misc

```
(* this is Eqn1 or the energy functional given in the paper *)
```

In[∘]:=

$$L = 2\pi R[s]\left(\frac{\kappa}{2}\left(\psi'[s] + \frac{\mathrm{Sin}[\psi[s]]}{R[s]} - \mathrm{Co}\right)^2 + \sigma - \frac{f}{a}\mathrm{Cos}[\alpha - \psi[s]]\right) + \pi R[s]^2 \mathrm{Sin}[\psi[s]]\left(\mathrm{Po} + \frac{f}{a^2}\mathrm{Sin}[\alpha - \psi[s]]\right) + \nu[s]\,(R'[s] - \mathrm{Cos}[\psi[s]]) + \eta[s]\,(z'[s] + \mathrm{Sin}[\psi[s]]);$$

The Hamiltonian is $H = -L + \dot{\psi}\frac{\partial L}{\partial \dot{\psi}} + \dot{R}\frac{\partial L}{\partial \dot{R}} + \dot{z}\frac{\partial L}{\partial \dot{z}} + \dot{\nu}\frac{\partial L}{\partial \dot{\nu}} + \dot{\eta}\frac{\partial L}{\partial \dot{\eta}}$

```
(* using the above expression and Eqn1 should yield the Hamiltonian below *)
```

In[∘]:=

$$H = \mathrm{FullSimplify}\left[-L + (\psi'[s]\,\partial_{\psi'[s]} L) + (R'[s]\,\partial_{R'[s]} L) + (z'[s]\,\partial_{z'[s]} L) + (\nu'[s]\,\partial_{\nu'[s]} L) + \left(\eta'[s]\,\partial_{\eta'[s]} L\right)\right]$$

Out[∘]=

$$-\frac{\pi R[s]^2\left(a^2\,\mathrm{Po} + f\,\mathrm{Sin}[\alpha - \psi[s]]\right)\mathrm{Sin}[\psi[s]]}{a^2} - \frac{\pi\kappa\,\mathrm{Sin}[\psi[s]]^2}{R[s]} + \mathrm{Sin}[\psi[s]]\,(2\,\mathrm{Co}\,\pi\kappa - \eta[s]) + \mathrm{Cos}[\psi[s]]\,\nu[s] + \pi R[s]\left(-\mathrm{Co}^2\kappa - 2\sigma + \frac{2 f\,\mathrm{Cos}[\alpha - \psi[s]]}{a} + \kappa\,\psi'[s]^2\right)$$

```
(*the above equation can be reduced through
 the following steps and then factoring 2 π κ *)
```

In[∘]:=

$$\kappa\pi R[s]\left(\psi'[s]^2 - \left(\frac{\mathrm{Sin}[\psi[s]]^2}{R[s]^2} + \mathrm{Co}^2 - 2\,\mathrm{Co}\frac{\mathrm{Sin}[\psi[s]]}{R[s]}\right)\right) - 2\pi\sigma R[s] + \pi\frac{2f}{a}R[s]\,\mathrm{Cos}[\alpha - \psi[s]] - \pi R[s]^2\,\mathrm{Sin}[\psi[s]]\left(\mathrm{Po} + \frac{f}{a^2}\mathrm{Sin}[\alpha - \psi[s]]\right) + \nu[s]\,\mathrm{Cos}[\psi[s]] - \eta[s]\,\mathrm{Sin}[\psi[s]];$$

In[∘]:=

$$\mathrm{alternateForm} = 2\pi\kappa\left(\frac{R[s]}{2}\left(\psi'[s]^2 - \left(\frac{\mathrm{Sin}[\psi[s]]}{R[s]} - \mathrm{Co}\right)^2\right) - \frac{\sigma}{\kappa}R[s] + \frac{f}{a\kappa}R[s]\,\mathrm{Cos}[\alpha - \psi[s]] - \frac{R[s]^2}{2}\mathrm{Sin}[\psi[s]]\left(\frac{\mathrm{Po}}{\kappa} + \frac{f}{a^2\kappa}\mathrm{Sin}[\alpha - \psi[s]]\right) + \frac{\nu[s]}{2\pi\kappa}\mathrm{Cos}[\psi[s]] - \frac{\eta[s]}{2\pi\kappa}\mathrm{Sin}[\psi[s]]\right);$$

