CIS 571 - Assignment 4

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Question 1

1. True
2. False
3. False
4. False; X_1 is conditionally independent of X_2
5. True

Question 2

Initial factors: $P(A), P(B|A), P(C|B), P(D|B), P(E|C, D)$.

$P(A)P(B|A) = P(A, B)$

A	B	$P(A, B)$
0	0	0.08
0	1	0.12
1	0	0.16
1	1	0.64

$P(C|B)P(D|B) = P(C, D|B)$

C	D	B	$P(C, D|B)$
0	0	0	0.48
1	0	0	0.32
0	1	0	0.12
1	1	0	0.08
0	0	1	0.36
1	0	1	0.24
0	1	1	0.24
1	1	1	0.16

$P(A, B, C, D) = P(A, B)P(C, D|B)$

A	B	C	D	$P(A, B, C, D)$
0	0	0	0	0.0384
0	0	0	1	0.0096
0	0	1	0	0.0256
0	0	1	1	0.0064
0	1	0	0	0.0432
0	1	0	1	0.0288
0	1	1	0	0.0288
0	1	1	1	0.0192
1	0	0	0	0.0768
1	0	0	1	0.0192
1	0	1	0	0.0512
1	0	1	1	0.0128
1	1	0	0	0.2304
1	1	0	1	0.1536
1	1	1	0	0.1536
1	1	1	1	0.1024

$P(A, B, C, D)P(E|C, D) = P(A, B, C, D, E)$

A	B	C	D	E	$P(A, B, C, D, E)$
0	0	0	0	0	0.00768
0	0	0	0	1	0.03072
0	0	0	1	0	0.00768
0	0	0	1	1	0.00192
0	0	1	0	0	0.01536
0	0	1	0	1	0.01024
0	0	1	1	0	0.00512
0	0	1	1	1	0.00128
0	1	0	0	0	0.00864
0	1	0	0	1	0.03456
0	1	0	1	0	0.02304
0	1	0	1	1	0.00576
0	1	1	0	0	0.01728
0	1	1	0	1	0.01152
0	1	1	1	0	0.01536
0	1	1	1	1	0.00384
1	0	0	0	0	0.01536
1	0	0	0	1	0.06144
1	0	0	1	0	0.01536
1	0	0	1	1	0.00384
1	0	1	0	0	0.03072
1	0	1	0	1	0.02048
1	0	1	1	0	0.01024
1	0	1	1	1	0.00256
1	1	0	0	0	0.04608
1	1	0	0	1	0.18432
1	1	0	1	0	0.12288
1	1	0	1	1	0.03072
1	1	1	0	0	0.09216
1	1	1	0	1	0.06144
1	1	1	1	0	0.08192
1	1	1	1	1	0.02048

2.1 P(B = 1|E = 1) ?

Eliminating $E = 0$, normalizing, and filtering out $B = 1$ instances:

A	B	C	D	E	$P(A, B, C, D \vert E = 1)$
0	0	0	0	1	0.06332
0	0	0	1	1	0.00396
0	0	1	0	1	0.02111
0	0	1	1	1	0.00264
0	1	0	0	1	0.07124
0	1	0	1	1	0.01187
0	1	1	0	1	0.02375
0	1	1	1	1	0.00792
1	0	0	0	1	0.12665
1	0	0	1	1	0.00792
1	0	1	0	1	0.04222
1	0	1	1	1	0.00528
1	1	0	0	1	0.37995
1	1	0	1	1	0.06332
1	1	1	0	1	0.12665
1	1	1	1	1	0.04222

Therefore, $P(B = 1|E = 1) = 0.07124 + 0.01187 + 0.02375 + 0.00792 + 0.37995 + 0.06332 + 0.12665 + 0.04222 = \mathbf{0.7269}$.

