Date:

Lecture 1

Objects + Rules

Ex: Arithmetic $\underbrace{\text{Number}}_{\text{Objects}} \rightarrow \underbrace{(+, -, *, \div \text{ etc})}_{\text{Rules}}$

n-ary Relationships

10 people (watch, shoes, Mobile)

watch = {•, •, •, •, •, •} 6

Shoes = {•, •, •, •, •, •} 6

Mobiles = {•, •, •, •, •, •} 6

ordered triplets (•, •, •)

{(w, s, M) • • •} $6 \times 6 \times 6 = 216$

subset {(•, •, •), • • •, (•, •, •)} 3-ary relationship

Random

Binary Relationship, Relationship

chairs, students

Input —[?]— Output (determined input plus rule)

Input, Output → Relationship

Input → Domain (chairs)

Output → Codomain (students)

Restriction: 1) Everything in the domain has some pair in the codomain

2) No element in domain gets more than one element in co-domain.

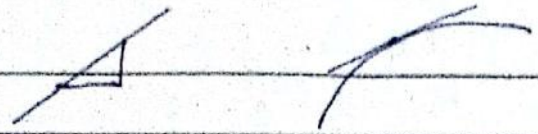
Objects → functions

function → (+, -, composition)

Chairs, → students → Rooms

$3+2=5-2=3$

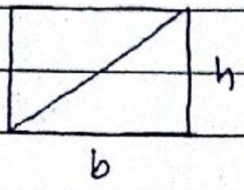
Derivative

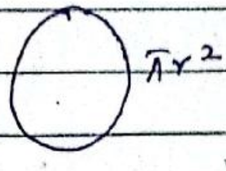


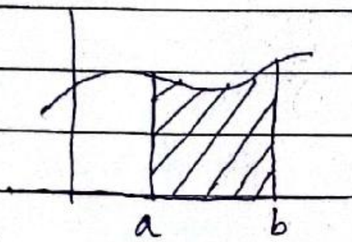
instanteuous slope

Rectangle area = $b \times h$

Triangle area = $\frac{1}{2} bh$







infinite rectangles area sum

$3 \times 4 = 12$ derivative like Multiplication

~~factors~~ Integration like factors.

Date:

Lecture 2

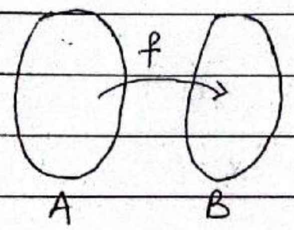
Relationships ⟶ 2 Sets ⟶ Functions

(Rule to relate elements of several Sets) (binary Relationships Relationships)

two sets

Functions: $f : A \to B$ ⟶ Codomain

function name domain



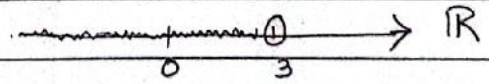
$a \to b$ (a, b)

f assigns each element of the domain to exactly one element of the Codomain.

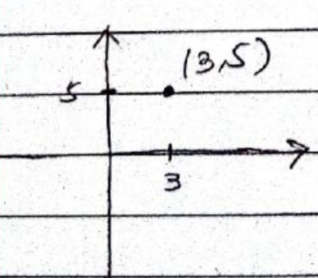
Calculus: $f : \mathbb{R} \to \mathbb{R}$

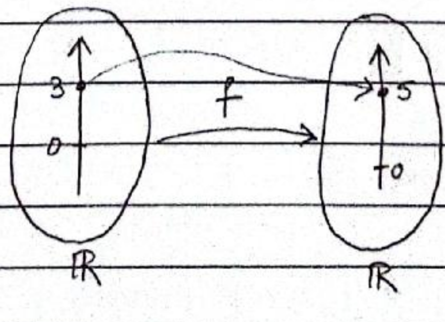
Notation: $\mathbb{R}$ (set of all real numbers)

$x < 3$



$f : \mathbb{R} \to \mathbb{R}$





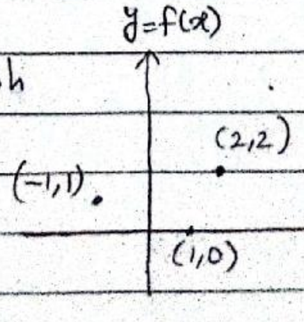
$(3, 5)$

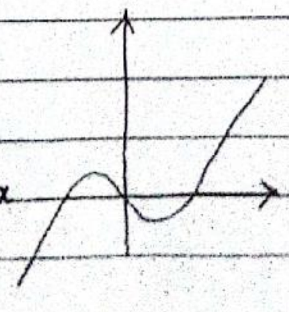
Representation $f : \mathbb{R} \to \mathbb{R}$

sets of Ordered pairs

① Table ② graph

x	$y = f(x)$
.	.
.	.





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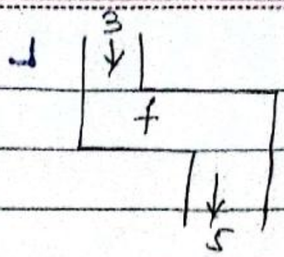
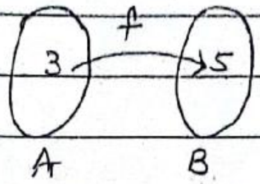
③ Machine diagram



④ Arrow diagram

$f: A \to B$



Rules (why?)

① Verbal description

② Algebric $3 \to 5$ b/c number, double minus 1

$y = 2x - 1$

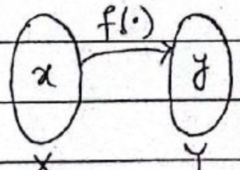
Algebric Notation: ex: $f(x) = 3x^2 - 2x + 7$

$f(\boxed{}) = 3\boxed{}^2 - 2\boxed{} + 7$

$f(a+h) = 3(a+h)^2 - 2(a+h) + 7$

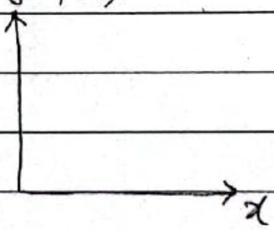
$= 3a^2 + 6ah + 3h^2 - 2a - 2h + 7$

x $\quad$ $y = f(x)$



Domain $\quad$ Codomain

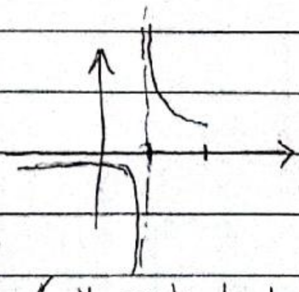
$y = f(x)$



$f: \mathbb{R} \to \mathbb{R}$ $\quad$ Natural domain

ex $f(x) = \dfrac{1}{x-1}$

$x \neq 1$

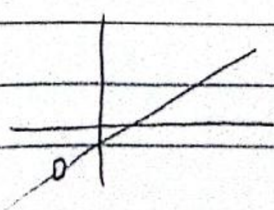


domain = natural domain (all reals but $x \neq 1$)

ex $f(x) = \sqrt{x}$ $\quad$ domain : $x \geq 0$

ex $f(x) = \dfrac{x^2 - 4}{x+2} = \dfrac{\cancel{(x+2)}(x-2)}{\cancel{(x+2)}} = x - 2$

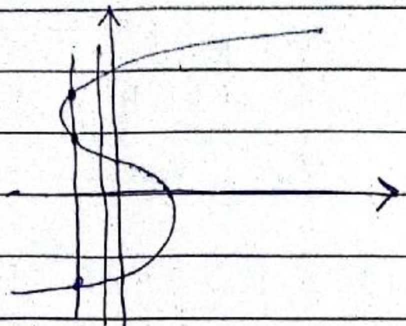
$x \neq -2$



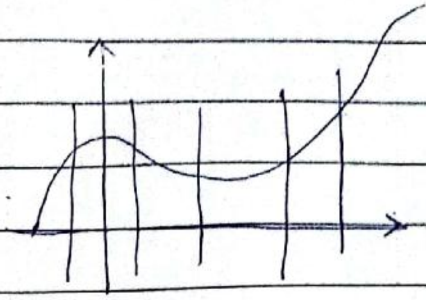
Date:

$f(x) = x-2, \quad x \neq -2$

Is a graph a function? Vertical line test

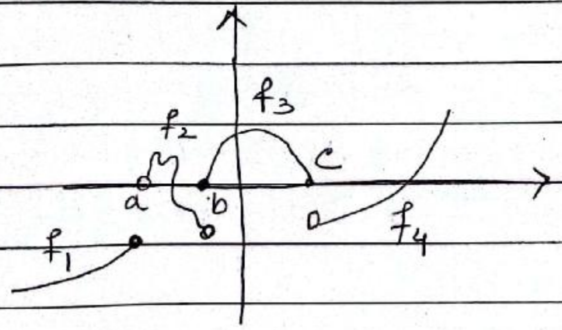


VS

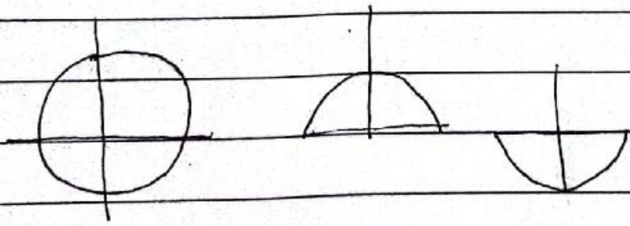


Types of functions

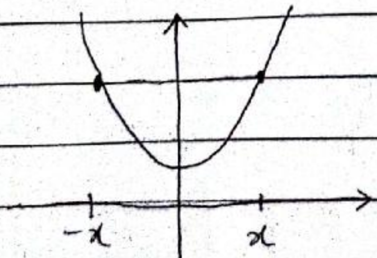
① piecewise defined



$$f(x) = \begin{cases} f_1(x) & x \leq a \\ f_2(x) & a < x < b \\ f_3(x) & b \leq x \leq c \\ f_4(x) & x > c \end{cases}$$



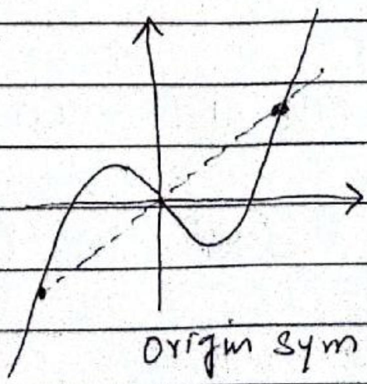
② by symmetry $\begin{pmatrix} y\text{-axis sym} \\ \text{origin sym} \end{pmatrix}$



y-axis sym

$f(-x) = f(x)$

even function



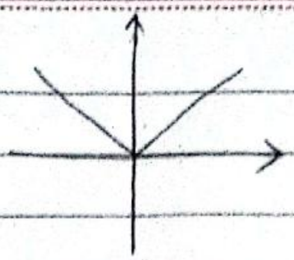
origin sym

$f(-x) = -f(x)$

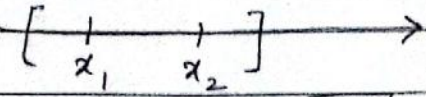
odd function.

Date:

$$|x| = \begin{cases} -x & x<0 \\ x & x \geq 0 \end{cases}$$



③ Inc/dec : If x_1, x_2 on an interval and $x_1 < x_2$



a) If $f(x_1) < f(x_2)$ call f $\boxed{\text{inc}}$

b) If $f(x_1) > f(x_2)$ call f $\boxed{\text{dec}}$

① Polynomial $P(x) = a_n x^n + \cdots + a_2x^2 + a_1 x + a_0$

degree $= 1 \rightarrow P(x) = ax+b$

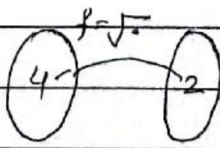
degree $= 2 \rightarrow P(x) = ax^2+bx+c$

degree $= 3 \rightarrow P(x) = ax^3+bx^2+cx+d$

degree $= 4 \rightarrow P(x) = ax^4+bx^3+cx^2+dx+e$

② Power $f(x) = x^P$ (P is some real

(i) $P = 0, 1, 2, 3, \cdots$ (Polynomial



(ii) $P = \frac{1}{n}$ $n = 2, 3, 4, \cdots$

$P = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

$f(x) = \sqrt{x} = x^{1/2}$ $\qquad \sqrt{4} = 2$ principal root

$= \sqrt[3]{x} = x^{1/3}$ $\qquad \sqrt[3]{-8} = -2$

$= \sqrt[P]{x} = x^{1/P}$

(iii) $P = -1$ $f(x) = \frac{1}{x}$ domain all $\mathbb{R}$ except $x=0$

③ Rationals $f(x) = \dfrac{\text{Polynomial}}{\text{Polynomial}}$ domain all $\mathbb{R}$ except den Poly $= 0$

④ Algebric $(+, -, \times, \div, \text{roots})$ $f(x) = \dfrac{\sqrt{x+4}}{x^3-3x} + 7x^7 + \pi$

⑤ Trignometric

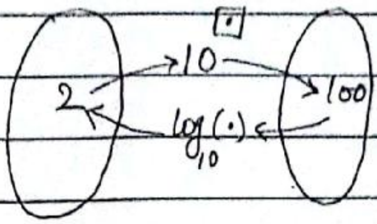
Date:

(6) Exponential $f(x) = a^x$

(7) logarithmic. $f(x) = \log_a x$

$y = \log_a x$

$a^y = x$



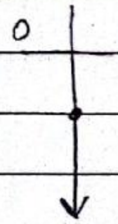
$\sqrt{x} = y$

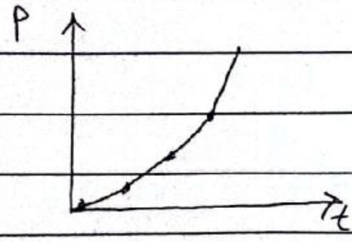
$y \cdot y = x$

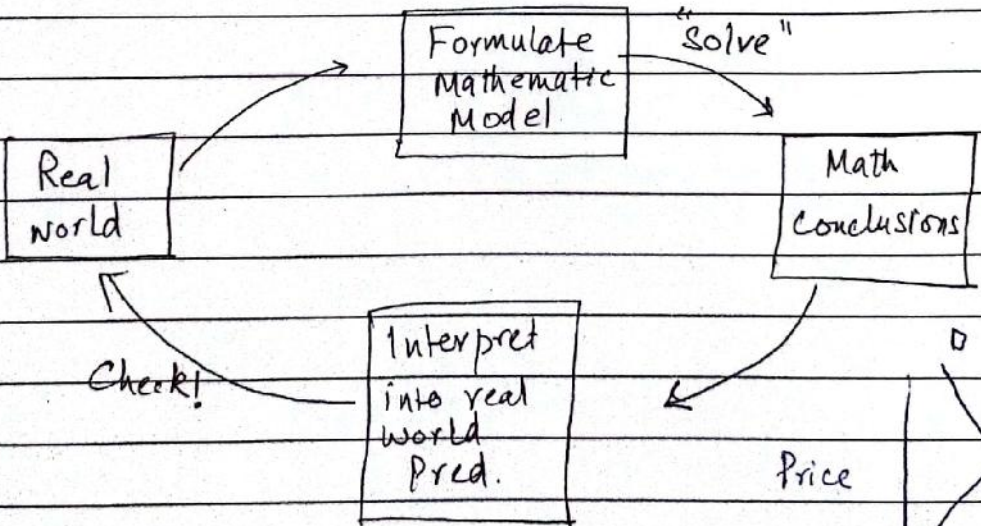
$y^2 = x$

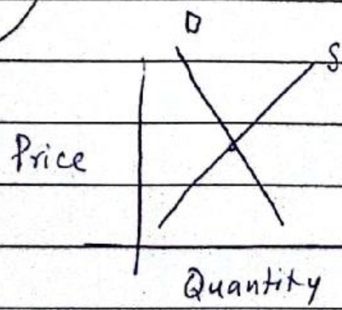
Point?

Math + Models









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LECTURE 3

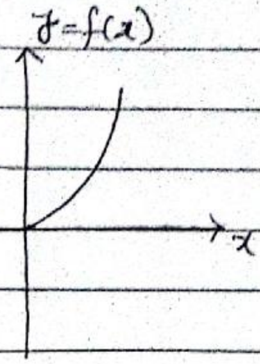
Functions → graph + Domain (Natural Domain) + Codomain (Range) + Algebra

$f: \mathbb{R} \to \mathbb{R}$

Domain → Natural Domain

Codomain → Range

$f(x) = x^2$



Domain: $\mathbb{R}$

Natural Domain: 0 to ∞

Codomain: $\mathbb{R}$

Range: 0 to ∞

Functions to know

1. $\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

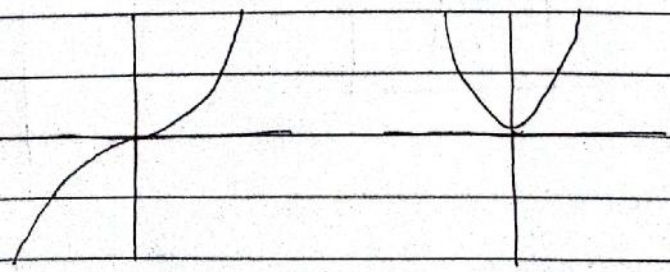
$\sqrt{x^2} = |x| = x$, "told" $x \geq 0$

$\sqrt{x^2} = |x| = -x$, "told" $x < 0$

2. $P(x)$ is a polynomial

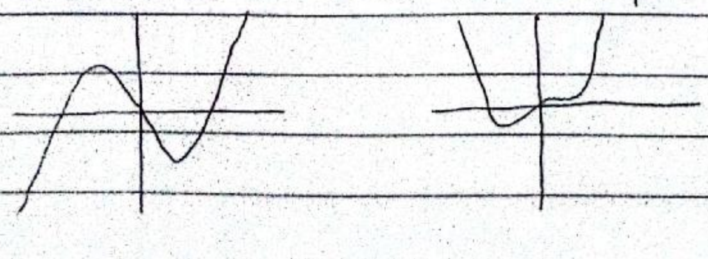
$P(x) = x^n, \quad n = 0, 1, 2, \dots$

Odd   Even



$P(x) = a_n x^n + \dots + a_1 x + a_0$

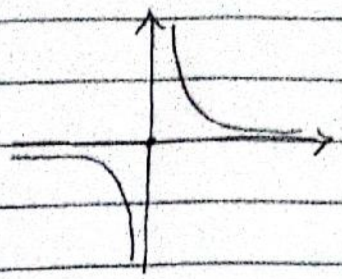
Other terms added will have bumps in between two end portions



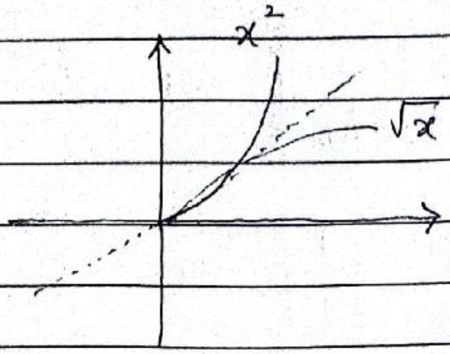
Date:

$P(x) = x^n, \quad n = -1$

$P(x) = \frac{1}{x}$



$P(x) = x^n, \quad n = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

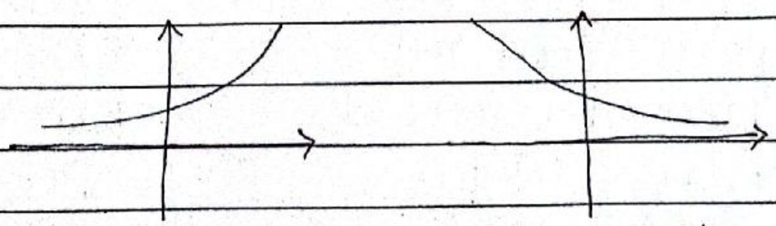


3 $\dfrac{P(x)}{q(x)} \leftarrow q(x) \overset{?}{=} 0$ $\qquad \dfrac{1}{(x+2)(x-2)}$

$$\frac{(x+2)}{(x+2)(x-2)} = \frac{1}{x-2}, \quad x \neq -2, 2$$

4 $\sin x, \cos x, \sec x, \operatorname{cosec} x, \tan x, \cot x$

5 a^x



$a > 1$ $\qquad 0 < a < 1$

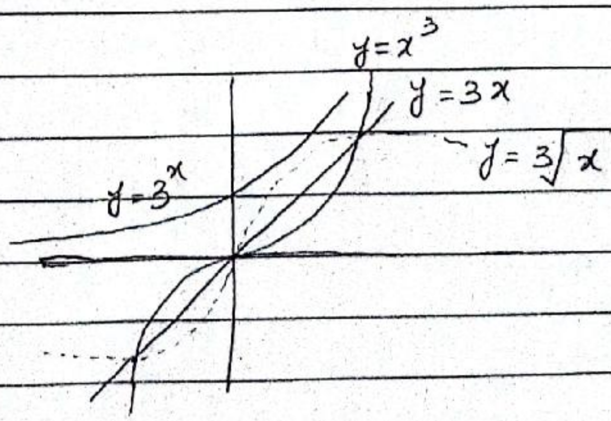
Ex

$y = 3x$

$y = 3^x$

$y = x^3$

$y = \sqrt[3]{x}$



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Operations on functions

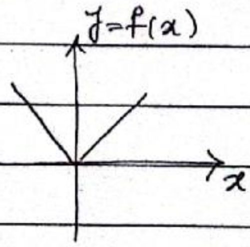
part 1: "same" function... except we translate and/or stretch it.

$f(x)$ vs $a \cdot f(b(x+c)) + d$

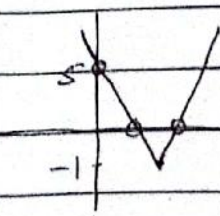
a, b, c, d are constants.

① $a f(x)$ ② $f(bx)$	stretch.
③ $f(x) + d$ ④ $f(x+c)$	translation

ex $|x|$



$3|x-2|-1$



part 2 Operations (+, −, *, ÷, composition)

$(f+g)(x) = f(x) + g(x)$

$(f-g)(x) = f(x) - g(x)$

$(f \cdot g)(x) = f(x) g(x)$

$(f/g)(x) = f(x)/g(x)$

$(f \circ g)(x) = f(g(x))$

$f(x) = x$

$g(x) = 4x$

$(f+g)(x) = 5x$

Identity and Inverse for an operation

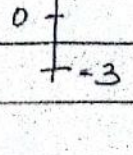
ex arithmetic: object = number, operation = addition

① Identity: does nothing to objects under operation

→ Additive identity: $3 + \boxed{0} = 3$

② Inverse: make an object into the identity

→ additive Inverse of 3 $3 + (-3) = 0$



$x = 2$

$$3x + 7 = 4$$
$$3x + 7 + (-7) = 4 + (-7)$$
$$3x + 0 = -3$$
$$\left(\frac{1}{3}\right) 3x = \frac{1}{3}(-3)$$
$$x = -1$$

sin(x) vs inverse sin(x).?

<u>under composition</u>

Identity function $I(x) = x$

Inverse function $(f \circ f^{-1})(x) = x$

$(f^{-1} \circ f)(x) = x$

<u>Modeling Example:</u> Cost (Miles Driven)

Month	Cost(c)	Dis (d)
1	\$380	480 mi
2	\$460	800 mi

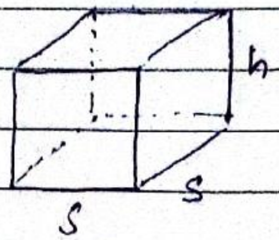
Linear Model: $c(d) = s \cdot d + y_0$

$$s = \text{slope} = \frac{460-380}{800-400} = \frac{8}{32} = \frac{1}{4} \frac{\$}{mi}$$

$$y - y_1 = s(x - x_1) \Rightarrow y - 380 = \frac{1}{4}(x - 480)$$
$$y - 380 = \frac{x}{4} - 120$$
$$y = \frac{1}{4}x + 260$$

Date:

Ex: An open rectangular box with volume of $2\,m^3$ has a square base. Express the surface area of the box as a function of the length of a side of the base



$V = hs^2 = 2\,m^3$

$SA = 4sh + s^2$

$h = \frac{V}{s^2} = \frac{2}{s^2}$

$SA(s) = 4s\frac{2}{s^2} + s^2$

$SA(s) = \frac{8}{s} + s^2$

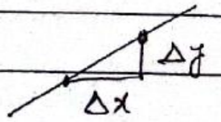
check

$s = 1 \Rightarrow h = 2$

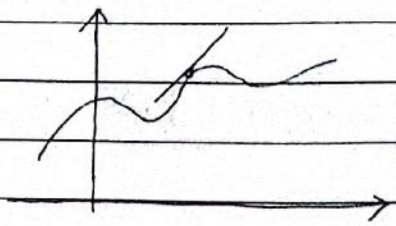
$SA = 8 + 1 = 9$

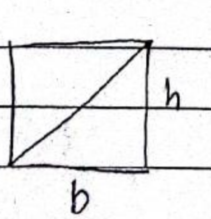
$SA(1) = \frac{8}{1} + 1 = 9$

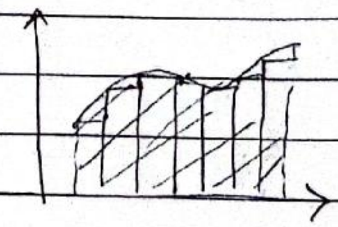
Calculus: two problems
① change? (slope)
② sum? (area)



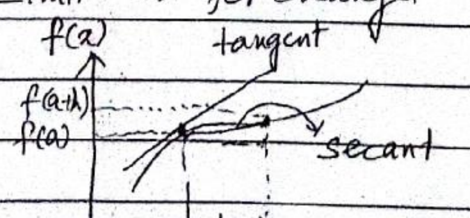
$m = \frac{\Delta y}{\Delta x}$







Limit → for change.



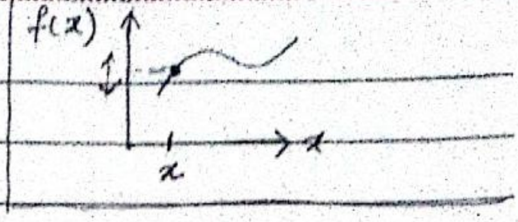
slope of secant $= \frac{f(a+h) - f(a)}{h}$

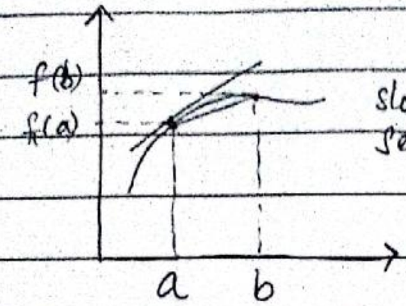
Let h get close to 0.

slope of secant → slope of tangent

Lecture 4

Tangents / Velocities (Change)



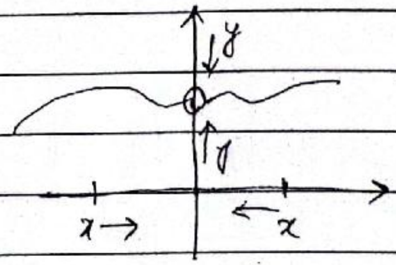


slope of secant $= \dfrac{f(b)-f(a)}{b-a}$ $\xrightarrow[\text{close to}]{\text{get}}$ slope of tangent @ b gets close to a

as b gets close to a
the secant gets close to a tangent

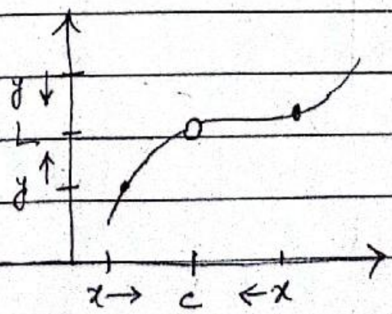
$\boxed{\text{Rough idea}}$ graph (ex) $\dfrac{1-\cos x}{x}$ as x gets close to 0?

x	$y = f(x) = \dfrac{1-\cos x}{x}$
-1	
-0.1	
-0.001	
-0.0001	
	? going to ?
0.0001	
0.001	
0.01	
0.1	
1	



Gets close ? ⟶ Limit

① Intuitive



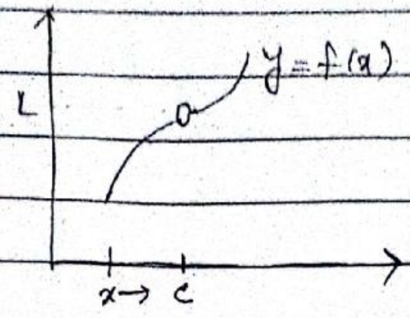
Left side $x \to c^-$

both $x \to c$

Right side $x \to c^+$

Date:

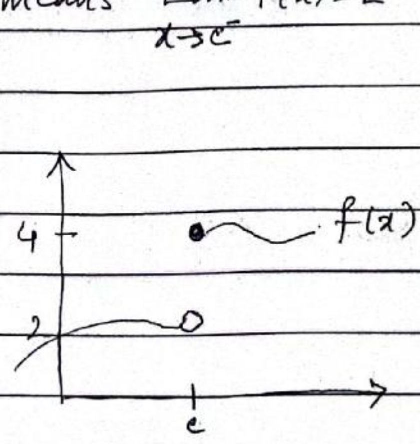
Def ① $\lim_{x \to c^-} f(x) = L$ mean $f(x)$ approaches the number L whenever x approach c from the left



② $\lim_{x \to c^+} f(x) = L$

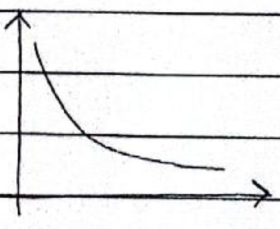
③ $\lim_{x \to c} f(x) = L$ means $\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$

Graphical idea:



$\lim_{x \to c^-} f(x) = 2$, $\lim_{x \to c^+} f(x) = 4$ $\lim_{x \to c} f(x)$ does not exist

Ex $\lim_{x \to 0^+} \frac{1}{x} = \infty$



x	y
1	1
0.1	10
0.01	100
0.001	1000

a) $x \to 0^+$

y is not approaching a number

y is getting Large without bound

call $y \to$ infinite $= \infty$

Note: q is $1, 10, 100, 1000, 10000, \dots$

$q \to \infty$ Large in the pos. direction

q is $-1, -10, -100, -1000, \dots$

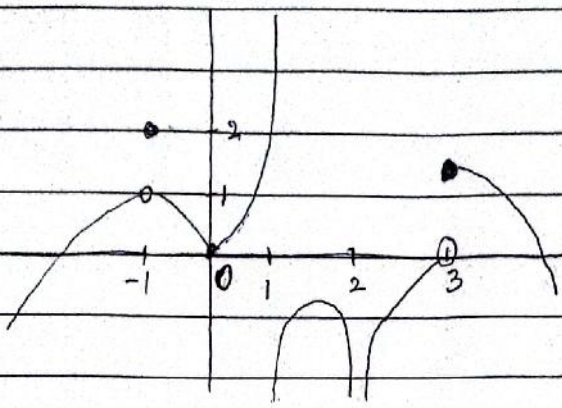
$q \to -\infty$ Large in the negative direction.

$\boxed{\text{Def}}$ $\lim_{x \to a^-} f(x) = \infty$ → means $f(x)$ is arbitarily large (positive direction) as 'x' goes to 'a' from left.

$\lim_{x \to a^+} f(x) = \infty$

$\lim_{x \to a} f(x) = \infty$

ex



$\lim_{x \to -1} f(x) = 1$, $f(-1) = 2$

$\lim_{x \to 0} f(x) = 0$, $f(0) = 0$

$\lim_{x \to 1^-} f(x) = +\infty$

$\lim_{x \to 1^+} f(x) = -\infty$

$\lim_{x \to 2} f(x) = -\infty$

ex $\lim_{x \to 2^+} \frac{x^2 - 2x}{x^2 - 4x + 4} = \lim_{x \to 2^+} \frac{x\cancel{(x-2)}}{(x-2)^{\cancel{2}}} = \lim_{x \to 2^+} \frac{x}{x-2}$

x	y
3	3
2.1	$2.1/0.1 = 21$
2.01	$2.1/0.01 = 201$
2.001	$2.001/0.001 = 2001$

$\lim_{x \to 2^+} \frac{x}{x-2} = +\infty$

Better way?

ex $2x + 1 = 0$ $x = -\frac{1}{2}$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x - 3 = 0$, $x - 2 = 0$

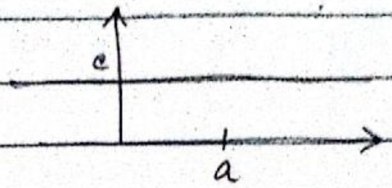
idea

1. two "easy" Limits
2. Laws that allow us to know "harder" problems are just variations of the two easy ones.