In[∘]:=

```
FullSimplify[H - alternateForm]
```

Out[∘]=

```
0
```

(* $\frac{\nu[s]}{2\pi\,\kappa}$ and $\frac{\eta[s]}{2\pi\,\kappa}$ can be reduced to $\frac{\nu[s]}{\kappa}$ and $\frac{\eta[s]}{\kappa}$ as ν and η are essentially lagrange multipliers so alternateForm becomes similar to Eqn3 in the paper*)

In[ ]:=

$$\text{eqn3} = 2\pi\kappa\left(\frac{R[s]}{2}\left(\psi'[s]^2 - \left(\frac{\text{Sin}[\psi[s]]}{R[s]} - \text{Co}\right)^2\right) - \frac{\sigma}{\kappa}R[s] + \frac{f}{a\,\kappa}R[s]\,\text{Cos}[\alpha - \psi[s]] - \frac{R[s]^2}{2}\text{Sin}[\psi[s]]\left(\frac{\text{Po}}{\kappa} + \frac{f}{a^2\,\kappa}\text{Sin}[\alpha - \psi[s]]\right) + \frac{\nu[s]}{\kappa}\text{Cos}[\psi[s]] - \frac{\eta[s]}{\kappa}\text{Sin}[\psi[s]]\right);$$

In[ ]:=

$$\text{eqnHpaper} = \frac{R[s]}{2}\left(\psi'[s]^2 - \left(\frac{\text{Sin}[\psi[s]]}{R[s]} - \text{Co}\right)^2\right) - \frac{\sigma}{\kappa}R[s] + \frac{f}{a\,\kappa}R[s]\,\text{Cos}[\alpha - \psi[s]] - \frac{R[s]^2}{2}\text{Sin}[\psi[s]]\left(\frac{\text{Po}}{\kappa} + \frac{f}{a^2\,\kappa}\text{Sin}[\alpha - \psi[s]]\right) + \frac{\nu[s]}{\kappa}\text{Cos}[\psi[s]] - \frac{\eta[s]}{\kappa}\text{Sin}[\psi[s]];$$

In[ ]:=

```
FullSimplify[(eqn3 / (2 π κ)) - eqnHpaper]
```

Out[ ]=

```
0
```

deriving Hamiltonian of the system

(*since $2\pi\kappa$ is constant and κ_G or the gaussian curvature term does not play a role in Helfrich model ... our energy functional can be written as *)

In[ ]:=

$$L = \left(\frac{R[s]}{2}\left(\psi'[s] + \frac{\text{Sin}[\psi[s]]}{R[s]} - \text{Co}\right)^2 + \frac{\sigma\,R[s]}{\kappa}\right) + \frac{R[s]^2}{2}\text{Sin}[\psi[s]]\left(\frac{\text{Po}}{\kappa}\right) + \frac{\nu[s]}{\kappa}(R'[s] - \text{Cos}[\psi[s]]) + \frac{\eta[s]}{\kappa}(z'[s] + \text{Sin}[\psi[s]]);$$

The Hamiltonian is $H = -L + \dot{\psi}\frac{\partial L}{\partial \dot{\psi}} + \dot{R}\frac{\partial L}{\partial \dot{R}} + \dot{z}\frac{\partial L}{\partial \dot{z}} + \dot{\nu}\frac{\partial L}{\partial \dot{\nu}} + \dot{\eta}\frac{\partial L}{\partial \dot{\eta}}$

In[ ]:=

$$H = \text{FullSimplify}\left[-L + (\psi'[s]\,\partial_{\psi'[s]}L) + (R'[s]\,\partial_{R'[s]}L) + (z'[s]\,\partial_{z'[s]}L) + (\nu'[s]\,\partial_{\nu'[s]}L) + (\eta'[s]\,\partial_{\eta'[s]}L)\right]$$

Out[ ]=

$$\frac{1}{2\,\kappa\,R[s]}\left(-\text{Po}\,R[s]^3\,\text{Sin}[\psi[s]] - \kappa\,\text{Sin}[\psi[s]]^2 + 2\,R[s]\,(\text{Sin}[\psi[s]]\,(\text{Co}\,\kappa - \eta[s]) + \text{Cos}[\psi[s]]\,\nu[s]) + R[s]^2\left(-\text{Co}^2\,\kappa - 2\,\sigma + \kappa\,\psi'[s]^2\right)\right)$$

(*eqn3 is the hamiltonian equation 3 given in the article*)

