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Using Markov Chains to Model Human Migration in a Network Equilibrium Framework

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Abstract—In this paper, we develop a multistage network equilibrium model of human migration. The model provides a general framework for describing the dynamics of populations, and takes advantage of the special network structure of the problem. Unlike earlier network equilibrium models of migration, the model allows for chain migration in that humans can move from location to location in a "chaining" fashion, which is more realistic from an application perspective, than simply allowing for migration from a given origin to a direct destination in a single step. For the purpose of modeling and analysis, we utilize finite state nonhomogeneous Markov chains to handle the migration. A stable population distribution is then shown to exist under certain assumptions.

Keywords—Migration, Networks, Markov chains, Economic equilibrium, Variational inequalities, Population dynamics.

1. INTRODUCTION

Sociologists and economists have long sought theoretical models for studying human mobility. Markov chain models have been developed and used by many researchers (see, e.g., [1–5]). In these models, various geographical locations are the states in Markov chains, and the transition probabilities are either empirically estimated or assumed to possess certain properties.

Most models of migration, in addition to the above characteristics, have also failed to use nonhomogeneous Markov chains, which have time-varying transition matrices and are more realistic. The main reason for using homogeneous Markov chain models was due to the fact that there were more detailed studies and results about them. One of the few models using nonhomogeneous Markov chains was Lipstein's [6] model on consumer behavior. He raised the need for such a model, but, nevertheless, failed to provide any qualitative analysis of limiting behavior.

In [7–9] a different approach to the study of human mobility—using network equilibrium models—was taken. In these models, the migration or the flow of population is determined by a set of equilibrium conditions that reflects the fact that no individual in the system has any incentive to change his or her location at the equilibrium states. These network equilibrium models contrast with the earlier empirical models in a significant way in that they are conceptual models that explicitly incorporate the utility functions and transaction/migration costs. Moreover, the

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migration flows and, consequently, the transition probabilities, can actually be explicitly computed, rather than assumed in advance, although this aspect of such network equilibrium models of migration has, heretofore, been unrealized.

In the network framework developed thus far, certain (economically meaningful) monotonicity assumptions can be made on the utility functions and the transaction costs, and, moreover, under these assumptions, the equilibrium conditions have a unique solution. The equilibrium conditions are then formulated as a variational inequality problem, which can be solved by a decomposition algorithm. For additional background and references to migration network equilibrium models, as well as applications of network equilibrium models to other problem domains, we refer the readers to the recent book by Nagurney [10].

Yet in the network equilibrium models considered there are only one-step models: neither chain migration nor dynamics of migration in a time span is allowed. In this paper, in contrast, we shall develop a model that describes chain migration, that is, permits human mobility from location to location in a "chaining" fashion, rather than simply from a given origin to another direct destination in a single step, and we shall provide a qualitative study of its limiting behaviors through the theory of nonhomogeneous Markov chains. We believe that such a model is more representative of the actual reality in which individuals alter their migration locations over a span of time.

The paper is organized as follows: Section 2 gives an example of chain migration based on the network equilibrium conditions, and proposes a multiperiod model of chain migration. Section 3 describes the connection of the multiperiod model to the nonhomogeneous Markov chain and presents the necessary notation and the existing results on nonhomogeneous Markov chains. Section 4 consists of the main result of the paper, the proof of the existence of a limiting population distribution for the multiperiod model. Finally, Section 5 concludes with some directions for future research.

2. THE IMPORTANCE OF CHAIN MIGRATION AND A GENERAL MULTIPERIOD MODEL

For convenience of the readership, we recall the network equilibrium conditions governing the migration model consisting of N locations considered in [9]:

$$u_i(p) + c_{ij}(f) \begin{cases} = u_j(p), & \text{if } f_{ij} > 0, \\ \geq u_j(p), & \text{if } f_{ij} = 0. \end{cases} \tag{1}$$

In the above, $f = (f_{ij} : (i,j) = (1,2), \dots, (N-1, N))$ is the vector of migration flows, where f_{ij} denotes the flow from location i to location j, $p = (p_1, \dots, p_N)$ is the vector of populations, where p_i denotes the population at location i, $u_i(p)$ is the utility function associated with locating at location i, and $c_{ij}(f)$ is the transaction, i.e., migration cost associated with moving from location i to location j. We can also group $u_i(p)$'s into a vector $u(p)$, and $c_{ij}(f)$'s into a vector $c(f)$.