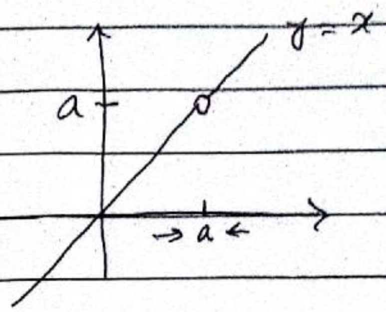
Date:

Two limits to Solve

① $\lim_{x \to a} c = c.$



② $\lim_{x \to a} x = a$



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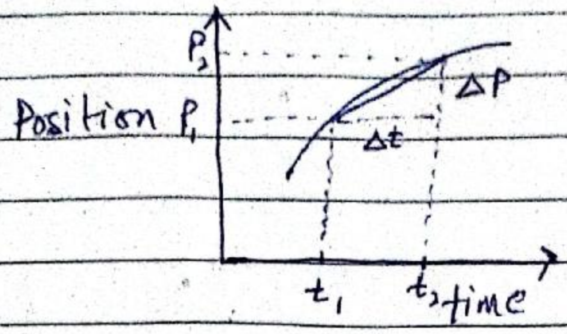
Lecture 5



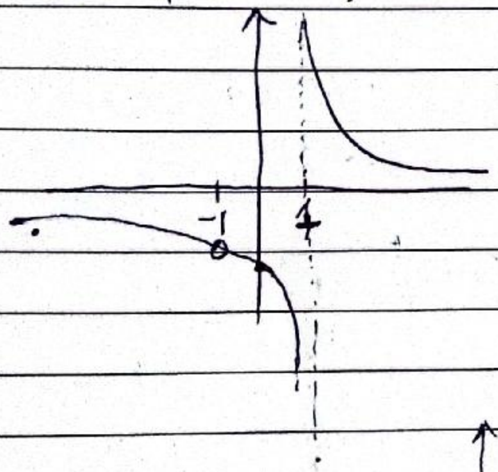
Average velocity

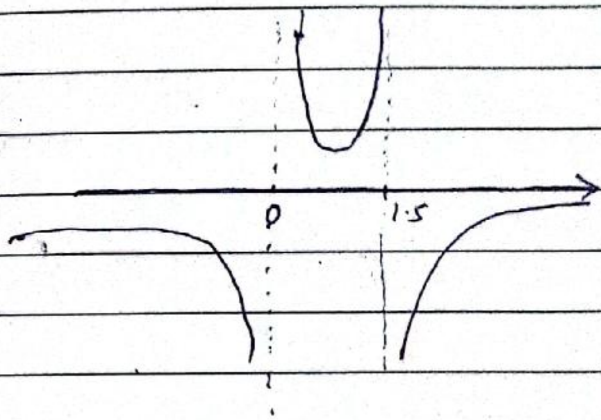
$$\Delta V = \frac{P_2 - P_1}{t_2 - t_1} = \frac{\Delta P}{\Delta t}$$

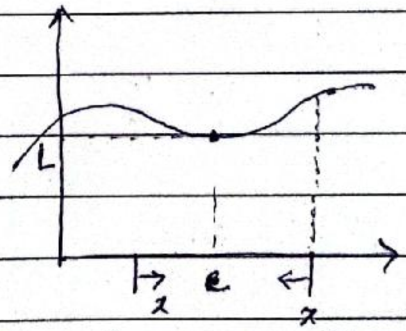
Hole (vs) Vertical Asymptote

$$\frac{x+1}{x^2-1} = \frac{x+1}{(x+1)(x-1)} = \frac{1}{x-1}, \; x \neq -1 \qquad \frac{x^2+1}{x(3-2x)}$$

Root of num (+) Root of denom







$\lim\limits_{x \to -c} f(x)$ (vs) $\lim\limits_{x \to c^+} f(x)$ (vs) $\lim\limits_{x \to c} f(x)$

both

$x = \frac{2}{3}$, $3x - 2 = 0$

$\boxed{\frac{2}{3}}$ $\quad 3x = 2$

$x = \frac{2}{3}$

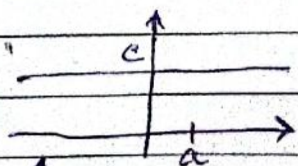
$\rightarrow x^2 - 4 = 0 \qquad a \cdot b = 0$

$(x-2)(x+2) = 0 \quad a = 0, b = 0$

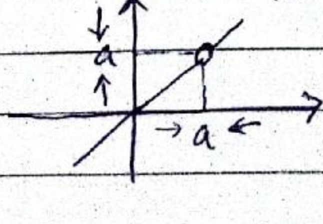
$x - 2 = 0, \; x + 2 = 0$

$x = 2 \quad x = -2$

$\lim\limits_{x \to a} c = c$



$\lim\limits_{x \to a} x = a$



Date:

(1) $\lim_{x \to a} f(x) \pm g(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

(ex) $\lim_{x \to 3} (x) + (\pi) = 3 + \pi$

(2) $\lim_{x \to a} f(x) \cdot g(x) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$

(ex) $\lim_{x \to -10} 3x = 3(-10) = -30$

(3) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ $\left(\lim_{x \to a} g(x) \neq 0\right)$

(ex) $\lim_{x \to 3} \frac{2x+4}{x-1} = \frac{2(3)+4}{3-1} = 5$

(4) $\lim_{x \to a} (f(x))^P = \left(\lim_{x \to a} f(x)\right)^P$, $P = 1, 2, 3, \cdots$
$P = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

$\lim_{x \to 2} \frac{(x)^{1/2} + 3x^2 - x}{3 - \frac{4}{x}} = \frac{(2)^{1/2} + 3(2)^2 - (2)}{3 - \frac{4}{2}}$

$= 10 + \sqrt{2}$

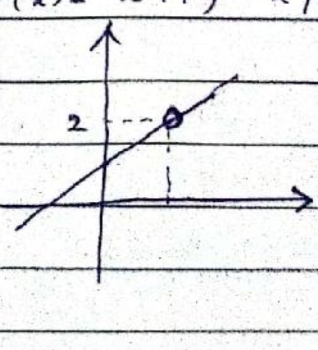
Not evaluation

(ex) $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

$f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$

$= \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$

$f(x) = x + 1, \quad x \neq 1$



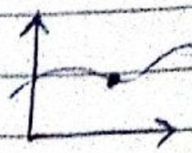
$= \lim_{x \to 1} x + 1, \quad x \neq 1$

$= 2$

Date:

<u>Substitution Rule</u>, $P(x)$ is a polynomial

$$\lim_{x \to a} P(x) = P(a)$$



(ex) $\displaystyle \lim_{x \to -1} 3x^2 - x + 2 = 3(-1)^2 - (-1) + 2 = 6$

(ex) $f(x) = 2x^2 - x + 1$

$f(\square) = 2\square^2 - \square + 1$

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{[2(3+h)^2 - (3+h) + 1] - [16]}{h}$$

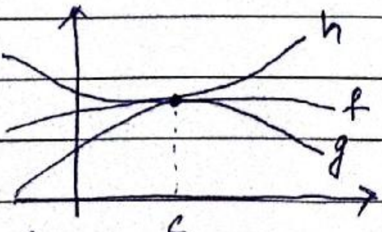
$$= \lim_{h \to 0} \frac{[2(9 + 6h + h^2) - 3 - h + 1] - 16}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{18} + 12h + 2h^2 - \cancel{2} - h - \cancel{16}}{h} = \lim_{h \to 0} \frac{\cancel{h}(11 + 2h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 11 + 2h, \quad h \neq 0$$

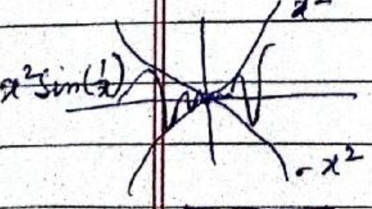
$$= 11$$

<u>Squeeze Theorem</u>



$g(x) \leq f(x) \leq h(x)$

$\displaystyle \lim_{x \to a} f(x) = ?$

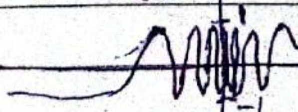


Know: $\displaystyle \lim_{x \to a} h(x) = L, \quad \lim_{x \to c} g(x) = L$

$\displaystyle \to \lim_{x \to c} f(x) = L$

$\boxed{\text{Typical}}$ $-1 \leq \sin(x) \leq 1$ (fact)

$\sin(\frac{1}{x})$

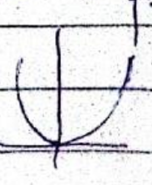


$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2, \quad x \neq 0$$

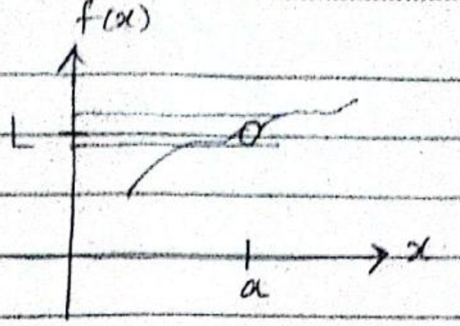
x^2



$\left.\begin{array}{l} \displaystyle \lim_{x \to 0} (-x^2) = -0^2 = 0 \\ \displaystyle \lim_{x \to 0} x^2 = 0^2 = 0 \end{array}\right\} \to \displaystyle \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

Date:

Lecture 6

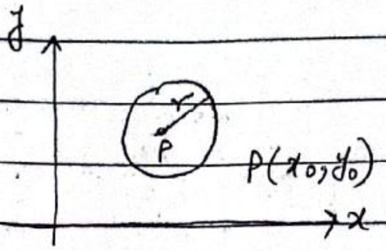


$\lim_{x \to a} f(x) = L$

means as x gets close to a
Then $f(x)$ gets close to L

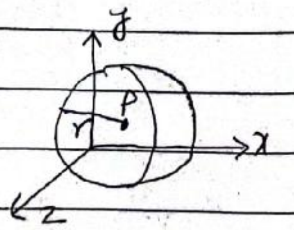
"gets close" into math (use a ball)

2D



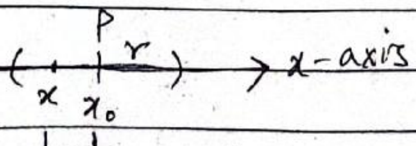
close to P is inside a circle of radius r,
$\left((x-x_0)^2+(y-y_0)^2\right)^{1/2} < r$

3D



$P(x_0, y_0, z_0)$
close to P is inside the ball
$\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{1/2} < r$

1D

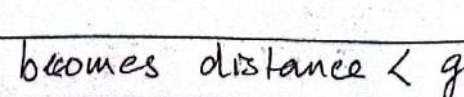


$|x-x_0|$ = distance

close to P is inside the "a ball"
(1D ball is an interval
$\left((x-x_0)^2\right)^{1/2} < r$
$|x-x_0| < r$

" gets close" becomes distance < given value

ex



$|x-x_0| < r$

as an interval 'x' is in ...

$|x-x_0| < r \rightarrow -r < x-x_0 < r$

$\rightarrow x_0 - r < x < x_0 + r$

so x is in $(x_0 - r, x_0 + r)$ Interval notation back to $\lim_{x \to a} f(x) = L$

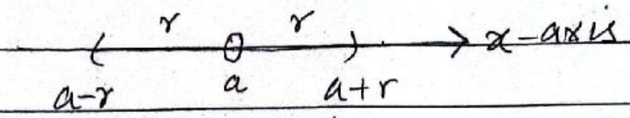
$\leftarrow$ x gets close to a.

$\boxed{1^{st}}$ gets close with $|x-a| < r$ type notation

$\boxed{2^{nd}}$ $x \neq a$ for a limit

x gets close to a, but $x \neq a$ means $0 < |x-a| < r$

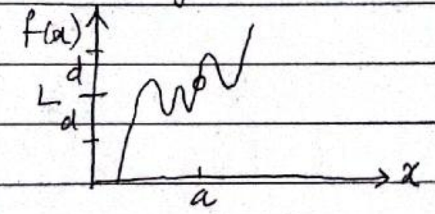
<u>Interval</u>



$(a-r, a)$ $\cup$ $(a, a+r)$

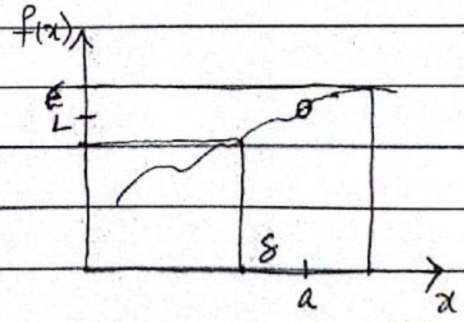
left side Right side

$\boxed{3^{rd}}$ $f(x)$ gets close to L



$|f(x) - L| < d$

<u>all together</u>



$\lim_{x \to a} f(x) = L$ as x gets close to a, Then f(x) gets close to L

"given to limit you" "for you to find given ε"

$\boxed{\text{Def}}$ $\lim_{x \to a} f(x) = L$ for any $\epsilon > 0$ there is a $\delta > 0$ such That if $\underline{0 < |x-a| < \delta}$, Then $|f(x) - L| < \epsilon$

x's with in δ distance of 'a' — f(x) is with in ε distance of L

Notation: $\boxed{\text{Def}}$ $\lim_{x \to a} f(x) = L$ is

$$\forall \epsilon > 0, \exists \delta > 0 \ni: (0 < |x-a| < \delta) \rightarrow (|f(x) - L| < \epsilon)$$

for all — there exists — such that — if , Then

Date:

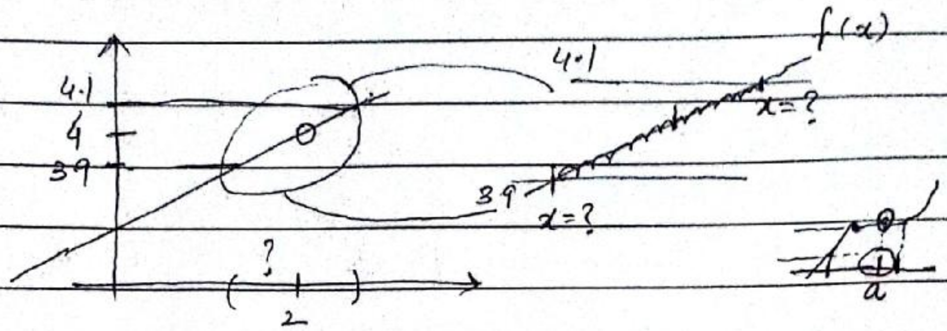
Graphical approach.
given $\epsilon = 0.1$, find $\delta = ?$

by intuition: $\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$, $x \neq 2$

$= 4$

$\underline{\epsilon, \delta}$

$\epsilon = 0.1$



quadratic general form $ax^2+bx+c=0$ solution.

$\lim_{x \to 2} x+2 = 4$ given $\epsilon > 0$ pick a $\delta > 0$
so that $0 < |x-2| < \delta \rightarrow |(x+2)-4| < \epsilon$

$\boxed{\text{Note}}$ Direct Proof is given the left side $(0 < |x-2| < \delta)$
show the right side $(|(x+2)-4| < \epsilon)$ must follow.

Goal: start at left: $0 < |x-2| < \delta$

{do algebra correctly} $\leftarrow$ valid: replace δ (any value > 0)
correct: add, sub, mult, divide
use inequalities

until right side is true $|(x+2)-4| < \epsilon$

① Scratch work 1st to study the right side
(ex) $|(x+2)-4| < \epsilon$
same as $|x-2| < \epsilon$ (compare to left $0 < |x-2| < \delta$)
obvious let $\delta = \epsilon$

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(2) Proof: $0 < |x-2| < \delta$ let $\delta = \epsilon$

so $0 < |x-2| < \epsilon$, so $0 < |\underbrace{x+2-4}_{=(-2)}| < \epsilon$

$\therefore$ $|(x+2)-4| < \epsilon$ is true

$|f(x) - L| < \epsilon$

(Ex) $\lim\limits_{x \to 1} \dfrac{2+4x}{3} = \dfrac{2+4(1)}{3} = 2$

$\boxed{\text{Prove}}$ means find δ so That

$0 < |x-1| < \delta \rightarrow \left|\left(\dfrac{2+4x}{3}\right) - 2\right| < \epsilon$

(1) Scratch work: mess with right side

$$\left|\left(\frac{2+4x}{3}\right) - 2\right| < \epsilon$$

is... $|(2+4x) - 6| < 3\epsilon$, is... $|4x-4| < 3\epsilon$

is.. $|x-1| < \frac{3}{4}\epsilon$

with left side $0 < |x-1| < \delta$ pick $\delta = \frac{3}{4}\epsilon$!

(2) $\boxed{\text{Proof}}$ $0 < |x-1| < \delta$ let $\delta = \frac{3}{4}\epsilon$

so $0 < |x-1| < \frac{3}{4}\epsilon$

Then $4|x-1| < 3\epsilon$

is.. $|4x-4| < 3\epsilon$

is.. $|(2+4x) - 6| < 3\epsilon$

is.. $\left|\dfrac{2+4x}{3} - 2\right| < \epsilon$

$\therefore$ $|f(x) - L| < \epsilon$ $\boxed{\text{done}}$

(Ex) $\lim\limits_{x \to 2} (x^2-4x+5) = 1$, Find δ such that $0 < |x-2| < \delta$ Then $|(x^2-4x+5)-1| < \epsilon$

$\boxed{\text{Prove}}$ $0 < |x-2| < \delta \rightarrow |(x^2-4x+5)-1| < \epsilon$

(1) Scratch work: right side $|(x^2-4x+5)-1| < \epsilon$

is.. $(x^2-4x+4|$ $< \epsilon$, is.. $|(x-2)(x-2)| < \epsilon$

is $|x-2||x-2| < \epsilon$ compare to left side $0 < |x-2| < \delta$

Date:

Let $\delta = \sqrt{\epsilon}$!

(2) $\boxed{\text{Proof}}$

$0 < |x-2| < \delta$ Let $\delta = \sqrt{\epsilon}$

so.. $|x-2| < \sqrt{\epsilon}$ Then $|x-2|\,|x-2| < \sqrt{\epsilon}\sqrt{\epsilon}$

so.. $|(x-2)(x-2)| < \epsilon$

so.. $|x^2-4x+4| < \epsilon$

so.. $|x^2-4x+5-1| < \epsilon$

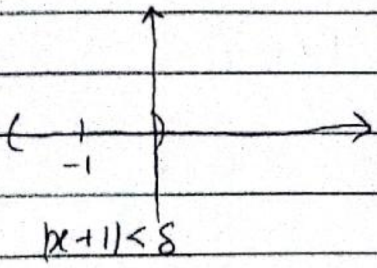
so.. $|(x^2-4x+5)-1| < \epsilon$

so. $|f(x) - 1| < \epsilon$ is done

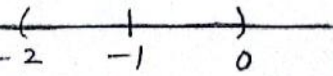
<u>Note</u>: If you have something like

$0 < |x+1| < \delta \rightarrow |x+1|\,|x-3| < \epsilon$ Right-side scratch work

$|x+1|\,|x-3| < |x+1| \cdot 5$



close to -1 δ is largest 1

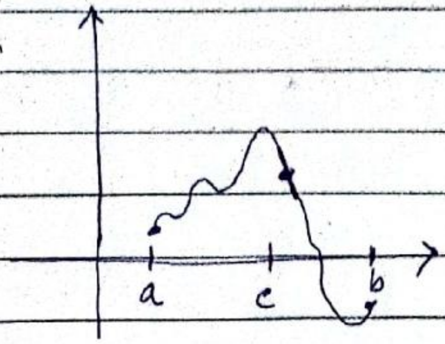


so $|x-3|$ @ $x=-2$ is 5

@ $x=0$ is 3

Lecture 7

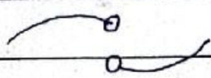
Continuity



$\boxed{\text{Def}}$ f is continuous @ $x=c$ if

$$\boxed{\lim_{x\to c} f(x) = f(c)}$$

① $\lim\limits_{x\to c} f(x)$ exist



② $f(c)$ exist

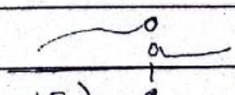
③ equal $\lim\limits_{x\to c} f(x) = f(c)$

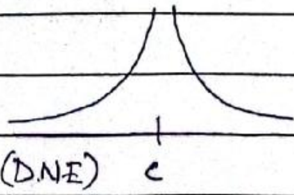
$\boxed{\text{If not}}$ $f(x)$ is discontinuous @ $x=c$.

Discontinuous

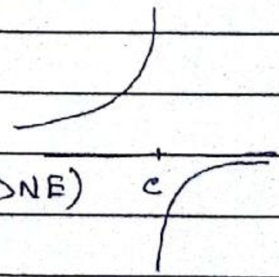
① $\lim\limits_{x\to c} f(x)$ does not exist

jump



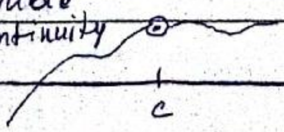


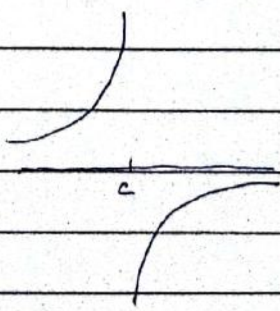
$\lim\limits_{x\to c} f(x) = +\infty$

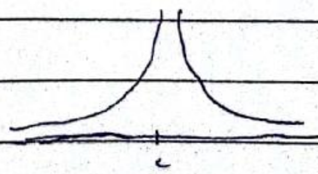


② $f(c)$ (D.N.E)

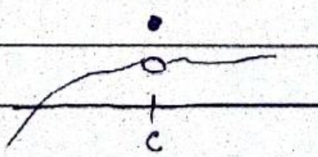
Removable discontinuity



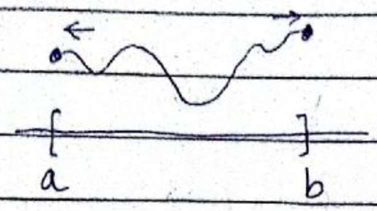




③ $\lim\limits_{x\to c} f(x) \neq f(c)$



Before is continuity @ a single point.
<u>But</u> functions have domains (<u>intervals</u>)



① $f(x)$ is continuous from the right when $\lim_{x \to a^+} f(x) = f(a)$

② $f(x)$ is cont from the left when $\lim_{x \to b^-} f(x) = f(b)$

③ $f(x)$ is cont on Interval (a,b) if $\lim_{x \to c} f(x) = f(c)$ for all c in (a,b)

all 3 above say f is cont. on $[a,b]$.

<u>Properties of Cont. functions</u>

$\boxed{\text{Th}^m}$ If $f(x)$ and $g(x)$ are cont @ $x = a$ then

① $(f \pm g)(x)$ is cont. @ $x = a$
② $(fg)(x)$ is cont. @ $x = a$
③ $(f/g)(x)$ is cont. @ $x = a$ (if $g(a) \neq 0$)

$\boxed{\text{So}}$

① $p(x)$ a polynomial is continuous on all real numbers. $\mathbb{R} = (-\infty, \infty)$

② $r(x) = \dfrac{p(x)}{q(x)}$ a rational is continuous on all reals except where $q(x) = 0$
$\mathbb{R} - \{x \mid q(x) = 0\} = \{x \mid q(x) \neq 0\}$

So.. poly, rationals are cont. on their domains.

Date:

$\boxed{Th^m}$ $f(x)$ is cont. on its domain if

① $f(x)$ is a polynomial
② $f(x)$ is a rational
③ $f(x)$ is a root function
④ $f(x)$ is a trig function.

Use $\boxed{1^{st}}$ Cont. is $\lim_{x\to c} f(x) = f(c)$

ex $\lim_{x\to 1}\left(\frac{x^2-1}{x+1} + \frac{1}{x} - \sin(x) + \tan(x)\right)$

$$= \frac{1^2-1}{1+1} + \frac{1}{1} - \sin(1) + \tan(1) = \tan(1) - \sin(1) + 1$$

$\boxed{\text{Composition}}$ $(f\circ g)(x) = f(g(x))$

$\boxed{Th^m}$ $\lim_{x\to a} g(x) = b$ and f is cont. @ b

Then

$$\lim_{x\to a}(f\circ g)(x) = \lim_{x\to a} f(g(x)) = f(b) = f\left(\lim_{x\to a} g(x)\right)$$

$\boxed{Th^m}$ If g is cont @ a and f is cont @ $g(a)$ then

$(f\circ g)(x) = f(g(x))$ is cont. @ a.

"a continuous function of a continuous function is continuous" (on its domain)

Intervals of continuity are really...

" what is the natural domain?

ex

$$f(x) = \overset{\to \text{cont. everywhere}}{3} - \frac{\tan(x)}{\sqrt{x}}$$

$\tan(x) \to$ cont $x \neq \frac{\pi}{2} \pm n\pi$

$\sqrt{x} \to$ cont on $x > 0$

cont on $x > 0$ and $x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

ex $f(x) = \sin(\cos(\sin(x)))$ composition of 3 cont. functions → $\boxed{\text{cont}}$ on $\mathbb{R}$

Date:

(ex) $f(x) = \sqrt{1+\frac{1}{x}}$ is cont. on $1+\frac{1}{x} \geq 0$

$1+\frac{1}{x} \geq 0 \rightarrow \frac{x+1}{x} \geq 0$

$\frac{x+1}{x} \rightarrow$ + $\;-1\;$ − $\;0\;$ + $\qquad (-\infty, -1] \cup (0, +\infty)$

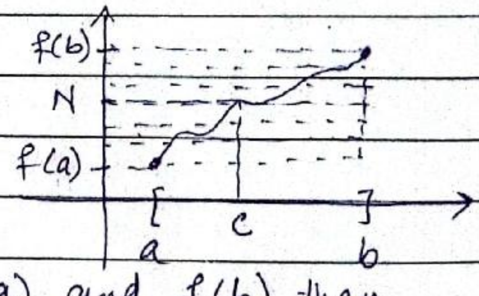
(ex) $f(x) = \sqrt{1+\frac{1}{x}}$

$\lim_{x \to 2} f(x) = \sqrt{1+\frac{1}{2}}$

So 1st use is to easily do this

$\boxed{2^{nd}}$ $\boxed{Q}$ does knowing something about the outside help us to know anything about the inside?

$\boxed{\text{Intermediate Value Theorem}}$



Know: ① f(x) is cont. on [a,b]
② $f(a) \neq f(b)$

N is any number between f(a) and f(b) then there is some $x=c$ in (a,b) such that $f(c) = N$.

using the Intermediate value theorem:

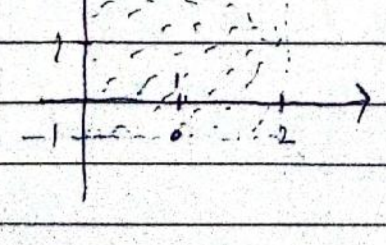
(ex) $\sqrt{2} = ?$ solve $x \cdot x = 2$

positive root of $x^2 = 2$

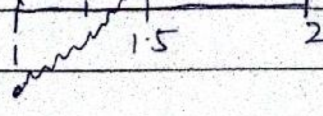
$f(x) = x^2 - 2$ $\qquad x^2 - 2 = 0$

b/c $x^2 - 2$ is cont. (poly)

then $x^2 - 2 = 0$ between $x=1$ and $x=2$



$(1.5)^2 - 2 = 2.25 - 2 = 0.25$



Date:

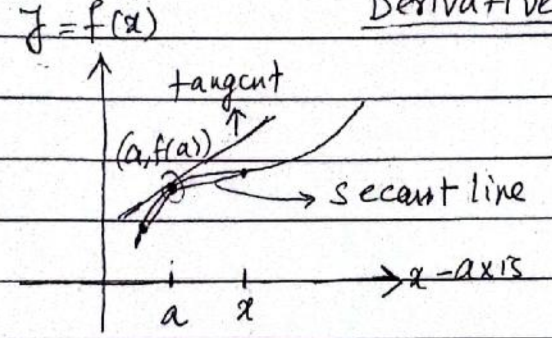
Lecture 8

$$\lim_{x \to a} f(x) = L$$

$\forall \epsilon > 0\ \exists \delta > 0$ (If $0 < |x-a| < \delta$, Then $|f(x) - L| < \epsilon$)

Calculus: change & Sum

Derivatives



Line: defined by two pieces of info
① point
② slope

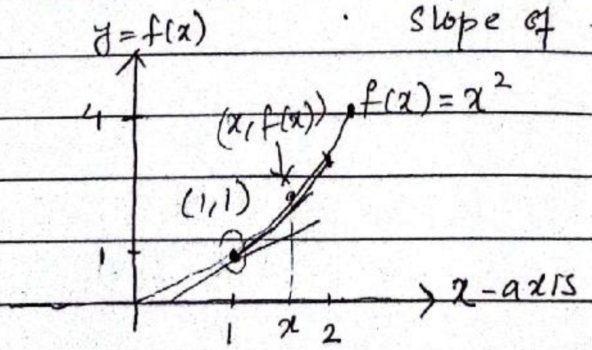
slope of secant is $m_{secant} = \dfrac{f(x) - f(a)}{x - a}$

slope of tangent is $m_{tangent} = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

instantenous slope

(ex) $\boxed{f(x) = x^2}$ slope of secant between $x=1$ and $x=2$

· slope of tangent @ $x=1$



$m_{secant} = \dfrac{4-1}{2-1} = 3$

$m_{tangent} = \lim_{x \to 1} \dfrac{f(x) - f(1)}{x - 1}$

$= \lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

$= \lim_{x \to 1} \dfrac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$

$= \lim_{x \to 1} x + 1 = 2$

→ equation of secant line? pt $(1,1)$, $m=3$

$y - 1 = 3(x-1)$ or $y - 4 = 3(x-2)$

$y - 1 = 3x - 3$ $\quad$ $y - 4 = 3x - 6$

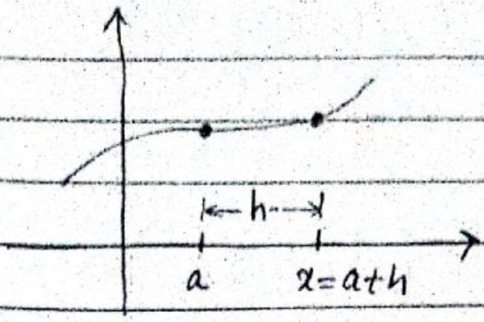
$y = 3x - 2$ $\quad$ $y = 3x - 2$

→ equation of tangent line? pt $(1,1)$, $m=2$

$y - 1 = 2(x-1) \Longrightarrow y = 2x - 1$

Date:

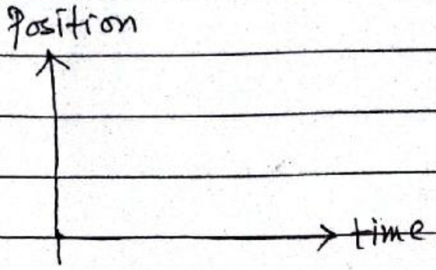
Notation:



$$m_{tangent} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}, \quad \begin{matrix} x-a=h \\ x=a+h \end{matrix}, \quad m_{tangent} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

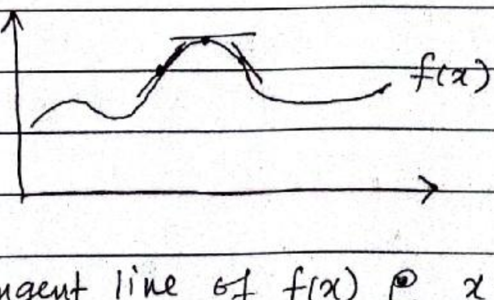
$\lim_{x\to a}$ same as $\lim_{h\to 0}$

Application:



$$\text{slope} = \frac{\text{change in position}}{\text{change in time}}$$ direction, average velocity

$$(\text{velocity @ } x=a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$



Slope of tangent line of $f(x)$ @ $x=a$

$$m_{tangent} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$f(x), a$

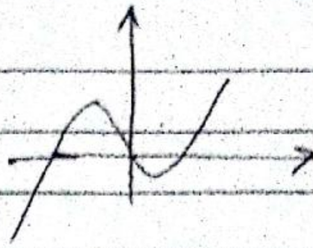
$$\boxed{\lim_{h\to 0} \frac{f(a+h)-fa)}{h}}$$ operator.

$m_{tangent}$

Date:

ex $f(\square) = \square^3 - \square$

$m_{tangent}$ at $x = a$



$$m_{tangent} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\left[(a+h)^3 - (a+h)\right] - \left[a^3 - a\right]}{h}$$

$$\rightarrow m_{tangent} = \lim_{h \to 0} \frac{\cancel{a^3} + 3a^2h + 3ah^2 + h^3 - \cancel{a} - h - \cancel{a^3} + \cancel{a}}{h}$$

$$m_{Tangent} = \lim_{h \to 0} \frac{\cancel{h}\left[3a^2 + 3ah + h^2 - 1\right)}{\cancel{h}}$$

$$= 3a^2 + 3a(0) + (0)^2 - 1 = 3a^2 - 1$$

b/c 'a' is any number it's a variable.

Slope @ x for $f(x) = x^3 - x$

$m_{tangent} = 3x^2 - 1$ (above)

slope of $f(x)$ @ x

$$m_{Tangent} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Takes a function in and spits function out.

it is an operator on functions...

$f(x)$

$$\boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$ Derivative

(Notation?)

<u>Derivative Notation:</u>

$D_x[f(x)]$, $\frac{d}{dx}[f(x)]$, $\frac{dy}{dx}$, $[f(x)]'$, y'

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Date:

(ex) $f(\square) = \sqrt{\square}$

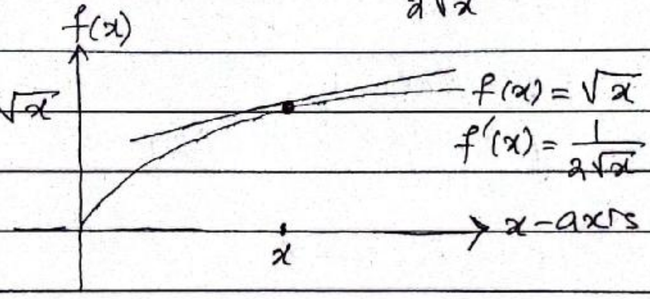
$f(x) = \sqrt{x}$ find $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} - \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} - \sqrt{x})} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$f(x) = \sqrt{x}$ $\qquad$ $f'(x) = \frac{1}{2\sqrt{x}}$



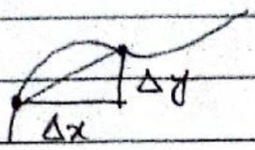
(Goal?) given $f(x) \longrightarrow \boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} \longrightarrow f'(x)$

x^P like quadratic polynomial.

Date:

Lecture 9

Derivatives



rate of change = slope of secant $= \dfrac{\Delta y}{\Delta x}$

instanteuous rate of change $= \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx}$

Derivative as limit $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Notations: $f'(x) = y' = \dfrac{dy}{dx} = \dfrac{d}{dx} y = \dfrac{d}{dx}[f(x)] = D[f(x)] = D_x[f(x)]$

b/c

$f(x)$

$\Downarrow$

$\boxed{\text{Derivative}}$

$\Downarrow$

$f'(x)$ is a function

Evaluate $f'(x)$ @ $x = a$

Notation: $f'(a) = \left.\dfrac{dy}{dx}\right|_{x=a}$

Ex $f(x) = \dfrac{x^2-1}{2x-3}$ find $f'(1) = \left.\dfrac{d}{dx}\left[\dfrac{x^2-1}{2x-3}\right]\right|_{x=1}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{(x+h)^2 - 1}{2(x+h) - 3} - \dfrac{x^2-1}{2x-3}}{h}$$

$$= \lim_{h \to 0} \frac{((x+h)^2 - 1)(2x-3) - (x^2-1)(2(x+h)-3)}{h(2(x+h)-3)(2x-3)}$$

$$= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2 - 1)(2x-3) - (x^2-1)(2x+2h-3)}{h(2x+2h-3)(2x-3)}$$

Date:

$$= \lim_{h\to 0} \frac{\cancel{2x^3} + 4x^2h + 2xh^2 - \cancel{2x} - \cancel{3x^2} - 6xh - 3h^2 + \cancel{3} - \cancel{2x^3} - 2x^2h + \cancel{2x} + \cancel{3x^2} - \cancel{3} + 2h}{h(2x+2h-3)(2x-3)}$$

$$= \lim_{h\to 0} \frac{2x^2h + 2xh^2 - 6xh + 2h - 3h^2}{h(2x+2h-3)(2x-3)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\left[2x^2 + 2xh - 6x + 2 - 3h\right]}{\cancel{h}(2x+2h-3)(2x-3)}$$

$$= \frac{2x^2 - 6x + 2}{(2x-3)(2x-3)} = \frac{2(x^2-3x+1)}{(2x-3)^2}$$

$$f'(x) = \frac{2(x^2-3x+1)}{(2x-3)^2}$$

$$\text{so } f'(1) = \frac{2(1-3+1)}{(2-3)^2} = -2$$

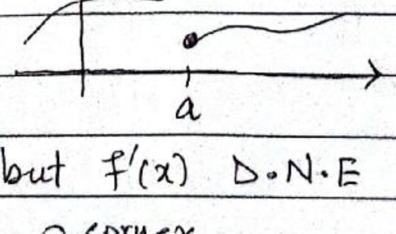
Differentiable functions

1. f is differentiable @ $x=a$ if $f'(a)$ exists
2. f is differentiable on an open interval if $f'(x)$ exists for all x in the open interval

When is f $\boxed{\text{NOT}}$ differentiable?

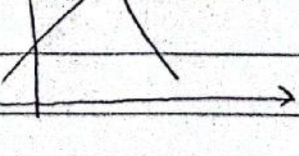
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

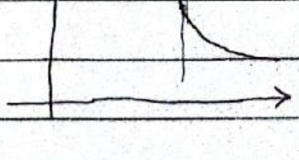
geometrically



@ discontinuities → not differentiable

f is continuous but $f'(x)$ D.N.E





Date:

$\boxed{Th^m}$ If f is differentiable @ $x=a$ then f is continuous @ $x=a$.

$\rightarrow$ $\boxed{\text{Smooth functions}}$

(2) Multiple Derivatives (higher-order derivatives)

$$\frac{d}{dx}[f(x)] = f'(x) \text{ is a function}$$

$$D_x[f'(x)] = \frac{d}{dx}\left[\frac{d}{dx}[y]\right] = \frac{d^2y}{dx^2} = D_x^2[f(x)]$$

n^{th} derivatives

$$f^{(n)}(x) = y^{(n)} = D_x^{(n)}[f(x)] = \frac{d^ny}{dx^n}$$

ex

$f(t) =$ position
$f'(t) =$ velocity
$f''(t) =$ acceleration
$f^{(3)}(t) =$ Jerk
$f^{(4)}(t) =$ snap
$f^{(5)}(t) =$ crackle
$f^{(6)}(t) =$ pop

$\boxed{\text{ex of formula}}$ $3x^2-4x+17=0$

$\rightarrow$ $ax^2+bx+c=0$

$\Downarrow$ $\boxed{\text{Algebra}}$

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad x = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Lecture 10

Rules / Formulas for derivatives

$$D_x[f(x)] = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$\boxed{\text{goal}}$ $D_x[\text{Polynomial}]$ $\quad a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Formulas for...

