Computation using Quantum Mechanics

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August 8, 2020

CLASSICAL v. QUANTUM

Classical Computer

Maxwell's Equations	Condensed Matter (Solid State)

Classical Computer

Circuits: Ohm's Law + Kirchoff's Laws ***AC, Resistance, Inductor, Capacitor***	**Electronics: Transistors** ***Switch***
Maxwell's Equations	**Condensed Matter (Solid State)**

Classical Computer

Digital Abstraction: Boolean Algebra

AND, OR, NOT

Circuits: Ohm's Law + Kirchoff's Laws

AC, Resistance, Inductor, Capacitor

Electronics: Transistors

Switch

Maxwell's Equations

Condensed Matter (Solid State)

Quantum Mechanics by Example

Time-independent 1-D Schrödinger Equation is

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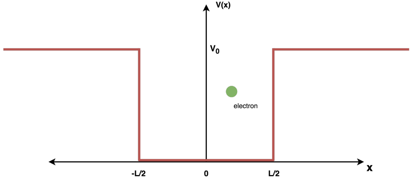
where
$\hbar$ is Plank's constant / 2π
m is the mass of the particle
ψ is a complex valued wavefunction
$V(x)$ is the potential energy at point x
E is the total energy of the particle.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$



$$\psi = \begin{cases} \psi_l, & \text{if } x \leqslant -L/2 \\ \psi_m, & \text{if } -L/2 \leqslant x \leqslant L/2 \\ \psi_r, & \text{if } x \geqslant L/2 \end{cases}$$

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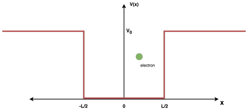

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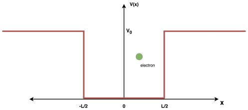

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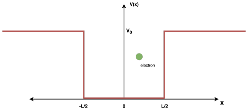

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Let $k = \frac{\sqrt{2mE}}{\hbar}$, we get $\frac{d^2\psi_m}{dx^2} = -k^2\psi_m$

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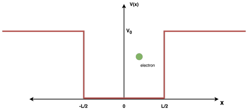

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So $\psi_m = A\cos(kx) + B\sin(kx)$

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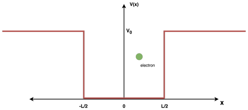

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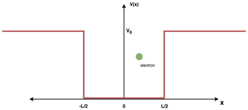
So $\psi_m = A\cos(kx) + B\sin(kx)$ for any $k \in \mathbb{R}$.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$



If $V_0 > E$,

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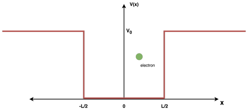

If $V_0 > E$, letting $\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$, gives us $\frac{d^2\psi_l}{dx^2} = \kappa^2\psi_l$

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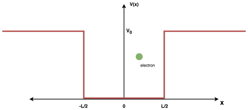

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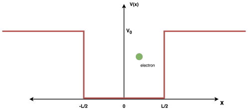

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So $\psi_l = Ce^{-\kappa x} + De^{\kappa x}$. Similarly, $\psi_r = Ee^{-\kappa x} + Fe^{\kappa x}$

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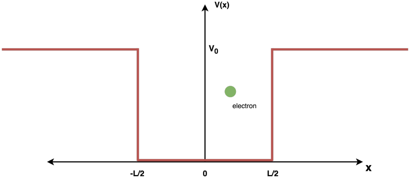

$$\psi = \begin{cases} \psi_l = Ce^{-\kappa x} + De^{\kappa x}, & \text{if } x \leqslant -L/2 \\ \psi_m = A\cos(kx) + B\sin(kx), & \text{if } -L/2 \leqslant x \leqslant L/2 \\ \psi_r = Ee^{-\kappa x} + Fe^{\kappa x}, & \text{if } x \geqslant L/2 \end{cases}$$

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$\frac{\psi_l}{dx}(\frac{-L}{2}) = \frac{\psi_m}{dx}(\frac{-L}{2})$,
and $\frac{\psi_m}{dx}(\frac{L}{2}) = \frac{\psi_r}{dx}(\frac{L}{2})$.

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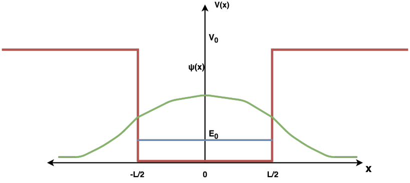

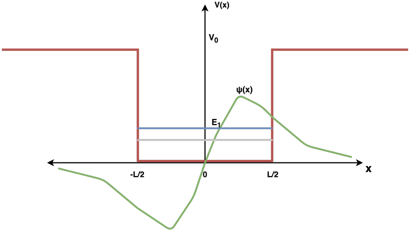

Quantum Computer

Quantum Mechanics

Quantum Computer

- Energy Levels in Harmonic Oscillators
- Location of Single Optical Photon in 2 Cavities
- Polarization of Photons
- Nuclear spin state of an ion in a magnetic field
- Spin of a Nucleus

- Laser pulses
- Magnetic fields
- Electric fields
- Beam splitters
- Phase shifters

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Quantum Computer

Quantum Gates and Circuits

Toffoli, Hadamard, Controlled NOT, X, Z, S, T

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Quantum Mechanics

QUANTUM DYNAMICS

Mathematical Modelling

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A complete complex inner product vector space.

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Unitary operators, U, obeying $UU^\dagger = U^\dagger U = I$.