In other words, equilibrium conditions (1) state that the net gain in utility associated with migrating from location i to location j, $u_j(p) - u_i(p)$, will exactly equal the cost, $c_{ij}(f)$, associated with migrating between this pair of locations, if migration occurs. There will be no migration between a pair of locations, that is, the flow f_{ij} will be equal to zero, if the cost associated with migration exceeds the net gain in utilities.

The network structure of the problem is as follows: nodes correspond to locations, with associated utilities, and arcs denote possible linkages between locations, with arc costs corresponding to migration costs.

The above conditions, however, define a one-step model, which does not permit chain migration. The following example illustrates the need for an extension of the conceptual network framework to handle chain or repeated migration. Consider two locations with linear utility functions and migration costs,

$$u_1(p) = -p_1 + 8, \quad u_2(p) = -p_2 + 14, \quad c_{12}(f) = 2f_{12}, \quad c_{21}(f) = 2f_{21}.$$

Suppose that the initial populations at time zero are $p_1^0 = 4$ and $p_2^0 = 2$, then initially we have $u_1(p^0) = 4 < u_2(p^0) = 12$.

Seeing a chance to improve their utilities, people will move from location 1 to location 2 until the net gain becomes zero; that is, the gain in utility is exactly offset by the cost of migration. This implies that, in equilibrium, (cf. conditions (1))

$$f_{21}^1 = 0, \quad f_{12}^1 = 2, \quad p_1^1 = 2, \quad p_2^1 = 4,$$

and

$$u_1(p^1) + c_{12}(f^1) = 6 + 4 = 10 = u_2(p^1).$$

In the above expressions, the superscript 1 indicates that the results are from the first step in an iterative sequence. However, the migration process certainly cannot be expected to stop here. For example, individuals remaining in location 1 who now enjoy a utility of 6 will still see the opportunity of improving their utilities (note that the utility associated with location 2 is 10), since the migration costs should not be treated as cumulative and the congestion effect of the previous migration will be replaced by the new congestion in the new stage. The migration, hence, continues into the second stage. The results from the equilibrium conditions give, after stage 2, the following flow and population distribution pattern:

$$f_{21}^2 = 0, \quad f_{12}^2 = 1, \quad p_1^2 = 1, \quad p_2^2 = 5,$$

and

$$u_1(p^2) + c_{12}(f^2) = 7 + 2 = 9 = u_2(p^2).$$

This process continues in the same fashion with the explicit expressions of migration flows at stage n given by: $f_{12}^n = 1/2^{n-2}, f_{21}^n = 0$. Clearly, there is a limiting population distribution given by: $p_1^* = 0$, $p_2^* = 6$.

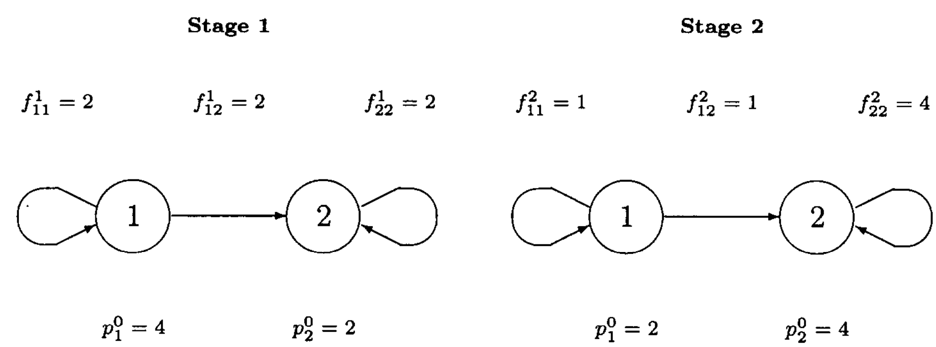

Figure 1. A transition map of the migration process.

See Figure 1 where a transition map is used to describe the migration process, with two associated transition matrices A_1 and A_2 given as follows:

$$A_1 = \begin{bmatrix} 2/4 & 2/4 \\ 0 & 2/2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 4/4 \end{bmatrix}.$$

This suggests that we should consider a sequence of equilibrium conditions or the equivalent variational inequalities, each of them, in turn, yielding the migration flows at a particular time period.

2.1. The Multiperiod Model

We now establish a multiperiod model for chain migration as follows.

At stage n, we let p_i^n denote the population at location i, and f_{ij}^n the migration flow from location i to location j.