2.2 P(A = 1|C = 0, E = 0) ?

Eliminating instances other than where $E = 0$ and $C = 0$, normalizing, and filtering out $A = 1$ instances:

A	B	C	D	E	$P(A, B, D \vert C = 0, E = 0)$
0	0	0	0	0	0.03113
0	0	0	1	0	0.03113
0	1	0	0	0	0.03502
0	1	0	1	0	0.09339
1	0	0	0	0	0.06226
1	0	0	1	0	0.06226
1	1	0	0	0	0.18677
1	1	0	1	0	0.49805

Therefore, $P(A = 1|C = 0, E = 0) = 0.06226 + 0.06226 + 0.18677 + 0.49805 = \mathbf{0.80934}$.

Question 3

3.1 Rejection Sampling

A is sampled first, and the random value is $0.320 \geq 0.2$, which leads to the sample $A = 1$.
B is sampled next, given that $A = 1$, and the random value is $0.037 < 0.2$ therefore $B = 0$. This is **inconsistent** with the observation $B = 1$, therefore it will be rejected.

A: 1, **B: 0** C: none D: none E: none **Rejected**

Next round, A is sampled with the random value of $0.303 \geq 0.2$ which leads to $A = 1$.
Afterwards, B is sampled with the random value of $0.318 \geq 0.2$ which leads to $B = 1$.
Next, C is sampled with the random value of $0.032 < 0.6$, which leads to $C = 0$.
D is sampled with the random value of $0.969 \geq 0.6$, which leads to $D = 1$.
Finally, E is sampled with the random value of $0.018 < 0.8$, which leads to $E = 0$. This is **inconsistent** with the observation $E = 1$, therefore it will be rejected.

A: 1, B: 1 C: 0 D: 1 **E: 0** **Rejected**

Not enough random samples exist for another round.

3.2 Likelihood Weighting

A is sampled first, and the random value is $0.249 \geq 0.2$, which leads to the sample $A = 1$, no change in weight: $w = 1$.
B is observed to be 1, weight changes: $w = w \cdot P(B = 1|A = 1) = 1 \cdot 0.8 = 0.8$.
Next, C is sampled with the random value of $0.052 < 0.6$, which leads to $C = 0$, no change in weight: $w = 0.8$.
D is sampled with the random value of $0.299 < 0.6$, which leads to $D = 0$, no change in weight: $w = 0.8$.
Finally, E is observed to be 1, weight changes: $w = w \cdot P(E = 1|C = 0, D = 0) = 0.8 \cdot 0.8 = 0.64$.

A: 1, B: 1 C: 0 D: 0 E: 1
Weight: 0.64 .

3.3 Gibbs Sampling

Current sample: A: 1, B: 0 C: 1 D: 1 **E: 1**

3.3.1 Non-evidence variable B is chosen. What would be the values after re-sampling?

$P(B|A = 1, C = 1, D = 1, E = 1)$:

A	B	C	D	E	$P(B\|A, C, D, E)$
1	0	1	1	1	0.889
1	1	1	1	1	0.111

With a random variable of $0.320 < 0.889$, B is assigned 0.
A: 1, *B: 0* C: 1 D: 1 **E: 1**

3.3.2 Non-evidence variable D is chosen. What would be the values after re-sampling?

$P(D|A = 1, B = 0, C = 1, E = 1)$: With a random variable of $0.037 < 0.889$, D is assigned 0.

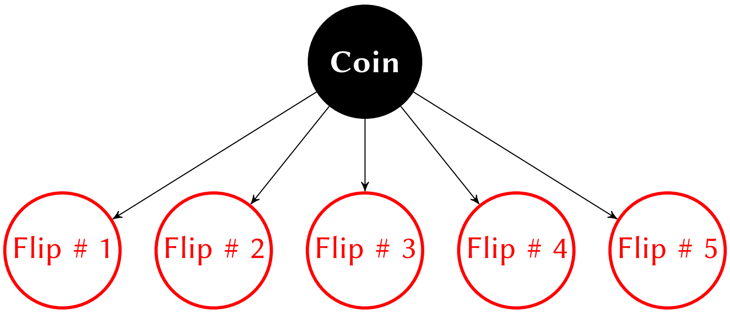
A	B	C	D	E	$P(D\|A, B, C, E)$
1	0	1	0	1	0.889
1	0	1	1	1	0.111