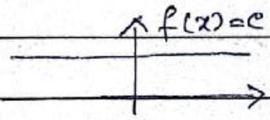
① $D_x[c]$ ④ $D_x[c f(x)]$

② $D_x[x]$ ⑤ $D_x[f(x)+g(x)]$

③ $D_x[x^n]$ $n = 2, 3, 4, \ldots$

① $D_x[\overset{f(x)=c}{c}] = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h}$

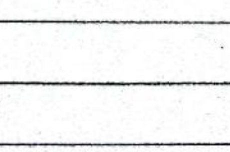
$= \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$



(ex) $D_x[\pi] = 0$

② $D_x[\overset{f(x)=x}{x}] = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)-x}{h}$

$= \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$



2 Formulas ① $D_x[\text{constant}] = 0$

② $D_x[x] = 1$

Background: Binomial Theorem

① $n! = n(n-1)(n-2)\cdots(1)$, $\quad 0! = 1$

$3! = 3 \cdot 2 \cdot 1$

② $(a+b)^n = (a+b)(a+b)(a+b)\cdots(a+b)$ $\quad n$-times

$$= \overset{1}{\frac{n!}{n!\,0!}} a^n + \overset{n}{\frac{n!}{(n-1)!\,1!}} a^{n-1} b^1 + \frac{n!}{(n-2)!\,2!} a^{n-2} b^2 + \cdots + \overset{1}{\frac{n!}{n!\,0!}} b^n$$

Date:

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

$$= \frac{3!}{3!\,0!}a^3 + \frac{3!}{2!\,1!}a^2b + \frac{3!}{1!\,2!}ab^2 + \frac{3!}{0!\,3!}b^3$$

$\downarrow$ 1 $\downarrow$ 3 $\downarrow$ 3 $\downarrow$ 1

1

1   1

1   2   1

1   3   3   1

(3) $D_x[x^n] = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$

$f(x) = x^n$

$$= \lim_{h\to 0} \frac{(\cancel{x^n} + nx^{n-1}h + \frac{n!}{(n-2)!\,2!}x^{n-2}h^2 + \dots + h^n) - \cancel{x^n}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(nx^{n-1} + \frac{n!}{(n-2)!\,2!}x^{n-2}h + \dots + h^{n-1})}{\cancel{h}}$$

$$= \lim_{h\to 0} nx^{n-1} + \underbrace{\frac{n!}{(n-2)!\,2!}x^{n-2}h}_{=0} + \dots + \underbrace{h^{n-1}}_{=0}$$

$$= nx^{n-1}$$

$\boxed{\text{Formula}}$ $D_x[x^n] = nx^{n-1}$

ex $D_x[x^4] = 4x^3$

(4) $D_x[cf(x)]$ assume $f'(x)$ exists.

$$= \lim_{h\to 0} \frac{[cf(x+h)] - [cf(x)]}{h} = \lim_{h\to 0} c\,\frac{f(x+h)-f(x)}{h}$$

$$= c\left[\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right] = cf'(x)$$

ex $D_x[3x^4] = 3\,D_x[x^4] = 3(4x^3) = 12x^3$

(5) $D_x[f(x)+g(x)]$ assume $f'(x), g'(x)$ exist

$$= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)] - [f(x)+g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}$$

Note:

$H(x) = f(x)+g(x)$

$H(x+h) = f(x+h)+g(x+h)$

Date:

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x)$$

$\boxed{\text{so}}$ $$D_x[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

ex $$\frac{d}{dx}\left[3 + 2x + 4x^2 - 5x^3\right]$$

$$= (3)' + (2x)' + (4x^2)' - (5x^3)'$$
$$= 0 + 2(1) + 4(2x) - 5\cdot(3x^2)$$
$$= 2 + 8x - 15x^2$$

Rational functions?

$$D_x\left[\frac{f(x)}{g(x)}\right] = ?$$

Products?

$$D_x[f(x)g(x)] = ?$$

6) $D_x[f(x)\, g(x)]$ $\qquad H(\square) = f(\square)\, g(\square)$

$$= \lim_{h \to 0} \frac{f(x+h)\, g(x+h) - f(x)\, g(x)}{h}$$

Remember

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

If we had 2

$$f(x+h)\, g(x) - f(x)\, g(x)$$
$$(f(x+h) - f(x))\, g(x)$$

$$= \lim_{h \to 0} \frac{f(x+h)\, g(x+h) - f(x+h)\, g(x) + f(x+h)\, g(x) - f(x)\, g(x)}{h}$$

$$= \lim_{h \to 0} \left[f(x+h)\, \frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h}\, g(x) \right]$$

$$= f(x)\, g'(x) + f'(x)\, g(x)$$

Product Rule $\quad D_x[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$

ex $\frac{d}{dx}\left[(2+x^3)(3-4x^2)\right]$

$= (2+x^3)'(3-4x^2) + (2+x^3)(3-4x^2)'$

$= (3x^2)(3-4x^2) + (2+x^3)(-8x)$

Quotient Rule $\quad D_x\left[\frac{f(x)}{g(x)}\right]$

$= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)\,g(x)\,h}$

$= \frac{1}{(g(x))^2} \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h}$

$= \frac{1}{(g(x))^2} \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h}$

$= \frac{1}{(g(x))^2} \lim_{h\to 0} g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}$

$= \frac{1}{(g(x))^2} \quad f'(x)g(x) - f(x)g'(x)$

$D_x\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

$D_x\left[x^{-3}\right] = D_x\left[\frac{1}{x^3}\right] = \frac{(1)'(x^3) - (1)(3x^2)}{(x^3)^2} = \frac{0 - 3x^2}{x^6} = \frac{-3x^2}{x^6}$

$= -\frac{3}{x^4} = -3x^{-4}$

$D_x[x^{-3}] = -3x^{-4}$

So power Rule $\quad D_x[x^n] = nx^{n-1} \qquad n = \ldots, -3, -2, -1, 1, 2, 3, \ldots$ (integers)

Note: $D_x[x^n] = n x^{n-1}$ works for all real numbers n.

Date:

$$D_x\left[\frac{x^2-x^{1/2}}{2x^3}\right] = D_x\left[\frac{1}{2}(x^2-x^{1/2})(x^{-3})\right]$$

Quotient

Product

$$\frac{\left(2x-\frac{1}{2}x^{-1/2}\right)2x^3-(x^2-x^{1/2})(6x^2)}{(2x^3)^2}$$

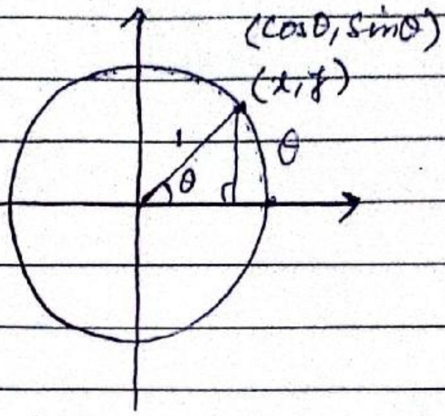
$$\frac{1}{2}\left[\left(2x-\frac{1}{2}x^{-1/2}\right)(x^{-3})+(x^2-x^{1/2})\cdot(-3x^{-4})\right]$$

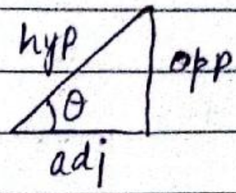
Date:

Lecture 11

Trig Functions

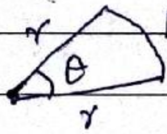
Note:





$c = (2\pi) r$

$\sin\theta = \frac{opp}{hyp}$, $\cos\theta = \frac{adj}{hyp}$



Arc $= \theta r$

$adj^2 + opp^2 = hyp^2$

$\cos^2\theta + \sin^2\theta = 1$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$ $\qquad \csc\theta = \frac{1}{\sin\theta}$

Know: $\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$$D_x[\sin x] = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

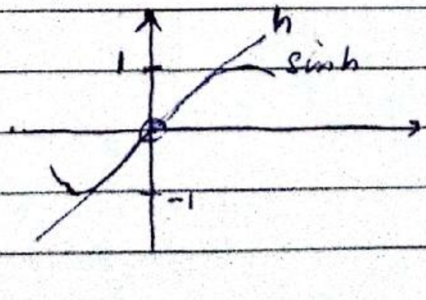
$$D_x[\sin x] = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

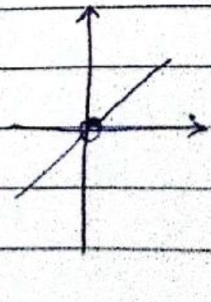
$$= \lim_{h\to 0} \sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right)$$

$$= \sin x \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

$\boxed{\lim_{h\to 0} \frac{\sin h}{h}}$ $\quad$ $\sin h$ vs h

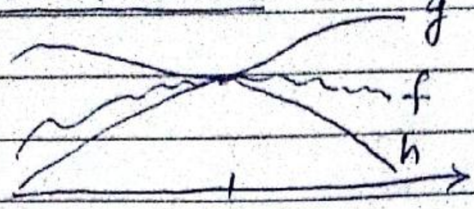
$\lim_{h\to 0} \frac{\sin h}{h} = 1 = ?$





Date:

Squeeze Theorem

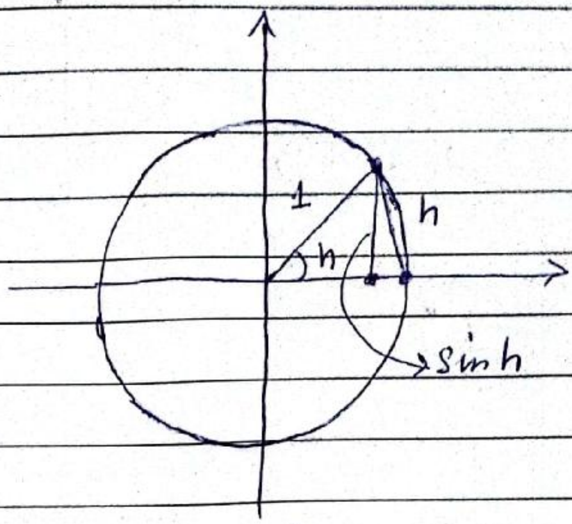


$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

$$\to \lim_{x \to a} f(x) = L$$

for $\lim_{h \to 0} \frac{\sin h}{h} = ? = 1$

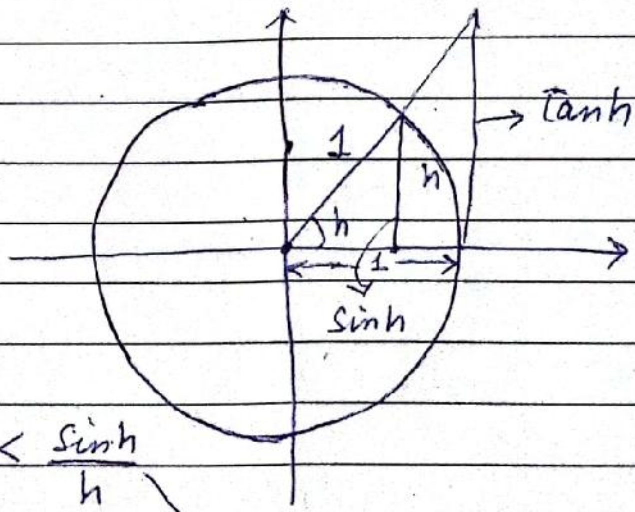
back to unit circle



$\sin h < \text{Secant} < h$

$\sin h < h$

so $\frac{\sin h}{h} < 1$ — (*)



$\tan h > h > \sin h$

$\tan h > h$

$\frac{\sin h}{\cos h} > h$

$\frac{\sin h}{h} > \cos h \Rightarrow \cos h < \frac{\sin h}{h}$ (**)

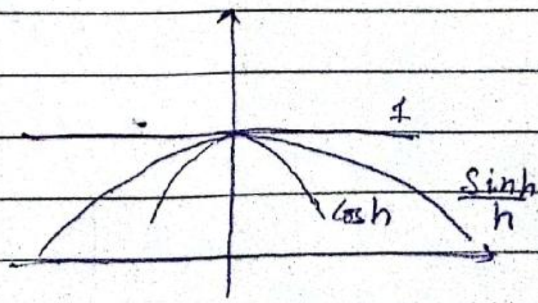
$$\cos h < \frac{\sin h}{h} < 1$$

so

$\lim_{h \to 0} \cos(h) = 1$

$\lim_{h \to 0} 1 = 1$

$\therefore \lim_{h \to 0} \frac{\sin(h)}{h} = 1$



Date:

$$\boxed{\lim_{h\to 0} \frac{\cos(h)-1}{h}}$$

$$\lim_{h\to 0} \frac{(\cos h-1)}{h}\frac{(\cos h+1)}{(\cos h+1)}$$

$$= \lim_{h\to 0} \frac{\cos^2 h-1}{h(\cos h+1)}$$

b/c $\sin^2 h + \cos^2 h = 1$

$$\cos^2 h - 1 = -\sin^2 h$$

$$= \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h+1)}$$

$$= -\lim_{h\to 0} \underbrace{\frac{\sin h}{h}}_{\to 1} \frac{\overbrace{\sin h}^{\to 0}}{\underbrace{\cos h+1}_{\to 2}}$$

$$= -\left(1\cdot\frac{0}{2}\right) = 0$$

$$D_x[\sin x] = \sin x \underbrace{\lim_{h\to 0}\frac{\cos h-1}{h}}_{\to 0} + \cos x \underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{\to 1}$$

$$\boxed{D_x[\sin x] = \cos x}$$

$$D_x[\cos(x)] = \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} = \text{similar as before}$$

$$\boxed{D_x[\cos x] = -\sin x}$$

We can use all our formulas now to also show

$D_x[\tan(x)] = \sec^2 x$

$D_x[\cot(x)] = -\csc^2 x$

$D_x[\sec(x)] = \sec x \tan x$

$D_x[\csc(x)] = -\csc x \cot x$

Note:

$$\lim_{\square\to 0}\frac{\sin(\square)}{\square} = 1 \quad \text{use this}$$

ex $\lim_{x\to 0} \frac{\sin 6x}{6x} = 1$

same as $6x \to 0$

Date:

ex $\lim_{x \to 0} \pi \left(\frac{\sin(\pi x)}{\pi x} \right) = \pi \cdot 1 = \pi$

ex $\lim_{x \to 0} \frac{\tan(3x)}{x} = \lim_{x \to 0} 3 \frac{\sin 3x}{3x} \cdot \frac{1}{\cos(3x)}$

$= 3 \cdot 1 \cdot \frac{1}{1} = 3$

Date:

Lecture 12

$\boxed{\text{Composition}}$ $D_x[(f\circ g)(x)] = D_x[f(g(x))]$

Make a formula?

(ex) $\frac{d}{dx}\left[\sqrt{\sin(x)+3}\right] = ?$

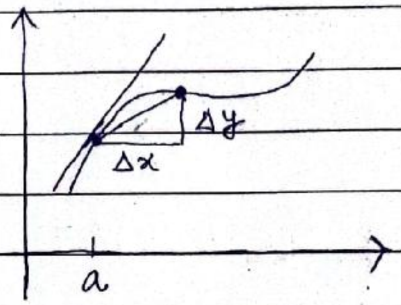
$D_x[f(g(x))]$

try $= \lim\limits_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} = ?$

(stuck)

Study work

① $y = f(x)$



slope of secant ≈ slope of tangent

$\frac{\Delta y}{\Delta x} \approx f'(a)$

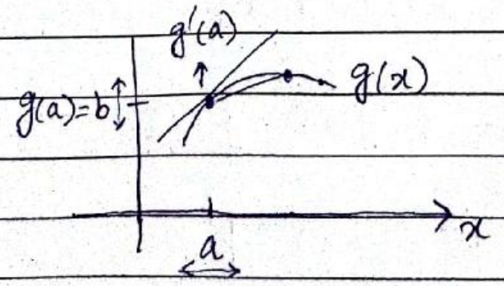
$\lim\limits_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = f'(a)$

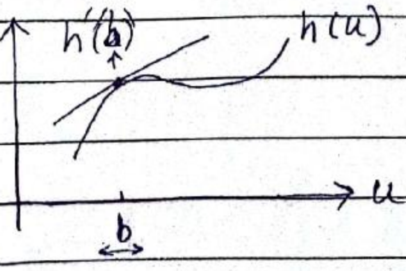
or $\frac{\Delta y}{\Delta x} - f'(a) = \varepsilon$ (error)

How far secant is from tangent

so as $\Delta x \to 0$ $\varepsilon \to 0$

composition $h(g(x)) = h(u)$ Let $u = g(x)$

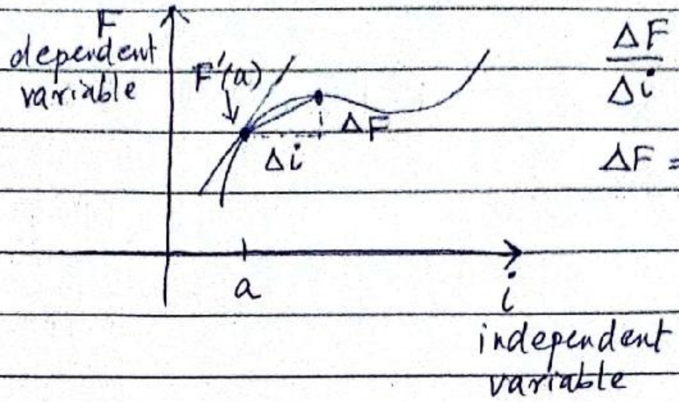




$\frac{dh}{dx} = ?$

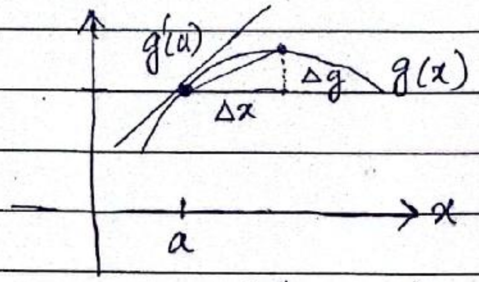
Date:

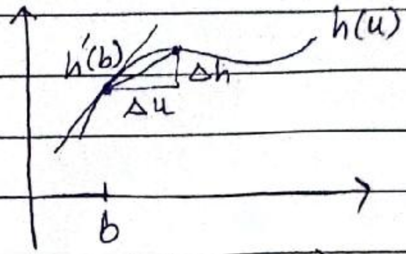
For any function



$$\frac{\Delta F}{\Delta i} - F'(a) = \epsilon$$

$$\Delta F = \left(F'(a) + \epsilon\right)\Delta i$$





$$\Delta g = \left(g'(a) - \epsilon_1\right)\Delta x \qquad \Delta h = \left(h'(b) + \epsilon_2\right)\Delta u$$

but $u = g(x)$

$\Delta u = \Delta g$

$$\Delta h = \left(h'(b) + \epsilon_2\right)\left(g'(a) + \epsilon_1\right)\Delta x$$

$$\frac{\Delta h}{\Delta x} = \left(h'(b) + \epsilon_2\right)\left(g'(a) + \epsilon_1\right)$$

as $\Delta x \to 0 \quad \epsilon_1 \to 0 \quad \epsilon_2 \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x} = h'(b)\, g'(a) \qquad b = g(a)$$

So $D_x\left[h(g(x))\right] = h'(g(x)) \cdot g'(x)$

Chain Rule $\quad D_x\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

Ex $\quad D_x\left[\sqrt{x+3x^2}\right] = D_x\left[(x+3x^2)^{1/2}\right]$

$$= \frac{1}{2}(x+3x^2)^{-1/2} \cdot \frac{d}{dx}\left[x + 3x^2\right]$$

$$= \frac{1}{2}(x+3x^2)^{-1/2}(1+6x)$$

$$= \frac{1+6x}{2\sqrt{x+3x^2}}$$

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ex $\frac{d}{dx}\left[\sin\left(\cos\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)\right)\right]$ b/c $\frac{d}{dx}\sin(x)=\cos(x)$

$= \cos\left(\cos\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)\right)\cdot\frac{d}{dx}\left[\cos\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)\right]$

$= \cos\left(\cos\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)\right)\cdot\left(-\sin\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)\right)\cdot\frac{d}{dx}\left[\sqrt{x^2+\frac{x+1}{x-1}}\right]$

$= -\cos\left(\cos\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)\right)\left(\sin\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)\right)\cdot\frac{1}{2}\left(x^2+\frac{x+1}{x-1}\right)^{-1/2}\cdot\frac{d}{dx}\left[x^2+\frac{x+1}{x-1}\right]$

$= \frac{-\cos\left(\cos\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)\right)\sin\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)}{2\sqrt{x^2+\frac{x+1}{x-1}}}\cdot\left(2x+\frac{(1)(x-1)-(x+1)(1)}{(x-1)^2}\right)$

ex $\frac{d}{dt}\left[\left(\frac{1-\cos(2t)}{1+\cos(2t)}\right)^4\right]$

$= 4\left(\frac{1-\cos(2t)}{1+\cos(2t)}\right)^3\frac{d}{dt}\left[\frac{1-\cos(2t)}{1+\cos(2t)}\right]$

$= 4\left(\frac{1-\cos(2t)}{1+\cos(2t)}\right)^3\cdot\frac{(1-\cos(2t))'(1+\cos(2t))-(1-\cos(2t))(1+\cos(2t))'}{(1+\cos(2t))^2}$

Note: $\frac{d}{dt}[1-\cos(2t)] = 0+\sin(2t)\cdot\frac{d}{dt}(2t) = 2\sin(2t)$

$\frac{d}{dt}[1+\cos(2t)] = -2\sin(2t)$

$= 4\left(\frac{1-\cos(2t)}{1+\cos(2t)}\right)^3\frac{2\sin(2t)(1+\cos(2t))+2(1-\cos(2t))\sin(2t)}{(1+\cos(2t))^2}$

$= 8\frac{(1-\cos(2t))^3}{(1+\cos(2t))^5}\left(\sin(2t)(1+\cos(2t))+\sin(2t)(1-\cos(2t))\right)$

$= 16\frac{(1-\cos(2t))^3\sin(2t)}{(1+\cos(2t))^5}$

Date:

chain Rule: $D_x[f(g(x))] = f'(g(x)) \cdot g'(x)$

play with notation:

$$g(x) = u$$

$$D_x[f(u)] = f'(u) \cdot u'$$

ex $\frac{d}{dx}\left[(g(x))^3\right] = 3(g(x))^2 \cdot g'(x)$

Let $y = g(x)$

$$\frac{d}{dx}\left[(y)^3\right] = 3(y)^2 \cdot y'$$

$$\frac{d}{dx}\left[y^3\right] = 3y^2 \frac{dy}{dx}$$

$$\frac{d}{dx}\left[x^3\right] = 3x^2$$

ex $D_x\left[\sin(y) + yx + x^3\right]$

$y = f(x)$

$$= \cos(y) \cdot y' + (1) y x + y(1) + 3x^2$$

$$= \cos(y) y' + x y' + y + 3x^2$$

Explicit Functions vs Implicit Functions

$y = f(x)$ $\qquad$ $y = f(x)$

$y = x^2 - \sin(x)$ $\qquad$ $2y - 3x + 8 = 0$

$y = \frac{x+1}{x-1}$ $\qquad$ $x^2 + xy - \sin(y^3) = 3x$

$y = \frac{3}{2}x - 4$

$D_x[y] = D_x\left[\frac{3}{2}x - 4\right] = \frac{3}{2}$

$D_x[2y - 3x + 8 = 0]$

$\frac{dy}{dx}$

$D_x[2y - 3x + 8] = D_x[0]$

$2\frac{dy}{dx} - 3 + 0 = 0$

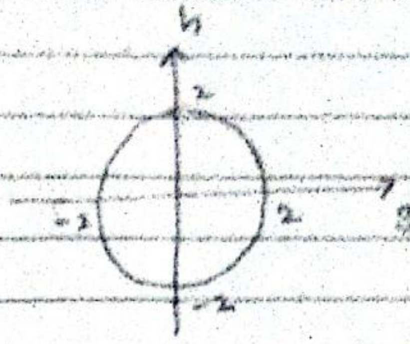
$\frac{dy}{dx} = \frac{3}{2}$

Date:

Lecture 13

Implicit function

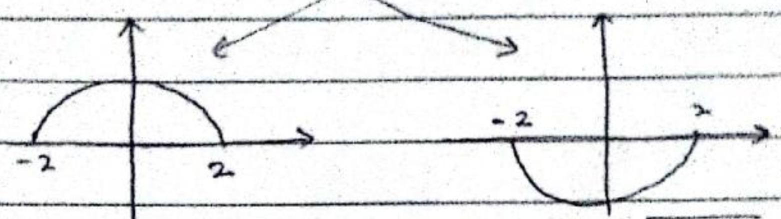
$$h^2 + s^2 = 4$$



s: Independent var.

h: Dependent var

$h(s) = ?$



$h = +\sqrt{4-s^2}$ $\qquad$ $h = -\sqrt{4-s^2}$

Restrict domain and range

use chain rule to find derivatives..

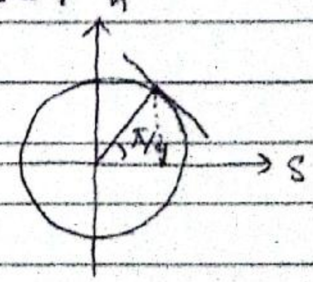
(ex) $h^2 + s^2 = 4$ $\qquad$ $\frac{dh}{ds} = ?$

$$D_s[h^2 + s^2] = D_s[4]$$

$$2h\frac{dh}{ds} + 2s = 0$$

$$\frac{dh}{ds} = -\frac{s}{h}$$

Point on $h^2 + s^2 = 4$



$$s = 2\cos(\pi/4)$$

$$h = 2\sin(\pi/4)$$

$$\frac{dh}{ds} = -\frac{2\cos(\pi/4)}{2\sin(\pi/4)} = -\cot(\pi/4) = -1$$

(ex) $x^2 + y^3 = 2xy$ $\qquad$ eqn of tangent line @ $(0.5118, 0.8447)$

slope? $\frac{dy}{dx} = ?$

$$D_x[x^2 + y^3] = D_x[2xy]$$

so $2x + 3y^2\frac{dy}{dx} = 2y + 2x\left(1 \cdot \frac{dy}{dx}\right)$

$$3y^2\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - 2x$$

$$\frac{dy}{dx} = \frac{2y - 2x}{3y^2 - 2x} \quad @ (0.5118, 0.8447)$$

eqn of tangent line

$$\frac{dy}{dx} = \frac{2(0.8447) - 2(0.5118)}{3(0.8447)^2 - 2(0.5118)} = m \Rightarrow \quad y - 0.8447 = m(x - 0.5118)$$

Date:

ex $x^2+xy+y^2=3 \qquad y''=?$

$$y''=\frac{d}{dx}\left[\frac{dy}{dx}\right]$$

$$D_x\left[x^2+(xy)+y^2\right]=D_x[3]$$

$$2x+\left[(1)(y)+(x)(1)\frac{dy}{dx}\right]+2y\frac{dy}{dx}=0$$

$$x\frac{dy}{dx}+2y\frac{dy}{dx}=-2x-y$$

$$\frac{dy}{dx}=-\frac{y+2x}{x+2y}$$

y'' $\quad \frac{d}{dx}\left[\frac{dy}{dx}\right].$

$$y''=-\frac{d}{dx}\left[\frac{y+2x}{x+2y}\right]=-\frac{\frac{d}{dx}[y+2x][x+2y]-[y+2x]\frac{d}{dx}[x+2y]}{(x+2y)^2}$$

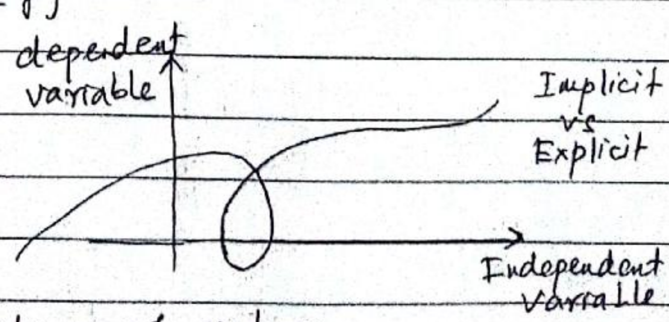
$$y''=-\frac{\left(1\cdot\frac{dy}{dx}+2\right)(x+2y)-(y+2x)\left(1+2\frac{dy}{dx}\right)}{(x+2y)^2}$$

Note: $\frac{dy}{dx}=-\frac{y+2x}{x+2y}$

$$y''=-\frac{\left(-\frac{y+2x}{x+2y}+2\right)(x+2y)-(y+2x)\left(1-2\left(\frac{y+2x}{x+2y}\right)\right)}{(x+2y)^2}$$

Applications

Idea: $y=f(x)$



Functions: relations/equations of values.

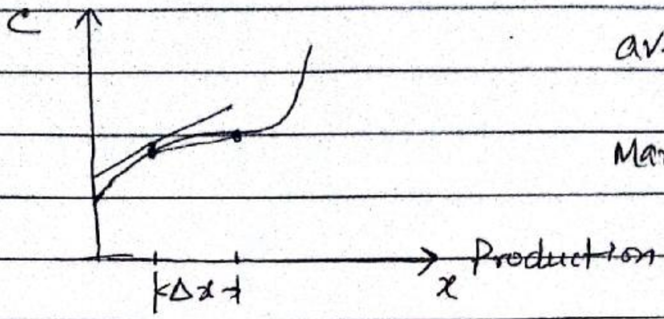
table	ind	dep
	a	$b \rightarrow (a,b)$

Derivatives: (change)

$\rightarrow \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ instantenous change.

average change over $\Delta x = \frac{\Delta y}{\Delta x}$

ex



ave cost $= \frac{\Delta c}{\Delta x}$

Marginal Cost : $\frac{dc}{dx}$

Physics: position $s = f(t)$

velocity $s' = f'(t)$ vs $\frac{\Delta s}{\Delta t}$ ave. velocity

Acceleration $s'' = f''(t)$

Looking Forward Force is what causes change in momentum

momentum: (mass)(velocity

$\frac{d}{dt}[m \cdot v] = mv' = m \cdot$ acceleration

Current (Electricity) Function charge $= Q(t)$

current $= \frac{dQ}{dt}$ ave. current $= \frac{\Delta Q}{\Delta t}$

Cost / Revenue / Profit (econ)

ave. cost $= \frac{\Delta c}{\Delta x}$ marginal cost $= \frac{dc}{dx}$

ex $F = \frac{GmM}{r^2}$ m —r— M

$X \propto Y$ directly proportional

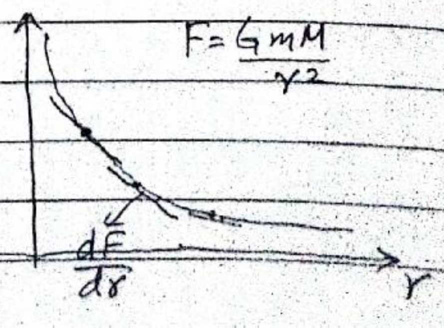
$X \propto \frac{1}{Y}$ inversly proportional

$\frac{dF}{dr} = D_r\left[\frac{GmM}{r^2}\right]$

$\frac{dF}{dr} = GmM \, D_r[r^2] = -\frac{GmM}{r^3}$

↑ change in instantenous force with respect to distance

$F = \frac{GmM}{r^2}$



Date:

Lecture 14

Application Problem 1:

Tank. volume of 5000 gallons.

Torricelli's Law: $V(t) = 5000\left(1 - \frac{1}{40}t\right)^2 \qquad t \in [0,40]$

rate of drain after 5 mins ? $V'(5)$

(1) $V'(t) = D_t\left[5000\left(1-\frac{1}{40}t\right)^2\right]$

$= 10000\left(1-\frac{1}{40}t\right) D_t\left(1-\frac{1}{40}t\right)$

$= -250\left(1-\frac{1}{40}t\right) \frac{gal}{min} \qquad \frac{\Delta V}{\Delta t}$

$V'(t) = (6.25t - 250) \frac{gal}{min} \qquad t \in [0,40]$

(2) $V'(5),\ V'(10),\ V'(20),\ V'(40)$ Summary

Application Problem 2:

$f = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$

L = length of string

T = Tension

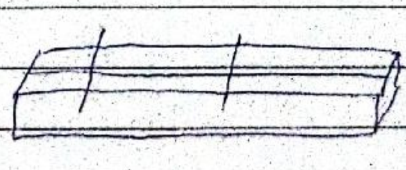
ρ = Linear density

Rate of change of f with respect to L (If T, ρ are fixed)

$f'(L) = \frac{d}{dL}\left[\frac{1}{2L}\sqrt{\frac{T}{\rho}}\right] = \frac{1}{2}\sqrt{\frac{T}{\rho}}\,\frac{d}{dL}\left[\frac{1}{L}\right] = \frac{1}{2}\sqrt{\frac{T}{\rho}}\,\frac{d}{dL}\left[L^{-1}\right]$

$= -\frac{1}{2}\sqrt{\frac{T}{\rho}}\,\frac{1}{L^2}$

$\frac{df}{dL} \propto -\frac{1}{L^2}$



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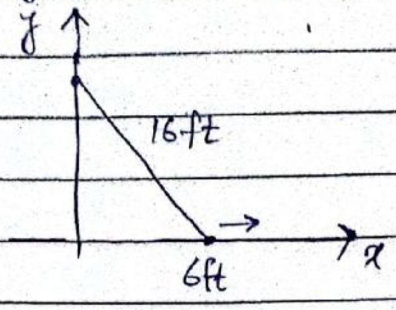
Related Rates

① given implicit functions in an equation.



$x^2 + y^2 = 16^2$ position equation

$\rightarrow x = f(t)\ ,\ y = f(t)$

Know $\frac{dx}{dt} = 3$ ft/sec

② Implicit Derivs to find $\boxed{\text{related rates eqn}}$

↳ with respect to ind. variable

$x^2 + y^2 = 16^2$ but $x = f(t)\ ,\ y = g(t)$

x, y are in ft

t is in seconds

$D_t[x^2 + y^2] = D_t[16^2]$

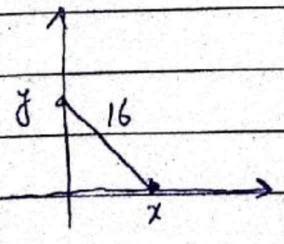
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\boxed{x\frac{dx}{dt} + y\frac{dy}{dt} = 0}$ Related Rates equation.

③ Use position and related rates eqn's to solve problems

[Q] How fast is the fall $\frac{dy}{dt} = ?$

$x(0) = 6$ ft , $\frac{dx}{dt} = 3$ ft/sec



$x^2 + y^2 = 16^2$

$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

$6^2 + y^2 = 16^2 \longrightarrow y = \sqrt{16^2 - 6^2}$

$6(3) + y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{-18}{\sqrt{16^2 - 6^2}}$ ft/sec.

Date:

Q $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ $\quad f'(0)$ exists or not

Note: When does $f'(a)$ exist?

$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists

continuous function $f(a) = \lim_{x \to a} f(x)$

is $f(x)$ continuous? $\lim_{x \to 0} x\sin\left(\frac{1}{x}\right)$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$\lim_{x \to 0^+} x\sin\left(\frac{1}{x}\right)$

x is positive

$-x \leq x\sin\left(\frac{1}{x}\right) \leq x$

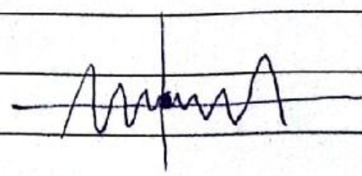
$\left.\begin{aligned} \lim_{x \to 0^+} -x &= 0 \\ \lim_{x \to 0^+} x &= 0 \end{aligned}\right]$ $\therefore \lim_{x \to 0^+} x\sin\left(\frac{1}{x}\right) = 0$

$\lim_{x \to 0^-} x\sin\left(\frac{1}{x}\right)$

x is negative

$-x \geq x\sin\left(\frac{1}{x}\right) \geq x$

$\left.\begin{aligned} \lim_{x \to 0^-} -x &= 0 \\ \lim_{x \to 0^-} x &= 0 \end{aligned}\right]$ $\therefore \lim_{x \to 0^-} x\sin\left(\frac{1}{x}\right) = 0$



$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\cancel{x}\sin\left(\frac{1}{x}\right) - 0}{\cancel{x}}$

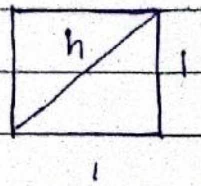
$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ D.N.E

so $f'(0)$ DNE

Date:

Lecture 15

① Linear Approximation (Polynomial Approximation)
② Error



$1^2 + 1^2 = h^2 \Rightarrow h^2 = 2 \rightarrow h \cdot h = 2$
(arithmetic) $h = \sqrt{2}$

Rational Number: $\frac{7}{5} = 7 \cdot \left(\frac{1}{5}\right)$

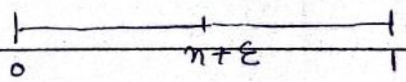
$\sqrt{2} \neq \frac{a}{b}$ $\quad \sqrt{x}$

Terminating $\frac{1}{2}$
Repeating $\frac{1}{3}$

Transcendental functions : $\sin(x)$, e^x, $\ln(x)$

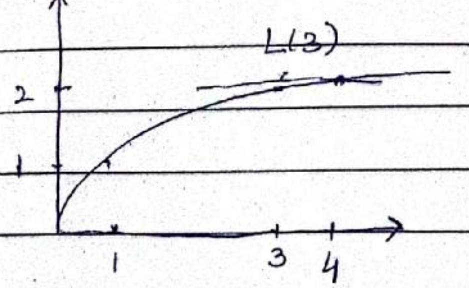
$\sin(1)$ transcend Algebra: finite sequence of algebra steps not possible.