In[ ]:=

$$\text{eqn3} = \frac{R[s]}{2}\left(\psi'[s]^2 - \left(\frac{\text{Sin}[\psi[s]]}{R[s]} - \text{Co}\right)^2\right) - \frac{\sigma}{\kappa}R[s] - \frac{R[s]^2}{2}\text{Sin}[\psi[s]]\left(\frac{\text{Po}}{\kappa}\right) + \frac{\nu[s]}{\kappa}\text{Cos}[\psi[s]] - \frac{\eta[s]}{\kappa}\text{Sin}[\psi[s]];$$

```
(*is my Hamiltonian the same as equation3?*)
```

In[ ]:= `FullSimplify[H - eqn3]`

Out[ ]=

0

```
(*indeed we get the same form for the
 Hamiltonian as in the article. till here it seems fine *)
```

derivation of bulk terms

In[ ]:=

$$L = \left(\frac{R[s]}{2}\left(\psi'[s] + \frac{Sin[\psi[s]]}{R[s]} - Co\right)^2 + \frac{\sigma R[s]}{\kappa}\right) + \frac{R[s]^2}{2} Sin[\psi[s]]\left(\frac{Po}{\kappa}\right) + \frac{\nu[s]}{\kappa}(R'[s] - Cos[\psi[s]]) + \frac{\eta[s]}{\kappa}(z'[s] + Sin[\psi[s]]);$$

In[ ]:=

$$eqn4 = \psi''[s] == \left(\frac{Cos[\psi[s]]\,Sin[\psi[s]]}{R[s]^2} - \frac{\psi'[s]}{R[s]} Cos[\psi[s]] + \frac{R[s]}{2}\frac{Po}{\kappa} Cos[\psi[s]] + \frac{\nu[s]}{R[s]\,\kappa} Sin[\psi[s]] + \frac{\eta[s]}{R[s]\,\kappa} Cos[\psi[s]]\right);$$

$$eqn5 = \nu'[s] == \frac{1}{2}(\psi'[s] - Co)^2 - \frac{Sin[\psi[s]]^2}{2\,R[s]^2} + \frac{\sigma}{\kappa} + R[s]\,Sin[\psi[s]]\left(\frac{Po}{\kappa}\right);$$

$$eqn6 = n'[s] == 0;$$

```
(* deriving equation 4: R'[s]=Cos[ψ[s]] *)
```

In[ ]:=

$$bulkterm1 = (First @@ Solve[-D[\partial_{\psi'[s]} L, s] + \partial_{\psi[s]} L == 0, \psi''[s]]) /. R'[s] \to Cos[\psi[s]]$$

Out[ ]=

$$\psi''[s] \to \frac{1}{2\,\kappa\,R[s]^2}\left(Po\,Cos[\psi[s]]\,R[s]^3 + 2\,\kappa\,Cos[\psi[s]]\,Sin[\psi[s]] + 2\,Cos[\psi[s]]\,R[s]\times\eta[s] + 2\,R[s]\,Sin[\psi[s]]\,\nu[s] - 2\,\kappa\,Cos[\psi[s]]\,R[s]\,\psi'[s]\right)$$

In[ ]:= `FullSimplify[eqn4[[2]] - Values@bulkterm1]`

Out[ ]=

0

```
(* deriving equation 5 *)
```

In[ ]:=
```
bulkterm2 = Block[{ν},
    ν'[s] → (ν'[s] /. (First @@ Solve[(∂R[s] L - D[∂R'[s] L, s]) == 0, ν'[s]]))
  ] // Simplify
```

Out[ ]=

$$\nu'[s] \to \frac{1}{2}\left(\text{Co}^2\,\kappa + 2\,\sigma + 2\,\text{Po}\,R[s]\,\text{Sin}[\psi[s]] - \frac{\kappa\,\text{Sin}[\psi[s]]^2}{R[s]^2} - 2\,\text{Co}\,\kappa\,\psi'[s] + \kappa\,\psi'[s]^2\right)$$

```
(* the above equation has dimensions so dividing sol2 with κ will give the non-
 dimensional result. this will yield the same result as equation 5*)
```

In[ ]:=
```
FullSimplify[(Values[bulkterm2] / κ) - eqn5[[2]]]
```

Out[ ]=

0

```
(* deriving equation 6 *)
```

In[ ]:=
```
bulkterm3 = First @@ Solve[(-D[∂z'[s] L, s] + ∂z[s] L) == 0, η'[s]]
```

