ and that $f(x, y)$ is a PD kernel, which makes $\ln(2)f(x, y)$ and therefore $e^{\ln(2)f(x,y)}$ PD kernels.

3. setting $X = \{1, 3\}$, the matrix of K would have a negative determinant, therefore it can not be positive semi-definite, hence K is not a PD kernel (Mercer's theorem).

4. Since the domain is $\mathbb{R}$, we are allowed to set $|x-y|^2 = (x-y)^2$. We have:

   $$\exp(-|x-y|^2) = \exp(-(x-y)^2) = \exp(2xy)\exp(-x^2)\exp(-y^2)$$

   Set $f(x,y) = \exp(-x^2)\exp(-y^2)$. It is easy to see that f is a PD kernel. Further, $\exp(2xy)$ is also a PD kernel, and the multiplication of two PD kernels is a PD kernel, which makes K a PD kernel.

5. Set $X = \{1, 2\}$. The matrix of K with respect to X has a negative determinant, hence it is not positive semi-definite and K is not a PD kernel.

6. Set $X = \{x_1, \ldots, x_n\}$ and assume that $\max_i x_i = N$.

   Take $V = [v_1, \ldots, v_n]^\top, v_i \in \mathbb{R}^N$, where $v_{ij} = 1$ for $j \in [x_i]$ and $v_{ij} = 0$ elsewhere.

   The matrix of K with respect to X is $VV^\top$, which is the Gram matrix of v_i's. We know that all Gram matrices are positive semi-definite, which implies that K is a PD kernel according to Mercer's theorem.

7. We can write $\max(x,y) = \frac{1}{\min(1/x,1/y)}$, so we have:

   $$\frac{\min(x,y)}{\max(x,y)} = \min(x,y).\min(1/x,1/y)$$

   As we showed above, min is a PD kernel, which implies that K is also a PD kernel.

8. Represent the prime factorization of x and y with vectors $X, Y \in \mathbb{R}^P$ where P is the largest prime that appears in the factorization of x and y. Set $X_p = p^j$ if p appears in the prime factorization of x with power j, and set $X_i = 1$ elsewhere, and define Y in the same manner.

   We can now define $\gcd(x,y) = \prod_{k=1}^{P} \min(X_k, Y_k)$. With this interpretation, we can see that K is a PD kernel, since min is a PD kernel and the multiplication of PD kernels is also a PD kernel.

9. Set $X = \{1, 2\}$. The matrix of K with respect to X has a negative determinant and is not positive semi-definite, which means that K is not a PD kernel.

10. we know that $\operatorname{lcm}(x, y) = \frac{xy}{\gcd(x,y)}$. We have:

$$\frac{\gcd(x, y)}{\operatorname{lcm}(x, y)} = \frac{\gcd^2(x, y)}{xy}$$

Setting $f(x, y) = \frac{1}{x}.\frac{1}{y}$, it is easy to check that f is PD, which implies that K is also PD.