Class Exercises: Probabilistic Analysis & Quicksort

Análisis y Diseño de Algoritmos

January 1, 2026

Exercise 1. Let X be a random variable that represents the number of heads in two tosses of a fair coin. What is the value of $E[X^2]$? What is the value of $E[X]^2$?

Exercise 2. Let X be a random variable that represents the sum of the results in the roll of n dice. What is the value of $E[X]$?

Exercise 3. Consider the problem of hiring n people (HIRE-ASSISTANT). Let X_i be the random variable that is 1 if the i-th candidate is hired and 0 if not. Find X_1, X_2, X_3 when $n = 3$ and the candidates are $\{1, 2, 3\}$. Find X_1, X_2, X_3, X_4 when $n = 4$ and the candidates are $\{1, 2, 3, 4\}$.

Exercise 4. In the pseudocode of HIRE-ASSISTANT, assuming the candidates are presented in a uniformly random manner, what is the probability that exactly one candidate is hired? What is the probability that n candidates are hired? What is the probability that 2 candidates are hired?

Exercise 5. Consider the following algorithm that determines the largest and smallest element of a vector $v[1 \ldots n]$ with distinct positive numbers.

LARGESTSMALLEST(v, n)
1: $largest = v[1]$
2: $smallest = v[1]$
3: **for** $i = 2$ **to** n
4: **if** $v[i] > largest$
5: $largest = v[i]$
6: **else**
7: **if** $v[i] < smallest$
8: $smallest = v[i]$
9: **return** $largest, smallest$

Assume the input of the algorithm is a permutation of 1 to n chosen uniformly from all permutations of 1 to n. What is the expected number of comparisons executed in line 7 of the algorithm? What is the expected number of assignments performed in line 8 of the algorithm?

Exercise 6. Illustrate the PARTITION operation on the array $A = [13, 19, 9, 5, 12, 8, 7, 4, 21, 2, 6, 11]$.

Exercise 7. Write a function that receives a vector with n letters A and B and, through swaps, moves all the A to the beginning of the vector. Your function should have an execution time of $O(n)$.

Exercise 8. What is the execution time of QUICKSORT when all the elements of the array A have the same value?

Exercise 9. Prove that $\max_{0 \leq q \leq n-1} \{q^2 + (n-q-1)^2\} = (n-1)^2$