Postulate (Quantum States)

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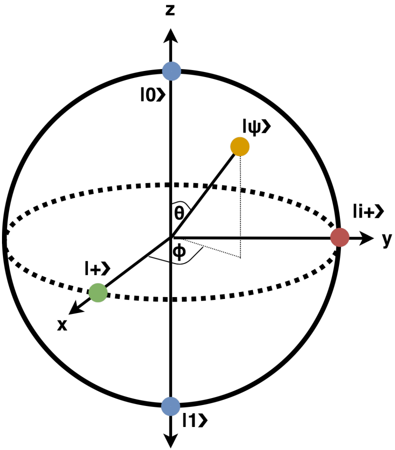
Definition (Quantum Bits (Qubits))

A qubit is the simplest quantum system and is represented by a 2-dimensional Hilbert space $\mathcal{H}$. A canonical basis spanning $\mathcal{H}$ is $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Making any arbitrary state vector $|\psi\rangle \in \mathcal{H}$ be written as $\alpha |0\rangle + \beta |1\rangle$, where $\alpha, \beta \in \mathbb{C}$ are called probability amplitudes constrained that $|\alpha|^2 + |\beta|^2 = 1$.


z
|0❯
|ψ❯
θ
|i+❯
y
ϕ
|+❯
x
|1❯

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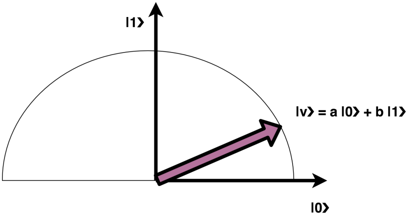

Postulate (Quantum Evolution)

A closed quantum system, $|\psi_{t=0}\rangle$, evolves according a unitary operator U (for time T) to reach the (new) state $|\psi_{t=T}\rangle$ written as $|\psi_{t=T}\rangle = U|\psi_{t=0}\rangle$.

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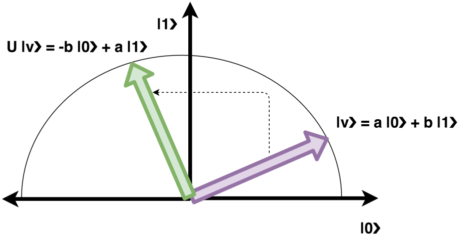
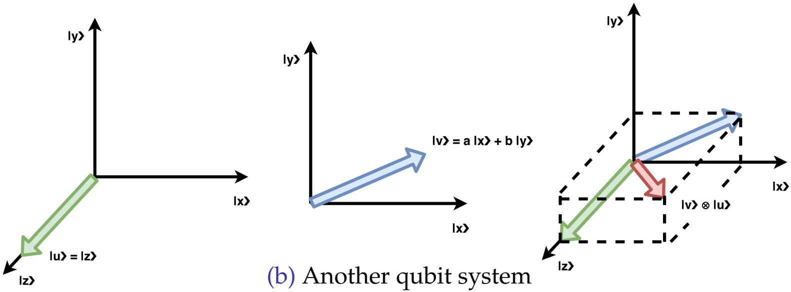


Here $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Postulate (Quantum state composition)

The state space representing a composition of multiple (possibly interacting) closed quantum systems is defined by the tensor product of the state spaces of the individual quantum systems, and is written as the product state $|\psi_1\rangle \otimes |\psi_2\rangle$*, where* $|\psi_1\rangle \in \mathcal{H}_1$ *and* $|\psi_2\rangle \in \mathcal{H}_2$ *and dimension* $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$ *is the product of their dimensions.*



(a) A qubit system $|u\rangle$.

(b) Another qubit system $|v\rangle$.

(c) A composition of both systems $|v\rangle \otimes |u\rangle$.

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \begin{pmatrix} c \\ d \end{pmatrix} \\ b \begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x & y \\ z & l \end{pmatrix} = \begin{pmatrix} a \begin{pmatrix} x & y \\ z & l \end{pmatrix} & b \begin{pmatrix} x & y \\ z & l \end{pmatrix} \\ c \begin{pmatrix} x & y \\ z & l \end{pmatrix} & d \begin{pmatrix} x & y \\ z & l \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ax & ay & bx & by \\ az & al & bz & bl \\ cx & cy & dx & dy \\ cz & cl & dz & dl \end{pmatrix}$$

Definition (Quantum Entanglement)

We say a quantum state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is entangled if it cannot be decomposed/factored into a tensor product of constituents of the sub-systems $\mathcal{H}_1$ and $\mathcal{H}_2$, namely $\forall\ |\psi_1\rangle \in \mathcal{H}_1, |\psi_2\rangle \in \mathcal{H}_2$

$$|\psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1\psi_2\rangle$$

Table: The four Bell Basis states represent the simplest maximally entangled two qubit systems

Symbol	Expansion in Tensor Product of Canonical Basis
$\|\Phi^+\rangle$	$\frac{1}{\sqrt{2}}\left(\|00\rangle_{AB} + \|11\rangle_{AB}\right)$
$\|\Phi^-\rangle$	$\frac{1}{\sqrt{2}}\left(\|00\rangle_{AB} - \|11\rangle_{AB}\right)$
$\|\Psi^+\rangle$	$\frac{1}{\sqrt{2}}\left(\|01\rangle_{AB} + \|10\rangle_{AB}\right)$
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$$(\alpha\,|0\rangle + \beta\,|1\rangle) \otimes (\gamma\,|0\rangle + \xi\,|1\rangle) =$$
$$\alpha\gamma\,|00\rangle + \alpha\xi\,|01\rangle + \beta\gamma\,|10\rangle + \beta\xi\,|11\rangle$$

Definition (No-Cloning Theorem)

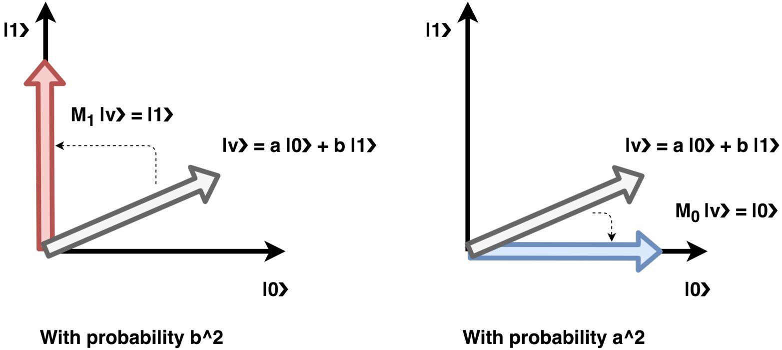
Creating an identical copy of an arbitrary unknown quantum state without destroying the original is impossible.