Assuming that the utility functions and migration cost functions do not change over time, the equilibrium conditions at stage n, following the derivation for the single step or stage model contained in [9], can be stated as a variational inequality problem (which we call VIn), $n = 1, 2, \ldots$, where

$$-u(p^n) \cdot (p^{n\prime} - p^n) + c(f^n) \cdot (f^{n\prime} - f^n) \geq 0, \tag{2}$$

for all $(p^{n\prime}, f^{n\prime})$ such that

$$f_{ij}^{n\prime} \geq 0, \quad \sum_{j \neq i} f_{ij}^{n\prime} \leq p_i^{n-1}, \tag{3}$$

$$p_i^{n\prime} = p_i^{n-1} - \sum_{j \neq i} f_{ij}^{n\prime} + \sum_{j \neq i} f_{ji}^{n\prime}. \tag{4}$$

As in the single stage model, the above constraint set puts an upper bound on the total amount of out-flows from a given location, and guarantees the conservation of the total population.

3. THE CONNECTION BETWEEN THE MULTIPERIOD MODEL AND A NONHOMOGENEOUS MARKOV CHAIN

Naturally, we would like to understand the limiting behavior of the population distribution p_i^n for the multiperiod model. First of all, such behavior does not imply that of the migration flows f_{ij}^n, for we can not exclude the possibility of both in-flow and out-flow at the same location, even at equilibrium. Hence, how do we know whether p_i^n converges or not? It seems that the transition probability from one location to the other should offer us useful information.

When we determine the migration flows from the equilibrium conditions, we are not interested in or even able to decide whether a particular individual migrates or not. What we do know, however, are the transition probabilities from one location to other locations. For an individual, his or her location at stage n can be considered as a random variable X^n. Therefore, it is most appropriate to construct the Markov transition matrix at stage n based on VIn, the variational inequality formulation of the equilibrium conditions. This matrix can be written as follows:

$$P^{(n-1,n)} = \begin{bmatrix} \frac{f_{11}^n}{p_1^{n-1}} & \frac{f_{12}^n}{p_1^{n-1}} & \cdots & \frac{f_{1N}^n}{p_1^{n-1}} \\ \frac{f_{21}^n}{p_2^{n-1}} & \frac{f_{22}^n}{p_2^{n-1}} & \cdots & \frac{f_{2N}^n}{p_2^{n-1}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{f_{N1}^n}{p_N^{n-1}} & \frac{f_{N2}^n}{p_N^{n-1}} & \cdots & \frac{f_{NN}^n}{p_N^{n-1}} \end{bmatrix} = \left[p_{ij}^{(n-1,n)}\right], \tag{5}$$

where $p_{ij}^{(n-1,n)} = f_{ij}^n / p_i^{n-1} = \text{Prob}\{X^n \text{ is at } j \mid X^{n-1} \text{ is at } i\}$ is the transition probability from location i to location j at stage n; $n = 1, 2, \ldots$, and due to the constraint (3),

$$f_{ii}^n = p_i^{n-1} - \sum_{j \neq i} f_{ij}^n \geq 0. \tag{6}$$

Thus, f_{ii}^n stands for the total amount of the population at location i that remains there at stage n. Also, if $p_i^{n-1} = 0$, then we define that $p_{ii}^{(n-1,n)} = 1$, $p_{ij}^{(n-1,n)} = 0$, for any $j \neq i$. Clearly, we see that $P^{(n-1,n)}$ is a stochastic matrix.

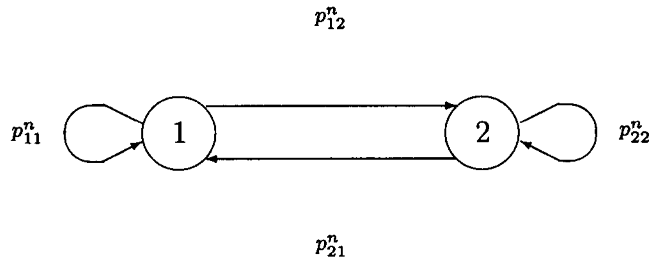

Figure 2. A transition map for two locations.

The multiperiod model is equivalent to the nonhomogeneous Markov chain determined by $P^{(n-1,n)}$. At every stage, the solution of the variational inequality problem VIn provides the transition probabilities between locations. On the other hand, the transition probabilities tell us the amount of flows between locations. See Figure 2 for an illustration using a transition map in the case of two locations.