A: 1, B: 0 C: 1 *D: 0* **E: 1**

Question 4

4.1 Coins

The problem can be formulated as the following Bayes Net: where variable C (Coin) can take



any of the values 1, 2, 3, and 4 with a probability of 0.25 each, and each variable F_i (Flip # i) can take either H or T, given the following probabilities:

C	F_i	$P(F_i \mid C)$	$P(F_i, C)$
1	H	0.5	0.125
1	T	0.5	0.125
2	H	0.25	0.0625
2	T	0.75	0.1875
3	H	0.75	0.1875
3	T	0.25	0.0625
4	H	0.625	0.15625
4	T	0.375	0.09375

Therefore, for each F_i, $P(C \mid F_i)$:

C	F_i	$P(C \mid F_i)$
1	H	0.23529
2	H	0.11765
3	H	0.35294
4	H	0.29412
1	T	0.26667
2	T	0.40
3	T	0.13333
4	T	0.20

Therefore, the probability that coin # 2 was chosen given that the outcome is HTHTH is:

$$\begin{aligned} P(C=2 \mid F_1=H, F2=T, F_3=H, F_4=T, F_5=H) &= \\ \frac{P(C=2 \mid F_1=H)P(C=2 \mid F_2=T)P(C=2 \mid F_3=H)P(C=2 \mid F_4=T)P(C=2 \mid F_5=H)}{\sum_{c \in 1,2,3,4} P(C=c \mid F_1=H, F2=T, F_3=H, F_4=T, F_5=H)} \\ &= \frac{0.11765 \cdot 0.4 \cdot 0.11765 \cdot 0.4 \cdot 0.11765}{0.0029861817852862025} \\ &= 0.08725. \end{aligned}$$

4.2

Since X and Y are independent variables, for any given values x and y in the domain, $P(X = x, Y = y) = P(X = x)P(Y = y)$.

10.

$$
\begin{aligned}
P(Y = 2) &= \sum_x P(X = x, Y = 2) = 0.25 + P(X = 1, Y = 2) \\
P(Y = 3) &= \sum_x P(X = x, Y = 3) = 0.125 + P(X = 1, Y = 3) \\
P(Y = 4) &= \sum_x P(X = x, Y = 4) = 0.5 + P(X = 1, Y = 4)
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\sum_y P(Y = y) = 1 \quad & (4.1) \\
= P(Y = 1) + 0.875 + P(X = 1, Y = 2) + P(X = 1, Y = 3) + P(X = 1, Y = 4) \quad & (4.2) \\
\implies P(Y = 1) + P(X = 1, Y = 2) + P(X = 1, Y = 3) + P(X = 1, Y = 4) = 0.125 \quad & (4.3) \\
\implies P(Y = 1) + P(X = 1)[P(Y = 2) + P(Y = 3) + P(Y = 4)] = 0.125 \quad & (4.4) \\
\implies P(Y = 1) + 0.1[P(Y = 2) + P(Y = 3) + P(Y = 4)] = 0.125 \quad & (4.5) \\
\implies 10P(Y = 1) + P(Y = 2) + P(Y = 3) + P(Y = 4) = 1.25 \quad & (4.6) \\
\implies 9P(Y = 1) = 0.25 \quad & (4.7) \\
\implies P(Y = 1) = \frac{1}{36} \quad & (4.8)
\end{aligned}
$$

Therefore, $\mathbf{P(X = 1, Y = 1) = P(X = 1)P(Y = 1) = \frac{1}{10}\frac{1}{36} = 1/360}$.

11.

$$
\begin{aligned}
&\frac{P(X = 2, Y = 2)}{P(X = 2, Y = 3)} = \frac{P(Y = 2)}{P(Y = 3)} = \frac{20}{10} \\
&\implies 0.5P(Y = 2) = P(Y = 3) \\
&\frac{P(X = 2, Y = 2)}{P(X = 2, Y = 4)} = \frac{P(Y = 2)}{P(Y = 4)} = \frac{5}{10} \\
&\implies 2P(Y = 2) = P(Y = 3) \\
&\sum_y P(Y = y) \qquad = P(Y = 1) + P(Y = 2) + 0.5P(Y = 2) + 2P(Y = 2) \\
&= \frac{1}{36} + 3.5P(Y = 2) = 1 \\
&\implies 3.5P(Y = 2) = \frac{35}{36} \\
&\implies \frac{35}{10}P(Y = 2) = \frac{35}{36} \\
&\implies P(Y = 2) = \frac{10}{36}
\end{aligned}
$$

Therefore, $\mathbf{P(X = 1, Y = 2) = P(X = 1)P(Y = 2) = \frac{1}{10}\frac{10}{36} = 1/36}$.