Postulate (Quantum Measurement)

An open quantum system, $|\psi_{\text{Pre}}\rangle$, interacts with the rest of the world in non-unitary evolution. Let us model this external system interacting with our quantum system by a collection of measurement operators (for our purposes these will be projectors) $\{M_b\}$ where b represents the measurement outcome and $\sum_b M_b^\dagger M_b = I$. The result of this interaction is the collapse of $|\psi_{\text{Pre}}\rangle$ to

$$|\psi_{\text{Post}}\rangle = \frac{M_b |\psi_{\text{Pre}}\rangle}{\sqrt{\langle\psi_{\text{Pre}}| M_b^\dagger M_b |\psi_{\text{Pre}}\rangle}}$$

for a specific b, where the probability for any particular b, $Pr[b] = \langle\psi_{\text{Pre}}| M_b^\dagger M_b |\psi_{\text{Pre}}\rangle$


|1❯
M1 |v❯ = |1❯
|v❯ = a |0❯ + b |1❯
|0❯
With probability b^2
|1❯
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Definition (Heisenberg Uncertainty Principle)

Let A, B be two Hermitian operators (aka. observables). Then the following inequality always holds:

$$(\Delta A)^2(\Delta B)^2 \geqslant \left(\langle\Psi| \frac{1}{2i}[A, B] |\Psi\rangle \right)^2$$

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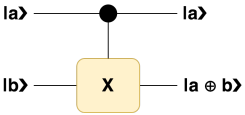
example: $\Delta x \Delta p \geqslant \frac{\hbar}{2}$

QUANTUM COMPUTATION

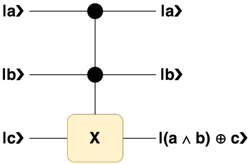
Quantum Gates

Pauli-X / NOT	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$
Pauli-Y / Rotation by π around y-axis	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto i \begin{pmatrix} -\beta \\ \alpha \end{pmatrix}$
Pauli-Z / Rotation by π around z-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$
Hadamard H	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix}$
$S = \sqrt{Z}$	$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha \\ i\beta \end{pmatrix}$
$T = \sqrt{S}$	$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$	$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha \\ e^{i\pi/4}\beta \end{pmatrix}$

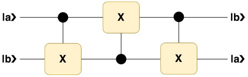
Quantum Gates



(a) Controlled NOT



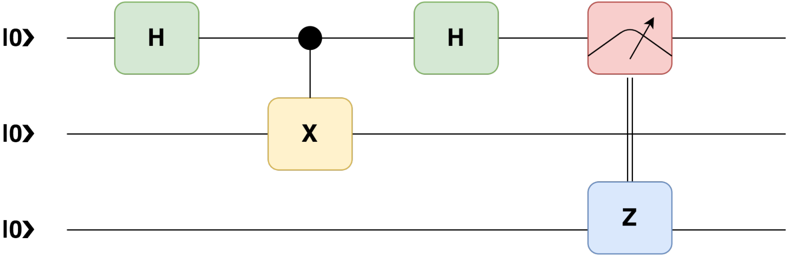
(b) Toffoli

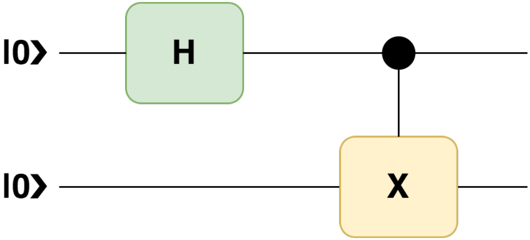


(c) Swap

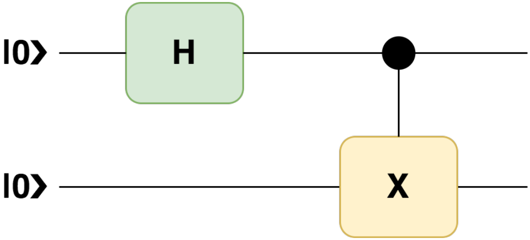
Universal Quantum Gate Sets

- CNOT and all single qubit gates.
- Toffoli, Hadamard, and S gates.
- CNOT, Hadamard, and T gates.

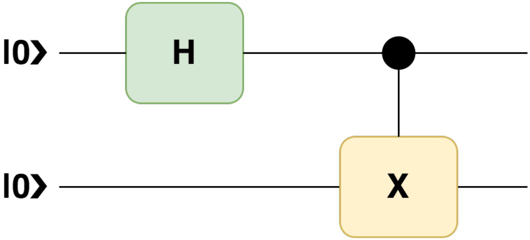

|0❯
|0❯
|0❯
H
X
H
Z



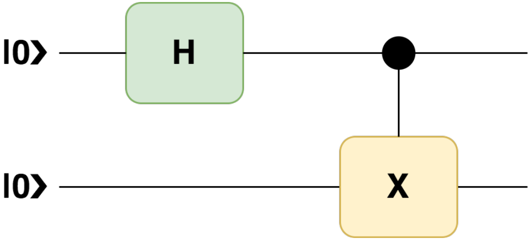
$|0,0\rangle$



$$|0,0\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle$$



$$|0,0\rangle \rightarrow \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes |0\rangle = \frac{|00\rangle+|10\rangle}{\sqrt{2}}$$



$$|0,0\rangle \rightarrow \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes |0\rangle = \frac{|00\rangle+|10\rangle}{\sqrt{2}} \rightarrow \frac{|00\rangle+|11\rangle}{\sqrt{2}}$$

Quantum Fourier Transformation

DFT: $y_k \equiv \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi ijk/N}$. Time $O(N \log N)$

QFT: $|j\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi ijk/N} |k\rangle$. Time $O(\log^2 N)$

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$$\begin{pmatrix} y_0 \\ y_1 \\ \dots \\ y_k \\ \dots \\ y_{N-1} \end{pmatrix}, \begin{pmatrix} x_0 \\ x_1 \\ \dots \\ \dots \\ \dots \\ x_{N-1} \end{pmatrix}$$

i is the imaginary number.