For completeness, and easy reference, we include here some standard results on the classification of stochastic matrices. (See, e.g., [11,12].)

I. For any stochastic matrix P, after relabeling the states or performing the same permutation on rows and columns, P has the following form:

$$\begin{bmatrix} Q & R_1 & R_2 & \cdots & R_k \\ O & P_1 & O & \cdots & O \\ O & O & P_2 & \cdots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & \cdots & P_k \end{bmatrix},$$

where $P_1, \dots, P_k$ are irreducible stochastic matrices. The row numbers of Q correspond to the transient states S_0, and the row numbers of the irreducible submatrix P_i correspond to a set of closed states S_i. S_i and P_i together form a sub-Markov chain.

II. After relabeling of the states, any irreducible periodic Markov chain has a transition matrix which can be expressed in the following form, where d denotes a period:

$$\begin{bmatrix} O & P_1 & O & \cdots & O \\ O & O & P_2 & \cdots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & \cdots & P_{d-1} \\ P_d & O & O & \cdots & O \end{bmatrix},$$

where $P_1, \dots, P_d$ are nonnegative matrices whose rows sum to unity, and if the set of row numbers of P_i is denoted by S_i, then P_i is a matrix of order $|S_i| \times |S_{i+1}|$. ($|S_i|$ stands for the size of S_i.)

III. An irreducible and aperiodic stationary Markov chain is weakly ergodic.

We now state our first result for the transition matrices for the multiperiod model. From this point on, we assume a very reasonable property for the migration costs:

$$c_{ij}(f) \begin{cases} > 0, & \text{if } f_{ij} > 0, \\ = 0, & \text{if } f_{ij} = 0. \end{cases} \tag{7}$$

The above assumption simply says that if there is no flow from i to j, then the corresponding cost is zero, and if a flow occurs from i to j, then there must be a positive cost. Continuity when $f_{ij} = 0$ is not assumed for the migration costs, since there can be fixed costs if the migration occurs, regardless of the amount of f_{ij}.

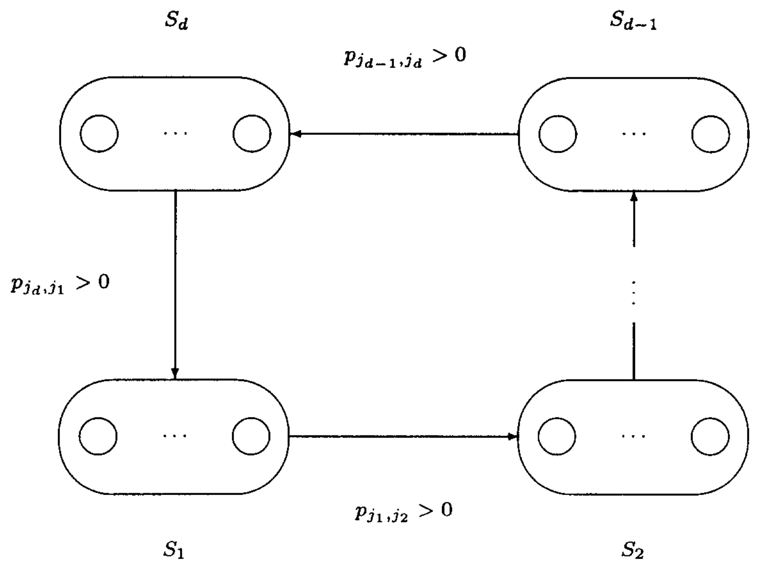

Figure 3. A loop map.

LEMMA 1. *At any stage n, $P^{(n-1,n)}$ in the multiperiod migration model is aperiodic.*

PROOF. For any fixed n, first we consider the case in which every pair of two locations is reachable to one another within stage n, that is, the transition matrix $P^{(n-1,n)}$ represents an irreducible (homogeneous) Markov chain. We can classify the locations into subsets of states (locations). Let $S_1, S_2, \dots, S_{d-1}, S_d$ be defined by the sets of row numbers of $P_1, P_2, \dots, P_{d-1}, P_d$. Then by the classification result **II**, we must have

$$p_{j_1,j_2} > 0,\ p_{j_2,j_3} > 0,\ \dots,\ p_{j_{d-1},j_d} > 0, \qquad p_{j_d,j_1} > 0, \tag{8}$$

for any $j_1 \in S_1, j_2 \in S_2, \dots, j_{d-1} \in S_{d-1}, j_d \in S_d$. See Figure 3 for an illustration.