Approximations



$f(n) = f(n+\varepsilon) = ?$

ex $\sqrt{3} = ? \rightarrow f(x) = \sqrt{x}$



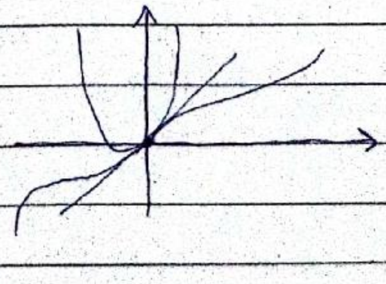
~~L(x) = ax+b~~

$L(x) = a + bx$ (Local Linear approximation)

$Q(x) = a + bx + cx^2$ (Local quad. approx)

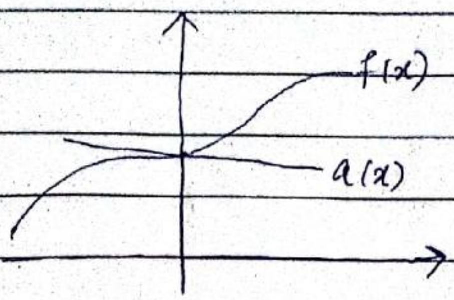
$Q(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$
(Local poly of deg 'n' approx)

ex near $x = 0$
① Match height
② ① + match slope
③ ①, ② + match $f^{(2)}(x)$



1. Same height $f(0) = a(0)$
2. Same 1^{st} derivative $f'(0) = a'(0)$
3. Same 2^{nd} derivative $f''(0) = a''(0)$

$\vdots$



ex $a(x) = c_0 + c_1 x$

$a'(x) = c_1$

know $\boxed{f(0)} = a(0) = \boxed{c_0}$

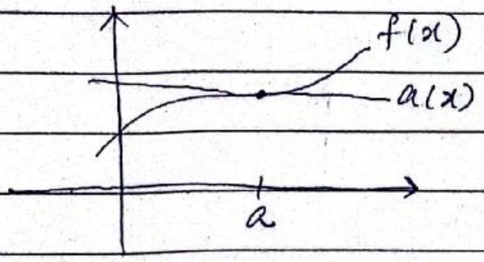
know $\boxed{f'(0)} = a'(0) = \boxed{c_1}$

So $f(x) \approx f(0) + f'(0)\,x$

move to $x = a$

approx near $x = a$

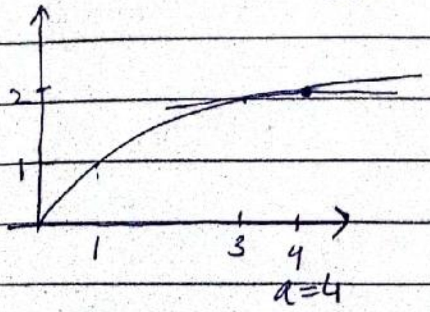
$f(x) \approx f(a) + f'(a)(x-a)$



$\sqrt{3} \approx ?$

$f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$



$f(x) \approx f(a) + f'(a)(x-a)$

$\sqrt{x} \approx \sqrt{4} + \dfrac{1}{2\sqrt{4}}(x-4)$

$\sqrt{x} \approx 2 + \dfrac{1}{4}(x-4)$

near $x=4$ $\sqrt{x} \approx 1 + \dfrac{x}{4}$

$\sqrt{3} \approx 1 + \dfrac{3}{4} = \dfrac{7}{4} = 1.75$

check: $\dfrac{7}{4} \cdot \dfrac{7}{4} = \dfrac{49}{16} = 3 + \dfrac{1}{16} \neq 3$

close!

Date:

Ex $\sin(x)$ near $x=0$ (Linear approx)

$$f(x) \simeq f(0) + f'(0)x$$

$$f(x) = \sin(x) \to f(0) = \sin(0) = 0$$
$$f'(x) = \cos(x) \to f'(0) = \cos(0) = 1$$
$$\sin(x) \simeq 0 + 1\cdot x = x$$
$$\sin(x) \simeq x \quad \text{near } x=0$$
$$\sin(0.1) \simeq 0.1 \qquad \text{remember } \lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

• Local quadratic: $a(x) = c_0 + c_1 x + c_2 x^2$

$$a'(x) = c_1 + 2c_2 x$$
$$a''(x) = 2c_2$$

Know: $a(0) = f(0) = c_0$

$$a'(0) = f'(0) = c_1$$
$$a''(0) = f''(0) = 2c_2 \to c_2 = \frac{f''(0)}{2}$$

near $x=0$ $\quad f(x) \simeq f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

near $x=a$ $\quad f(x) \simeq f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

• Polynomial of deg n

Near $x=0$ $\quad f(x) \simeq f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$

Near $x=a$ $\quad f(x) \simeq f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

ex 3^{rd} degree polynomial for $\sin(x)$ near $x=0$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$f(x) = \sin(x) \quad f(0) = 0$

$f'(x) = \cos(x) \quad f'(0) = 1$ $\qquad$ so

$f''(x) = -\sin(x) \quad f''(0) = 0$ $\qquad \sin(x) \simeq x - \frac{1}{6}x^3$

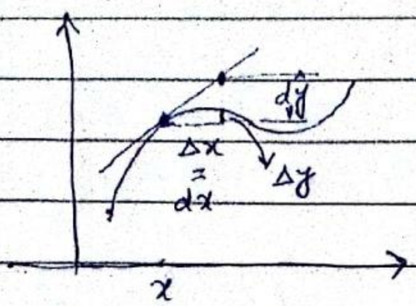
$f'''(x) = -\cos(x) \quad f'''(0) = -1$

Error propagation $\quad x = n \pm \text{error}$

$f(x) \approx f(n + \text{error})$

real ans $\pm$ error

approx ans $\pm$ error



$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = f'(x)$$

differential $dy = f'(x)\,dx$

so $\Delta x = dx$ but $\Delta y = f(x+\Delta x) - f(x)$

$dy = f'(x)\,dx$

and $dy \approx \Delta y$

$\boxed{\text{Error}}$ $\Delta x = dx$ is error in measure of independent variable

Δy is true error in outcome of function

$dy = f'(x)\,dx$ is approx error that you can calculate

$(\Delta y \approx dy)$

ex $\quad c = \pi d$, measure $d = \boxed{3\text{in}} \pm \boxed{0.1}$ in

so $\to x \qquad dx = \Delta x$

error in circumf? $\quad \Delta c \approx dc = c'(0.1)$

$c = \pi d \qquad c' = \pi$

$c = 3\pi \pm$ error $\qquad c'(3) = \pi$

$c = 3\pi \pm 0.1\pi = 3\pi \pm 0.314$ in

Relative Error: $\dfrac{\Delta y}{y} \approx \dfrac{dy}{y}$ $\quad$ ex above $c = 3\pi \pm 0.314$

rel. error: $\dfrac{0.314}{3\pi} \approx 0.033$

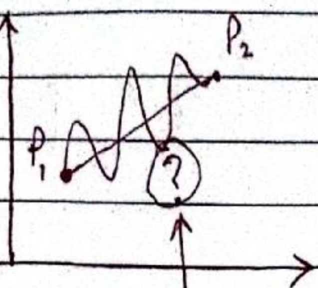
3.3%

to $x \quad \dfrac{0.1}{3} \approx 3.3\%$

Date:

Lecture 16

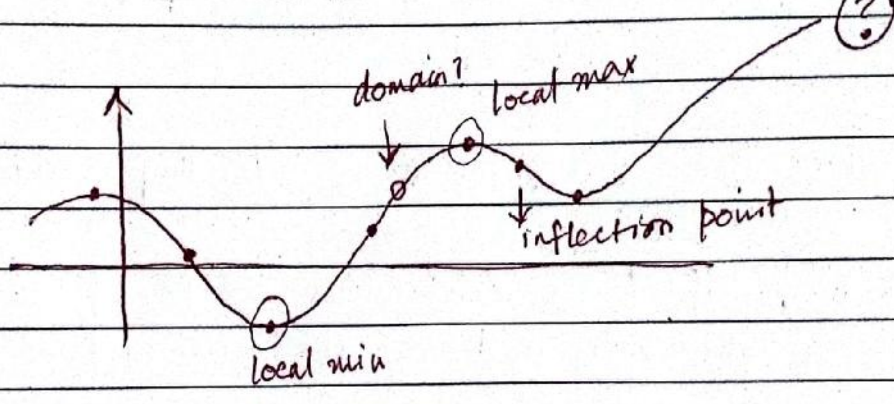
x	$y=f(x)$
Pairs	



use calculus

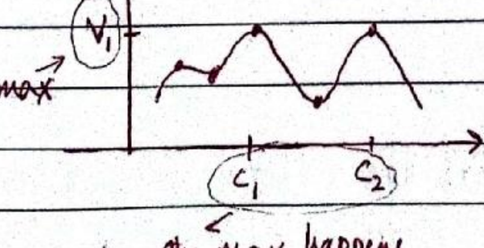
Table

x	y
x_1	y_1
x_2	y_2

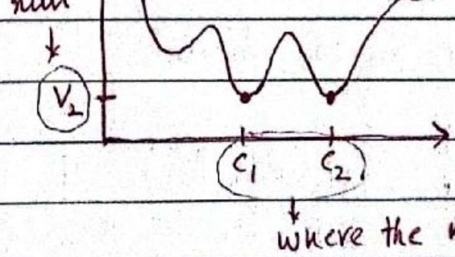


Exterme Values (Global, Local)

max min



where the max happens

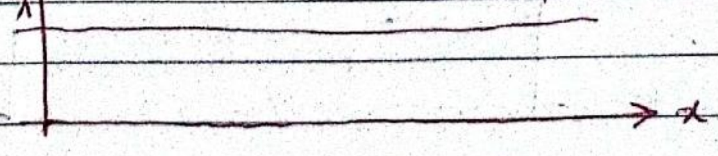


where the min happens

Max or Min of a function

max $=\pi$ occurs @ all real values of x

min $=\pi$ occurs @ all real values of x



Def ① $\boxed{f(c)}$ is an absolute max

max where global of Domain D

if $f(c) \geq f(x)$ for all x.

② $\boxed{f(c)}$ is an absolute min of f if $f(c) \leq f(x)$ for all x.

Min where

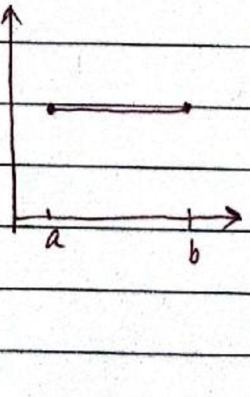
Local ① $f(c)$ is a local max of f if $f(c) \geq f(x)$ near c.

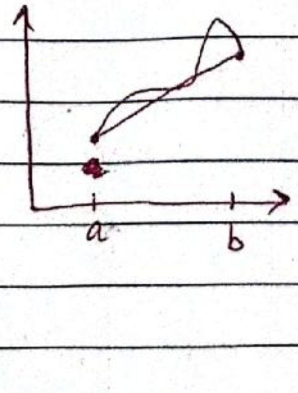
$|x-c| < \delta$

② $f(c)$ is a local min of f if $f(c) \leq f(x)$ near c.

Where? $\rightarrow$ 1st Question is do they exist?

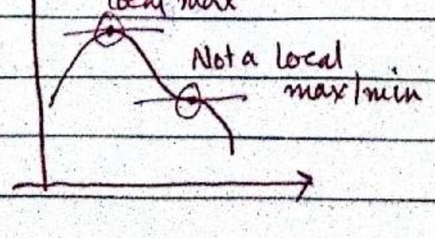
$\boxed{Th^m}$ If f is continuous on $[a,b]$ then $f(x)$ has an abs. max and an abs min on $[a,b]$

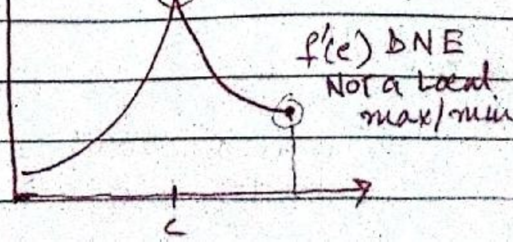




Now on to where?

$\boxed{Th^m}$ (Fermat's Th^m) If $f(x)$ has a local min or max @ $x=c$ and $f'(c)$ exists, Then $f'(c)=0$





Date: ___________

Where to look? critical numbers

all x such that $f'(x) = 0$

or $f'(x) =$ DNE (corners)

check?

(1) Abs Extrema on $[a, b]$

a) Find $f'(x)$

b) Find all c_i where $f'(c_i) = 0$

or $f'(c_i)$ DNE (corner)

critical Numbers

c) $f(a) = \boxed{?}$ } end points

$f(b) = \boxed{?}$

$f(c_i) = \boxed{?}$] critical points

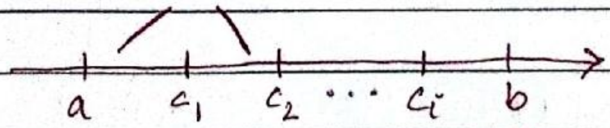
max? = Max

min? = Min

(2) Local Extrema

1st derivative

$f'(x)$

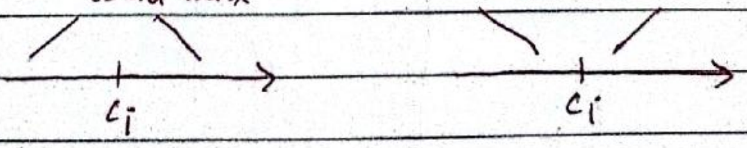


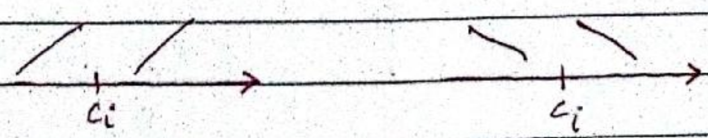
[f(any number in interval)] >0? <0?

local max

local min

f'



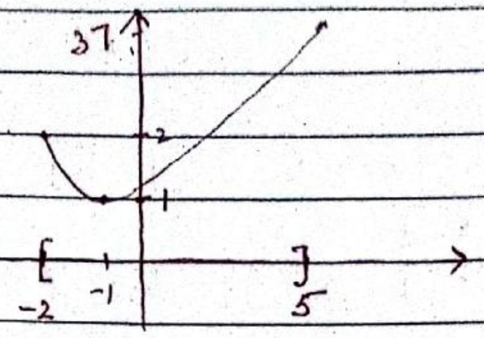


Nothing

Nothing

$f(x) = 1 + (x+1)^2 \qquad x \in [-2, 5]$

abs min = 1
@ $x = -1$
abs max = 37
@ $x = 5$
Local min = 1
near $x = -1$



$f(x) = (x+1)^2 + 1$

$f(x) = 1 + (x+1)^2$

$f'(x) = 2(x+1)(1) = 2x + 2$

critical numbers $\quad f'(x) = 0 \rightarrow x = -1$

$f'(x)$ DNE $\rightarrow$ never

$\boxed{\text{abs Extrema}}$

$f(-2) = 2$

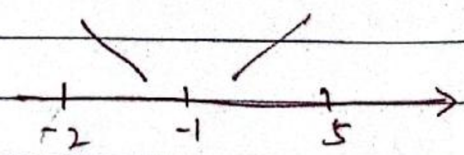
$f(5) = 37$

$f(-1) = 1$

abs min = 1 @ $x = -1$

abs max = 37 @ $x = 5$

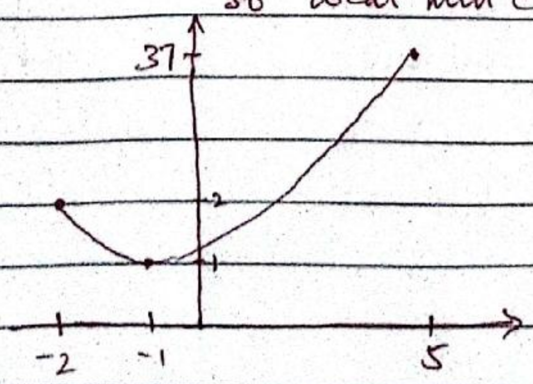
$\boxed{\text{Local Extrema}}$



$f'(x) = 2x + 2 \qquad f'(-3/2) = -1, \quad f'(0) = 2$

so local min @ $x = -1$ of $f(-1) = 1$

$\boxed{\text{Know}}$



Date:

ex $f(x) = \frac{x}{x^2-x+1}$ $[0,3]$

Natural Domain $x^2-x+1 \neq 0$

$x^2-x+1=0 \rightarrow x = \frac{1 \pm \sqrt{\underbrace{1-4}_{-3}}}{2}$

No real numbers

$$f(x) = \frac{x}{x^2-x+1}$$

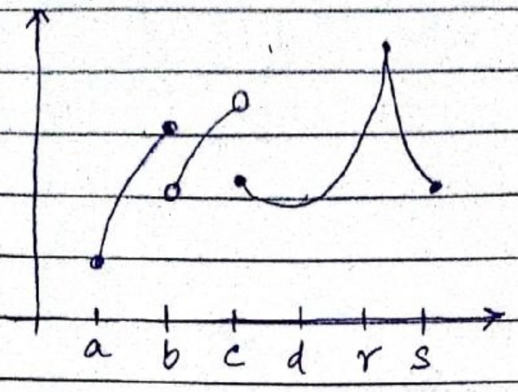
$$f'(x) = \frac{(1)(x^2-x+1)-(x)(2x-1)}{(x^2-x+1)^2}$$

Simplify! $f'(x)=0$

$f'(x)$ DNE never

Date:

Lecture 17



abs max = $f(r)$ @ $x=r$

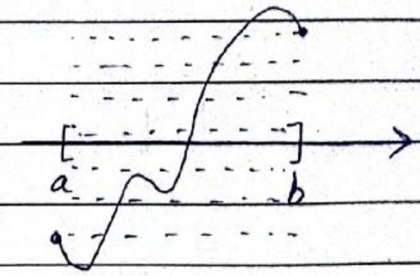
abs min = $f(a)$ @ $x=a$

Local min = $f(d)$ @ $x=d$

Local max = $f(r)$ @ $x=r$

Intermediate Value theorem

f is continuous on $[a,b]$



$\boxed{\text{Rolle's Th}^m}$

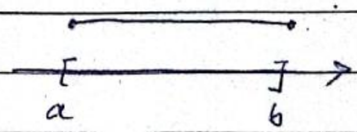
① $f(a)=f(b)$

② f is continuous on $[a,b]$

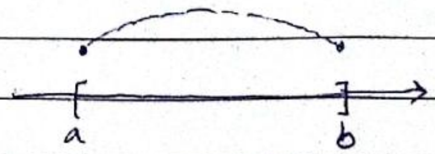
③ f' exists on (a,b)

Then there is some c in (a,b) such that $f'(c)=0$

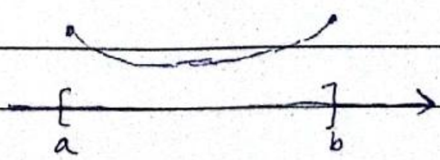
$\boxed{\text{case 1}}$ $f(x)=c$



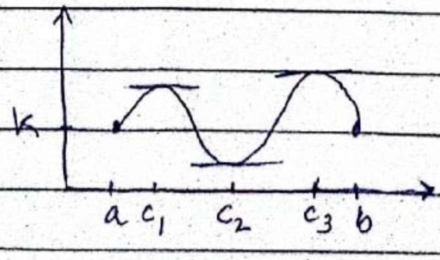
$\boxed{\text{case 2}}$ $f(x)>f(a)$ for some x in (a,b)



$\boxed{\text{case 3}}$ $f(x)<f(a)$ for some x in (a,b)

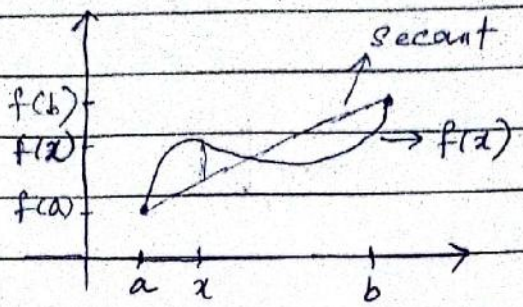


ex

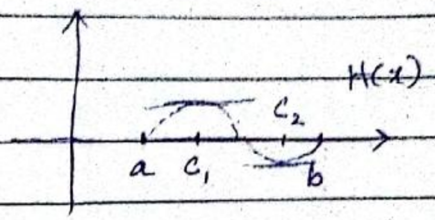


$f'(c_i)=0$

Date:



$H(x) = f(x) - \text{secant line}$

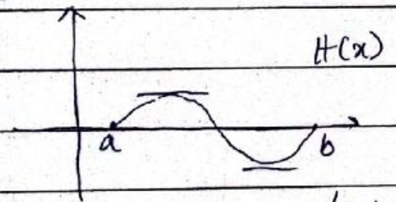


Secant line: $y - f(a) = \dfrac{f(b)-f(a)}{b-a}(x-a)$

$$y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

$$H(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

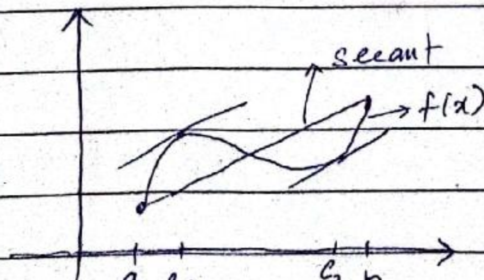
so



Rolle's Applies

so $\quad 0 = H'(c) = f'(c) - \dfrac{f(b)-f(a)}{b-a}$

so $\quad f'(c) = \dfrac{f(b)-f(a)}{b-a}$



$f'(c) = \underbrace{\dfrac{f(b)-f(a)}{b-a}}_{\text{average velocity}}$

$\boxed{\text{Mean Value Th}^m}$

① f is continuous on $[a,b]$

② f' exists on (a,b) (f is differentrable)

Then there is a $c \in (a,b)$ such that

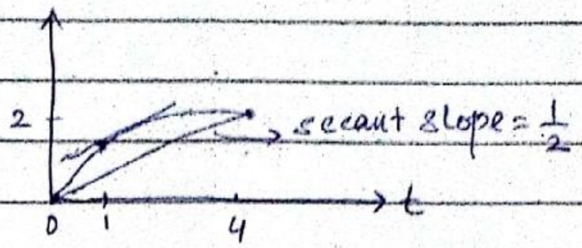
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

or $\quad f(b) - f(a) = f'(c)(b-a)$

Date:

(ex) $f(x) = x^{1/2}$ on $[0,4]$

$f'(x) = \frac{1}{2} x^{-1/2}$



$f'(c) = \frac{1}{2} \rightarrow \frac{1}{2\sqrt{2}} = \frac{1}{2} \rightarrow c = 1$

@ $x=1$ $\quad f'(1) = \frac{f(4)-f(0)}{4-0}$ $\quad$ Mean Value Th^m

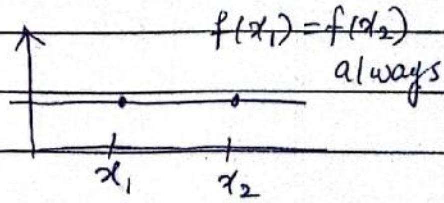
Another Application of Mean Value Th^m

If you take a derivative.... can you undo it?

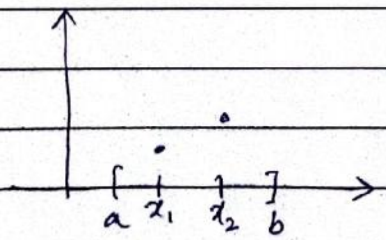
If $f'(x) = g'(x) \rightarrow$ what do I know about f, g?

Th^m $\quad$ if $f'(x) = 0$ for all $x \in (a,b)$

$\rightarrow f(x) = \text{constant}$



Pf



any $x_1 < x_2$ in (a,b)

Then MVT

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

$$f(x_2) - f(x_1) = \underbrace{f'(c)}_{=0}(x_2 - x_1)$$

$$f(x_2) - f(x_1) = 0$$

$f(x_2) = f(x_1)$ always.

so $f(x) = \text{constant}$

Corollary If $f'(x) = g'(x)$ for all x in (a,b)

Then $(f-g)(x) = \text{constant}$ on all x in (a,b)

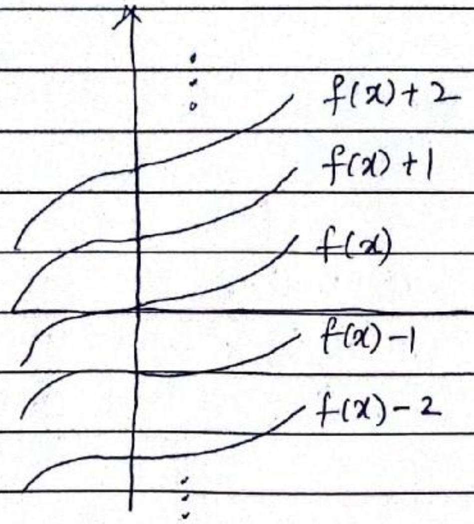
$\hookrightarrow f(x) - g(x) = c$ (a constant)

$\hookrightarrow f(x) = g(x) + c$

$\boxed{\text{Pf}}$ $(f-g)(x) = f(x) - g(x)$ → know $f'(x) = g'(x)$

$(f-g)'(x) = f'(x) - g'(x) = 0$

so $(f-g)(x) = \text{constant}$



Derivative $\Longrightarrow$ $f'(x)$

Using Derivatives for graphing

$f(x)$ points / position / asymptotes / Domain

$\hookrightarrow f'(x)$ slope / max / min

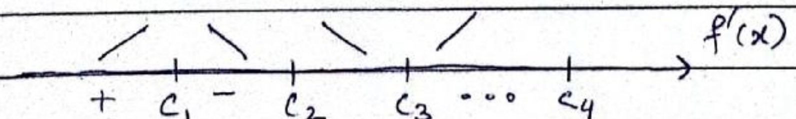
$f''(x)$ curvature / max / min

$\boxed{f'(x)}$

① $f'(x) = 0$, $f'(x)$ DNE (corner / vert. asymptote)

critical numbers

②

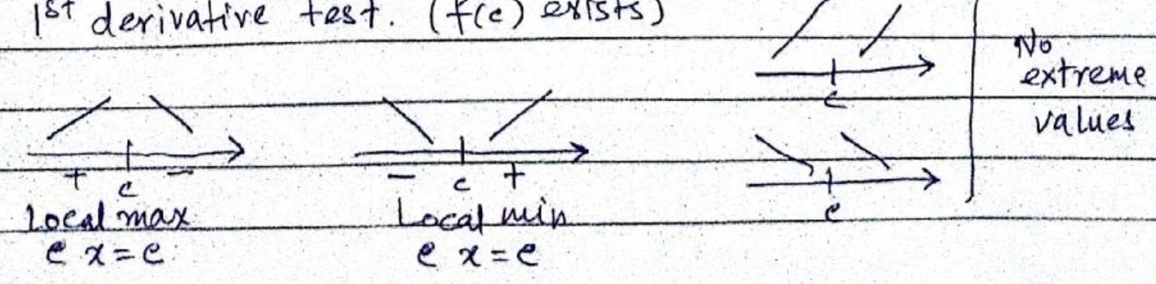


$\boxed{\text{inc/dec } f(x)}$ $f(x)$ inc. if $f'(x) > 0$

$f(x)$ dec. if $f'(x) < 0$

③ 1st derivative test. ($f(c)$ exists)

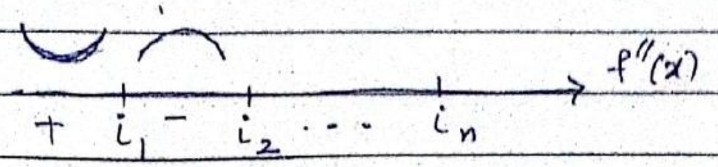
$f'(x)$



Date:

$\boxed{f''(x)}$

1. $f''(x) = 0$ or DNE <u>possible inflection point</u>

2. 

   $f''(x) > 0$ Concave up $\cup$
   $f''(x) < 0$ concave down $\cap$

3. <u>Concavity switches</u> $\rightarrow$ <u>Inflection point</u>

4. 2^{nd} <u>Derivative test</u>
   c is a critical number, $f'(c) = 0$
   $f''(c) > 0$, $\cup \rightarrow$ local minimum
   $f''(c) < 0$, $\cap \rightarrow$ local maximum

Date: ____________

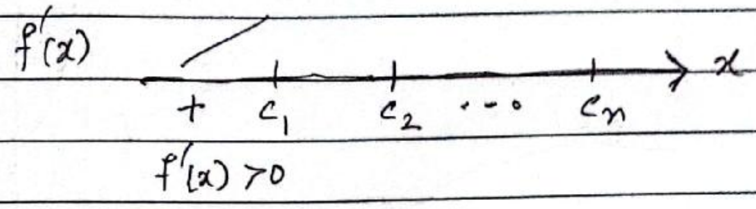
Lecture 18

Summary

$\boxed{y=f(x)}$ use for domain, intercepts, asymptotes, table of values

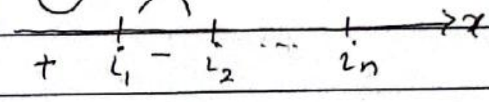
~~slopes~~ slopes $\boxed{f'(x)}$ use inc, dec, extrema, 1^{st} deriv test

critical numbers



Concavity $\boxed{f''(x)}$ use concave up, concave down, Inflections

2^{nd} derivative test $f''(x)$



ex $f(x)$; find $f'(x)$, find $f''(x)$

$\boxed{f(x)}$ ① domain all reals but $x \neq 1$, $x \neq -1$ (vertical asymptotes)

② intercepts were $(0,2)$ y-int

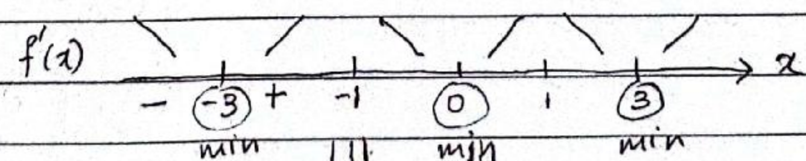
$(-4,0)$, $(-2,0)$, $(3,0)$ x-int

③ Right horizontal asymptote of $y=1$

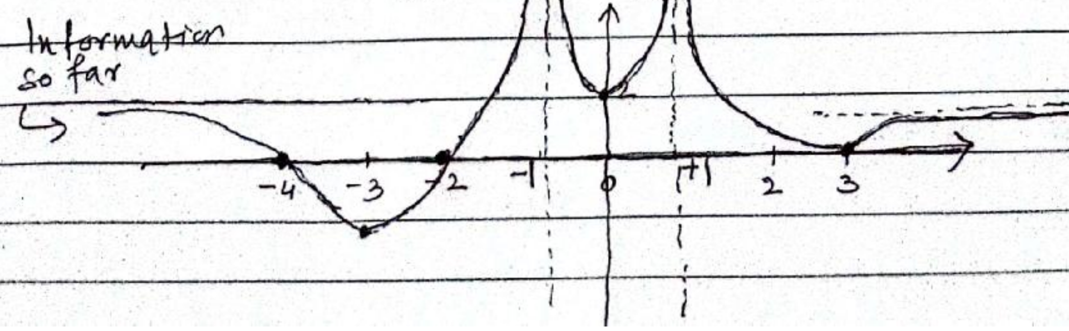
Left horizontal asymptote of $y=2$

$\boxed{f'(x)}$ critical numbers of $x=-3$, $x=3$, $x=0$ are $f'(x)=0$

$x=1$, $x=-1$ are $f'(x)$ DNE

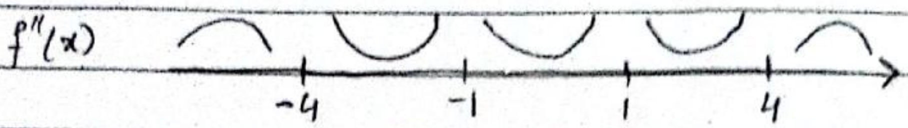


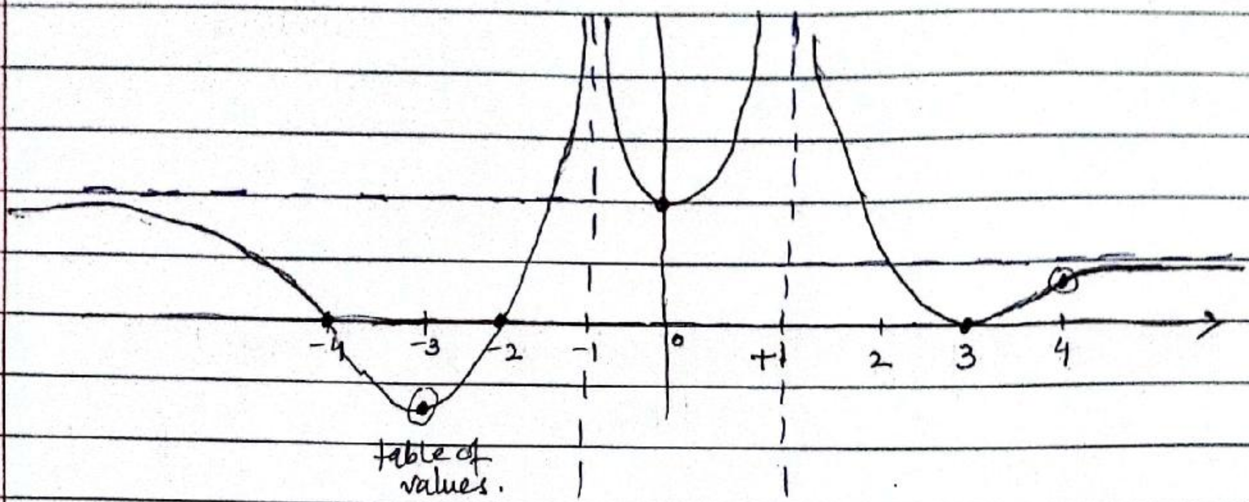
Information so far



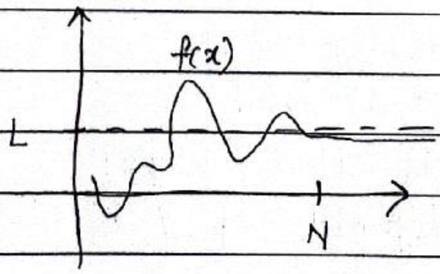
$\boxed{f''(x)}$ $\quad f''(x) = 0$ @ $x = -3$, $x = 4$

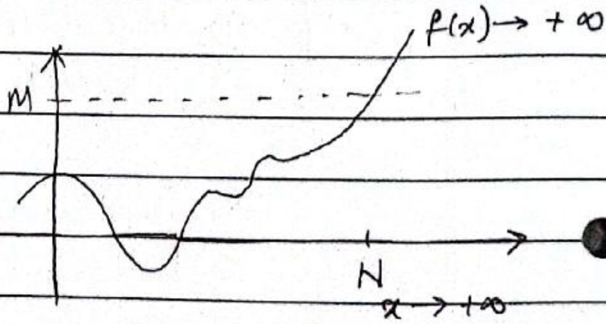
$f''(x)$ DNE $\quad x = -1$, $x = 1$





Limits @ infinity





$\lim\limits_{x \to +\infty} f(x) = L$

$\lim\limits_{x \to -\infty} f(x) = L$

$\lim\limits_{x \to +\infty} f(x) = +\infty$

$\lim\limits_{x \to +\infty} f(x) = -\infty$

$\lim\limits_{x \to -\infty} f(x) = +\infty$

$\lim\limits_{x \to -\infty} f(x) = -\infty$

③

Date:

Def: $\lim\limits_{x \to +\infty} f(x) = L$

$\forall \epsilon > 0 \quad \exists N$ if $x > N$, then $|f(x) - L| < \epsilon$

$\forall$ → for all, $\exists$ → There exists

Def: $\lim\limits_{x \to -\infty} f(x) = L$

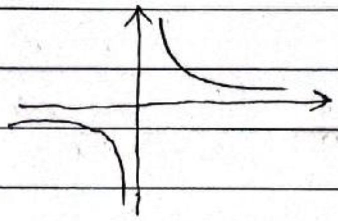
$\forall \epsilon > 0 \quad \exists N$ if $x < N$, then $|f(x) - L| < \epsilon$

Def: $\lim\limits_{x \to +\infty} f(x) = +\infty$

$\forall M \quad \exists N$ if $x > N$, $f(x) > M$

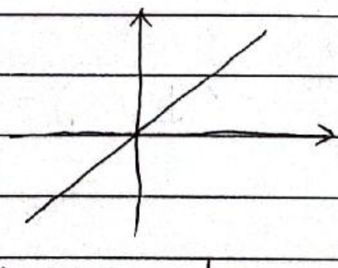
$\lim\limits_{x \to +\infty} c = c$, $\quad \lim\limits_{x \to -\infty} c = c$

$\lim\limits_{x \to +\infty} \frac{1}{x} = 0$



$\lim\limits_{x \to -\infty} \frac{1}{x} = 0$

$\lim\limits_{x \to +\infty} x = +\infty$



$\lim\limits_{x \to -\infty} x = -\infty$

$\boxed{\text{Note:}}$ $\boxed{\text{Th}^{m}}$ ① $\lim\limits_{x \to +\infty} \frac{1}{x^r} = \lim\limits_{x \to +\infty} \left(\frac{1}{x}\right)^r = 0$

② $\lim\limits_{x \to -\infty} \frac{1}{x^r} = \lim\limits_{x \to -\infty} \left(\frac{1}{x}\right)^r = 0$, if negatives are OK for $\frac{1}{x^r}$

$\boxed{\text{Application}}$ If $\lim\limits_{x \to +\infty} f(x) = L$ call $y = L$ a right $\boxed{\text{horz. asym}}$

If $\lim\limits_{x \to -\infty} f(x) = L$ call $y = L$ a left $\boxed{\text{horz. asym}}$

Date:

ex $f(x) = \dfrac{x^2-1}{x^2+1}$, $\displaystyle\lim_{x\to+\infty} \frac{x^2-1}{x^2+1} = \lim_{x\to+\infty} \frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(1+\frac{1}{x^2}\right)} = \lim_{x\to+\infty} \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}$

$$= \frac{1-0}{1+0} = 1$$

ex $\displaystyle\lim_{x\to-\infty} \frac{x^3-1}{x^2+1} = \lim_{x\to-\infty} \frac{x^3(1-1/x^3)}{x^2(1+1/x^2)} = \lim_{x\to-\infty} x\,\frac{(1-\overbrace{1/x^3}^{=0})}{(1+\underbrace{1/x^2}_{=0})} = -\infty(1) = -\infty$

ex $\displaystyle\lim_{x\to-\infty} \overbrace{\sqrt{4x^2+3x}}^{+\infty} - \overbrace{2x}^{+\infty} = +\infty$

ex $\displaystyle\lim_{x\to-\infty} \left(\sqrt{4x^2+3x}+2x\right)\frac{\left(\sqrt{4x^2+3x}-2x\right)}{\left(\sqrt{4x^2+3x}-2x\right)} = \lim_{x\to-\infty} \frac{4x^2+3x-4x^2}{\sqrt{4x^2+3x}-2x}$

$$= \lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+3x}-2x} = \lim_{x\to-\infty} \frac{x\,(3)}{|x|\sqrt{4+3/x}-2x}$$

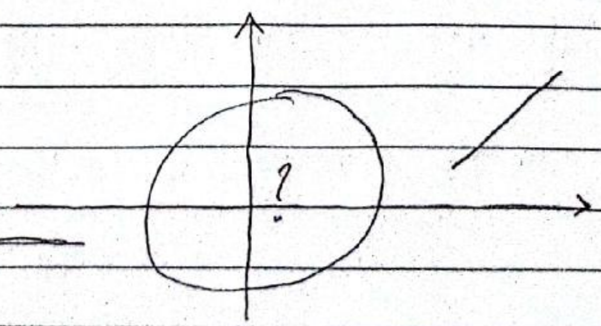
$$= \lim_{x\to-\infty} \frac{x\,(3)}{-x\left(\sqrt{4+3/x}\right)-2x} = \lim_{x\to-\infty} \frac{\cancel{x}(3)}{\cancel{x}\left(-\sqrt{4+3/x}-2\right)}$$

$$= -3/4$$

So $f(x) = \sqrt{4x^2+3x} + 2x$

left horiz asymp of $y = -3/4$

but $\displaystyle\lim_{x\to+\infty} \overbrace{\sqrt{4x^2+3x}}^{+\infty} + \overbrace{2x}^{+\infty} = +\infty$



Date: ____________

Lecture 19

Graph $f(x) = \sqrt{4x^2+3x} + 2x$

Domain: $4x^2+3x \geq 0$

$x(4x+3) \geq 0$

$x = 0,\ x = -3/4$

ok | bad | ok
$+ \quad -3/4 \quad - \quad 0 \quad +$

$(-\infty, -3/4] \cup [0, +\infty)$

Intercepts:

(1) y-int Let $x = 0$

$y = f(0) = \sqrt{4(0)^2+3(0)} + 2(0) = 0$

$(0,0)$

(2) x-int Let $y = f(x) = 0$

$0 = \sqrt{4x^2+3x} + 2x$

$\sqrt{4x^2+3x} = -2x$

$\cancel{4x^2} + 3x = \cancel{4x^2}$

$3x = 0 \rightarrow x = 0 \rightarrow (0,0)$

Asymptotes $x \to -\infty,\ f(x) \to -3/4$

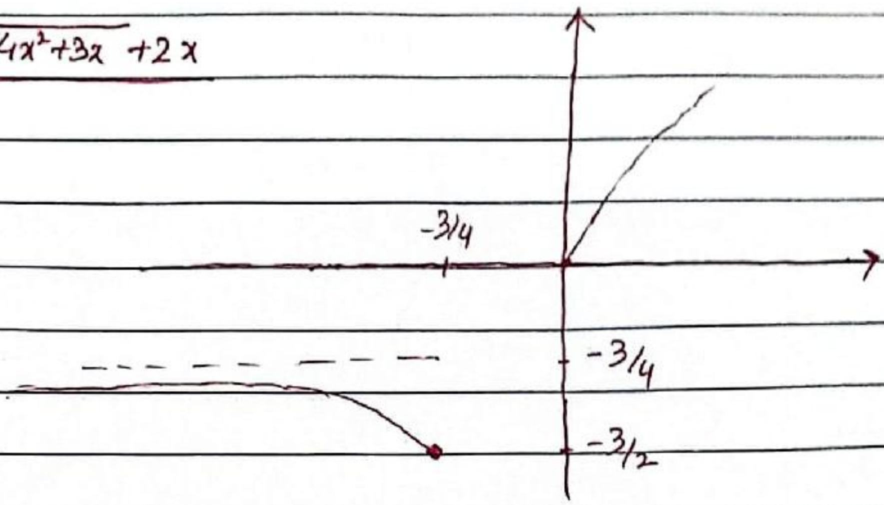
$x \to +\infty,\ f(x) \to +\infty \qquad f(x) \sim 4x$

$f(x)$: Domain, Intercepts, Table of values, asymptotes
$f'(x)$: Critical Numbers/points, Inc/dec, extrema
$f''(x)$: Possible Inflection points, concave up/down, extrema
Simplify $\rightarrow$ Graph.