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Differentiation in time domain $\frac{df(t)}{dt}$ becomes multiplication by frequency $i2\pi f$.
Convolution in time is multiplication in frequency:
$\text{QFT}(x * y) = \text{QFT}(x) \cdot \text{QFT}(y)$

Let $N = 2^n$, $|j\rangle = |j_1, j_2, .., j_n\rangle$, and $0.j_1 j_2 ... j_n = j/2^n$. Then

$$|j\rangle \to \frac{1}{2^n} \sum_{k=0}^{2^n-1} e^{2\pi i jk/2^{n/2}} |k\rangle = \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} ... \sum_{k_n=0}^{1} e^{2\pi i j(\sum_{l=1}^{n} k_l 2^{-l})} |k_1 ... k_n\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} ... \sum_{k_n=0}^{1} \bigotimes_{l=1}^{n} e^{2\pi i j k_l 2^{-l}} |k_l\rangle = \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[\sum_{k_1=0}^{1} e^{2\pi i j k_l 2^{-l}} |k_l\rangle \right]$$

$$= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[|0\rangle + e^{2\pi i j 2^{-l}} |1\rangle \right]$$

$$= \frac{\left(|0\rangle + e^{2\pi i 0.j_n} |1\rangle\right)\left(|0\rangle + e^{2\pi i 0.j_{n-1}j_n} |1\rangle\right)..\left(|0\rangle + e^{2\pi i 0.j_1..j_n} |1\rangle\right)}{2^{n/2}}$$

$$\text{Let } R_k \equiv \begin{pmatrix} 1 & 0 \\ 0 & e^{i(2\pi/2^k)} \end{pmatrix}, R_1 = Z, R_2 = S \text{ and } R_3 = T$$

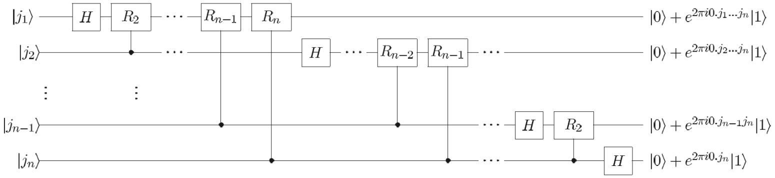

Image source from Mike and Ike.

QUANTUM ALGORITHMS

DEUTSCH'S ALGORITHM

Deutsch's Algorithm

Consider All functions $f : \{0,1\} \rightarrow \{0,1\}$:

Function #1: $f(x) = 0$ (the constant zero function)

Function #2: $f(x) = 1$ (the constant one function)

Function #3: $f(x) = x$ (the identity function)

Function #4: $f(x) = \overline{x}$ (the inverse function)

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Functions #1 and #2 are constant functions, functions #3 and #4 are called balanced functions.

Deutsch's Algorithm (cont.)

Imagine I pick one of the 4 functions, f, uniformly at random. How many queries do you need to determine if f is constant or balanced?

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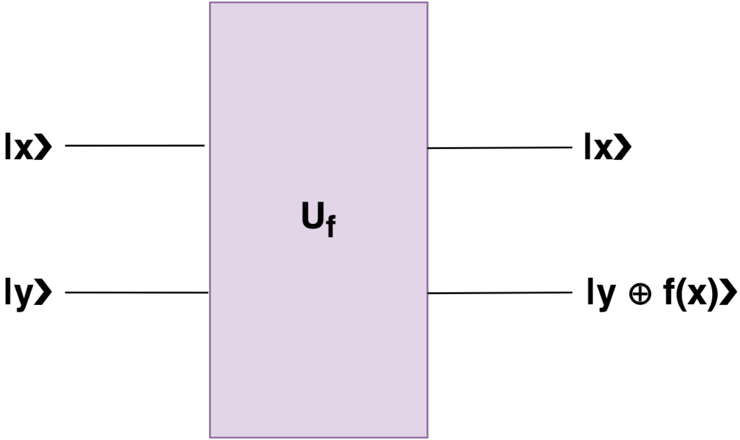
Quantumly you would need only 1 call!

Quantum-izing a Function



Deutsch's Algorithm

Recall $H|0\rangle = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

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Furthermore, $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which is a kind of Fourier Transform.

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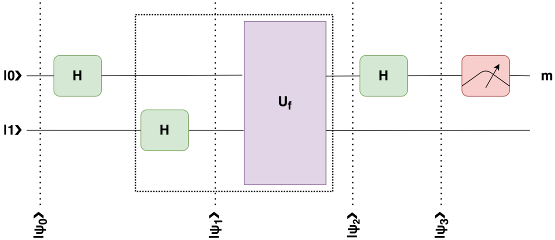
Finally we denote by $|\pm\rangle = 1/\sqrt{2}(|0\rangle \pm |1\rangle)$.

Deutsch's Algorithm (cont.)