Out[ ]=

$\eta'[s] \to 0$

In[ ]:=
```
FullSimplify[Values[bulkterm3] - eqn6[[2]]]
```

Out[ ]=

0

deriving membrane shape equation

In[ ]:=

$$\text{eqn4} = \psi''[s] == \left(\frac{\text{Cos}[\psi[s]]\,\text{Sin}[\psi[s]]}{R[s]^2} - \frac{\psi'[s]}{R[s]}\,\text{Cos}[\psi[s]] + \frac{R[s]}{2}\,\frac{\text{Po}}{\kappa}\,\text{Cos}[\psi[s]] + \frac{\nu[s]}{R[s]\,\kappa}\,\text{Sin}[\psi[s]] + \frac{\eta[s]}{R[s]\,\kappa}\,\text{Cos}[\psi[s]]\right);$$

$$\text{eqn5} = \nu'[s] == \frac{1}{2}\,(\psi'[s] - \text{Co})^2 - \frac{\text{Sin}[\psi[s]]^2}{2\,R[s]^2} + \frac{\sigma}{\kappa} + R[s]\,\text{Sin}[\psi[s]]\,\left(\frac{\text{Po}}{\kappa}\right);$$

```
eqn6 = n'[s] == 0;
```

```
(* --------------------------------------------------------procedure
 to derive membrane shape equation ------
 ----------------------------------------------------------------- *)
```

[redacted] H is conserved along the contour because L does not explicitly depend on the arc length s, i.e., $H \equiv 0$. Now rewrite Eq. (**3**) as $\eta = \eta(\psi, \dot\psi, R, \nu)$ and insert it into Eq. (**4**), which then rewritten as $\nu = \nu(\psi, \dot\psi, \ddot\psi, R)$. We take the first derivative of $\nu = \nu(\psi, \dot\psi, \ddot\psi, R)$ with respect to arc length s, equal it to Eq. (**5**). After eliminating $\dot\nu$, we have the shape equation of the membrane surface as Eq. (**7**):

```
(*lets rewrite equation 3 or the hamiltonian in
 terms of η given that H is 0 as it is a conserved quantity *)
```

In[ ]:=
```
sub1 = FullSimplify[Solve[eqn3 == 0, η[s]][[1, 1]]]
```

Out[ ]=

$$\eta[s] \to \text{Co}\,\kappa + \text{Cot}[\psi[s]]\,\nu[s] - \frac{\text{Po}\,R[s]^3 + \kappa\,\text{Sin}[\psi[s]] + \text{Csc}[\psi[s]]\,R[s]^2\left(\text{Co}^2\,\kappa + 2\,\sigma - \kappa\,\psi'[s]^2\right)}{2\,R[s]}$$

```
(*we replace the definition of η above in equation 4 and re-
 write equation in terms of ν *)
```

In[ ]:=
```
sub2 = Solve[(eqn4 /. Rule → Equal) /. sub1, ν[s]][[1, 1]]
```

Out[ ]=

$$\nu[s] \to \frac{1}{2\,R[s]\,(\text{Cos}[\psi[s]]\,\text{Cot}[\psi[s]] + \text{Sin}[\psi[s]])}\,\text{Sin}[\psi[s]]\,\Big(-\kappa\,\text{Cos}[\psi[s]] - 2\,\text{Co}\,\kappa\,\text{Cot}[\psi[s]]\,R[s] + \text{Co}^2\,\kappa\,\text{Cot}[\psi[s]]\,\text{Csc}[\psi[s]]\,R[s]^2 + 2\,\sigma\,\text{Cot}[\psi[s]]\,\text{Csc}[\psi[s]]\,R[s]^2 + 2\,\kappa\,\text{Cot}[\psi[s]]\,R[s]\,\psi'[s] - \kappa\,\text{Cot}[\psi[s]]\,\text{Csc}[\psi[s]]\,R[s]^2\,\psi'[s]^2 + 2\,\kappa\,\text{Csc}[\psi[s]]\,R[s]^2\,\psi''[s]\Big)$$

```
(*now we take the first derivative of ν with respect to s *)
```

In[ ]:=

$$\text{sub3} = \text{Simplify}\left[D\left[\frac{\text{sub2}[[2]]}{\kappa}, s\right] /.\ R'[s] \to \text{Cos}[\psi[s]]\right]$$

