Advanced Optimization
Homework 1

Autumn 1400
Due date: Aban 12th

Analytical Questions

1. Consider the function $f(x_1, x_2) = \left(x_1 + x_2^2\right)^2$. At the point $x^T = (1, 0)$ we consider the search direction $p^T = (-1, 1)$. Show that p is a descent direction and find all minimizers of the following problem at each iteration t.
$$\min_{\alpha > 0} f\left(x^t + \alpha p^t\right)$$

2. In the update rule of Newton method,
$$x_{k+1} = x_k - \alpha_k \left(\nabla^2 f(x_k)\right)^{-1} \nabla f(x_k)$$
one way to obtain the step-size is by solving the following sub-problem,
$$\alpha_k = \arg\min_{\alpha \geq 0} f\left(x_k - \alpha_k \left(\nabla^2 f(x_k)\right)^{-1} \nabla f(x_k)\right).$$
For the objective function $f(x) = \frac{1}{2} x^T Q x - x^T b$ where $Q = Q^T \succ 0$, show if with this approach the algorithm would be able to reaches the optimal point x^* where $\nabla f(x^*) = 0$, in just one iteration or not.

3. For minimizing $f(x) = \|x\|^\beta$ while $\beta > 1$ with pure Newton method, show that for what value of β the algorithm converges to the optimum point? What happens if $\beta \leq 1$?
Hint : $(I + BA)^{-1} B = B(I + AB)^{-1}$

Computer Questions

1. For the function,
$$f(x) = 100\left(x_2 - x_1^2\right)^2 + (1 - x_1)^2$$
   - Compute the gradient $\nabla f(x)$ and Hessian $\nabla^2 f(x)$.
   - Star from $x_0 = (-4, 10)^\top$, minimize these functions with steepest descent and Newton methods while using backtracking line search. Make comparison between the results. Is there a way to use the benefits of these two methods at the same time?
   - Report function value, distance to optimal point and step-size in 3 separate plots. Explain the results.
   - Try different strategies for initial step-size.

2. Minimize the below continuous function,
$$f(x) = x^3 - 60x^2 + 900x + 100$$
with Simulated Annealing while considering following terms:
   - Variable x is represented as a string of 5 bits.
   - The neighborhood defined as randomly flipping just one bit.
   - The initial point is 10011. $(x = 19, f(x) = 2399)$
   - Analyze two scenarios:
     - initial temperature T_0 equals to 500
     - initial temperature T_0 equals to 100
   - Cooling coefficient is equal to 0.9.

   Report the results thoroughly.