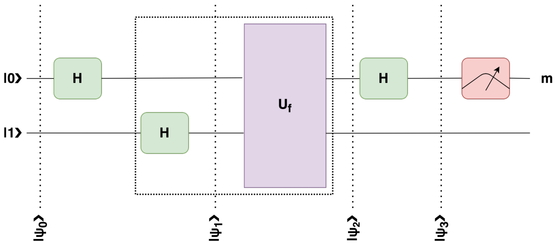


$$|\psi_0\rangle = |0,1\rangle$$

$$|\psi_1\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$|\psi_2\rangle = |+\rangle \otimes \left(\frac{|0 \oplus f(+)\rangle - |1 \oplus f(+)\rangle}{\sqrt{2}}\right)$$

Deutsch's Algorithm (cont.)



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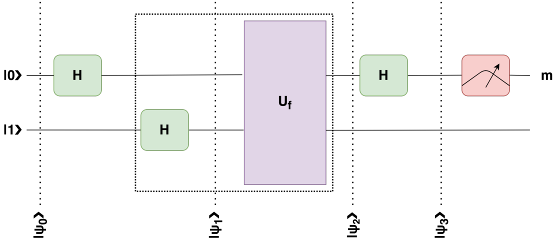
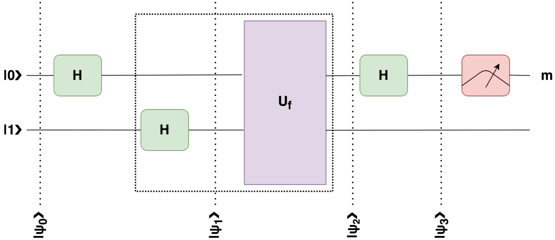


$$|\psi_2\rangle = |+\rangle \otimes \left(\frac{|0 \oplus f(+)\rangle - |1 \oplus f(+)\rangle}{\sqrt{2}} \right) = |+\rangle \otimes \left(\frac{|f(+)\rangle - |\overline{f(+)}\rangle}{\sqrt{2}} \right)$$

Deutsch's Algorithm (cont.)



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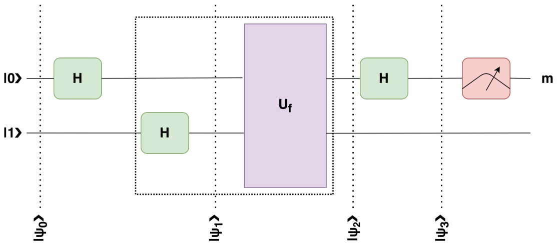
Deutsch's Algorithm (cont.)



$$|\psi_2\rangle = |+\rangle \otimes \left(\frac{|0 \oplus f(+)\rangle - |1 \oplus f(+)\rangle}{\sqrt{2}}\right) = |+\rangle \otimes \left(\frac{|f(+)\rangle - |\overline{f(+)}\rangle}{\sqrt{2}}\right) =$$
$$(-1)^{f(+)}\,|+\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \left(\frac{(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

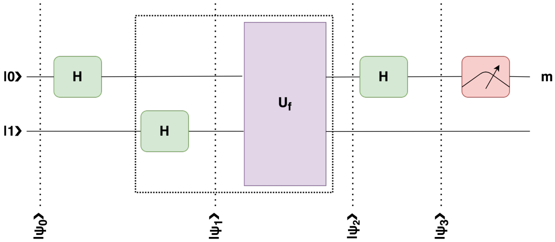
Deutsch's Algorithm (cont.)

$$|\psi_2\rangle = \left(\frac{(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

If f was constant then $(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle = \pm(|0\rangle + |1\rangle)$

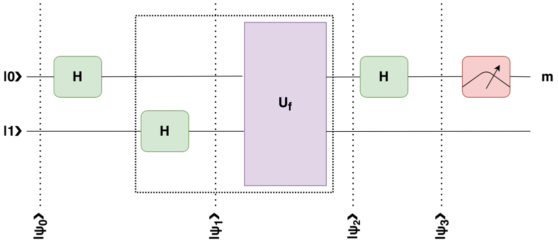
If f was balanced then $(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle = \pm(|0\rangle - |1\rangle)$

Deutsch's Algorithm (cont.)



$$|\psi_3\rangle = \begin{cases} \pm|0\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right), & \text{if } f \text{ constant} \\ \pm|1\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right), & \text{if } f \text{ balanced} \end{cases}$$

Deutsch's Algorithm (cont.)



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If $m = 0$ after measurement we conclude that f was constant, otherwise we say it was balanced.

Deutsch-Jozsa Algorithm

Consider All functions $f : \{0,1\}^n \rightarrow \{0,1\}$, where a balanced function is one where half inputs go to zero, while constant function all inputs go to either zero or one. This is a Promise problem.

Classically you would need

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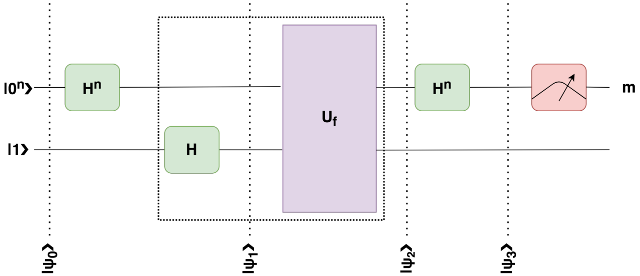
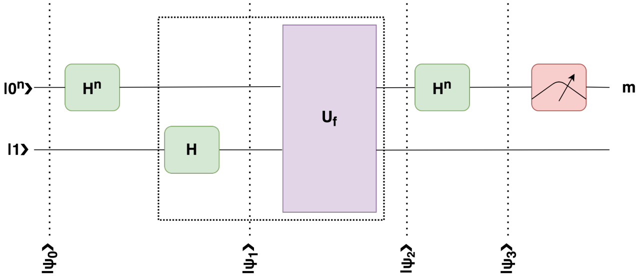
Classically you would need $\frac{2^n}{2} + 1 = 2^{n-1} + 1$ **calls.**

Quantumly you still need only 1 call! That is an exponential speedup.