Out[ ]=

$$\frac{1}{8\,\kappa\,R[s]^2}\Big(\kappa\,\text{Sin}[2\,\psi[s]]^2 + \kappa\,R[s]\,(\text{Sin}[\psi[s]] - 3\,\text{Sin}[3\,\psi[s]])\,\psi'[s] + 2\,R[s]^2\,\Big(-4\,\text{Co}\,\kappa\,\text{Cos}[2\,\psi[s]]\,\psi'[s] + \kappa\,(-1 + 3\,\text{Cos}[2\,\psi[s]])\,\psi'[s]^2 + 2\,\text{Cos}[\psi[s]]\,\Big(\left(\text{Co}^2\,\kappa + 2\,\sigma\right)\text{Cos}[\psi[s]] + 4\,\kappa\,\text{Sin}[\psi[s]]\,\psi''[s]\Big)\Big) - 4\,R[s]^3\,\text{Sin}[\psi[s]]\,\Big(\left(\text{Co}^2\,\kappa + 2\,\sigma\right)\psi'[s] - \kappa\,\psi'[s]^3 - 2\,\kappa\,\psi^{(3)}[s]\Big)\Big)$$

```
(*now we equate sub3 with eqn 5 and solve
 for ψ⁽³⁾ to get equation 7 or membrane equation*)
```

In[ ]:=
```
membraneEqn = (Solve[sub3 == eqn5[[2]], ψ⁽³⁾[s]][[1, 1]] // FullSimplify) /. Rule → Equal
```

Out[ ]=

$$\psi^{(3)}[s] == \frac{1}{8\,\kappa\,R[s]^3}\Big(-\kappa\,(5\,\text{Sin}[\psi[s]] + \text{Sin}[3\,\psi[s]]) + 2\,\kappa\,(1 + 3\,\text{Cos}[2\,\psi[s]])\,R[s]\,\psi'[s] + 4\,R[s]^3\,\Big(2\,\text{Po} + \left(\text{Co}^2\,\kappa + 2\,\sigma\right)\psi'[s] - \kappa\,\psi'[s]^3\Big) + 4\,R[s]^2\,\Big(\text{Sin}[\psi[s]]\,\Big(\text{Co}^2\,\kappa + 2\,\sigma + \kappa\,\psi'[s]\,(-4\,\text{Co} + 3\,\psi'[s])\Big) - 4\,\kappa\,\text{Cos}[\psi[s]]\,\psi''[s]\Big)\Big)$$

```
(*below is equation 7 and we need to compare it with our result above*)
```

In[◦]:=

```
eqn7 = ψ'''[s] == -1/2 ψ'[s]^3 - (2 Cos[ψ[s]])/R[s] ψ''[s] +
    (3 Sin[ψ[s]])/(2 R[s]) ψ'[s]^2 + ((3 Cos[ψ[s]]^2 - 1)/(2 R[s]^2) - (2 Co Sin[ψ[s]])/R[s]) ψ'[s] -
    (Cos[ψ[s]]^2 + 1)/(2 R[s]^3) Sin[ψ[s]] + (σ/κ + Co^2/2) ψ'[s] + (σ/κ + Co^2/2) Sin[ψ[s]]/R[s] + Po/κ;
```

In[◦]:=

```
TraditionalForm[eqn7]
```

Out[◦]//TraditionalForm=

$$\psi^{(3)}(s) = \frac{\left(\frac{\mathrm{Co}^2}{2} + \frac{\sigma}{\kappa}\right)\sin(\psi(s))}{R(s)} + \left(\frac{\mathrm{Co}^2}{2} + \frac{\sigma}{\kappa}\right)\psi'(s) + \psi'(s)\left(\frac{3\cos^2(\psi(s)) - 1}{2\,R(s)^2} - \frac{2\,\mathrm{Co}\sin(\psi(s))}{R(s)}\right) + \frac{\mathrm{Po}}{\kappa} - \frac{2\,\psi''(s)\cos(\psi(s))}{R(s)} + \frac{3\,\psi'(s)^2\sin(\psi(s))}{2\,R(s)} - \frac{\sin(\psi(s))\left(\cos^2(\psi(s)) + 1\right)}{2\,R(s)^3} - \frac{1}{2}\psi'(s)^3$$

In[◦]:=

```
FullSimplify[membraneEqn[[2]] - eqn7[[2]]]
```

Out[◦]=

```
0
```