Date:

On the base of information till obtained.

x	$y=f(x)=\sqrt{4x^2+3x}+2x$
0	0
$-3/4$	$-3/2$



1st and 2nd derivative calculation

$$f(x)=\sqrt{4x^2+3x}+2x$$

$$f'(x)=\frac{1}{2}(4x^2+3x)^{-1/2}(8x+3)+2$$

$$=\frac{8x+3}{2\sqrt{4x^2+3x}}+2=\frac{8x+3+4\sqrt{4x^2+3x}}{2\sqrt{4x^2+3x}}$$

$$\boxed{f'(x)=\frac{8x+3+4\sqrt{4x^2+3x}}{2\sqrt{4x^2+3x}}}$$

$$f''(x)=\frac{16\sqrt{4x^2+3x}-(8x+3)^2(4x^2+3x)^{-1/2}}{4(4x^2+3x)}$$

$$f''(x)=\frac{16(4x^2+3x)-(8x+3)^2}{4(4x^2+3x)^{3/2}}=\frac{\cancel{64x^2}+\cancel{48x}-\cancel{64x^2}-\cancel{48x}-9}{4(4x^2+3x)^{3/2}}$$

$$\boxed{f''(x)=\frac{-9}{4(4x^2+3x)^{3/2}}}$$

Date:

Critical Numbers using $f'(x) = \dfrac{8x+3+4\sqrt{4x^2+3x}}{2\sqrt{4x^2+3x}}$

$f'(x) = 0$

$8x+3+4\sqrt{4x^2+3x} = 0$

$\sqrt{4x^2+3x} = -2x - \frac{3}{4}$

$4x^2+3x = 4x^2+3x+\frac{9}{16}$

$0 = \frac{9}{16}$ NEVER

$f'(x) \neq 0$

$f'(x)$ DNE

$4x^2+3x=0$

$x(4x+3)=0$

$x=0$ $\quad x=-3/4$

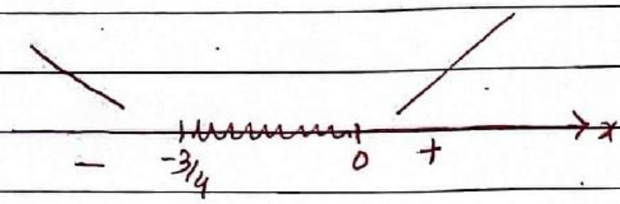
$(0,0)$ $\quad (-3/4, -3/2)$

vertical tangents

$f'(x)$ DNE

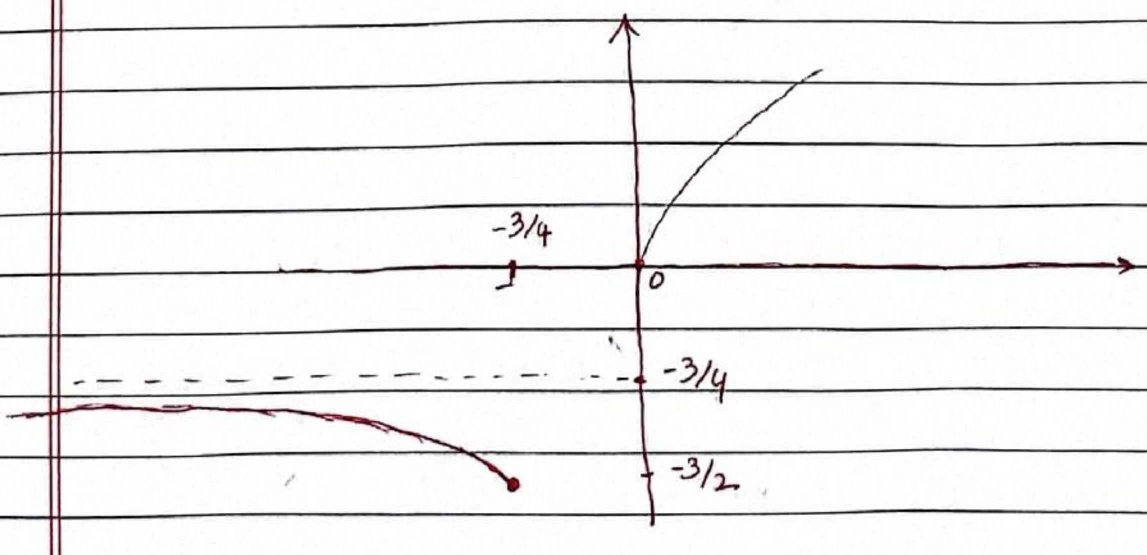
1st Derivative test

$f'(x) = \dfrac{8x+3+4\sqrt{4x^2+3x}}{2\sqrt{4x^2+3x}}$



2nd Derivative

$f''(x) = \dfrac{-9}{4(4x^2+3x)^{3/2}} < 0$ always so concave down every where!



Date:

Ex $f(x) = (x^2 - x - 6)^3 = (x-3)^3(x+2)^3$

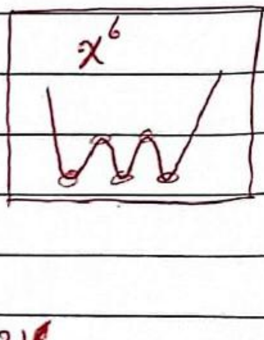
1st derivative $f'(x)$

$$f'(x) = 3(x^2-x-6)^2(2x-1) = 3(x-3)^2(x+2)^2(2x-1)$$

2nd derivative $f''(x)$

$$\begin{aligned} f''(x) &= \left[6(x^2-x-6)(2x-1)\right](2x-1) + 3(x^2-x-6)^2[2] \\ &= 6(x-3)(x+2)(2x-1)^2 + 6(x-3)^2(x+2)^2 \\ &= 6(x-3)(x+2)\left[(2x-1)^2 + (x-3)(x+2)\right] \\ &= 6(x-3)(x+2)\left[4x^2-4x+1+x^2-x-6\right] \\ &= 6(x-3)(x+2)(5x^2-5x-5) \\ &= 30(x-3)(x+2)(x^2-x-1) \end{aligned}$$

Domain: $f(x) = (x-3)^3(x+2)^3$
all reals.



$y = f(x) = (x-3)^3(x+2)^3$

Intercepts: y-int (Let $x=0$)

$$f(0) = (0-3)^3(0+2)^3 = (-27)(8) = -216$$

$$(0, -216)$$

x-int (Let $y=0$)

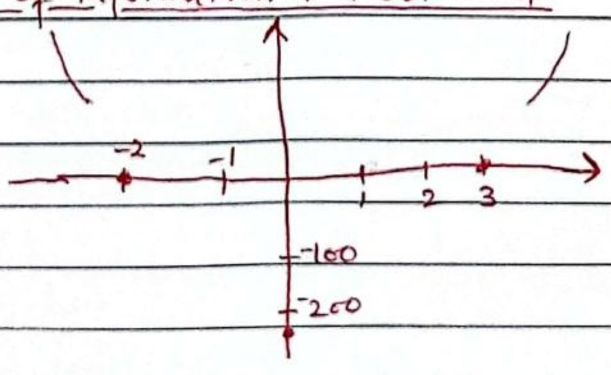
$$0 = (x-3)^3(x+2)^3$$

$$x = 3, \quad x = -2$$

$$(3,0), \quad (-2,0)$$

Asymptotes: No vertical, No oblique asymptotes, (x^6)

On the base of information till obtained



Date:

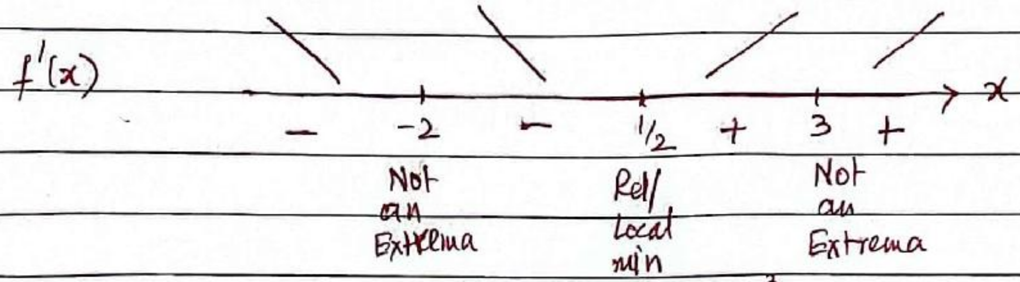
Critical Numbers using $f'(x) = 3(x-3)^2(x+2)^2(2x-1)$

$f'(x)=0$ $\qquad$ $f'(x)$ DNE NEVER

$3(x-3)^2(x+2)^2(2x-1)=0$

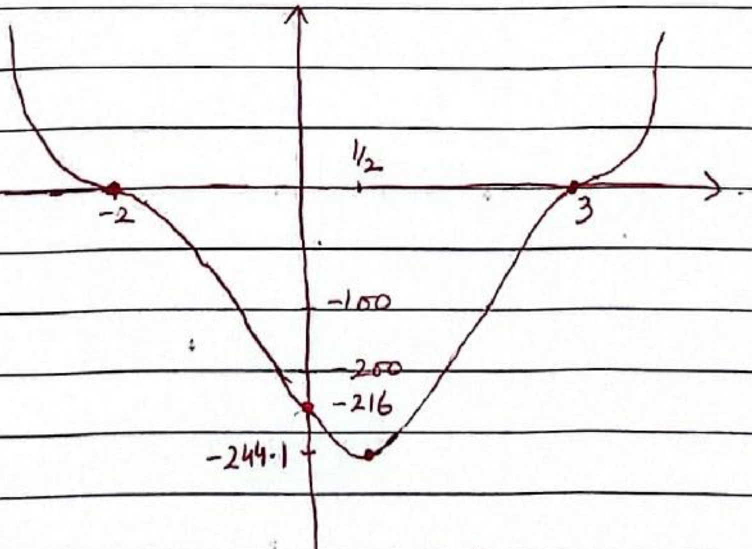
$x=3, \quad x=-2 \quad x=1/2$

1st Derivative test $\quad f'(x) = 3(x-3)^2(x+2)^2(2x-1)$



$f(x) = (x-3)^3(x+2)^3$

x-int	y-int	slope zero	Rel/Local Min	Slope zero
(3,0)	(0,-216)	(-2,0)	(1/2, -244.1	(3,0)
(-2,0)				



Possible inflection points using $f''(x) = 30(x-3)(x+2)(x^2-x-1)$

$f''(x)=0$ $\qquad$ $f''(x)$ DNE Never

$30(x-3)(x+2)(x^2-x-1)=0$

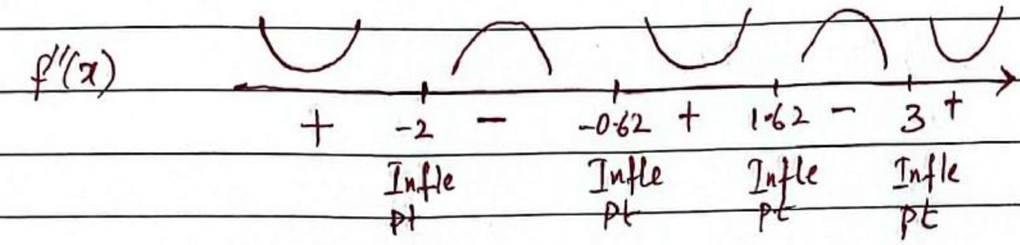
$x=3, \ x=-2, \ x=\dfrac{1\pm\sqrt{5}}{2}$

$x=3, \ x=-2, \ x=1.62, \ x=-0.62$

⑥

Date:

2^{nd} derivative test $\quad f''(x) = 30(x-3)(x+2)(x^2-x-1)$

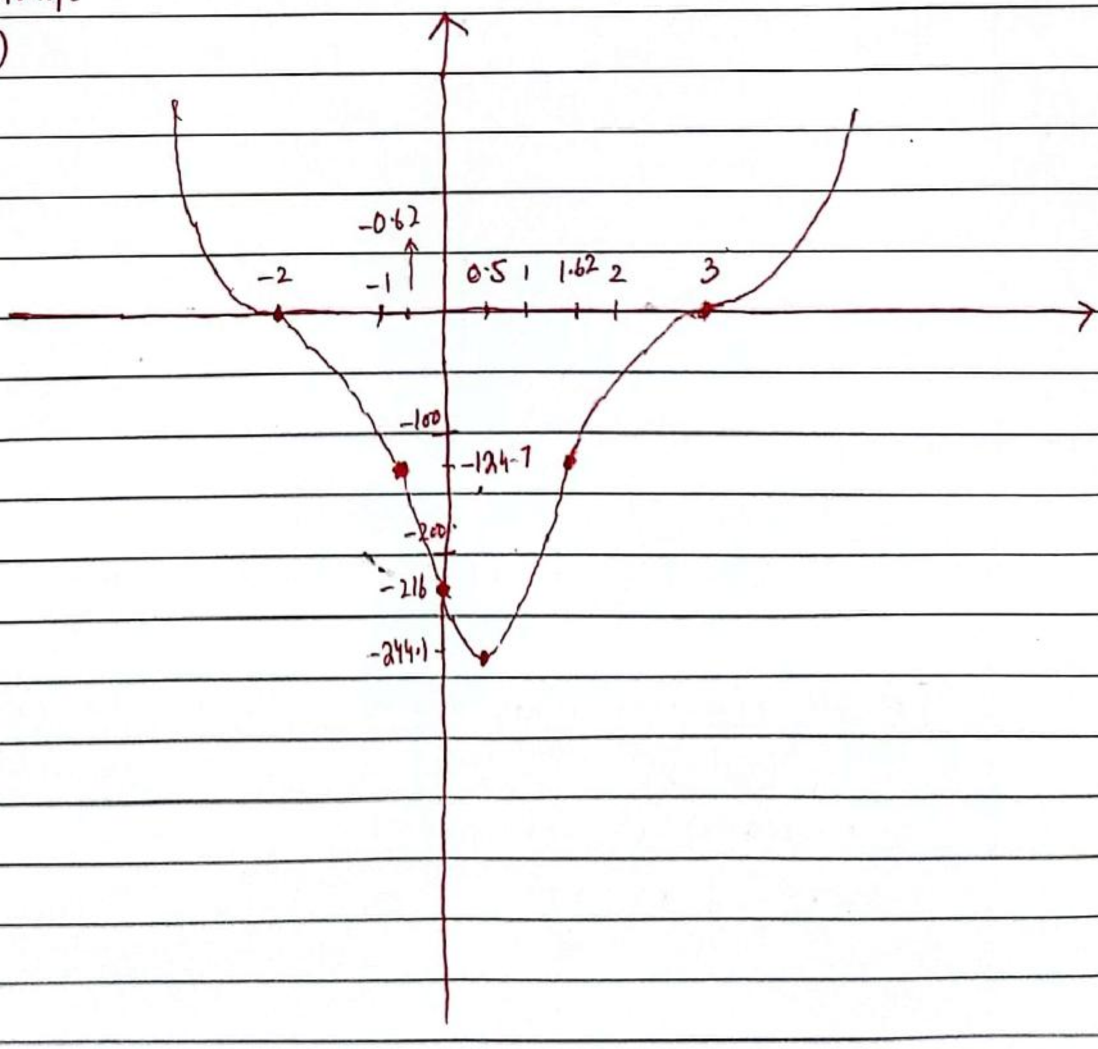


$f(x) = (x-3)^3(x+2)^3$

x-Int	y-Int	Zero Slope & inflection Pt	Inflection pt	Abs Rel local min	Inflec pt
(-2,0)	(0,-216)	(-2,0)	(-0.62,-124.7)	(0.5,-244.1)	(1.62,-124.7)
(3,0)					

Zeroslope & inflectionpt

(3,0)

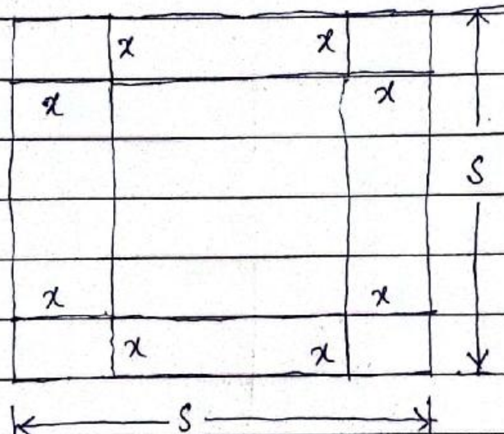


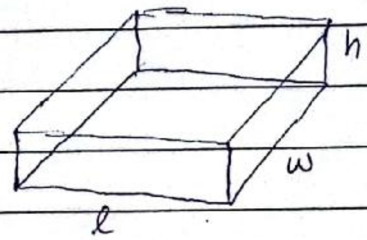
Lecture 20

Optimization Problems

1. Understand the problem (Read)
2. Draw!
3. Symbols, equations, functions (Pick, useful labels)
4. What quantity are you optimizing?
   → Find its function
5. Make it a function of one variable (Domain?)
6. Find Extrema.

ex.





$V = l \cdot w \cdot h$

$V = (S-2x)^2 x$

$h = x$

$0 \le x \le S/2$ $\quad l = S - 2x$ $\quad$ S is a constant

$w = S - 2x$

$$\boxed{V(x) = x(S-2x)^2 \qquad 0 \le x \le S/2, \qquad S \text{ is a constant}}$$

Yes, this has an abs max and abs min.

where is abs max? check endpoints + critical Numbers

$V'(x) = (1)(S-2x)^2 + (x)(2)(S-2x)(-2)$

$V'(x) = (S-2x)[(S-2x) - 4x]$

$V'(x) = (S-2x)(S-6x)$

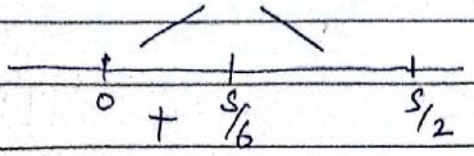
Critical Numbers: $V'(x) = 0$ $\qquad$ $V'(x)$ DNE Never

$S - 2x = 0, \quad S - 6x = 0$

$x = S/2, \quad x = S/6$

1st Derivative test

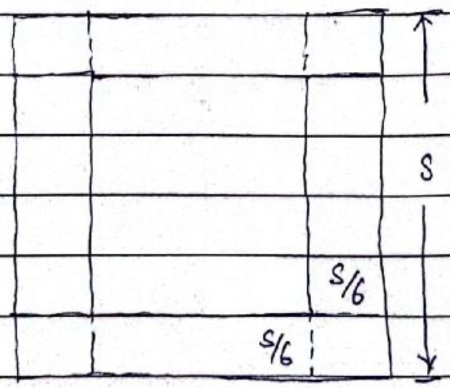
$V'(x)$

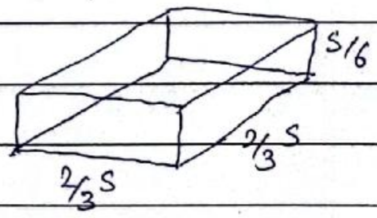


Rel/loc max
only one → abs max here

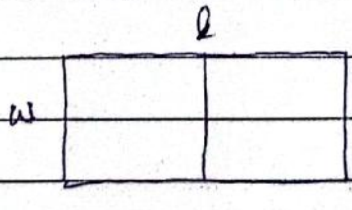
Abs max @ $x = S/6$
of $V = \frac{2}{27} S^3$

$$V = (S-2x)^2 x$$
$$V = \left(S - \frac{S}{3}\right)^2 \left(\frac{S}{6}\right)$$
$$V = \left(\frac{2}{3} S\right)^2 \left(\frac{S}{6}\right)$$
$$V = \frac{2}{27} S^3$$





(ex) Area of a rectangular field = $3\,ft^2$



Minimize cost of fence
Area = $l \cdot w$
Fence = $l + l + w + w + w$
Cost = (fixed Price) (Fence)

$$3ft^2 = l \cdot w \Rightarrow l = \frac{3ft^2}{w}$$

$$C = c(2l + 3w)$$
$$C(w) = c\left(\frac{6ft^2}{w} + 3w\right)$$

Minimize $C(w) = c\left(\frac{6ft^2}{w} + 3w\right)$, $0 < w < \infty$ Domain $(0, +\infty)$

Lecture 21

Optimization Problems

ex

$$Y = \frac{KN}{1+N^2}, \quad K \text{ is a positive constant}$$

$Y \propto \dfrac{N}{1+N^2}$

1. $N=0 \quad Y=0$
2. $\displaystyle\lim_{N\to+\infty} \frac{N}{1+N^2} = \lim_{N\to+\infty} \frac{N}{N^2\left(\frac{1}{N^2}+1\right)}$

$$= \lim_{N\to+\infty} \underbrace{\frac{1}{N}}_{=0} \; \underbrace{\frac{1}{\left(\frac{1}{N^2}+1\right)}}_{=1}$$

$$= 0 \cdot 1 = 0$$

Maximize Yield $\quad Y(N) = \dfrac{KN}{1+N^2} \quad$ Domain: $(0, +\infty)$

→ look for critical Numbers (rel. extrema)

$$Y'(N) = \frac{K(1+N^2) - KN(2N)}{(1+N^2)^2} = K\frac{(1-N^2)}{(1+N^2)^2}$$

$Y'(N) = 0$

$1 - N^2 = 0$

$N^2 = 1$

$N = \pm 1$

$\to N = +1$ b/c $\boxed{N=-1}$ doesn't apply to this problem

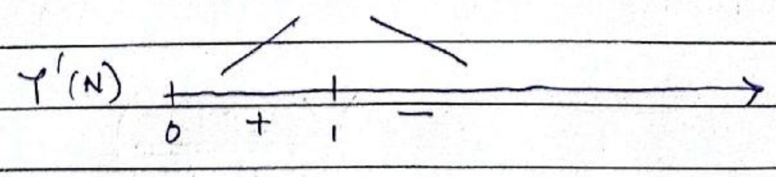
$Y'(N)$ DNE

$(1+N^2)^2 = 0$

$1+N^2 = 0$

$N^2 = -1$

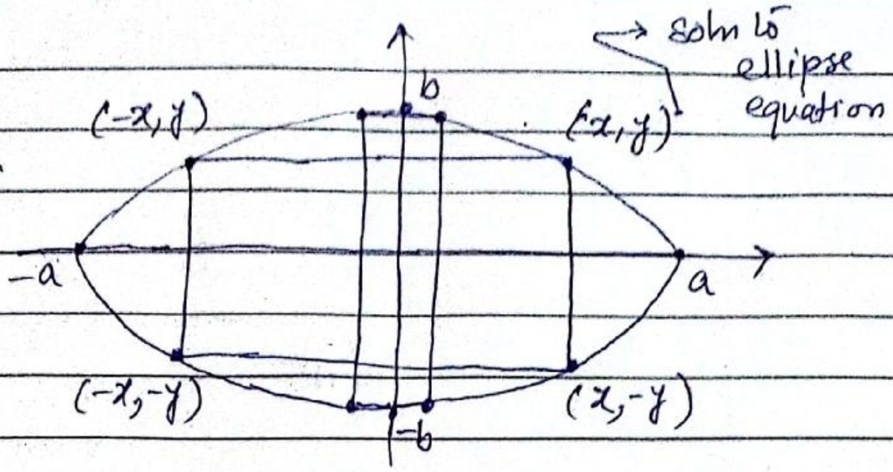
× $\boxed{N = \pm 2}$ Never



Rel max @ $N=1$ only one rel. extrema so abs max @ $N=1$ of $Y(1) = \frac{1}{2}K$

Date:

ex $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



Area of rectangle: $A = (2x)(2y) = 4xy$

→ Maximize area means find abs. max of A

but A has two variables x, y. use

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ to substitute for y.

$y^2 = \left(1 - \dfrac{x^2}{a^2}\right) b^2$ $\qquad \sqrt{b^2} = |b|$

$y = \sqrt{\left(1 - \dfrac{x^2}{a^2}\right) b^2} = b\sqrt{1 - \dfrac{x^2}{a^2}}$

$y = \dfrac{b}{a}\sqrt{a^2 - x^2}$

Make $A(x) = 4xy = \left(\dfrac{4b}{a} x\right)\left(\sqrt{a^2 - x^2}\right)$ Domain: $[0, a]$

critical numbers: $A'(x) = 0$ or $A'(x)$ DNE

$A'(x) = \dfrac{4b}{a}\sqrt{a^2 - x^2} - \left(\dfrac{4b}{a} x^2\right)\left(\dfrac{1}{\sqrt{a^2 - x^2}}\right)$

$= \dfrac{4b}{a}\left[\dfrac{\sqrt{a^2 - x^2}}{1} - \dfrac{x^2}{\sqrt{a^2 - x^2}}\right]$

$= \dfrac{4b}{a}\left[\dfrac{a^2 - x^2 - x^2}{\sqrt{a^2 - x^2}}\right] = \dfrac{4b}{a} \dfrac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}$

Date: ____________

$A'(x) = 0$

$a^2 - 2x^2 = 0$

$2x^2 = a^2$

$x^2 = \dfrac{a^2}{2}$

$x = \pm \dfrac{a}{\sqrt{2}}$

Not in Domain

$\boxed{x = \dfrac{a}{\sqrt{2}}}$

$A'(x)$ DNE

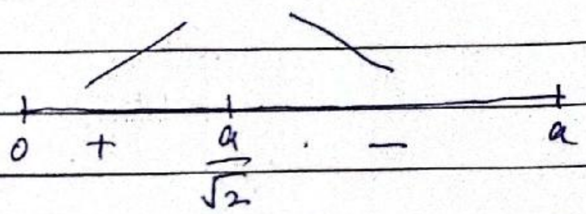
$a^2 - x^2 = 0$

$x^2 = \pm a$

Not In domain

$\boxed{x = a}$

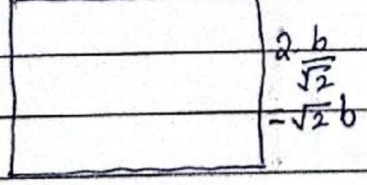
$A'(x) = \dfrac{4b}{a} \dfrac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}$

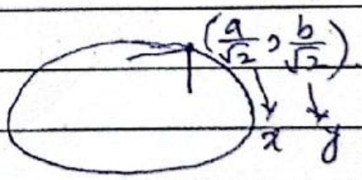


Rel/Loc max here only one

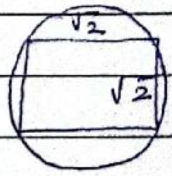
→ abs max @ $x = \dfrac{a}{\sqrt{2}}$

$\dfrac{2a}{\sqrt{2}} = \dfrac{2a}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}a$





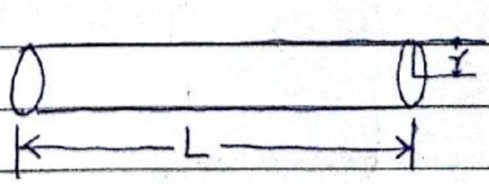
$a = b = 1$



$\boxed{A = 2ab}$ ← max area

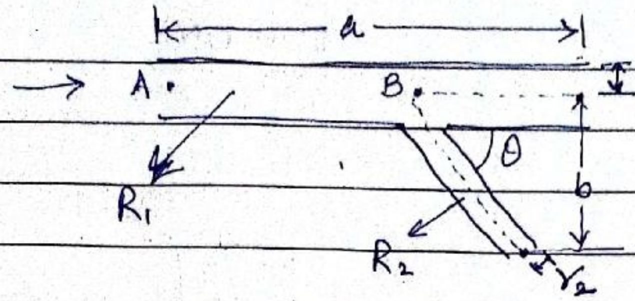
ex Poiseuille

$R = C \dfrac{L}{r^4}$



$R \propto L$ $\quad R \propto \dfrac{1}{r^4}$

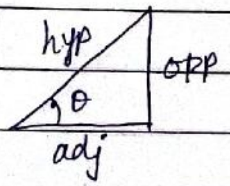
Date:



$$r_2 = \frac{r_1}{2}$$

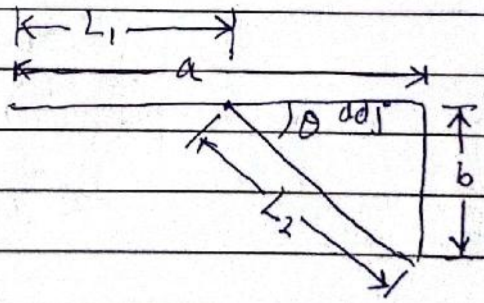
$$R = c\frac{L}{r^4} \rightarrow R = c\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\,\mathrm{cosec}\,\theta}{r_2^4}\right)$$

$$R = \underbrace{c\,\frac{a - b\cot\theta}{r_1^4}}_{R_1} + \underbrace{c\,\frac{b\csc\theta}{r_2^4}}_{R_2}$$



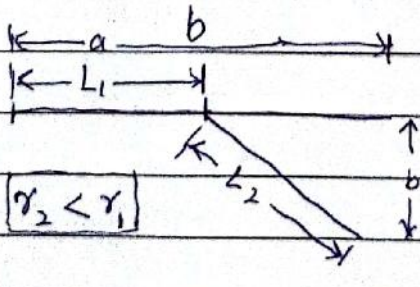
$\sin\theta = \frac{opp}{hyp}$, $\csc\theta = \frac{hyp}{opp} = \frac{1}{\sin\theta}$

$\tan\theta = \frac{opp}{adj}$, $\cot\theta = \frac{adj}{opp} = \frac{1}{\tan\theta}$



$$\mathrm{cosec}\,\theta = \frac{L_2}{b} \Rightarrow b\,\mathrm{cosec}\,\theta = L_2$$

$$\cot\theta = \frac{adj}{b} \Rightarrow adj = b\cot\theta \Rightarrow L_1 = a - b\cot\theta$$



$\boxed{r_2 < r_1}$

$$R = c\,\frac{L_1}{r_1^4} + c\,\frac{L_2}{r_2^4}$$

$L_1 = a - b\cot\theta$, $L_2 = b\csc\theta$

Date:__________

$\boxed{\text{Show}}$ min. resistance is when $\cos\theta = \left(\dfrac{r_2}{r_1}\right)^4$

$$R(\theta) = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right) = c\left[\frac{a}{r_1^4} - \frac{b}{r_1^4}\cot\theta + \frac{b}{r_2^4}\csc\theta\right]$$

Domain: $\theta \in (0, \pi/2)$

$\boxed{\text{critical Numbers}}$

$$R'(\theta) = c\left[0 + \frac{b}{r_1^4}\csc^2\theta - \frac{b}{r_2^4}\csc\theta\cot\theta\right]$$

$$= cb\csc\theta\left[\frac{1}{r_1^4}\csc\theta - \frac{1}{r_2^4}\cot\theta\right]$$

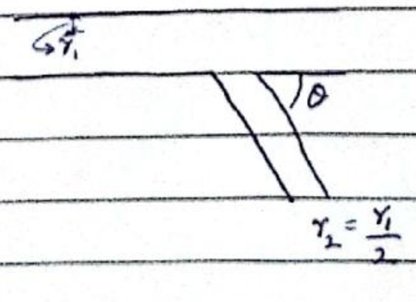
$R'(\theta) = 0$

$$\frac{1}{r_1^4}\frac{1}{\sin\theta} - \frac{1}{r_2^4}\frac{\cos\theta}{\sin\theta} = 0$$

$$\frac{1}{\sin\theta}\left[\frac{1}{r_1^4} - \frac{1}{r_2^4}\cos\theta\right) = 0$$

$$\frac{1}{r_2^4}\cos\theta = \frac{1}{r_1^4}$$

$$\cos\theta = \left(\frac{r_2}{r_1}\right)^4$$



$$\cos\theta = \left(\frac{\frac{1}{2}r_1}{r_1}\right)^4 = \frac{1}{16}$$

$$\theta = \cos^{-1}\left(\frac{1}{16}\right)$$

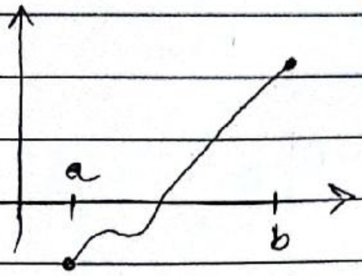
$$= 86^\circ$$

Date:

Lecture 22

Application: Newton's Method

Finding Roots



Bisection Method:

① $f(a)\cdot f(b) < 0$ — $f(a), f(b)$ are on opposite sides of x-axis.