Deutsch-Jozsa Algorithm



Deutsch-Jozsa Algorithm



Try to follow the same analysis as a homework.

GROVER'S ALGORITHM

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Given an unordered array of m elements, find a particular element.

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Quantumly, we can find the element in $\sqrt{m}$ queries.

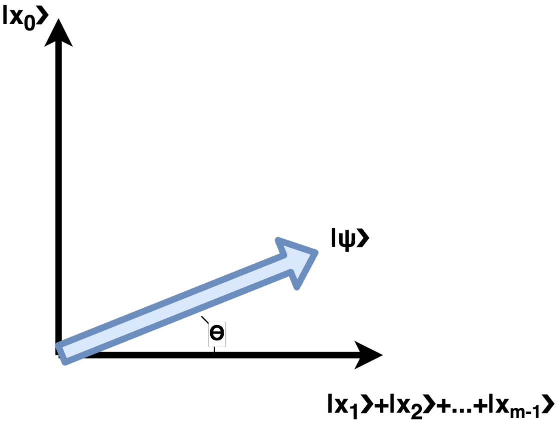
Grover's Algorithm (cont.)

Imagine we find a function $f : \{0,1\}^n \rightarrow \{0,1\}$ with exactly one x_0 representing the index of the desired element s.t.

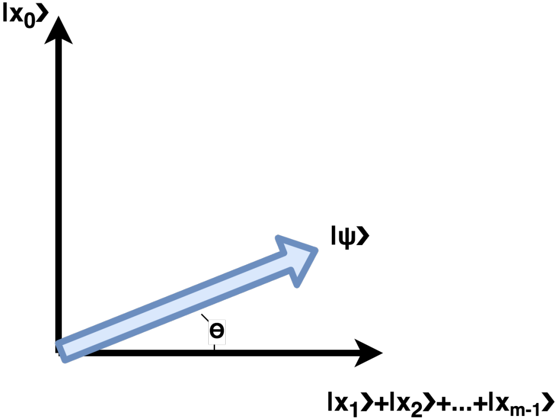
$$f(x) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{otherwise} \end{cases}$$

Quantum-izing f gives us the unitary $U_f \left|x, y\right\rangle \rightarrow \left|x, f(x) \oplus y\right\rangle$ where $x \in \{0,1\}^n$ and $y \in \{0,1\}$.

Grover's Algorithm (cont.)

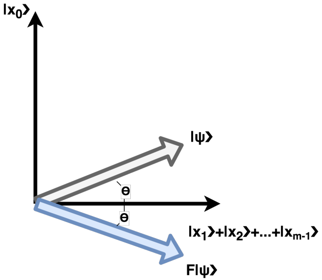


Grover's Algorithm (cont.)

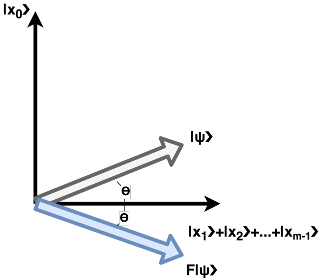


Recall number qubits n equals $\log(m)$, where m is number of elements in array.

Grover's Algorithm (cont.)

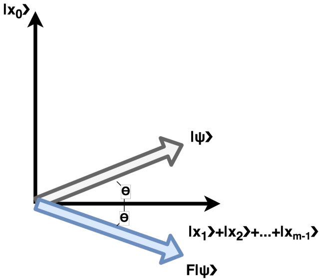


Grover's Algorithm (cont.)



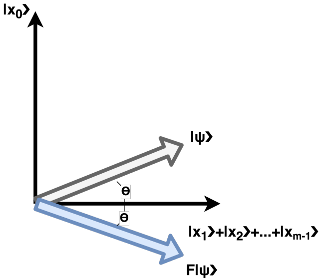
$$U_f(I^{\otimes n} \otimes H)\,|x, 1\rangle$$

Grover's Algorithm (cont.)



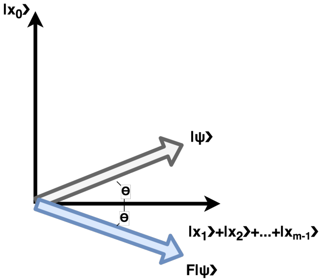
$$U_f(I^{\otimes n} \otimes H)\,|x,1\rangle = U_f\,|x\rangle \left(\frac{|0\rangle - |1\rangle}{2}\right)$$

Grover's Algorithm (cont.)



$$U_f(I^{\otimes n} \otimes H)\left|x,1\right\rangle = U_f\left|x\right\rangle\left(\frac{|0\rangle - |1\rangle}{2}\right) = \left|x\right\rangle\left(\frac{|f(x)\rangle - |\overline{f(x)}\rangle}{2}\right)$$

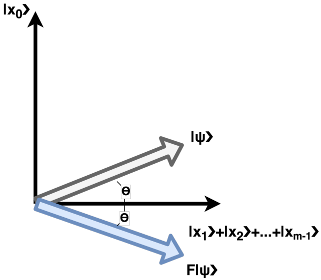
Grover's Algorithm (cont.)



$$U_f(I^{\otimes n} \otimes H)\,|x,1\rangle = U_f\,|x\rangle \left(\frac{|0\rangle - |1\rangle}{2}\right) = |x\rangle \left(\frac{|f(x)\rangle - |\overline{f(x)}\rangle}{2}\right) =$$

$$(-1)^{f(x)}\,|x,-\rangle.$$

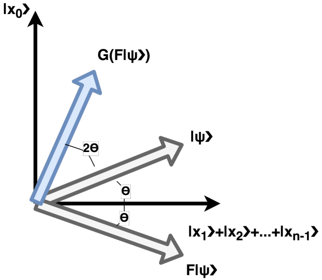
Grover's Algorithm (cont.)



