Introduction

Analysis & Design of Algorithms

1 de enero de 2026

Exercise 1. The *floor* of a real number x, denoted by $\lfloor x \rfloor$, is the maximum integer that is less or equal to x. The *ceil* of a real number x, denoted by $\lceil x \rceil$, is the minimum integer that is greater or equal to x.
Prove the following inequalities, given that x, y are arbitrary real numbers, n is an arbitrary integer number, and a, b are arbitrary nonzero integers

(a) $\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x + y \rfloor \le \lfloor x \rfloor + \lfloor y \rfloor + 1$

(b) $\lceil x \rceil + \lceil y \rceil - 1 \le \lceil x + y \rceil \le \lceil x \rceil + \lceil y \rceil$

(c) $(n-1)/2 \le \lfloor n/2 \rfloor \le n/2$

(d) n, $n/2 \le \lceil n/2 \rceil \le (n+1)/2$

(e) n, $n = \lfloor n/2 \rfloor + \lceil n/2 \rceil$

(f) $x \in \mathbb{R}, a, b \in \mathbb{Z}$, $\lceil \frac{\lceil x/a \rceil}{b} \rceil = \lceil \frac{x}{ab} \rceil$

Exercise 2. Prove by induction that

(a) $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$, for all positive integers n.

(b) $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$, for all positive integers n.

(c) $n \le 2^{n/2}$, for all positive integers $n \ge 4$.

Exercise 3.

(a) Find a simple formula of the sum $\sum_{k=1}^{n}(2k-1)$

(f) Evaluate the product $\prod_{k=1}^{n}(2 \cdot 4^k)$

(b) Find a simple form of $\prod_{k=2}^{n}(1 - 1/k^2)$

Exercise 4.

(a) Prove by induction that, for some constant c we have that $\sum_{k=0}^{n}(3^k) \le c3^n \quad \forall n \in \mathbb{Z}^+$

(b) Show that $\sum_{k=1}^{n} k \ge (n/2)^2$.

(c) Show that for some constant c, it holds $\sum_{k=1}^{n}(1/k^2) \leq c \quad \forall n \in \mathbb{Z}^+$.

(d) Find an upper bound for the sum $\sum_{k=0}^{\lfloor \lg n \rfloor} \lceil n/2^k \rceil$

Exercise 5. How many leaves has a full binary tree with n nodes? Use induction as one of the proofs

Exercise 6. Show that the height of a complete binary tree with k leaves is$\lg k$.