On the other hand, if f solves the equilibrium conditions for the migration problem at stage n, we have $f_{j_1j_2} \times f_{j_2j_3} \times \cdots f_{j_dj_{d+1}} = 0$, for any $j_1, j_{d+1} \in S_1, j_2 \in S_2, \dots, j_d \in S_d$. For if $f_{j_kj_{k+1}} > 0$, for $k = 1, 2, \dots, d$, then from the equilibrium conditions, we have

$$u_{j_k}(p) + c_{j_kj_{k+1}}(f) = u_{j_{k+1}}(p), \tag{9}$$

for $k = 1, 2, \dots, d$.

This implies that

$$u_{j_1}(p)+c_{j_1j_2}(f) = u_{j2}(p) = u_{j_3}(p)-c_{j_2j_3}(f) = u_{j_{d+1}}(p)-c_{j_2j_3}(f)-c_{j_3j_4}(f)-\cdots-c_{j_dj_{d+1}}(f). \tag{10}$$

Since $f_{j_{d+1}j_1} = 0$ (another consequence from **II**), we have that $u_{j_{d+1}}(p) \le u_{j_1}(p) + c_{j_{d+1}j_1}(f) = u_{j_1}(p)$, and, hence,

$$c_{j_1j_2}(f) + c_{j_2j_3}(f) + \cdots + c_{j_dj_{d+1}}(f) \le 0. \tag{11}$$

Therefore, we have arrived at a contradiction. Note here that we have used our assumption on the cost functions.

It is now clear that $P^{(n-1,n)}$ cannot represent a periodic irreducible Markov chain.

In the case where $P^{(n-1,n)}$ is reducible as in the classification result **I**, the above argument still holds, because we can consider each of the irreducible sub-Markov chains separately.

The proof is complete.

The above lemma simply guarantees that there can be no loop at any stage of the migration process. At equilibrium, one cannot expect people in location A to migrate to a location B if people migrate to location A.

4. THE EXISTENCE OF A LIMITING POPULATION DISTRIBUTION FOR THE MULTIPERIOD MODEL

In this section, we shall establish the existence of a limiting population distribution under certain assumptions on the structure of the Markov chain model.

First of all, we assume that there can be no transient state of all the locations. We do not lose any generality with this convenient assumption. For even if there are transient states in the earlier stages, the probability that the process stays at a transient state tends to zero as the number of stages tends to infinity. Therefore, by eliminating those locations where there are no existing populations and no incoming migration, we obtain a set of locations increasingly free of transiency.

Second, we assume that the structure of the transition matrices $P^{(n-1,n)}$ does not change with n. Specifically, we assume that if a subset of locations is closed in a period, then the same subset remains closed throughout the time span. Here, a subset of locations is called closed if the one-step transition probability between any state from the subset and any state outside of the subset is zero.

Now we provide some lemmas that lead to the existence result. Let $P^{(n-1,n)}$ be written as

$$P^{(n-1,n)} = \begin{bmatrix} P_1^{(n-1,n)} & O & \cdots & O \\ O & P_2^{(n-1,n)} & \cdots & O \\ \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & P_k^{(n-1,n)} \end{bmatrix}, \tag{12}$$

where $P_1^{(n-1,n)}, \dots, P_k^{(n-1,n)}$ are irreducible stochastic matrices. The row numbers of the irreducible submatrix $P_i^{(n-1,n)}$ correspond to a set of closed states S_i. S_i and P_i together form a sub-Markov chain.

LEMMA 2. *For a fixed n, and for $i = 1, 2, \dots, k$, $P_i^{(n-1,n)}$ is weakly ergodic.*

PROOF. By Lemma 1, $P_i^{(n-1,n)}$ is aperiodic and, hence, the finite homogeneous Markov chain it represents is irreducible, aperiodic, and consists of only nonnull persistent states. By the basic theory of Markov chains, $P_i^{(n-1,n)}$ is ergodic, or equivalently, weakly ergodic.

