CS 621 - Assignment 6

Ali Hassani

December 2, 2022

1

Let HAMPATH be the following problem: Given an undirected graph G, does G have a Hamilton path? Recall that a Hamilton path is one that visits every node exactly once. The HAMCYCLE problem has the same input but is asking whether the graph has a Hamilton cycle.

- Argue that both problems are in NP.
- Show carefully that HAMCYCLE $\leq_m^p$ HAMPATH.

Answer

Argue that both problems are in NP.

Given any graph $G = (V, E)$ with $|V| = n$ vertices, the length of any solution to HAMCYCLE is exactly n, and the length of any solution to HAMCYCLE is $n - 1$. The upper bound for any of these two is n^2 (maximum number of possible edges). As a result, given a witness, the solution can be verified in polynomial time with respect to input size n.

Show carefully that HamCycle $\leq_m^p$ HamPath.

Given any graph $G = (V, E)$, we define $f(.)$ as a function that constructs graph $f(G) = G' = (V', E')$, where:

$$V' = V \cap \{x', y, y'\}$$
$$E' = E \cap \{\{x', a\} \mid \{x, a\} \in E\} \cap \{\{y, x\}, \{y', x'\}\}$$

and x is the least-degree vertex in G. (Technically x can be an arbitrary vertex, but specifying a property helps create a well-defined f.) The function f takes graph G, cleaves vertex x (clones it and its edges into a new vertex x'), and adds two additional vertices y and y', where there's an edge between y and x, and another between y' and x'.

Through this definition, given any graph $G \in \textsc{HamCycle}$, we know that $G \in \textsc{HamPath}$ by definition (if a graph has a Hamilton cycle, it has a Hamilton path). Adding the three new vertices does not break that.

Proof: Consider only the Hamilton cycle within the original graph G. x and two edges connecting to it must be included in the cycle. Let's name the vertices on the other end of those two edges p and q. Therefore $\{x, p\}$ and $\{x, q\}$ are in the the cycle. Now when we introduce x' into the graph, we can simply swap one of those two edges with the corresponding x' one, and this breaks the cycle, and creates a path. Adding the two new vertices y and y' only expands that path by two vertices. As a result, adding the three new vertices and their specified edges creates a graph with a Hamilton path, given that the original graph had a Hamilton cycle.

Therefore, $G \in \textsc{HamCycle} \implies f(G) \in \textsc{HamPath}$.

Now assume that for some graph G, $f(G) \in \textsc{HamPath}$, but $G \notin \textsc{HamCycle}$. Given that assumption, we know the terminal vertices. It's the y and y' that the function f introduces. If we remove those two, we still have the cycle, and the terminal vertices are x and x'. If x and x' are connected with an edge, we've constructed a Hamilton cycle. This is the same as removing x'. If we remove x' however, we end up with the original graph, G. Therefore $G \in \textsc{HamCycle}$, and that is a contradiction.

As a result, $f(G) \in \textsc{HamPath} \implies G \in \textsc{HamCycle}$.

Therefore, we can conclude that $G \in \textsc{HamCycle} \iff f(G) \in \textsc{HamPath}$.

Therefore, $\textsc{HamCycle} \leq_m^p \textsc{HamPath}$.

A Long answer to part 1

This was my original attempt in arguing both problems are in NP by showing they reduce to TSP, and TSP is in NP.

Argue that both problems are in NP.

Consider the traveling salesman problem (TSP). TSP is NP-complete, and therefore $TSP \in NP$. Based on the closure theorem, NP is closed under $\leq_m^p$. Therefore, given any problem A that poly time many-one reduces to TSP ($A \leq_m^p TSP$), we can conclude that $A \in NP$. We claim that $\textsc{HamCycle} \leq_m^p TSP$:

Given any graph $G = (V, E)$, we define $h(G)$ as a weighted graph with weight function W so that given any two vertices x and y in V, $W(x, y) = 1$ IFF $\{x, y\} \in E$, and otherwise $W(x, y) = 0$. If $G \in \textsc{HamCycle}$, then $h(G) \in TSP$ with a path of exactly length $|V|$. If $h(G) \in TSP$, that would mean that there exists a cycle visiting all vertices exactly once by definition, therefore $G \in \textsc{HamCycle}$.

As a result, $\textsc{HamCycle} \leq_m^p TSP$ is valid, therefore we can conclude that $\textsc{HamCycle} \in NP$.

We also claim that $\textsc{HamPath} \leq_m^p \textsc{HamCycle}$, and if this claim is true, based on it and the fact that $\textsc{HamCycle} \leq_m^p TSP$, we can claim that $\textsc{HamPath} \leq_m^p TSP$.

Given any graph $G = (V, E)$, we define $g(.)$ as a function that constructs graph $g(G) = (V', E')$, where:

$$V' = V \cap \{x\}$$
$$E' = E \cap \{\{x, a\} \mid a \in V\} \ .$$

If graph G has a Hamilton path from some vertex a to another vertex b, adding x and edges $\{x, a\}$ and $\{x, b\}$ turns it into a cycle, and because $x \notin G$, the cycle is Hamilton cycle in $g(G)$. If $g(G)$ does not have a Hamilton cycle, G must also not have a Hamilton path. If $g(G)$ didn't have a Hamilton cycle, but G had a Hamilton path, and p and q were two endpoints, $g(G)$ would introduce x, and edges between x and p and x and q, therefore $g(G)$ has a Hamilton cycle, which is a contradiction.

Therefore $\textsc{HamPath} \leq_m^p \textsc{HamCycle}$, and as a result $\textsc{HamPath} \leq_m^p TSP$, and therefore $\textsc{HamPath} \in NP$.