② Let $c = \dfrac{a+b}{2}$

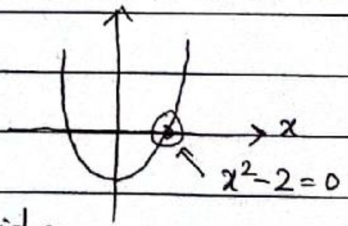
$\rightarrow$ check $f(a)\cdot f(c) < 0$? / $f(c)\cdot f(b) < 0$? — whichever is Yes $\rightarrow$ new $[a,b]$

Loop

Problem: slow

ex $\sqrt{2} \approx ?$

$y = f(x) = x^2 - 2$



$f(x) = x^2 - 2$

$f(1) = 1^2 - 2 = -1$
$f(2) = 2^2 - 2 = 2$ — opposite sides

$c = \dfrac{1+2}{2} = 1.5$

$f(1.5) = (1.5)^2 - 2 = 0.25$

between 1 and 1.5

$c = \dfrac{1+1.5}{2} = 1.25$

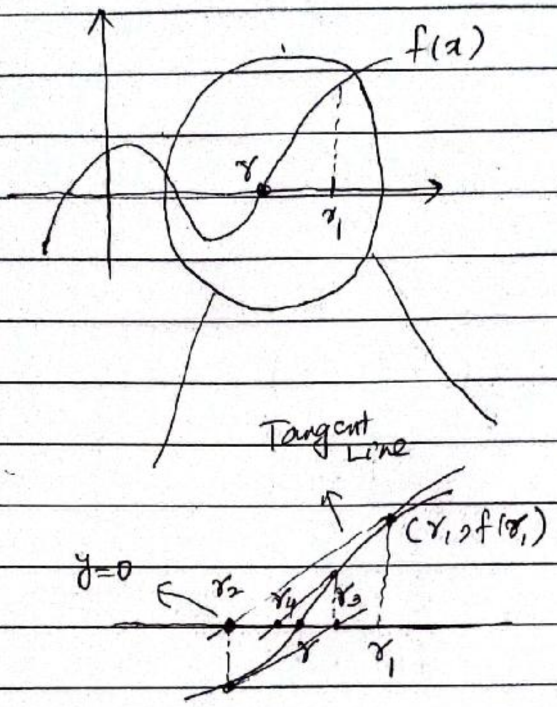
$f(1.25) = -0.4375$

between 1.25 and 1.5

⋮

$\boxed{\text{Faster way ?}}$ Newton's Method

$r_1 = 1^{st}$ guess at root r



Tangent line equation

$$y - f(r_1) = f'(r_1)(x - r_1)$$

Tangent line root

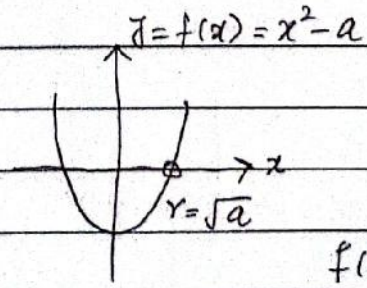
$$0 - f(r_1) = f'(r_1)(r_2 - r_1)$$

$$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)}$$

Newton's Method Algorithm

$$\boxed{\begin{array}{l} r_1 = \text{good guess} \\ r_{n+1} = r_n - \dfrac{f(r_n)}{f'(r_n)} \\ \lim\limits_{n\to+\infty} r_n = r \end{array}}$$

$$\lim_{n\to+\infty} r_n = r$$



ex $\sqrt{a} \simeq ?$

$$f(x) = x^2 - a \ , \ f'(x) = 2x$$

<u>Newton's Method</u> r_1 = good guess

$$r_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)} \Rightarrow r_{n+1} = r_n - \frac{r_n^2 - a}{2r_n}$$

$$r_{n+1} = r_n - \left(\frac{r_n^2}{2r_n} - \frac{a}{2r_n}\right)$$

$$r_{n+1} = r_n - \frac{1}{2}r_n + \frac{a}{2r_n} \Rightarrow r_{n+1} = \frac{1}{2}r_n + \frac{a}{2r_n} \Rightarrow r_{n+1} = \frac{1}{2}\left(r_n + \frac{a}{r_n}\right)$$

$\sqrt{a}$ Algorithm $\quad r_1 = \text{good guess}$

$$r_{n+1} = \frac{1}{2}\left(r_n + \frac{a}{r_n}\right)$$

so $\sqrt{2}$ $\quad r_1 = 1$

$$r_{n+1} = \frac{1}{2}\left(r_n + \frac{2}{r_n}\right)$$

so $\sqrt{3}$ $\quad r_1 = 1$

$$r_{n+1} = \frac{1}{2}\left(r_n + \frac{3}{r_n}\right)$$

Newton's Method $\quad r_1 = \text{good ones} \leftarrow$ stay away from $f'(x) = 0$

$$r_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)}$$

ex) Rs 18000 → Now

Later → 60 payments of Rs 375 (Monthly) (Rs. 22500)

↑ 5 years

$$\text{Present Value} = \text{Payment}\left[\frac{1-(1+r)^{-n}}{r}\right]$$

n = number of payments.

r = interest rate

$$18000 = 375\left[\frac{1-(1+r)^{-60}}{r}\right]$$

$$48 = \frac{1-(1+r)^{-60}}{r}$$

$$48r = 1-(1+r)^{-60}$$

$$48r(1+r)^{60} = (1+r)^{60} - 1$$

$$48r(1+r)^{60} - (1+r)^{60} + 1 = 0 \quad \text{Find root!}$$

Newton's Method

$$f(r) = 48r(1-r)^{60} - (1+r)^{60} + 1$$

$$f'(r) = 12(1+r)^{59}(244r - 1)$$

$$r_1 = 0{\cdot}01$$

$$r_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)}$$

$$r_{n+1} = r_n - \frac{48r_n(1+r_n)^{60} - (1+r_n)^{60} + 1}{12(1+r_n)^{59}(244r_n - 1)}$$

Antiderivatives

$$D_x[f(x)] = f'(x)$$

How to undo derivatives? $\boxed{\text{Know derivatives very well}}$

$\boxed{\text{Def}}$ $F(x)$ is an antiderivative of $f(x)$ on an interval I if $D_x[F(x)] = f(x)$ for all $x \in I$.

ex $\frac{1}{3}x^3 + \pi$ is an antiderivative of x^2 b/c

$$D_x\left[\frac{1}{3}x^3 + \pi\right] = x^2 + 0 = x^2$$

$\boxed{\text{Th}^m}$ If $F(x)$ is an antiderivative of $f(x)$ then $F(x) + c$, c is any arbitrary constant, is the most general antiderivative.

Find antiderivative? Know $D_x[F(x) + c] = f(x)$

Rewrite $A_x[f(x)] = F(x) + c$

b/c
1. $D_x[\sin(x)] = \cos(x)$

   $A_x[\cos(x)] = \sin x + c$
2. $D_x[\tan x] = \sec^2 x$

   $A_x[\sec^2 x] = \tan x + c$
3. $D_x[x^n] = nx^{n-1}$

   $A_x[nx^{n-1}] = x^n + c$

Date:

$$A_x[3x^2] = x^3 + c$$

b/c $D_x[x^3+c] = 3x^2 = 3x^{3-1}$

power in front, take 1 from power is equal to new power

$$D_x[x^3] = 3x^2$$

$$A_x[3x^2] = \frac{3}{3}x^{2+1} = x^3 + c$$

$$A_x[x^n] = \frac{1}{n+1}x^{n+1} + c$$

b/c $D_x\left[\frac{1}{n+1}x^{n+1} + c\right] = x^n$

$$A_x[\cos x - \csc x \cot x + x^3]$$

$$= \sin x + \csc x + \frac{1}{4}x^4 + c$$

Lecture 23

Antiderivatives

(1) $A_x[\cos x] = \sin x + c$

b/c $D_x[\sin x + c] = \cos x$

(2) $A_x[\sin x] = -\cos x + c$

b/c $D_x[-\cos x + c] = \sin x$

(3) $A_x[\sec^2 x] = \tan x + c$

b/c $D_x[\tan x + c] = \sec^2 x$

(4) $A_x[\sec x \tan x] = \sec x + c$

b/c $D_x[\sec x + c] = \sec x \tan x$

(5) $A_x[\csc^2 x] = -\cot x + c$

b/c $D_x[-\cot x + c] = \csc^2 x$

(6) $A_x[\csc x \cot x] = -\csc x + c$

b/c $D_x[-\csc x + c] = \csc x \cot x$

(7) $A_x[x^n] = \frac{1}{n+1} x^{n+1} + c$

$(n \neq -1)$ $D_x\left[\frac{1}{n+1} x^{n+1} + c\right] = x^n$

(8) $A_x[f(x) + g(x)] = A_x[f(x)] + A_x[g(x)]$

(9) $A_x[K f(x)] = K A_x[f(x)]$

(ex) $A_x[4x + 3x^2 - 2] = \cancel{4x\,x^2}\ 4A_x[x] + 3A_x[x^2] - 2A_x(x^0)$

$= 4\cdot\frac{1}{2}x^2 + 3\cdot\frac{1}{3}x^3 - 2x + c$

$+c_1 \quad +c_2 \quad +c_3 \quad c = c_1 + c_2 + c_3$

(ex) $A_t[8\sqrt{t} - \sec t \tan t] = 8A_t[t^{1/2}] - A_t[\sec t \tan t] = 8\cdot\frac{2}{3}t^{3/2} - \sec t + c$

$= \frac{16}{3}t^{3/2} - \sec t + c = \frac{16}{3}(\sqrt{t})^3 - \sec t + c$

Date:

D_t[Position function] = velocity function

D_t[velocity function] = acceleration function.

says: (1) A_t[acceleration function] = velocity func + $\boxed{c_1}$

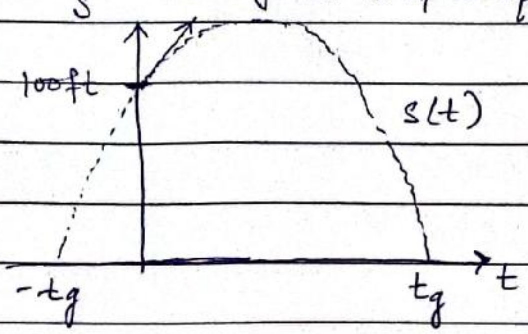
find this by 1 velocity measurement

(2) A_t[velocity function] = Position function + $\boxed{c_2}$

find this by 1 position measurement

velocity @ t=0 of 50mph

$\boxed{\text{ex}}$



ignore air resistance

acceleration of gravity

$\bigcirc \downarrow$ gravity $\sim -32 \frac{ft/sec}{sec}$

Force = gravity = m · acc

$$a(t) = -32$$

$$v(t) + c = A_t[\text{accel}] = A_t[-32]$$

$$v(t) = -32t + c_1$$

$$v(0) = c_1$$

$$V_0 = 73 \text{ ft/sec} = c_1$$

$t=0$ $V_0 = 50 \frac{\text{miles}}{\text{hr}} \cdot \frac{1\text{hr}}{3600\text{sec}} \cdot \frac{5280\text{ft}}{1\text{mile}}$

$V_0 \approx 73$ ft/sec

$\boxed{v(t) = -32t + 73}$ ← ft/sec

$\boxed{\text{Now}}$ position = A_t[velocity]

$$s(t) = A_t[-32t + 73] = -16t^2 + 73t + c_2$$

@ $t=0$ $S_0 = 100$ft $\quad s(0) = S_0 = 100 = c_2$

$$\boxed{s(t) = -16t^2 + 73t + 100}$$

When is $s = 0$? (on ground)

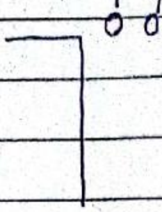
$$0 = -16t^2 + 73t + 100 \rightarrow t = \frac{-73 \pm \sqrt{73^2 + 4 \times 16 \times 100}}{-32}$$

Date:

at $t=0$, $V_0 = 48$ ft/sec

$t=1$, $V_0 = 24$ ft/sec

$V_0 =$ initial velocity

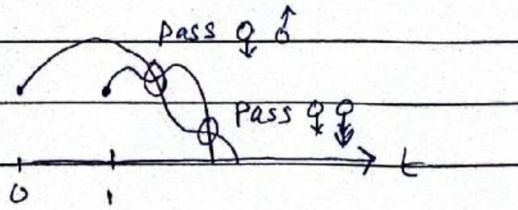


do the balls $\underbrace{\text{pass each}}$ other?

OO] same height @ same time?

Velocity?

$$S(t) = -\frac{1}{2}gt^2 + V_0 t + S_0$$



$$S_1(t) = -16t^2 + 48t$$

$$S_2(t) = -16(t-1)^2 + 24(t-1)$$

$$\left.\begin{aligned} S_1(t) &= -16t^2 + 48t \\ S_2(t) &= -16t^2 + 56t - 40 \end{aligned}\right] \quad t \geq 1$$

$$\cancel{-16t^2} + 48t = \cancel{-16t^2} + 56t - 40$$

$$8t = 40$$

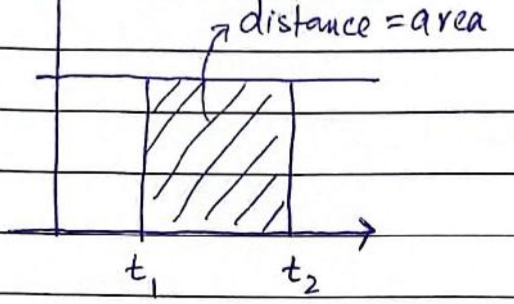
$$\boxed{t = 5 \text{ sec}}$$

Date:

Lecture 24

Integral Calculus

Area: ex distance = rate × time

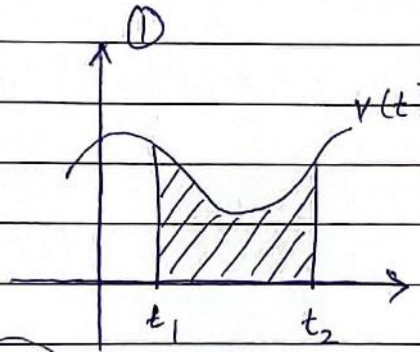


Antiderivatives

$A_x[f(x)] = F(x) + C$

when $D_x[F(x)] = f(x)$

(1) Area



Area = distance

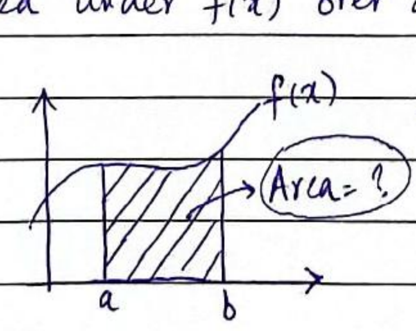
geometry: $\boxed{l}\, w$ $\quad A = lw$

r $\quad A = \pi r^2$

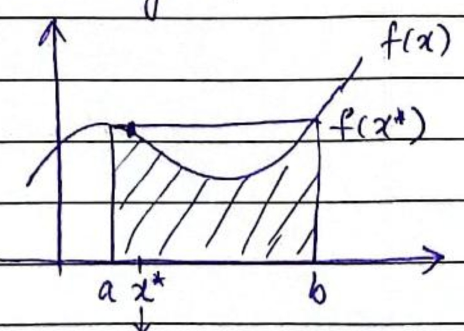
(2) Antiderivative

$\boxed{\text{Fundamental th}^m \text{ of Calculus}}$

(1) Area under $f(x)$ over an interval.



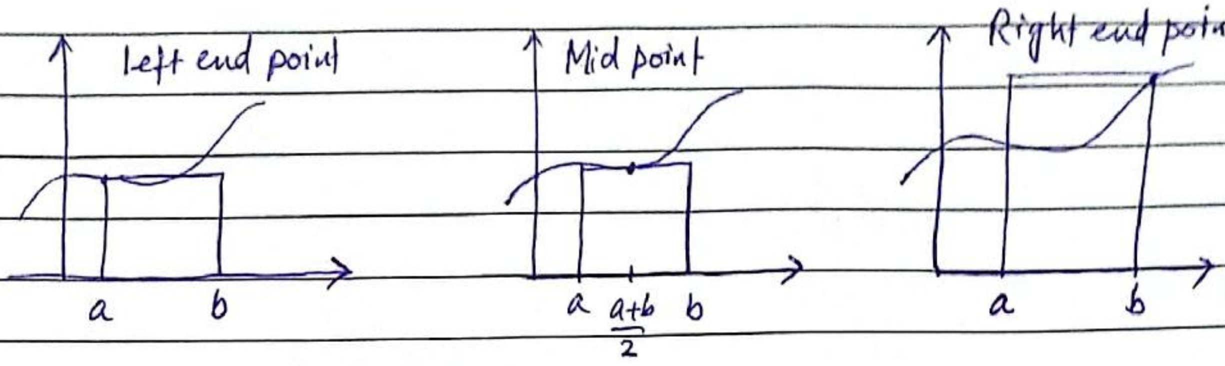
→ can we approximate using rectangles?



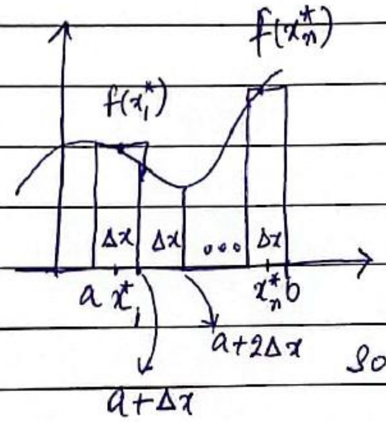
can be any x between a, b

Date: ________

Or take specific x^*



To make better approximates $\longrightarrow$ use more rectangles



want n-rectangles
use divide up $[a,b]$ n-times

$$\Delta x = \frac{b-a}{n}$$

$$A \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

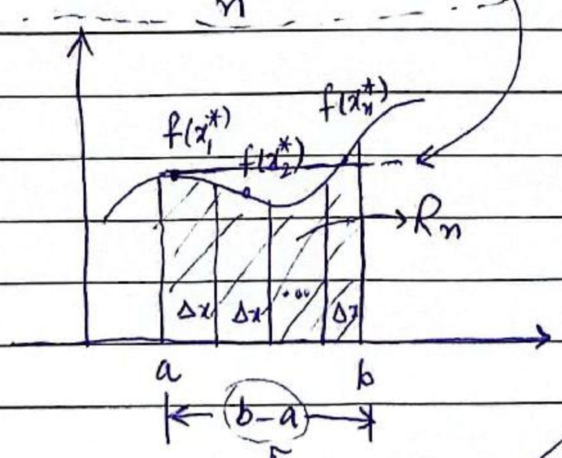
so $A \approx \underbrace{\left[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)\right] \Delta x}_{R_n}$

Now $\lim\limits_{n\to\infty} R_n = A$

$$R_n = \left[(f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)\right] \Delta x$$

b/c $\Delta x = \dfrac{b-a}{n}$

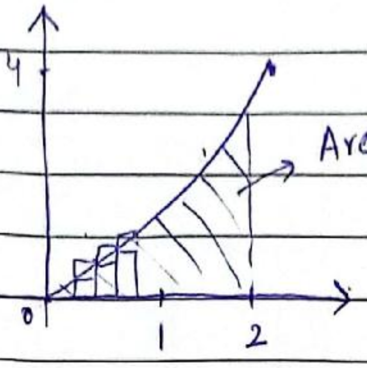
$$R_n = \frac{(f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*))}{n}(b-a)$$



Pick:
1. $x_i^* =$ left
2. $x_i^* =$ Right
3. $x_i^* =$ Midpoint

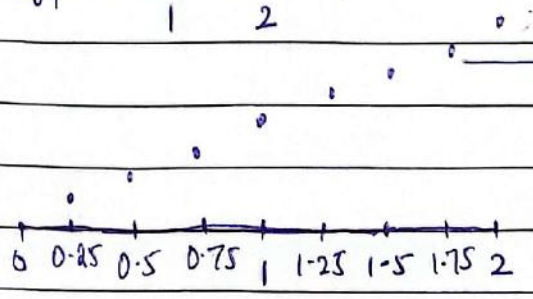
Date:

ex



Area under $f(x) = x^2$ over $[0, 2]$

$n = 8 \quad \Delta x = \frac{2-0}{8} = \frac{1}{4}$



$\longrightarrow f(1.75) = (1.75)^2$

$$A = \lim_{n\to\infty} \left[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*) \right] \Delta x$$

$\boxed{\text{New Notation}}$ sequences, Sums

sequence: $a_1, a_2, a_3, a_4, \ldots = \{a_i\}$

if $\quad a_n = f(n) \quad = \{f(n)\}$

So $f(x_1^*), f(x_2^*), \ldots \quad = \{f(x_i^*)\}$

Sum = add up a sequence

$$f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*) = \sum_{i=1}^{n} f(x_i^*)$$

ex $\displaystyle\sum_{i=1}^{10} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2$

ex $\displaystyle\sum_{i=1}^{100} i = 1 + 2 + 3 + \cdots + 100 = \frac{100(101)}{2} = 50(101) = 5050$

$1 + 2 + 3 + 4 + \cdots + 98 + 99 + 100$

$\underline{100 + 99 + 98 + 97 \cdots + 3 + 2 + 1}$

$101 + 101 + 101 + 101 + \cdots + 101 + 101 + 101 = 100(101)$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Area under $f(x)$ over $[a, b]$

① $\Delta x = \frac{b-a}{n}$

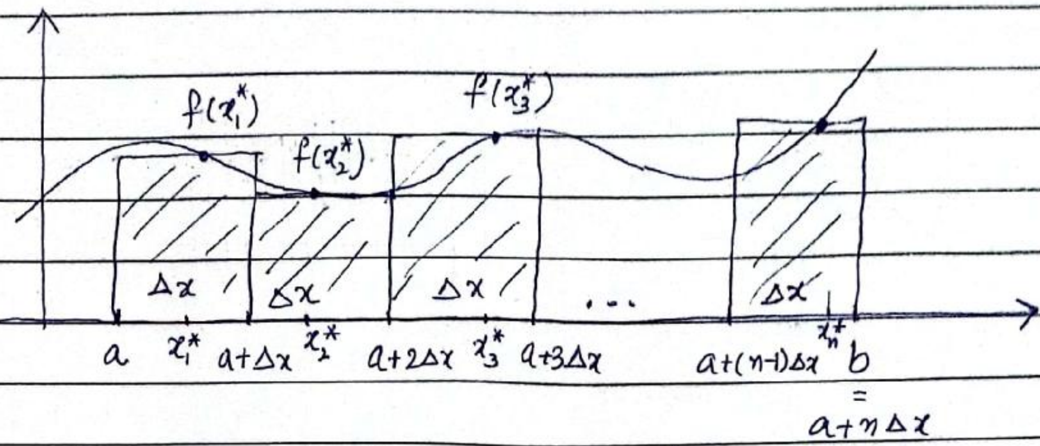
② $\text{Area} = \displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$

Date:

Lecture 25

Integral Calculus

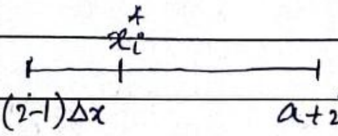
n- intervals: $\Delta x = \dfrac{b-a}{n}$



Area $\simeq f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$

Area $\simeq \left(\dfrac{f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)}{n}\right)(b-a)$

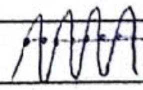
x_i^* are $\boxed{\text{any}}$ value in their interval.



Left end point approx: $x_i^* =$ left end point

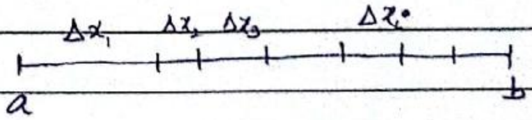
Right end point approx: $x_i^* =$ right end point

Mid point approx : $x_i^* = a + (i - \frac{1}{2})\Delta x$



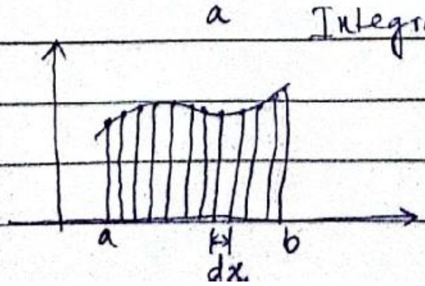
$$\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$\text{Area} = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$



Leibniz Notation $\displaystyle\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$

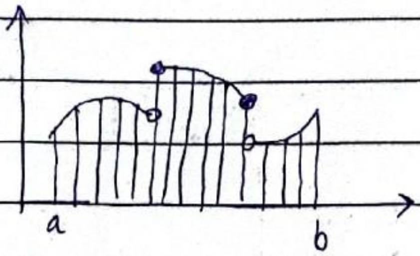
Integral, Definite Integral

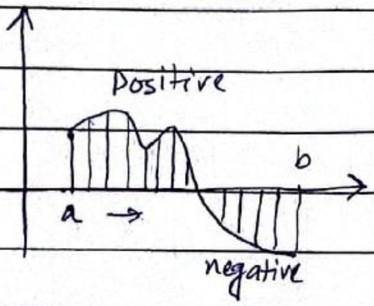


(Integrable)

$\boxed{Th^m}$ If f is continuous on $[a,b]$ or has at most a finite number of jump discontinuites then

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i \quad \underline{\text{EXISTS}}$$





$$\int_a^b f(x)\,dx = \text{Net signed area}$$

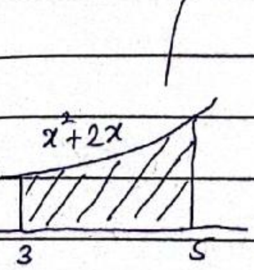
or

Net area

$\boxed{Th^m}$ If f is integrable then $\Delta x = \dfrac{b-a}{n}$

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

<u>Notations</u>



$$= \boxed{\int_3^5 (x^2+2x)\,dx} = \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$\lim_{n\to+\infty} \sum_{i=1}^{n} (x_i^2+2x_i)\,\Delta x$$

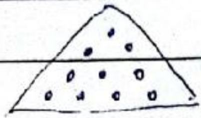
$$\Delta x = \frac{5-3}{n}$$

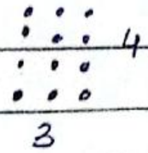
Date:

Sums to Know

① $\sum_{i=1}^{n} c = \underbrace{c + c + c + \cdots + c}_{n} = n.c$

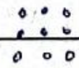
② $\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

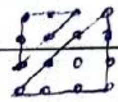


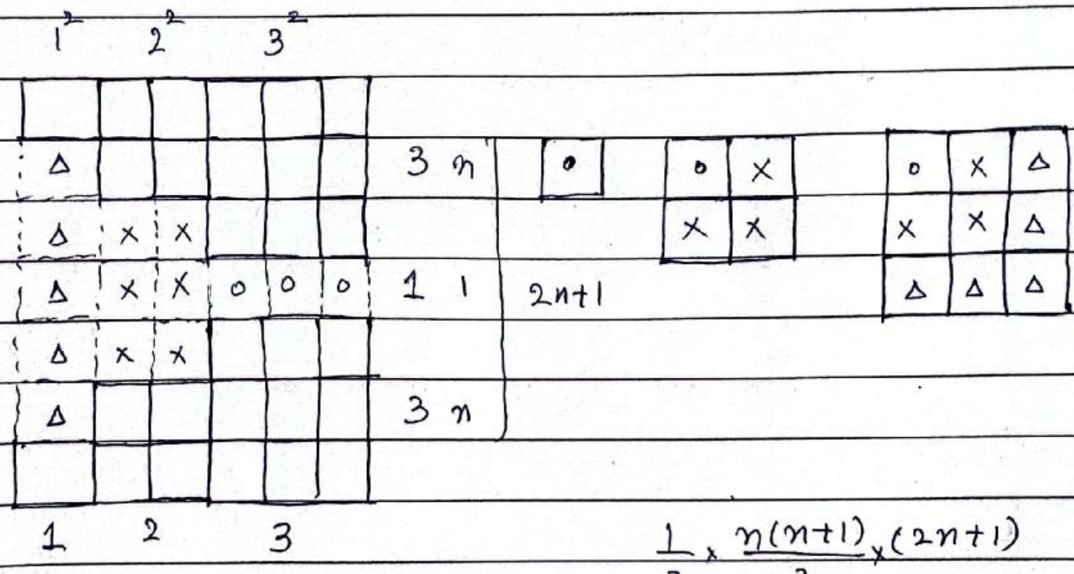


$\frac{4(3)}{2} = 6$

③ $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$







$\frac{1}{3} \times \frac{n(n+1)}{2} \times (2n+1)$

$1 + 2 + 3 + \cdots n = \frac{n(n+1)}{2}$

④ $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{(n+1)n}{2}\right)^2$

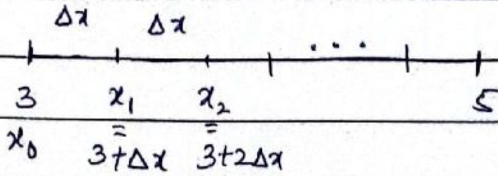
⑤ $\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$

⑥ $\sum_{i=1}^{n} a_i \pm b_i = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Date:

ex $\int_3^{5} (x^2+2x)\,dx = \lim_{n\to+\infty} \sum_{i=1}^{n} (x_i^2+2x_i)\,\Delta x$

$\Delta x = \frac{5-3}{n} = \frac{2}{n}$



Right end points: $x_i = 3 + i\Delta x = 3 + \frac{2}{n} i$

$$\int_3^5 (x^2+2x)\,dx = \lim_{n\to+\infty} \frac{2}{n} \sum_{i=1}^{n} \left(3+\frac{2}{n}i\right)^2 + 2\left(3+\frac{2}{n}i\right)$$

$$= \lim_{n\to+\infty} \frac{2}{n} \sum_{i=1}^{n} \left(9+\frac{12}{n}i+\frac{4}{n^2}i^2\right) + \left(6+\frac{4}{n}i\right)$$

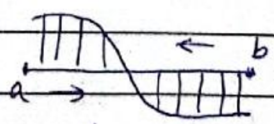
$$= \lim_{n\to+\infty} \frac{2}{n} \sum_{i=1}^{n} \left(15+\frac{16}{n}i+\frac{4}{n^2}i^2\right)$$

$$= \lim_{n\to+\infty} \frac{2}{n} \left[15n + \frac{16}{n}\frac{n(n+1)}{2} + \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to+\infty} \left[30 + 16\frac{n(n+1)}{n^2} + \frac{4}{3}\frac{n(n+1)(2n+1)}{n^3}\right]$$

$$= 30 + 16\cdot 1 + \frac{4}{3}\cdot 2 = 46 + \frac{8}{3} = 48 + \frac{2}{3} = 48.\bar{6}$$

$\boxed{\text{Properties}}$ $\int_a^b f(x)\,dx$



① $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

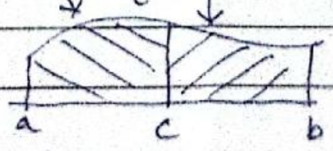
④ $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

② $\int_a^a f(x)\,dx = 0$

⑤ $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$

③ $\int_a^b c\,dx = c(b-a)$

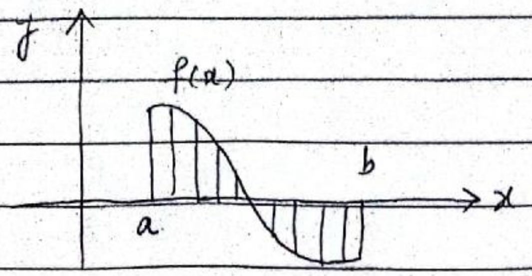
⑥ $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

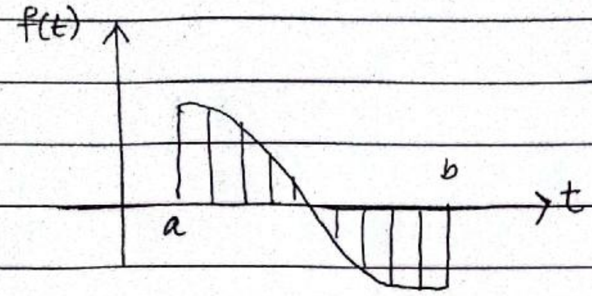


Lecture 26

Area: Net signed area

$$\text{Area} = \int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$





$$\int_a^b f(t)\,dt = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^{n} f(t_i^*)\Delta t_i$$

Fundamental Theorem of Calculus

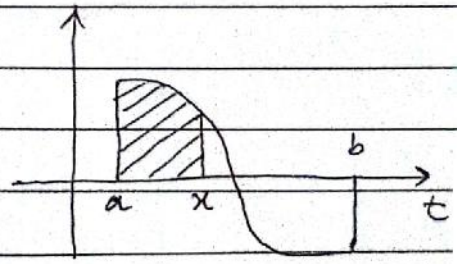
Parts: ① do an area problem → antiderivatives

② do an antiderivative problem → Area

Part 1 Net Signed Area Accumulator

f is continuous on $[a,b]$

$$A(x) = \underbrace{\int_a^x f(t)\,dt}_{\text{Area under } f \text{ over } [a,x]}$$



Properties of $A(x)$? $\quad A(\square) = \int_a^{\square} f(t)\,dt$

① $A(a) = \int_a^a f(t)\,dt = 0$

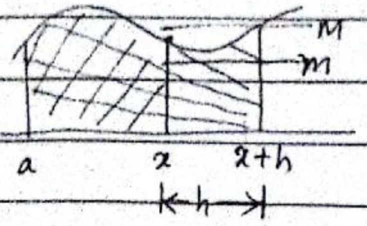
② $A(x)$ is a function... what is $D_x[A(x)] = ?$

Date:

$$D_x[A(x)] = D_x\left[\int_a^x f(t)\,dt\right] = ?$$

$$D_x[A(x)] = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

so $$D_x[A(x)] = \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$



$$= \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

$$m \le f(t) \le M$$

$$mh \le \int_x^{x+h} f(t)\,dt \le Mh$$

$$m \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le M$$

as $h \to 0$ $m \to f(x)$

$h \to 0$ $M \to f(x)$

so $$D_x[A(x)] = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} = f(x)$$

b/c $D_x[A(x)] = f(x)$

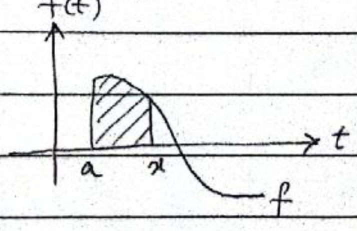
says $A(x)$ is an antiderivative of $f(x)$

$\boxed{\text{Fundamental Th}^m \text{ of Calculus part 1}}$

f is continuous on $[a,b]$ Let $A(x) = \int_a^x f(t)\,dt$

① $A(x)$ is an antiderivative of $f(x)$ b/c $D_x[A(x)] = f(x)$

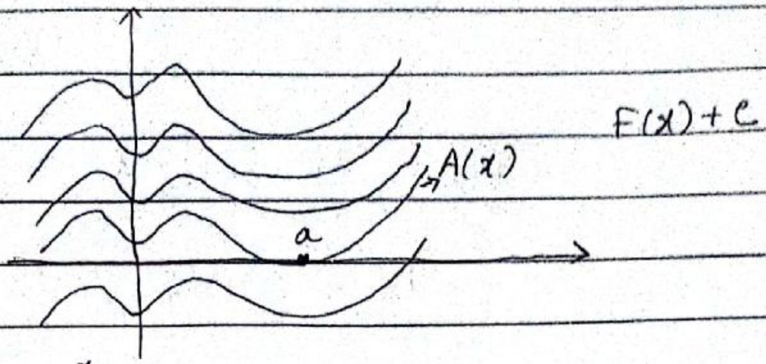
② $A(a) = 0$



Date:

How does $A(x)$ relate to $A_x[f(x)]$?

ex $\quad A_x[f(x)] = F(x) + c$



Use? $\quad A(x) = \int_a^x f(t)\,dt$ is a way to "find" antiderivatives that can't be done by normal ways.