12.

$$\begin{aligned}
&\frac{P(X=2,Y=3)}{P(X=2,Y=4)} = \frac{P(Y=3)}{P(Y=4)} = \frac{5}{20} \\
&\implies 4P(Y=3) = P(Y=4) \\
&\sum_y P(Y=y) \qquad = P(Y=1) + P(Y=2) + P(Y=3) + 4P(Y=3) \\
&= \frac{1}{36} + \frac{10}{36} + 5P(Y=3) = 1 \\
&\implies 5P(Y=3) = \frac{25}{36} \\
&\implies P(Y=3) = \frac{5}{36} \\
&\implies P(Y=4) = \frac{20}{36}
\end{aligned}$$

Therefore, $\mathbf{P(X=1,Y=3) = P(X=1)P(Y=3) = \frac{1}{10}\frac{5}{36} = 5/360}$.

13. $P(Y=4) = \frac{20}{36}$, therefore:

$\mathbf{P(X=1,Y=4) = P(X=1)P(Y=4) = \frac{1}{10}\frac{20}{36} = 1/18}$.

14.

$$\begin{aligned}
&\frac{P(X=2,Y=2)}{P(X=3,Y=2)} = \frac{P(X=2)}{P(X=3)} = \frac{15}{10} \\
&\implies \frac{2}{3}P(X=2) = P(X=3) \\
&\frac{P(X=2,Y=2)}{P(X=4,Y=2)} = \frac{P(X=2)}{P(X=4)} = \frac{12}{10} \\
&\implies \frac{5}{6}P(X=2) = P(X=3) \\
&\sum_x P(X=x) \qquad = P(X=1) + P(X=2) + \frac{2}{3}P(X=2) + \frac{5}{6}P(X=2) \\
&= \frac{1}{10} + \frac{15}{6}P(X=2) = 1 \\
&\implies \frac{15}{6}P(X=2) = \frac{9}{10} \\
&\implies \frac{75}{30}P(X=2) = \frac{27}{30} \\
&\implies P(X=2) = \frac{27}{75} = 9/25
\end{aligned}$$

Therefore, $\mathbf{P(X=2,Y=1) = P(X=2)P(Y=1) = \frac{9}{25}\frac{1}{36} = 1/100}$.

15.

$$\frac{P(X=3,Y=2)}{P(X=4,Y=2)} = \frac{P(X=3)}{P(X=4)} = \frac{12}{15}$$
$$\implies \frac{5}{4}P(X=3) = P(X=4)$$
$$\sum_x P(X=x) = P(X=1)+P(X=2)+P(X=3)+\frac{5}{4}P(X=3)$$
$$= \frac{1}{10}+\frac{9}{25}+\frac{9}{4}P(X=3) = 1$$
$$= \frac{10}{100}+\frac{36}{100}+\frac{9}{4}P(X=3) = 1$$
$$= \frac{23}{50}+\frac{9}{4}P(X=3) = 1$$
$$\implies \frac{9}{4}P(X=3) = \frac{27}{50}$$
$$\implies P(X=3) = \frac{12}{50} = \frac{6}{25}$$
$$\implies P(X=4) = \frac{3}{10}$$

Therefore, $\mathbf{P(X=3,Y=1)=P(X=3)P(Y=1)=\frac{6}{25}\frac{1}{36}=1/150}$.

16. $P(X=4)=\frac{3}{10}$, therefore:

$\mathbf{P(X=4,Y=1)=P(X=4)P(Y=1)=\frac{3}{10}\frac{1}{36}=1/120}$.