$$U_f(I^{\otimes n} \otimes H)\,|x,1\rangle = U_f\,|x\rangle\left(\frac{|0\rangle - |1\rangle}{2}\right) = |x\rangle\left(\frac{|f(x)\rangle - |\overline{f(x)}\rangle}{2}\right) = (-1)^{f(x)}\,|x,-\rangle.$$

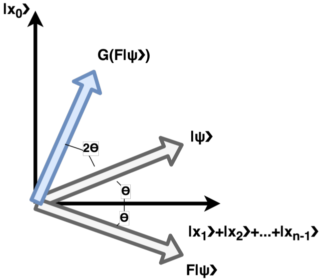
$$\cos(\theta) = \langle\psi|x_1 x_2 .. x_{m-1}\rangle = \sqrt{\frac{m-1}{m}}.$$

Grover's Algorithm (cont.)



$$[(-\mathbf{I}+\mathbf{2A})\otimes I](-1)^{f(x)}\,|x,-\rangle$$

Grover's Algorithm (cont.)



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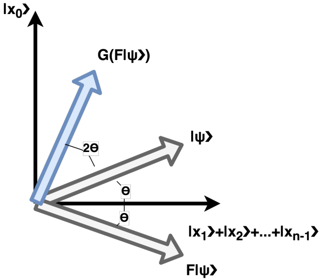

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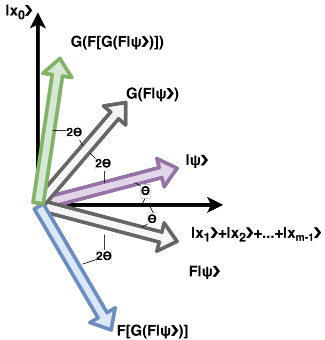

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Grover's Algorithm Summary

$$\left[\left(H^{\otimes n} (2 \left| 0^n \right\rangle \left\langle 0^n \right| - I^{\otimes n}) H^{\otimes n} \otimes I \right) U_f (I^{\otimes n} \otimes H)) \right]^{\sqrt{n}} (H^{\otimes n} \otimes I) \left| 0^{\otimes n}, 1 \right\rangle$$

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$\cos(\theta) = \sqrt{(m-1)/m}$, so $\sin(\theta) \approx \theta = \sqrt{1/m}$, then time complexity is $\frac{\pi}{2} \cdot \frac{1}{2\theta} = \frac{\pi}{2 \cdot 2}\sqrt{m} = O(\sqrt{m})$

SIMON'S ALGORITHM

Simon's Algorithm

Suppose we are given $f : \{0,1\}^n \to \{0,1\}^n$ and promised that $\exists$ a period $c \in \{0,1\}^n$ s.t. $\forall x, y \in \{0,1\}^n, f(x) = f(y)$ iff $x = y \oplus c$.

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Find c.

Simon's Algorithm (cont.)

Observe: if $c = 0^n$ then f is one-to-one. Otherwise, f is two-to-one, namely $f(x_1) = y = f(x_2)$ when $x_1 = x_2 \oplus c$

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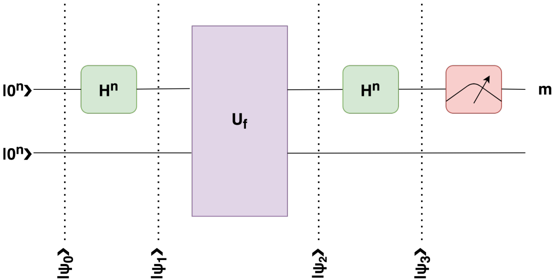

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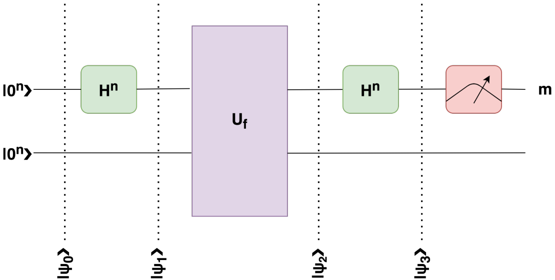

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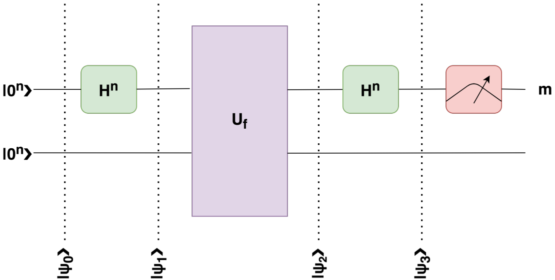

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Simon's Algorithm (cont.)

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$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & 1\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ 1\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & -1\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Simon's Algorithm (cont.)

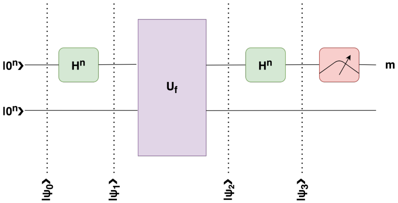
Notice that we are promised that $|z, f(x)\rangle = |z, f(x \oplus c)\rangle$. Then in $|\psi_3\rangle = \frac{\sum_{z\in\{0,1\}^n}\sum_{x\in\{0,1\}^n}(-1)^{\langle z,x\rangle}|z,f(x)\rangle}{\sqrt{2^n}}$, the coefficient of $|z, f(x)\rangle$ is

$$\frac{(-1)^{\langle z,x\rangle} + (-1)^{\langle z,x\oplus c\rangle}}{2} = \frac{(-1)^{\langle z,x\rangle} + (-1)^{\langle z,x\rangle\oplus\langle z,c\rangle}}{2}$$

$$\frac{(-1)^{\langle z,x\rangle} + (-1)^{\langle z,x\rangle}(-1)^{\langle z,c\rangle}}{2} = (-1)^{\langle z,x\rangle}\frac{(1 + (-1)^{\langle z,c\rangle})}{2}$$

if $\langle z, c\rangle = 1$ then the coefficient is 0 (destructive interference).
If $\langle z, c\rangle = 0$ it will be ± 1. (recall inner product is mod 2)

Simon's Algorithm (cont.)