→ $f(x) = \sin(x), \quad A_x[\sin x] = -\cos x + c$

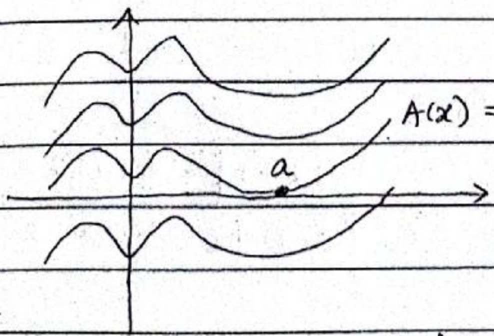
$f(x) = e^{-x^2}, \quad A_x[e^{-x^2}] = ?$

$A(x) = \int_0^x e^{-t^2}\,dt$ is an antiderivative of e^{-x^2}

$A(4) = \int_0^4 e^{-t^2}\,dt$

$\boxed{\text{Part 2}}$ ex $A_x[f(x)] = F(x) + c$

F(x)+c



$A(x) = \int_a^x f(t)\,dt$

says

$F(x) = A(x) + c$

Consider: $F(b) - F(a)$

$= (A(b) + c) - (A(a) + c)$

$= A(b) - A(a) = \int_a^b f(t)\,dt - \int_a^a f(t)\,dt$

$= \int_a^b f(t)\,dt =$ area under f over $[a, b]$

Date:

$\boxed{\text{Fundamental Th}^m \text{ of Calculus part 2}}$

f is cont on $[a,b]$

$$A_x[f(x)] = F(x) \qquad \text{Let } c=0$$

then $$\int_a^b f(t)\,dt = F(b) - F(a)$$

ex) Area under $f(x) = x^3$ over $[0,1]$

$$\int_0^1 x^3\,dx = \frac{1}{4}(1)^4 - \frac{1}{4}(0)^4 = \frac{1}{4}$$

$$A_x[x^3] = \frac{1}{4}x^4 + c$$

$\boxed{\text{Fundamental Th}^m \text{ of Calculus}}$ $f(x)$ is continuous on $[a,b]$

① (Use areas to create antiderivatives.)

$$A(x) = \int_a^x f(t)\,dt \quad \text{then} \quad \text{a) } D_x[A(x)] = f(x)$$
$$\text{b) } A(a) = 0$$

② (Use antiderivative to find areas)

$$A_x[f(x)] = F(x) + c \qquad \text{Let } (c=0)$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

using part 2 $\quad \int_a^b f(x)\,dx = \text{Area under } f \text{ over } [a,b]$

$$\int_a^b f(x)\,dx = ? \qquad \text{① } A_x[f(x)] = F(x) + c$$
$$\text{② } \int_a^b f(x)\,dx = F(b) - F(a)$$

<u>Notation</u> ① Indefinite Integral Notation:

$$A_x[f(x)] = \int f(x)\,dx = F(x) + c$$

<u>Now:</u> $$\int_a^b f(x)\,dx = \left[\underbrace{\int f(x)\,dx}_{A_x[f(x)]}\right]\Bigg|_{x=a}^{x=b} = \left[F(x)\right]\Bigg|_{x=a}^{x=b} = F(b) - F(a)$$

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ex $\int_1^3 (x^{-2/3}+1)\,dx = \left[\int (x^{-2/3}+1)\,dx\right]\Big|_{x=1}^{x=3}$

$= \left[3x^{1/3} + x\right]\Big|_{x=1}^{x=3} = \left(3(3)^{1/3} + 3\right) - \left(3(1)^{1/3} + 1\right)$

$= 3^{4/3} - 1$

Lecture 27

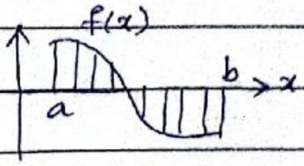
Integral Calculus

1. Antiderivatives (By Knowing Derivative Rules...)

Notation a) $A_x[f(x)] = F(x) + C$

b) $\int f(x)\,dx = F(x) + C$ (Indefinite Integral)

2. Areas (Net Signed Areas)



Two ways to find areas...

a) $$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

b) If you can find $f(x)$'s antiderivative $F(x)$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Antiderivative / Indefinite Integral notation

Basis of all our Integration rules

$$\int f(x)\,dx = F(x) + C$$

because

$$D_x[F(x) + C] = f(x)$$

Date:

Table

1. $\int c f(x)\,dx = c\int f(x)\,dx$

2. $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

3. $\int K\,dx = Kx + c$

4. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$

5. $\int \sin x\,dx = -\cos x + c$

6. $\int \cos x\,dx = \sin x + c$

7. $\int \sec^2 x\,dx = \tan x + c$

8. $\int \csc^2 x\,dx = -\cot x + c$

9. $\int \sec x \tan x\,dx = \sec x + c$

10. $\int \csc x \cot x\,dx = -\csc x + c$

How to use this table?

1. $\int f(x)\,dx = ?$ → a) is $f(x)$ in the table?

(no)

→ (b) it actually is in the table, you need to do Algebra/ Trig/ Arithmetic first.

ex $\int (x+3)\,dx = \frac{1}{2}x^2 + 3x + c$

ex $\int x(x+1)\,dx = \int (x^2 + x)\,dx = \frac{x^3}{3} + \frac{x^2}{2} + c$

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(2) $\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]\Big|_{x=a}^{x=b}$

use table here

(ex) $\int \frac{1+\sqrt{x}+x}{\sqrt{x}}\,dx = \int\left(\frac{1}{\sqrt{x}}+\frac{\sqrt{x}}{\sqrt{x}}+\frac{x}{\sqrt{x}}\right)dx$

$= \int\left(x^{-1/2}+1+x^{1/2}\right)dx = 2x^{1/2}+x+\frac{2}{3}x^{3/2}+c$

check $D_x\left[2x^{1/2}+x+\frac{2}{3}x^{3/2}+c\right] = x^{-1/2}+1+x^{1/2}$

(ex) $\int_0^{\pi/4}\left(\frac{1+\cos^2\theta}{\cos^2\theta}\right)d\theta = \left[\int\left(\frac{1+\cos^2\theta}{\cos^2\theta}\right)d\theta\right]\Big|_{\theta=0}^{\theta=\pi/4}$

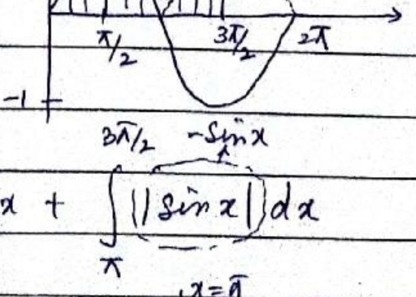
$= \int_0^{\pi/4}\left(\frac{1}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}\right)d\theta = \int_0^{\pi/4}\left(\frac{1}{\cos^2\theta}+1\right)d\theta$ but $\frac{1}{\cos^2\theta} = \sec^2\theta$

$= \int_0^{\pi/4}(\sec^2\theta+1)\,d\theta = \left[\tan\theta+\theta\right]\Big|_{\theta=0}^{\theta=\pi/4}$

$= \left(\tan\frac{\pi}{4}+\frac{\pi}{4}\right)-\left(\tan(0)+0\right) = \left(1+\frac{\pi}{4}\right)-(0+0) = 1+\frac{\pi}{4}$

(ex) $\int_0^{3\pi/2} |\sin x|\,dx$

sin x vs |sin x|



$\int_0^{3\pi/2}|\sin x|\,dx = \int_0^{\pi}\underbrace{|\sin x|}_{\sin x}\,dx + \int_{\pi}^{3\pi/2}\underbrace{|\sin x|}_{-\sin x}\,dx$

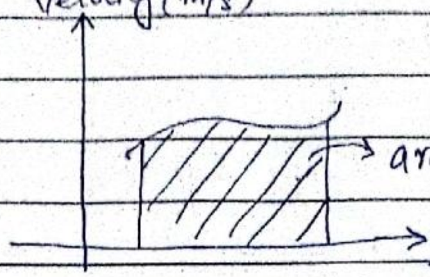
$= \int_0^{\pi}\sin x\,dx - \int_{\pi}^{3\pi/2}\sin x\,dx = \left[-\cos x\right]\Big|_{x=0}^{x=\pi} - \left[-\cos x\right]\Big|_{x=\pi}^{x=3\pi/2}$

$= \left[(-\cos(\pi))-(-\cos(0))\right] + \left[(\cos(\tfrac{3\pi}{2}))-\cos(\pi)\right] = [1+1]+[0+1] = 3$

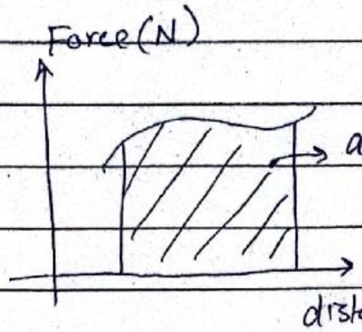
Date:

Applications | Why are Net Signed Areas $\int_a^b f(x)\,dx$ interesting?

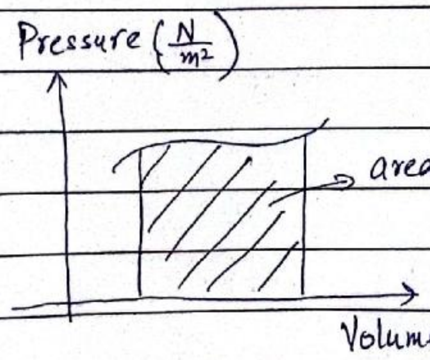
(ex)

① 

area $= \left(\frac{m}{s}\right)(s) =$ meters (Distance

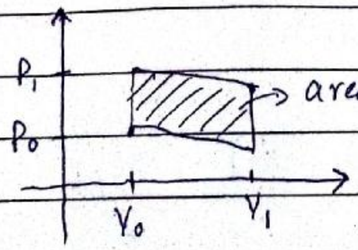
② 

area $= N \cdot m =$ Work

③ 

area $= \left(\frac{N}{m^2} \cdot m^3\right) = N \cdot m =$ Work

PV graph



area = Work   Stirling Cycle

<u>Net change Theorem</u>

$$D_x[F(x)] = F'(x)$$

$F(x)$

$$\int_a^b F'(x)\,dx = \underbrace{F(b) - F(a)}_{\text{Net change of } F(x)}$$

$F'(x)$: ~~Rate of~~ change of $F(x)$

Area under the rate of change of F(x)

Date:

(ex) $V(t)$ is volume of water in a pool @ time t
$V'(t)$ is rate at which water is draining from pool.

$$\underbrace{\int_{t_1}^{t_2} V'(t)\,dt}_{\substack{\text{~~Area under~~} \\ \text{the flow } (V') \text{ of} \\ \text{water over } [t_1, t_2]}} = \underbrace{V(t_2) - V(t_1)}_{\text{Net change in volume of water.}}$$

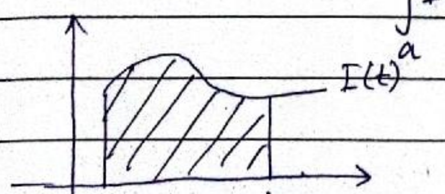
$s(t)$: position
$v(t) = s'(t)$: velocity
$a(t) = v'(t) = s''(t)$: acceleration

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

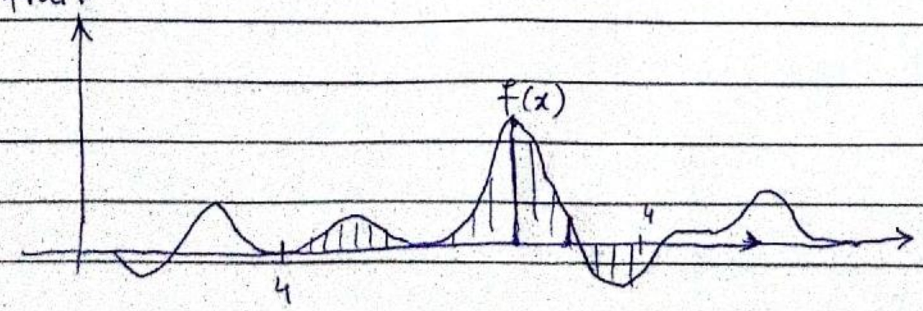
$$\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$$

(ex) $D_t[\underset{\substack{\uparrow \\ \text{charge}}}{Q(t)}] = I(t) \leftarrow$ current

$$\int_a^b I(t)\,dt = \underbrace{Q(t_2) - Q(t_1)}_{\text{Net change in charge from } t_1 \text{ to } t_2}$$



(ex) $f(x)$ is slope of a walking trail x miles from the start of the trail.



Date:

$\int_1^4 f(x)\,dx = ?$

$A_x[f(x)] = F(x) \leftarrow$ altitude or height

$\uparrow$

slope $\dfrac{\Delta \text{height}}{\Delta \text{distance}} = \dfrac{\text{altitude change}}{\text{walk length}}$

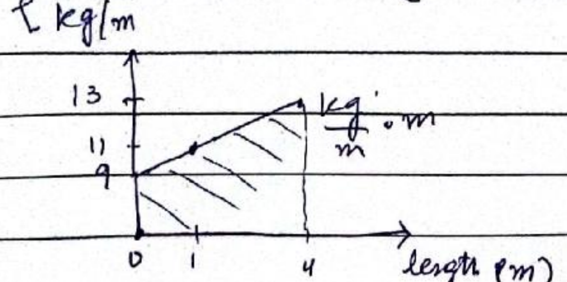
From 1 mile

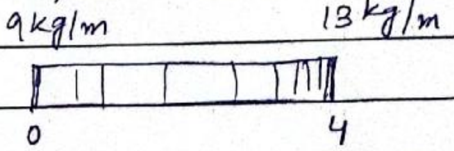
$$\int_1^4 f(x)\,dx = F(4) - F(1)$$

Net change in altitude or height from mile 1 to mile 4

Ex Linear Density of a metal rod 4 m long is

$\rho(x) = 9 + 2\sqrt{x}$ kg/meter.





total kg?

$$\int_0^4 (9 + 2x^{1/2})\,dx = 9x + \frac{4}{3}x^{3/2} \Big|_{x=0}^{x=4}$$

$$= \left(9(4) + \frac{4}{3}(4)^{3/2}\right) - (0)$$

$$= 36 + 10 + \frac{2}{3} = 46 + \frac{2}{3} \text{ kg}$$

Date:

Lecture 28

Antiderivatives

1. Table of Known derivatives → Antiderivatives
2. Algebra/Trig

$$D_x\left[f(g(x))\right] = f'(g(x))\, g'(x).$$

ex $\quad D_x\left[\sin(x^2+1)\right] = \cos(x^2+1)(2x)$

ex $\quad D_x\left[(x^2+x)^{1/2}\right] = \frac{1}{2}(x^2+x)^{-1/2}(2x+1)$

$\boxed{1^{st}\text{ example}}$ $\quad \int \cos(x^2+1)\cdot 2x\, dx = \sin(x^2+1) + c$

$\boxed{2^{nd}\text{ example}}$ $\quad \int \frac{1}{2}(x^2+x)^{-1/2}(2x+1)\, dx = \sqrt{x^2+x} + c$

Chain Rule $\quad D_x\left[f(g(x))\right] = f'(g(x))\cdot g'(x)$

$$\int f'(g(x))\cdot g'(x)\, dx = f(g(x)) + c$$

$\boxed{\text{Tech: Substitution Method}}$

$\int f(g(x))\, g'(x)\, dx \qquad$ Let $u = g(x)$

$\qquad du = g'(x)\, dx$

$\int f(u)\, du = F(u) + C$

$\qquad = F(g(x)) + c$

where F is antiderivative of f

Ex $\quad \int \cos(x^2+x)(2x+1)\, dx \qquad$ Let $u = x^2+x$

$\qquad du = (2x+1)\, dx$

$\int \cos(u)\, du = \sin(u) + c$

$\qquad = \sin(x^2+x) + c$

(ex) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \sin\sqrt{x} \cdot \frac{1}{\sqrt{x}}\,dx$

Let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

$2\int \sin(u)\,du = -2\cos(u) + c = -2\cos(\sqrt{x}) + c$

(ex) $\int x\sqrt{x+2}\,dx$ Let $u = x+2 \Rightarrow x = u-2$

$du = dx$

$= \int (u-2)\sqrt{u}\,du = \int (u-2)(u)^{1/2}\,du = \int \left(u^{3/2} - 2u^{1/2}\right)du$

$= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + c = \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + c$

(ex) $\int x^2\sqrt{x+2}\,dx$ Let $u = x+2 \Rightarrow x = u-2$

$du = dx$

$= \int (u-2)^2\sqrt{u}\,du = \int \left(u^2 - 4u + 4\right)u^{1/2}\,du \;\ldots$

(ex) $\int_0^1 (3t-1)^{50}\,dt = \left[\int (3t-1)^{50}\,dt\right]\Bigg|_{t=0}^{t=1}$

$\int (3t-1)^{50}\,dt$ Let $u = 3t-1$

$du = 3\,dt \Rightarrow dt = \frac{du}{3}$

$\frac{1}{3}\int u^{50}\,du = \frac{u^{51}}{153} + c$

$\left[\frac{(3t-1)^{51}}{153} + c\right]\Bigg|_{t=0}^{t=1} = \left(\frac{1}{153}2^{51} + c\right) - \left(\frac{1}{153}(-1)^{51} + c\right)$

$= \frac{1}{153}\left(2^{51} + 1\right)$

ex $\displaystyle\int_{t=0}^{t=1} (3t-1)^{50}\,dt$ Let $u = 3t-1$ $\quad t=0 \to u=-1$

$du = 3\,dt \quad t=1 \to u=2$

$\frac{du}{3} = dt$

$$= \frac{1}{3}\int_{u=-1}^{u=2} u^{50}\,du = \frac{1}{153}u^{51}\Big|_{u=-1}^{u=2} = \left(\frac{1}{153}2^{51}\right) - \left(\frac{1}{153}(-1)^{51}\right)$$

$$= \frac{1}{153}\left[2^{51}+1\right]$$

ex $\displaystyle\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$, Let $u = 1+\sqrt{x} \Rightarrow \sqrt{x} = u-1$

$x=0 \to u=1$

$x=1 \to u=2$

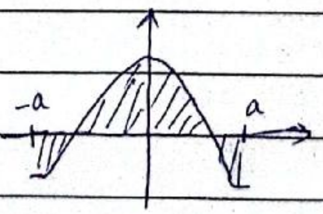
$du = \frac{1}{2\sqrt{x}}dx \Rightarrow dx = 2\sqrt{x}\,du = 2(u-1)du$

$$= 2\int_{u=1}^{u=2} \frac{(u-1)}{u^4}\,du = 2\int_1^2 (u^{-3}-u^{-4})du = 2\left(-\frac{1}{2}u^{-2} + \frac{1}{3}u^{-3}\right)\Big|_{u=1}^{u=2}$$

$$= \left[2\left(\frac{-1}{8}+\frac{1}{24}\right)\right] - \left[2\left(\frac{-1}{2}+\frac{1}{3}\right)\right] = \left(\frac{-3}{12}+\frac{1}{12}\right) - \left(\frac{-3}{3}+\frac{2}{3}\right)$$

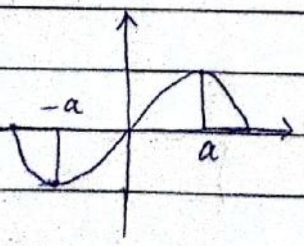
$$= \left(\frac{-1}{6}\right) + \frac{2}{6} = \frac{1}{6}$$

$\boxed{\text{Note:}}$ ① f is an even function (Y-axis sym)



$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

② f is an odd function (Origin Sym)



$$\int_{-a}^{a} f(x)\,dx = 0$$

ex $\displaystyle\int_{-100}^{100} \sin(x) = 0$

ex $\displaystyle\int_{-\pi}^{\pi} (\sin x + x^3 - x^5)\,dx = 0$

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ex $\displaystyle\int_{-\pi/2}^{\pi/2} (\cos(x) + \sin(x))\,dx$

$$= \int_{-\pi/2}^{\pi/2} \cos(x)\,dx + \int_{-\pi/2}^{\pi/2} \sin(x)\,dx = 2\int_0^{\pi/2} \cos(x)\,dx + 0$$

$$= 2\sin x \Big|_{x=0}^{x=\pi/2} = 2(1) - 2(0) = 2$$

ex $\displaystyle\int_0^1 x\sqrt{1-x^4}\,dx$ Let $u = 1 - x^4 \Rightarrow x^2 = \sqrt{1-u}$

$du = -4x^3 dx \Rightarrow -\frac{1}{4x^2}\,du = x\,dx$

$x=0 \to u=1$

$x=1 \to u=0$

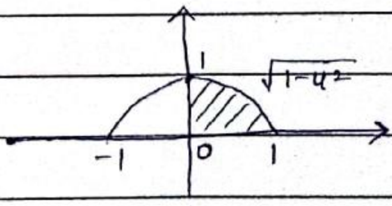
$$= -\frac{1}{4}\int_{u=1}^{u=0} \frac{\sqrt{u}}{\sqrt{1-u}}\,du$$

? (Try something else?)

Try again $\displaystyle\int_0^1 x\sqrt{1-x^4}\,dx$ Let $u = x^2$ $\quad x=0 \to u=0$, $x=1 \to u=1$

$du = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$

$$= \frac{1}{2}\int_{u=0}^{u=1} \sqrt{1-u^2}\,du$$



$$= \frac{1}{2}\left(\frac{1}{4}\pi\right) = \frac{\pi}{8}$$

Date:

Lecture 29

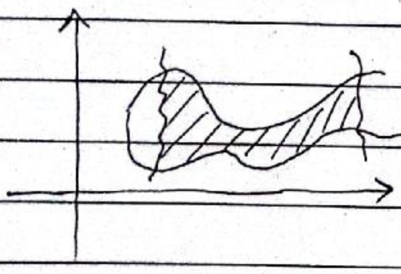
$\boxed{\text{Know}}$ ① $\int f(x)\,dx = F(x) + C$
   ↖ Antiderivative of $f(x)$

② $\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]\Big|_{x=a}^{x=b} = F(b) - F(a)$

③ Antiderivative? a) Table?
b) Algebra/Trig?
c) Substitution Method?

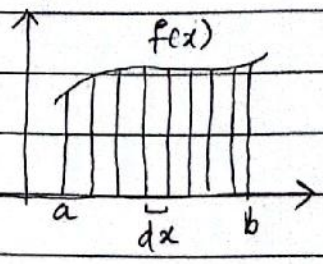
Applications

Area between curves (always positive)

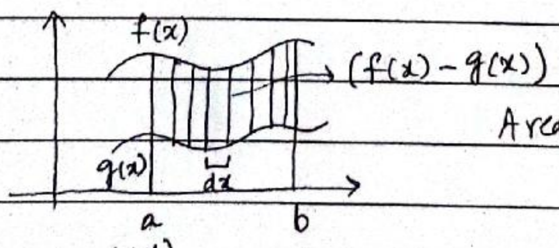


compare to $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$

$f(x) \geq 0$

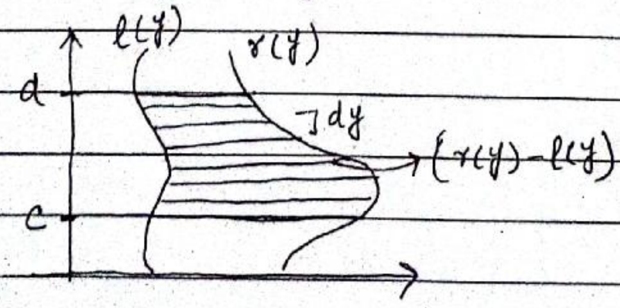


vs $f(x) \geq g(x)$



$$\text{Area} = \int_a^b (f(x) - g(x))\,dx$$

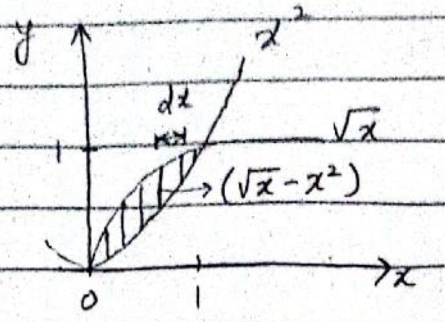
vs



$$\text{Area} = \int_c^d (r(y) - l(y))\,dy$$

Date:

ex Area between $\sqrt{x}$ and x^2 ?



Cross?

$\sqrt{x} = x^2$

$x = x^4$

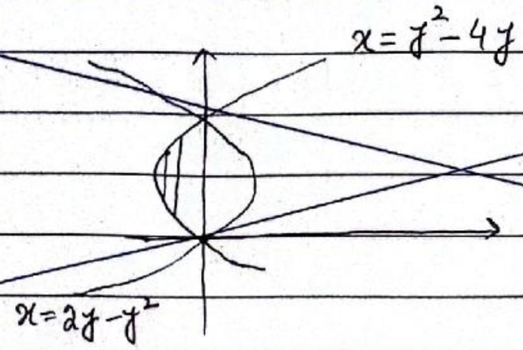
$x^4 - x = 0$

$x(x^3-1) = 0$

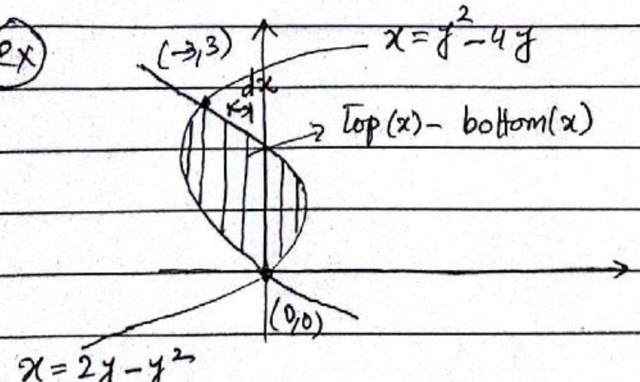
$x=0,\ x=1$

$$\int_0^1 (\sqrt{x}-x^2)\,dx = \frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \Big|_{x=0}^{x=1} = \left(\frac{2}{3}-\frac{1}{3}\right) - (0) = \frac{1}{3}$$

ex



ex

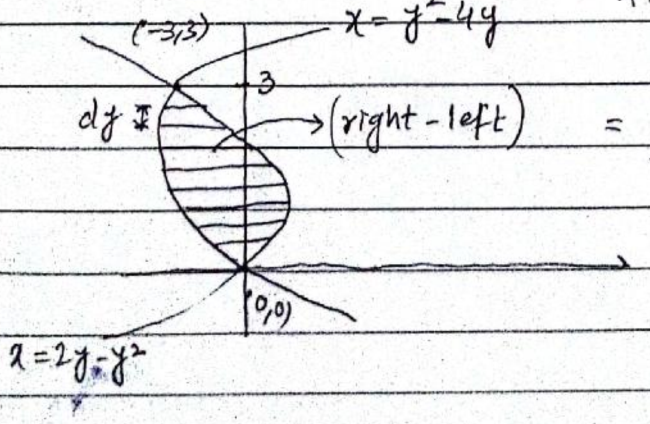


Cross? $y^2 - 4y = 2y - y^2$

$2y^2 - 6y = 0$

$2y(y-3) = 0$

$y=0,\ y=3$



$$\text{Area} = \int_0^3 \left((2y-y^2) - (y^2-4y)\right) dy$$

$$= \int_0^3 (-2y^2+6y)\,dy = \left(\frac{-2}{3}y^3 + 3y^2\right)\Big|_{y=0}^{y=3}$$

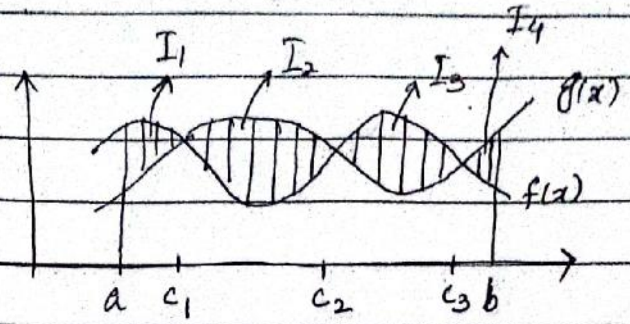
$$= \left(\frac{-2}{3}(3)^3 + 3(3)^2\right) - (0) = 9$$

Date:

(1) $f(x) \geq g(x)$ $\quad$ $\text{Area} = \int_a^b (f(x) - g(x))\, dx$

(2) Don't know? $\quad$ $\text{Area} = \int_a^b |f(x) - g(x)|\, dx$

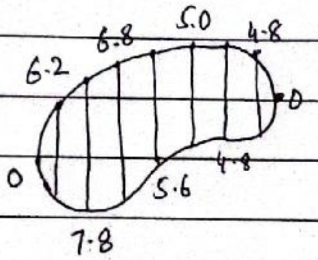
ex



$$\text{Area} = \int_a^b |f(x) - g(x)|\, dx$$

$$\text{Area} = \int_a^{c_1} (f-g)\, dx + \int_{c_1}^{c_2} (g-f)\, dx + \int_{c_2}^{c_3} (f-g)\, dx + \int_{c_3}^{b} (g-f)\, dx$$

ex



$\Delta x = 2$

Area $\simeq$ (1) left end point

(2) mid point.

Length: 0, 6.2, 7.8, 6.8, 5.6, 5.0, 4.8, 4.8, 0

left end pts: $\text{Area} \simeq \left(\dfrac{0+6.2+7.8+6.8+5.6+5.0+4.8+4.8}{8}\right) 16$

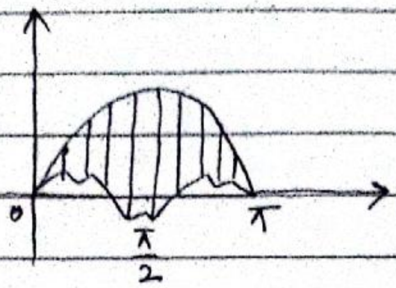
Midpoints: $\text{Area} \simeq \left(\dfrac{6.2+6.8+5.0+4.8}{4}\right) 16$

	1	4	2	4	2	4	2	4	1
Simpson's:	0	6.2	7.8	6.8	5.6	5.0	4.8	4.8	0

$$\text{Area} \simeq 16\left(\frac{0 + 4\times 6.2 + 2\times 7.8 + 4\times 6.8 + 2\times 5.6 + 4\times 5.0 + 2\times 4.8 + 4\times 4.8 + 0}{24}\right)$$

Date:

(ex) $y = \cos^2 x \sin x$, $y = \sin x$, between $0, \pi$



Cross ?

$\cos^2 x \sin x = \sin x$

$\sin x (\cos^2 x - 1) = 0$

$\sin x = 0$ $\quad$ $\cos^2 x = 1$

$0, \pi$ $\quad$ $\cos x = 1$ $\quad$ $\cos x = -1$

$x = 0$ $\quad$ $x = \pi$

$$\text{Area} = \int_0^\pi (\sin x - \cos^2 x \sin x)\, dx$$

$$= \int_0^\pi \sin x\, dx - \int_0^\pi (\cos x)^2 \sin x\, dx$$

$$= (-\cos x)\Big|_{x=0}^{x=\pi} - \int_0^\pi (\cos x)^2 \sin x\, dx$$

$$= ((-(-1)) - (-1)) - \int_0^\pi (\cos x)^2 \sin x\, dx$$

$$= 2 - \int_0^\pi (\cos x)^2 \sin x\, dx$$

Let $u = \cos x$ $\quad$ $x = 0 \rightarrow u = 1$

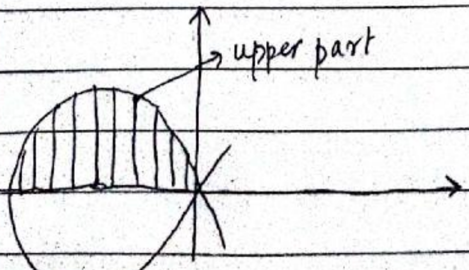
$du = -\sin x\, dx$ $\quad$ $x = \pi \rightarrow u = -1$

$$= 2 + \int_{u=1}^{u=-1} u^2\, du = 2 + \frac{1}{3} u^3 \Big|_{u=1}^{u=-1}$$

$$= 2 + \left(\left(-\frac{1}{3}\right) - \left(\frac{1}{3}\right)\right) = 2 - \frac{2}{3} = \frac{4}{3}$$

(ex) $y^2 = x^2 (x+3) \rightarrow y = +\sqrt{x^2(x+3)}$ $\quad$ if $y = 0 \rightarrow x = 0, x = -3$

$\rightarrow y = -\sqrt{x^2(x+3)}$



$$\text{Area of Loop} = 2 \int_{-3}^{0} (\text{upper part})\, dx$$

$-x$ in $[-3, 0]$

$$\text{Area} = 2 \int_{-3}^{0} \sqrt{x^2(x+3)}\, dx = 2 \int_{-3}^{0} |x| \sqrt{x+3}\, dx$$

$$\text{Area} = -2 \int_{-3}^{0} x \sqrt{x+3}\, dx = \text{use substitution.}$$

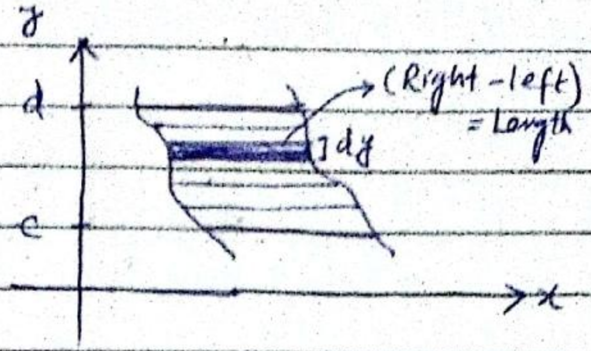
Date:

Lecture 30

① Area between Curves

$$\text{Area} = \int_{y=c}^{y=d} \text{(Length)}\, dy$$

↳ function of y



$$\text{Area} = \int_{x=a}^{x=b} \boxed{\text{Area's}} = \int_{x=a}^{x=b} (\text{Length})\,(dx)$$

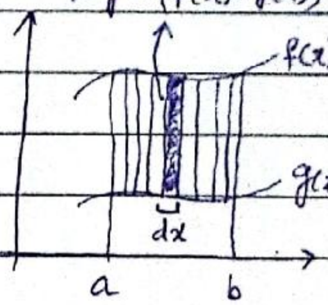
Area's → (length)(dx)

Slice in x-direction from a to b

Length → function of x

$x=a$: slices in x direction

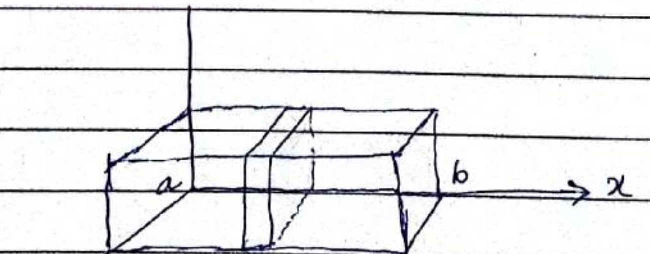
$\text{length} = (f(x) - g(x))$

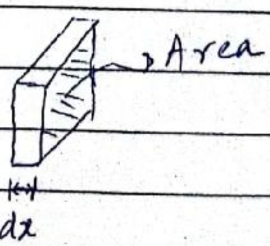


Can we also find Volumes? (Slice)

$$\text{Volume} = \int_a^b (\text{Area})\, dx$$

Area → function of x ; $A(x)$





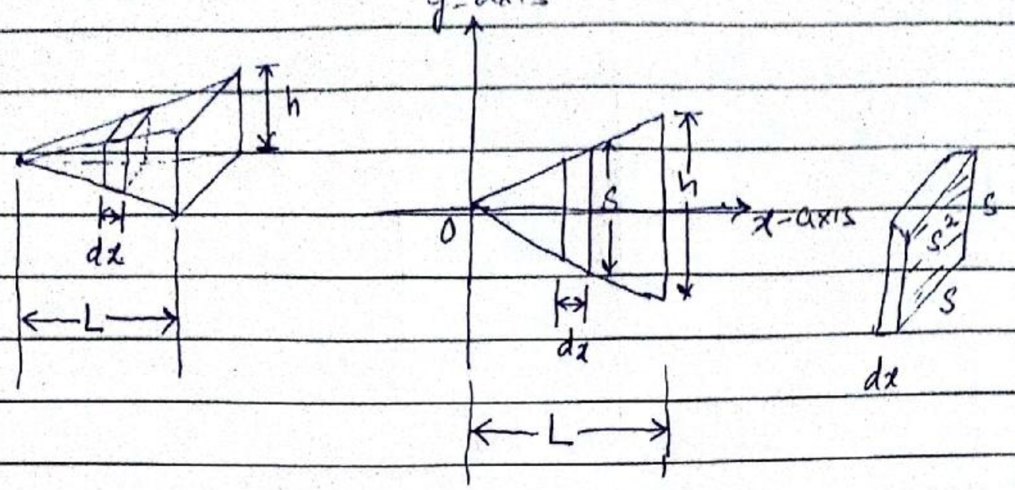
$V = (\text{Area})\, dx$

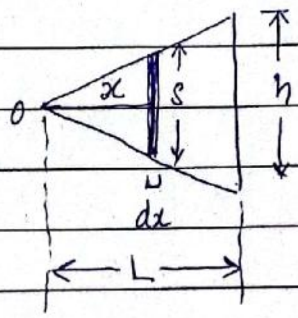
So Area by slicing $\int_a^b \text{Area}(x)\, dx$

slicing in x-direction from $x=a$ to $x=b$

Date:____________

Ex





$$\frac{L}{h} = \frac{x}{s}$$

$$\frac{x}{s} = \frac{L}{h} \Rightarrow s = \frac{h}{L}x$$

$x = 0 \rightarrow s = 0$

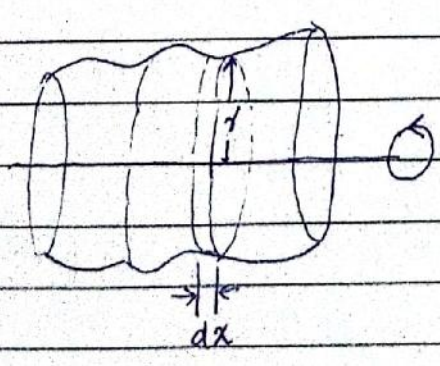
$x = L \rightarrow s = h$

$$V = \int_0^L \underbrace{Area(x)}_{s^2}\, dx$$

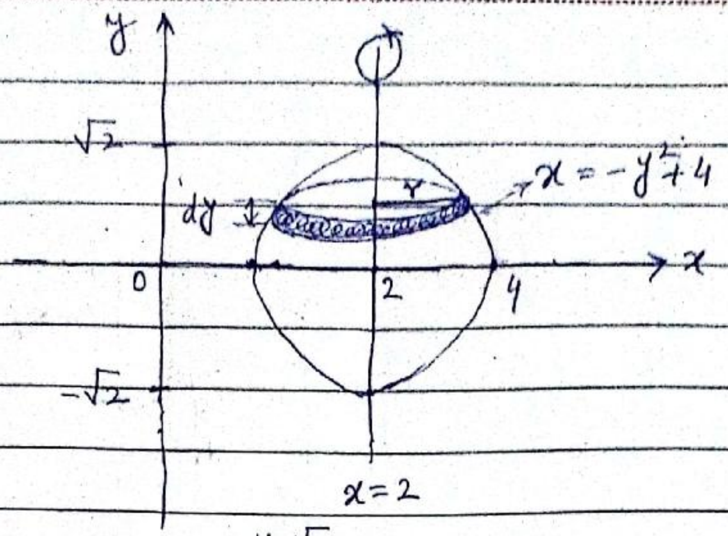
$$V = \int_{x=0}^{x=L} \left(\frac{h}{L}x\right)^2 dx$$

$$= \frac{h^2}{L^2}\int_0^L x^2\,dx = \frac{h^2}{3L^2}\, x^3 \Big|_0^L = \frac{1}{3}h^2 L$$