LEMMA 3. *If P_i is a stochastic matrix, and $(P_i^{(n-1,n)})^m$ converges to $(P_i)^m$ uniformly with respect to m as $n \to \infty$, then P_i is weakly ergodic, for $i = 1, 2, \dots, k$.*

PROOF. Let $\delta(P_i)$ be the ergodic coefficient of P_i. (See [12] for a definition.) For any ϵ, there is an integer $n > 0$ so that $\|(P_i^{(n-1,n)})^m - (P_i)^m\| < \epsilon$ for any m due to uniform convergence. Since $P_i^{(n-1,n)}$ is weakly ergodic, we have, for any ϵ, that there is an integer m such that (see [12, Theorem V.3.1]) $\delta((P_i^{(n-1,n)})^m) < \epsilon$, and, hence,

$$\left|\delta\left(\left(P_i^{(n-1,n)}\right)^m\right) - \delta\left((P_i)^m\right)\right| \le \left\|\left(P_i^{(n-1,n)}\right)^m - (P_i)^m\right\| < \epsilon. \tag{13}$$

(See [12, Lemma V.4.1 and Lemma V.4.3].) Therefore, $\delta((P_i)^m) < 2\epsilon$, which implies the weak ergodicity of P_i. (cf. [12, Theorem V.3.1].)

LEMMA 4. *If $(P_i^{(n-1,n)})^m$ converges to $(P_i)^m$ uniformly with respect to m as $n \to \infty$, then $P_i^{(n)} = P_i^{(0,1)} \cdot P_i^{(1,2)} \cdots P_i^{(n-1,n)}$, the i^{th} block of the n-step transition matrix, is strongly ergodic.*

PROOF. By Lemma 3, P_i is weakly ergodic. Hence, from [12, Theorem V.4.5], the conclusion is immediate.

LEMMA 5. *The population distribution satisfies the following identity:*

$$\begin{aligned}[p_1^n, p_2^n, \dots, p_N^n] &= \left[p_1^{n-1}, p_2^{n-1}, \dots, p_N^{n-1}\right] \cdot P^{(n-1,n)} \\ &= \left[p_1^0, p_2^0, \dots, p_N^0\right] \cdot P^{(0,1)} \cdot P^{(1,2)} \dots P^{(n-1,n)} \\ &= \left[p_1^0, p_2^0, \dots, p_N^0\right] \cdot P^{(n)}\end{aligned} \tag{14}$$

PROOF. The identities are straightforward from the definition of the matrices.

PROPOSITION 1. *If $(P_i^{(n-1,n)})^m$ converges to $(P_i)^m$ uniformly with respect to m as $n \to \infty$, then for $j = 1, 2, \dots, N$, as $n \to \infty$, $p_j^n \to \overline{p}_j$, where $\overline{p}_j$ depends only on the initial total population of the closed class to which j belongs.*

PROOF. Let p be the total initial population, and let $p^i = \sum_{j \in S_i} p_j^0$ be the total population in the closed class S_i. Taking the limit on the following identity, which can be obtained from Lemma 5,

$$\left(\frac{1}{p}\right)[p_1^n, p_2^n, \dots, p_N^n] = \left(\frac{1}{p}\right)\left[p_1^0, p_2^0, \dots, p_N^0\right] \cdot P^{(n)},$$

the strong ergodicity of $P_i^{(n)} (i = 1, 2, \dots, k)$ implies that for $j = 1, 2, \dots, N$, $p_j^n \to \overline{p}_j$, as $n \to \infty$, where $\overline{p}_j$ depends only on p^i, which is the total population of S_i, the closed class that contains j.

This existence result shows that, given the structure of the transition matrices, if the one-step transition matrix stabilizes, then we can expect the population distribution to stabilize. Furthermore, the limiting population distribution is independent of the distributions *within* each closed class. In the special case when there is only one closed class, the limiting distribution is independent of the entire initial distribution.

5. CONCLUSIONS

In this paper, we have proposed a multistage network equilibrium model for human migration. The model formalizes a sequence of equilibrium conditions as equivalent variational inequality problems. We then established the connection of such a model to the theory of Markov chains, focusing on the behavior of the long-run population distribution. Under certain conditions, we showed that the stability of the one-step transition matrix guarantees the stability of the n-step transition matrix. The framework comprised of the combination of network equilibrium models with the theory of Markov chains developed here appears to be a promising new approach for the study of human migration.

There are further issues that could be considered. For example, one could carry out the computations for the solutions of variational inequalities in the sequence, in order to understand the theoretically proven limiting behavior of the proposed multistage model. One could also add more details to the current model, such as the incorporation of a birth rate and a death rate. Finally, the development of a continuous model would be an interesting research direction, where one might find a more compact form as well as an additional pool of theoretical results.

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