Measuring at the end will collapse our n-qubits into one of the states with $\langle z, c \rangle = 0$.

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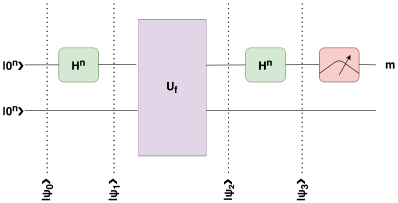

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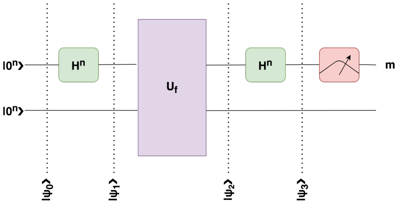

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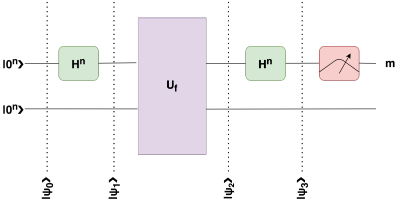

Measuring at the end will collapse our n-qubits into one of the states with $\langle z, c \rangle = 0$. So $m = z$. Thus $\langle m, c \rangle = 0$ is a linear equation. Since c has n-bits, we run Simon's $O(n)$ times, get $O(n)$ equations in n unknowns, and solve them classically.

SHOR'S ALGORITHM

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e.g.: $\gcd(42, 56) = 14$, then $\frac{42}{56} = \frac{3 \cdot 14}{4 \cdot 14} = \frac{3}{4}$.

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e.g. g=7, N=15, $g^2 = 3 \cdot N + 4$, $g^3 = 22 \cdot N + 13$, but $g^4 = 160 \cdot N + 1$.

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Notice $g^{p'+p} = g^{p'} \cdot g^p$. Multiplying $g^{p'}$ by g^p, multiplies RHS by $1 \mod N$ because $g^p \equiv 1 \mod N$.

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$\rightarrow$ measure $\left|r\right\rangle$ reading say r_0, which gives a superposition of the first register $\sum_{q'} \left|q'\right\rangle$ with q' being all powers that share the same remainder.

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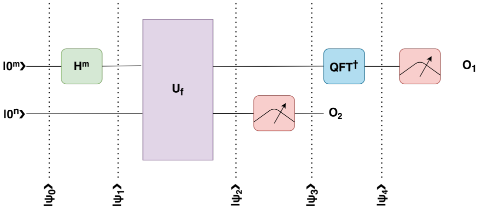
Shor's Algorithm (cont.)



$$|\psi_0\rangle = |0^m, 0^n\rangle$$

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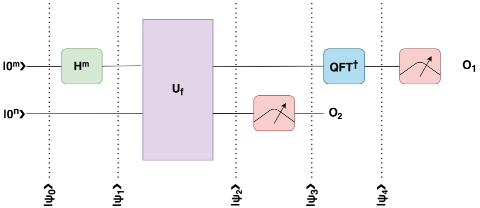
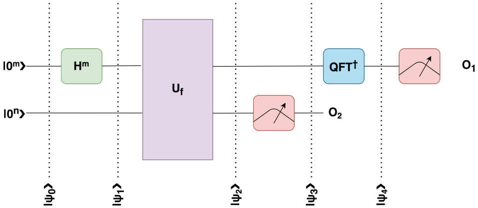


$$|\psi_0\rangle = |0^m, 0^n\rangle \qquad |\psi_1\rangle = \frac{\sum_{q\in\{0,1\}^m} |q, 0^n\rangle}{\sqrt{2^m}}$$

$$|\psi_2\rangle = \frac{\sum_{q\in\{0,1\}^m} |q, g^q \bmod N\rangle}{\sqrt{2^m}}$$

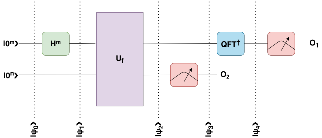
Shor's Algorithm (cont.)



Assuming $O_2 = g^v \bmod N$. $|\psi_3\rangle = \dfrac{\sum_{g^{q'} \equiv g^v \bmod N} |q', O_2\rangle}{\sqrt{2^m/r}}$

$= \dfrac{\sum_{j=0}^{(2^m/r)-1} |q_0 + j \cdot p, O_2\rangle}{\sqrt{2^m/r}}$ with $q_0 = \mathsf{min}_t \left[g^t \equiv a^v \bmod N \right]$

Shor's Algorithm (cont.)



The QFT† removes the offset q_0, and changes the period from $r \to 2^m/r$. Now measuring gives $O_1 = c \cdot 2^m/r$ for some c. Dividing by 2^m which we know, we get c/r which can be reduced to an irreducible fraction enabling us to extract r.

Full Shor's Algorithm

(1) Randomly pick $1 < g < N$. Compute $\text{GCD}(g, N)$. If $\text{GCD}(g, N) \neq 1$ then g is a factor return it.
(2) Find period of function $f_{a,N}(x)$ using previous circuit.
(3) If p is odd or $p^r \equiv -1 \bmod N$ then start over.
(4) Compute $\text{GCD}(g^{p/2} + 1, N)$ and $\text{GCD}(g^{p/2} - 1, N)$. At least one of these two GCDs is a factor if not both.

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Introspection

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That's All Folks!