Solids of Revolution → Slice $\boxed{\text{along}}$ axis of Revolution - circle $= \pi r^2$
perpendicular



ex



$x=2$

$2=-y^2+4$

$y^2=2$

$\boxed{y=\pm\sqrt{2}}$

$y=0$

$x=0^2+4$

$x=4$

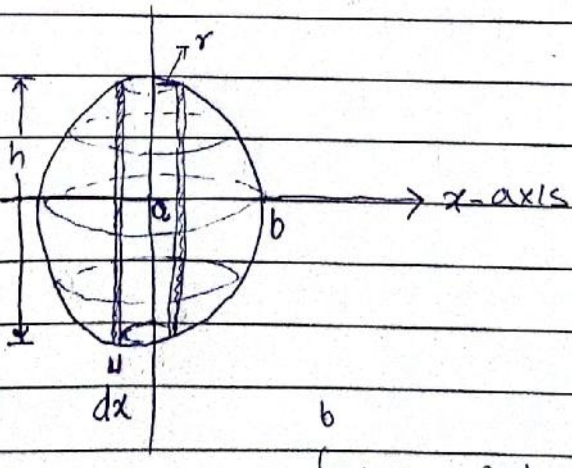
$$V = 2\int_{y=0}^{y=\sqrt{2}} \underbrace{Area(y)}_{\pi r^2}\,dy$$

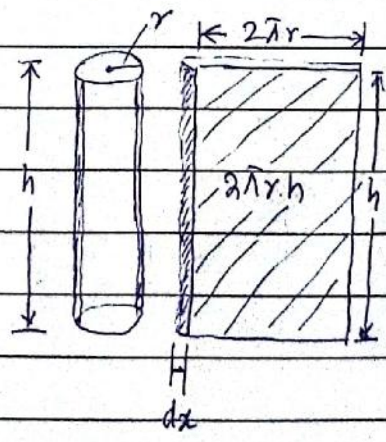
$$r = (-y^2+4)-(2) = (-y^2+2)$$

$$V = 2\int_0^{\sqrt{2}} \pi(-y^2+2)^2 dy = 2\pi\int_0^{\sqrt{2}} (-y^2+2)^2 dy$$

$$= 2\pi\int_0^{\sqrt{2}} (y^4-4y^2+4)\,dy \;\cdots$$

Shells



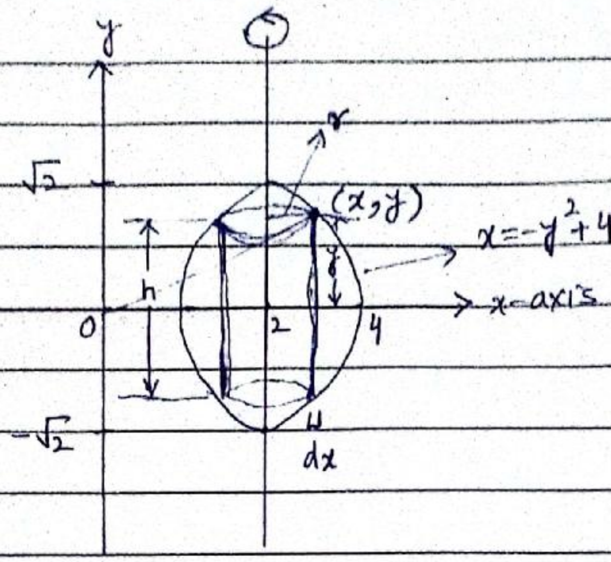


$$V = \int_a^b \underbrace{Area(x)}_{2\pi r h}\,dx$$

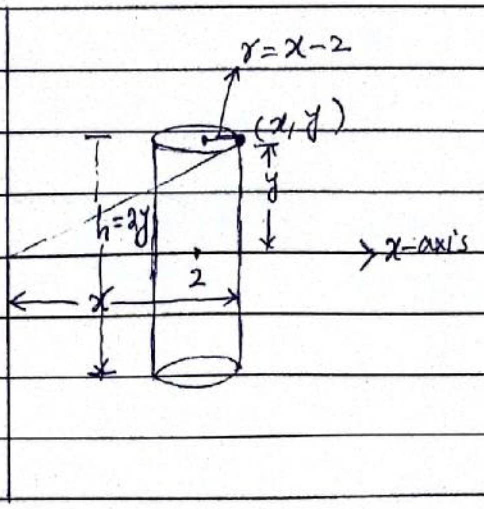
$2\pi r h \rightarrow$ function of 'x'

Date:

Ex



$x=-y^2+4 \Rightarrow y=\pm\sqrt{4-x}$



$h=2y=2\sqrt{4-x}$

$r=x-2$

$V=\int_2^4 \underbrace{Area(x)}_{2\pi rh}\, dx$ → function of x

$V=\int_2^4 2\pi r h\, dx$

$V=\int_2^4 2\pi(x-2)(2\sqrt{4-x})\,dx$

Let $u=4-x \Rightarrow 2-u=x-2$, $du=-dx$

$x=2 \to u=2$, $x=4 \to u=0$

$V=-4\pi\int_2^0 (2-u)\,u^{1/2}\,du$

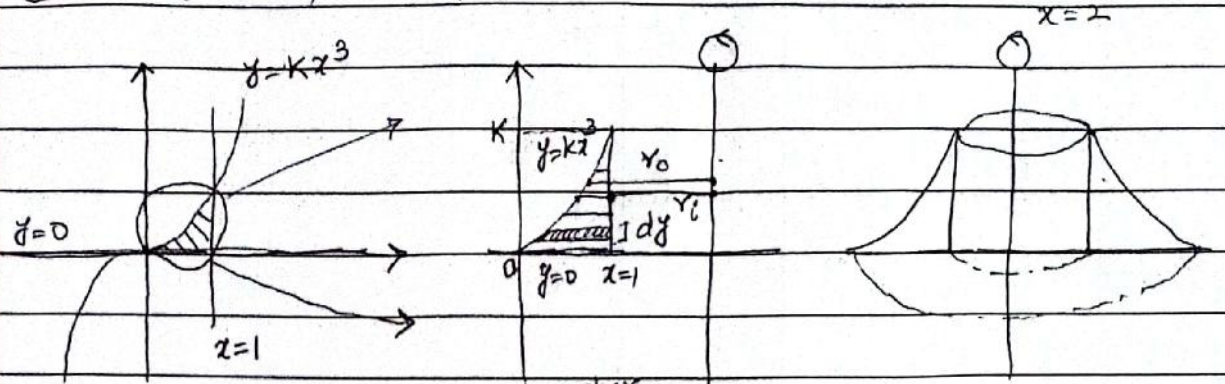
$V=4\pi\int_0^2 (2-u)\,u^{1/2}\,du \ldots$

Date:__________

Lecture 31

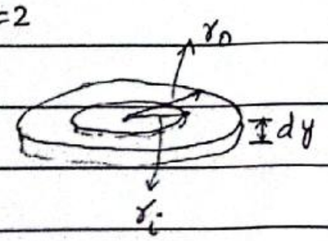
ex $y = kx^3,\ y=0,\ x=1$ about $x=2$

$\boxed{\text{Washer}}$



$$V = \int_a^b \overset{\text{washer}}{\boxed{A(y)}}\, dy$$

$$A(y) = \underbrace{\pi r_o^2}_{\text{outer radius}} - \underbrace{\pi r_i^2}_{\text{inner radius}}$$



$y = kx^3$

$x = \left(\frac{y}{k}\right)^{1/3}$

$$V = \int_{y=0}^{y=k} \pi\left(2-\left(\frac{y}{k}\right)^{1/3}\right)^2 - \pi(2-1)^2\, dy$$

$$V = \pi \int_0^k \left(\left(2 - \frac{1}{k^{1/3}} y^{1/3}\right)^2 - 1\right) dy$$

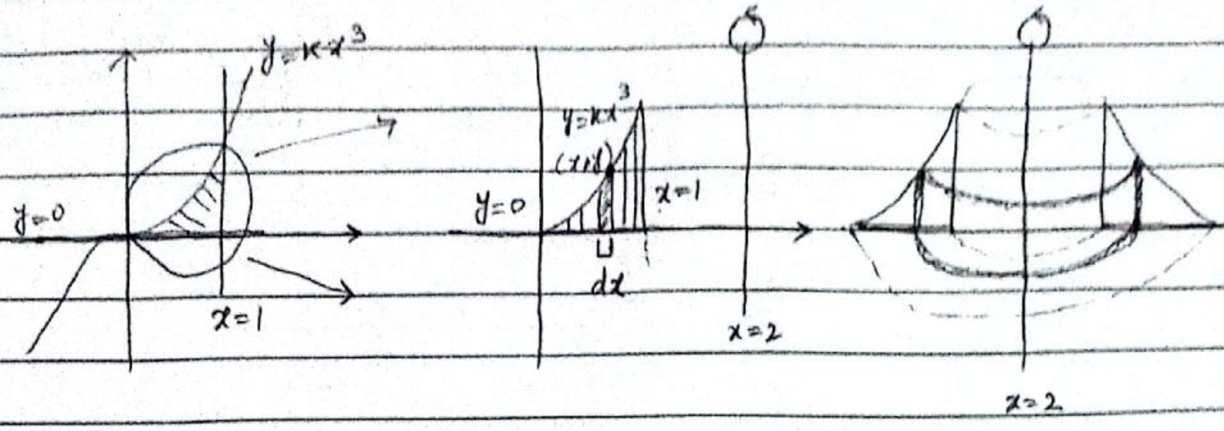
$\boxed{k=27}$

$$V = \pi \int_0^{27} \left(\left(2 - \frac{1}{3} y^{1/3}\right)^2 - 1\right) dy$$

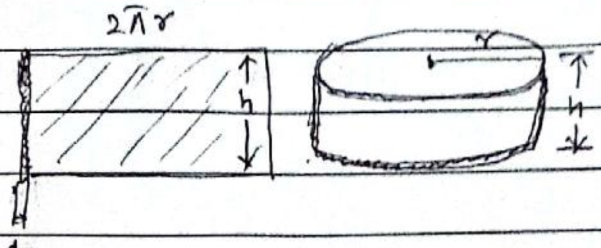
$$V = \pi \int_0^{27} \left(3 - \frac{4}{3} y^{1/3} + \frac{1}{9} y^{2/3}\right) dy$$

$$V = \pi\left[\cancel{3^4} - \cancel{3^4} + \frac{1}{5}3^4\right] = \frac{81}{5}\pi$$

$\boxed{\text{Shells}}$



$$V = \int_a^b \underbrace{A(x)}_{2\pi r h}\, dx$$



$r = 2 - x$

$h = y = kx^3$

$$V = \int_{x=0}^{x=1} 2\pi(2-x)(kx^3)\,dx = 2k\pi \int_0^1 (2x^3 - x^4)\,dx$$

$$= 2k\pi\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{5}k\pi$$

$k = 27 \Rightarrow V = \frac{81}{5}\pi$

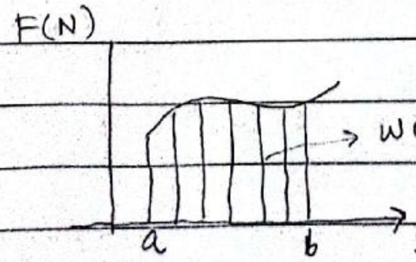
$$\int f(x)\,dx = F(x) + c$$

$$\int_a^b f(x)\,dx = \boxed{F(x)}\Big|_{x=a}^{x=b} = F(b) - F(a)$$

$\boxed{\text{Apps}}$ ① Area ② Volume ③ Work

Work = Force . displacement

O → Force → change in momentum (mass x velocity)

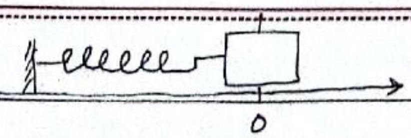


→ work = Area.

$$W = \int_a^b \underbrace{F(s)}_{\text{Force}}\, ds$$

Date:

$\boxed{\text{example}}$ Spring



spring at rest (Natural Displacement)

$F_{spring} \propto S$

spring constant

$\rightarrow F(S) = K\,S$

$S = 2\text{cm}; \quad F = 3N$ (test)

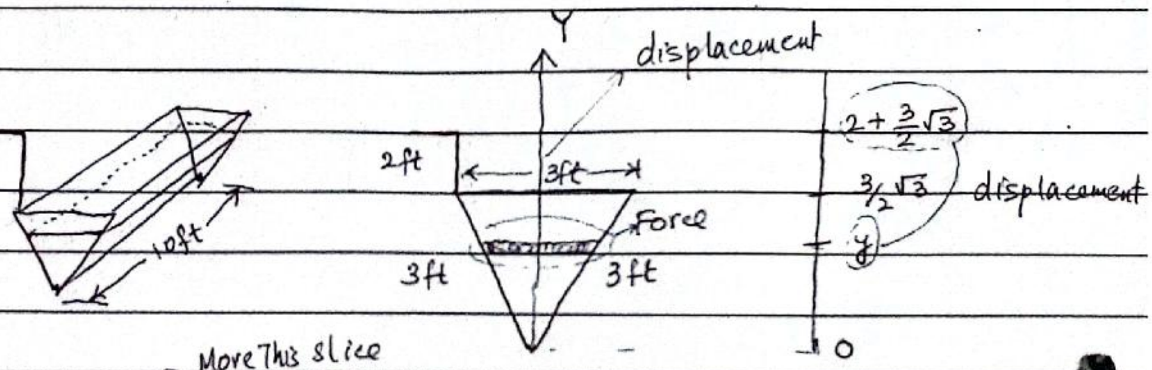
$3N = K(0.02m) \rightarrow K = 150\ \frac{N}{m}$

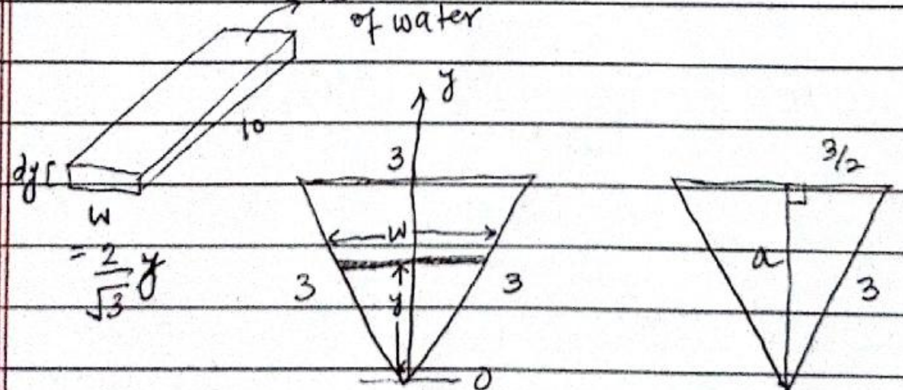
$F(S) = 150\,S$

work to move it from rest to 20 cm

$$W = \int_0^{0.2} 150\,s\,ds = 75s^2\Big|_{s=0}^{s=0.2} = 75(0.2)^2 - 0 = 3\,Nm = 3J$$

Ex





$a^2 = 3^2 - \left(\frac{3}{2}\right)^2$

$a^2 = 9\left(\frac{3}{4}\right)$

$a = \frac{3}{2}\sqrt{3}$

$$\frac{y}{w} = \frac{\frac{3}{2}\sqrt{3}}{3} \Rightarrow w = \frac{2}{\sqrt{3}}y$$

Force = weight = Volume × Volume density

= Volume × ρ → physical constant (lb/ft^3)

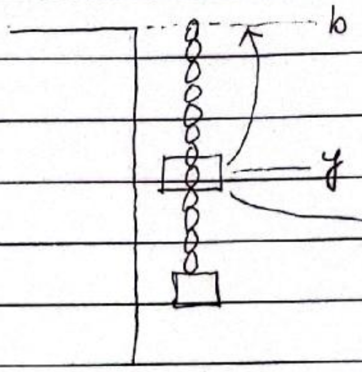
$\rho = \frac{\text{weight}}{\text{Volume}} \left(\frac{lb}{ft^3}\right)$

Date:

$$\text{Work} = \int_0^{\frac{3}{2}\sqrt{3}} \underbrace{\underbrace{\left((10)\left(\tfrac{2}{\sqrt{3}}y\right)dy\right)}_{\text{Vol}} \cdot \rho}_{\text{weight = Force}} \cdot \underbrace{\left(2+\tfrac{3}{2}\sqrt{3}-y\right)}_{\text{displacement}}$$

$$= \int_0^{\frac{3}{2}\sqrt{3}} \frac{20\rho}{\sqrt{3}}\, y \left(2+\tfrac{3}{2}\sqrt{3}-y\right) dy$$

(ex) Chain

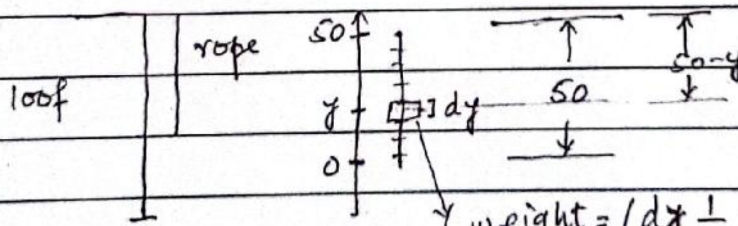


$\overbrace{}^{b-y}$

$\text{Work}_i = \text{weight} \times \text{displacement}$

(ex) 50 ft rope @ $\frac{1}{2}$ lb/ft

hanging off building: Work to lift it up



$\text{weight} = \left(dy \tfrac{1}{2} \tfrac{\text{lb}}{\text{ft}}\right) = \tfrac{1}{2}dy$

$$\text{Work} = \int_0^{50} \left(\tfrac{1}{2}dy\right)(50-y) = \int_0^{50} \tfrac{1}{2}(50-y)\,dy = \cdots$$

(ex)
coiled rope @ bottom
work = 2500 ft lb.

Date:

Lecture 32

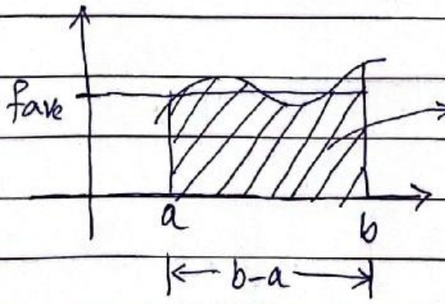
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x, \qquad \Delta x = \frac{b-a}{n}$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} (b-a) \sum_{i=1}^{n} \frac{f(x_i^*)}{n}$$

$$\int_a^b f(x)\,dx = (b-a) \underbrace{\lim_{n\to\infty} \left[\frac{\sum_{i=1}^{n} f(x_i^*)}{n} \right]}_{f_{ave}}$$

finite number
This is the arithmetic average

<u>Def</u>: $f_{ave} = \frac{1}{b-a} \int_a^b f(x)\,dx$



Area $= \int_a^b f(x)\,dx$

Area $= f_{ave} \times (b-a)$

$\boxed{\text{Mean Value th}^m \text{ for Integrals}}$

f is continuous on $[a,b]$ then there exists a $x=c$ such that

$$f(c) = f_{ave} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

or $$(b-a)\,f(c) = \int_a^b f(x)\,dx$$

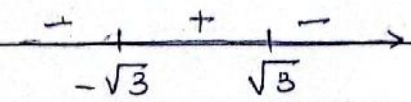
$\boxed{\text{Apps}}$ → fave? ex) $f(x) = \frac{x}{\sqrt{3-x^2}}$ over $[1,3]$

$$f_{ave} = \frac{1}{(b-a)} \int_a^b f(x)\,dx$$

Date:

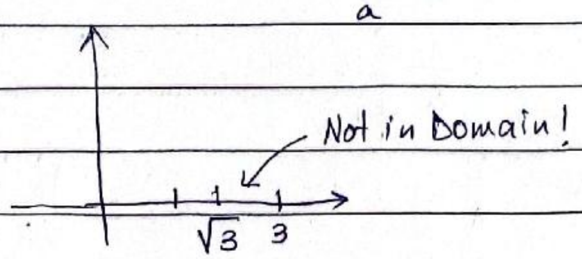
1^{st} Domain $\quad 3-x^2>0 \rightarrow (-\sqrt{3},\sqrt{3})$

$(\sqrt{3}+x)(\sqrt{3}-x)>0$



2^{nd} $\quad f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx \quad [a,b]=[1,3]$

Not in domain



(ex) $\quad f(x) = \frac{x}{\sqrt{3+x^2}}$ over $[1,3]$

1^{st} Domain $\quad 3+x^2>0$ $\boxed{\text{True}}$

2^{nd} $\quad f_{ave} = \frac{1}{3-1}\int_1^3 \frac{x}{\sqrt{3+x^2}}\,dx$

$$f_{ave} = \frac{1}{2}\int_1^3 \frac{x}{\sqrt{3+x^2}}\,dx = \frac{1}{4}\int_4^{12} u^{-1/2}\,du$$

Let $u = 3+x^2 \qquad x=1 \rightarrow u=4$

$du = 2x\,dx \qquad x=3 \rightarrow u=12$

$$= \frac{1}{2}u^{1/2}\Big|_{u=4}^{u=12} = \frac{1}{2}\left(\sqrt{12}-\sqrt{4}\right)$$

$$= \frac{1}{2}\left(2\sqrt{3}-2\right) = \sqrt{3}-1$$

By Mean Value thm of integrals

$x=c \qquad \frac{c}{\sqrt{3+c^2}} = \sqrt{3}-1 \rightarrow \frac{c^2}{3+c^2} = \underbrace{4-2\sqrt{3}}_{K}$

$\rightarrow c^2 = (4-2\sqrt{3})(3+c^2) \rightarrow c^2 = K(3+c^2)$

$\rightarrow c^2 - Kc^2 = 3K \rightarrow c = \pm\sqrt{\frac{3K}{1-K}}$

$$\boxed{c = \sqrt{\frac{3(4-2\sqrt{3})}{2\sqrt{3}-3}}}$$

ex) fare ? $f(t) = \cos^4(t)\sin(t) \quad [0, \pi]$

$$f_{ave} = \frac{1}{\pi - 0}\int_0^{\pi} (\cos t)^4 \sin t \, dt$$

Let $u = \cos t$ $\quad t=0 \;\; u=1$

$du = -\sin t \, dt \quad t=\pi \;\; u=-1$

$$= -\frac{1}{\pi}\int_1^{-1} u^4 \, du$$

$$f_{ave} = \frac{1}{\pi}\int_{-1}^{1} u^4 du = \frac{1}{\pi}\left(\frac{1}{5}u^5\right)\Big|_{u=-1}^{u=1}$$

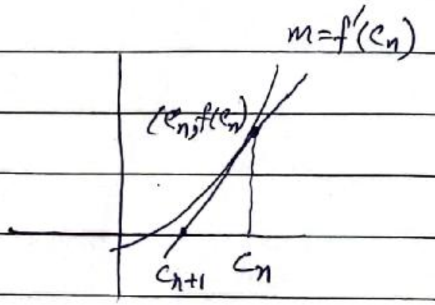
$$= \frac{1}{\pi}\left[\frac{1}{5} - \left(-\frac{1}{5}\right)\right] = \frac{2}{5\pi}$$

$c = ?$ $\quad f(c) = \frac{2}{5\pi} \quad \cos^4(c)\sin(c) = \frac{2}{5\pi}$

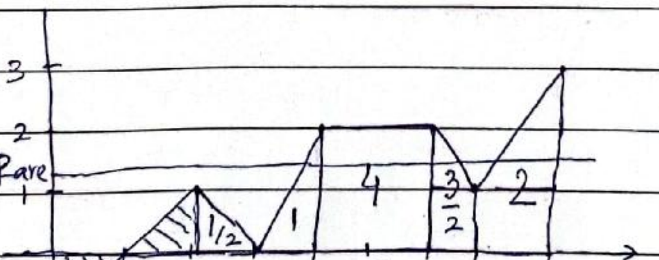
$\rightarrow \underbrace{\cos^4(c)\sin(c) - \frac{2}{5\pi}}_{f(c)=0} = 0$

① guess c_1

② $C_{n+1} = C_n - \frac{f(C_n)}{f'(C_n)}$



ex)



fare over $[0, 8]$

$$f_{ave} = \frac{1}{8}\boxed{\int_0^8 f \, dx}$$

↳ Net signed area

$$f_{ave} = \frac{1}{8}\left[\frac{1}{2} + 1 + 4 + \frac{3}{2} + 2\right] = \frac{9}{8}$$

ex) Rod with Linear density of $\frac{12}{\sqrt{x+1}} \frac{kg}{m}$, 8m long

Density$_{ave}$ = ?

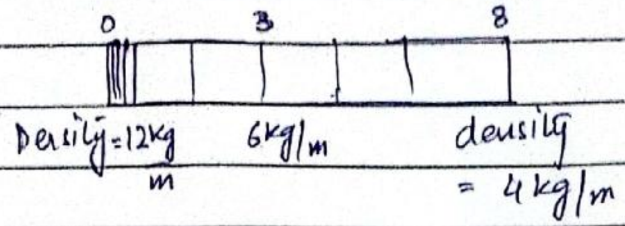
$$f_{ave} = \frac{1}{8-0}\int_0^8 \frac{12}{\sqrt{x+1}} dx$$

Let $u = x+1$ $\quad x=0 \rightarrow u=1$

$du = dx \quad x=0 \rightarrow u=9$

$$f_{ave} = \frac{1}{8}\int_1^9 \frac{12}{\sqrt{u}} du = \frac{3}{2}\int_1^9 u^{-1/2} du = 3u^{1/2}\Big|_{u=1}^{u=9}$$

$$= 3(\sqrt{9}-\sqrt{1}) = 6\ \frac{kg}{m}$$



$$density(x) = \frac{12}{\sqrt{x+1}}$$

when $c = ?$ is density 6 kg/m

$$\frac{12}{\sqrt{x+1}} = 6 \longrightarrow 2 = \sqrt{x+1} \Longrightarrow x = 3\,m$$

ex) average velocity from t_1 to t_2

$$V_{ave} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} v(t)\,dt$$

b/c $v(t) = s'(t)$

derivative of position

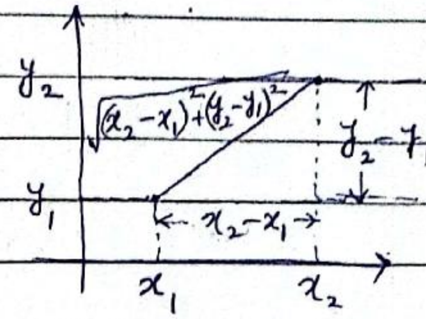
$$V_{ave} = \frac{1}{t_2 - t_1}\, s(t)\Big|_{t_1}^{t_2}$$

$$V_{ave} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

Lecture 33

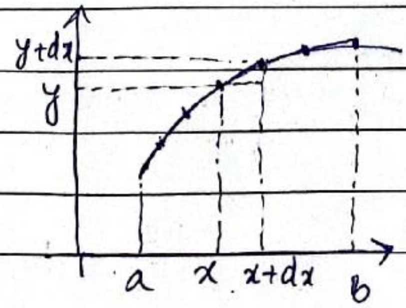
Arc length

$$L = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$



$y = f(x)$ is a differentiable function.

$$dL = \sqrt{(x+dx-x)^2 + (y+dx-y)^2}$$



$$dL = \sqrt{dx^2 + dy^2}$$

$$dL = \sqrt{(dx^2+dy^2)\frac{dx^2}{dx^2}}$$

$$dL = \sqrt{\frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}}\, dx$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = [f'(x)]^2$$

$$dL = \sqrt{1 + [f'(x)]^2}\, dx$$

$$L = \int_a^b dL = \int_a^b \sqrt{1+[f(x)]^2}\, dx \qquad a \le x \le b$$

OR

$$L = \int_c^d \sqrt{1+[f'(y)]^2}\, dy, \qquad x = f(y),\ c \le y \le d$$

(Ex) Find the length of the arc of $y = x^{3/2}$ between the points $(1,1)$ and $(4,8)$

$$y = x^{3/2}$$

$$y' = \frac{3}{2}x^{1/2}$$

$$L = \int_1^4 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx = \int_1^4 \sqrt{1+\frac{9}{4}x}\,dx$$

Let $u = 1+\frac{9}{4}x$ $\quad x=1 \to u = 13/4$, $x=4 \to u=10$

$du = \frac{9}{4}dx \to dx = \frac{4}{9}du$

$$= \frac{4}{9}\int_{13/4}^{10} \sqrt{u}\,du = \frac{4}{9}\left(\frac{2}{3}u^{3/2}\right)\Bigg|_{u=13/4}^{u=10}$$

$$= \frac{8}{27}\left(10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right) \simeq 7.6337$$

<u>Area of a Surface of Revolution</u>

$$dL = \sqrt{1+[f'(x)]^2}\,dx$$

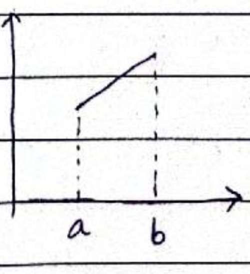
$$L = \int_a^b dL = \int_a^b \sqrt{1-[f'(x)]^2}\,dx$$

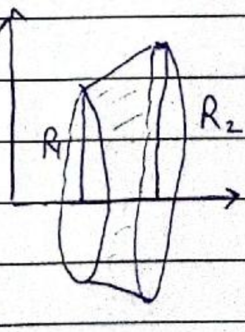
<u>Basic</u> $y = f(x)$ for $a \le x \le b$ is a line segment

conical frustum

$\pi(R_1+R_2)s$

$\pi(f(a)+f(b))L$

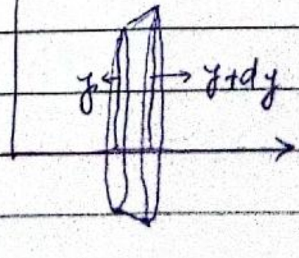




$dA_s = \pi(y+(y+dy))\,dL$

$dA_s = 2\pi y\,dL + \pi\,dy\,dL$

$dA_s = 2\pi y\,dL \quad \because dy\,dL = 0$



Date:

The area of the surface of revolution formed by rotating the graph $y=f(x)$ for $a \le x \le b$ about the x-axis is

$$SA = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\, dx$$

The area of the surface of revolution formed by rotating the graph $x=f(y)$ for $a \le y \le b$ about the x-axis is

$$SA = 2\pi \int_a^b y\sqrt{1+[f'(y)]^2}\, dy$$

If we rotate the arc about the y-axis, the radii of the small frusta are x and $x+dx$ so the surface area of the small frusta are

$$dA_S = 2\pi x\, dL$$

The area of the surface of revolution formed by rotating the graph $y=f(x)$ for $a \le x \le b$ about the y-axis is

$$SA = 2\pi \int_a^b x\sqrt{1+[f'(x)]^2}\, dx$$

The area of the surface of revolution formed by rotating the graph $x=f(y)$ for $a \le y \le b$ about the y-axis is

$$SA = 2\pi \int_a^b f(y)\sqrt{1+[f'(y)]^2}\, dy$$

Ex) The curve $y=f(x)=\sqrt{4-x^2}$, $-1 \le x \le 1$ is rotated about the x-axis. Find the area of the resulting surface.

$$f'(x) = \frac{-x}{\sqrt{4-x^2}}$$

$$SA = 2\pi \int_{-1}^{1} \sqrt{4-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\, dx$$

$$SA = 2\pi \int_{-1}^{1} \sqrt{4-x^2+x^2}\, dx = 4\pi \int_{-1}^{1} dx$$

$$SA = 4\pi \int_{-1}^{1} dx = 8\pi$$

Lecture 34

Inverse functions for compositions

Math: Objects + Operations

undo an operation: ex

$3x + 7 = 1$

$3x + 7 + (-7) = 1 + (-7)$

-7 is 7's additive Inverse

$3x + 0 = -6$

0 is additive identity

1. Notation: $(f \circ g)(x) = f(g(x))$

2. Identity of composition of functions: $I(x)$

want: $f(I(x)) = f(x)$

$I(f(x)) = f(x)$

Let $I(x) = x$, $I(\square) = \square$

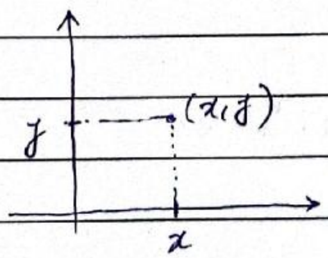
check: $I(x^3) = x^3$

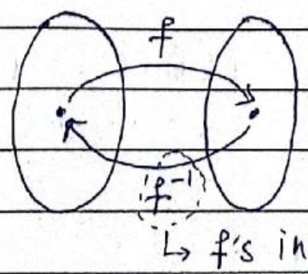
$(I(x))^3 = x^3$

$(I \circ f)(x) = (f \circ I)(x) = f(x)$

- Can $f(x)$ have an inverse ?

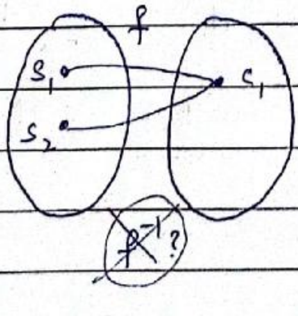
$y = f(x)$



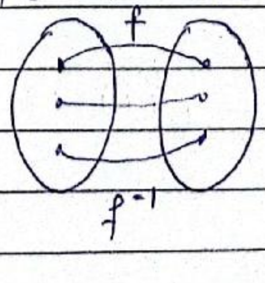


f's inverse function.

- When can f^{-1} exist?

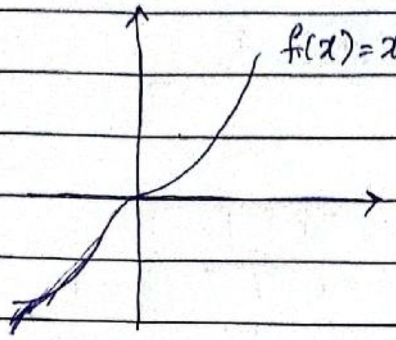


f's one to one



Date:

f^{-1} exists if f is one to one (passes horz. line test)



$f(x)=x^3$ has an inverse $f^{-1}(x)=?$

Inverse Functions

horz. line test

Existence: If f is one-to-one and onto (bijection) Then f^{-1} exists.

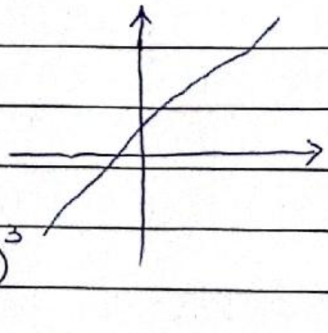
Find f^{-1}

① $y=f(x)$

② interchange x, y

③ $y=f^{-1}(x)$

ex $\quad f(x)=x^{1/3}+1$



① see that f^{-1} must exist

② $x=y^{1/3}+1 \xrightarrow{\text{algebra}} y=(x-1)^3$

③ $f^{-1}(x)=(x-1)^3$

$$f(f^{-1}(x))=\left((x-1)^3\right)^{1/3}+1=x$$

$$f^{-1}(f(x))=\left((x^{1/3}+1)-1\right)^3=x$$

ex $\quad f(x)=x^{1/3}+x$

① passes horz. line test (one-to-one) so f^{-1} exists.

② $y=x^{1/3}+x \xrightarrow[f^{-1}]{\text{Find}} \boxed{x=y^{1/3}+y} \xrightarrow{\text{algebra}} y=f^{-1}(x)$

$y=f^{-1}(x)$ exists $\boxed{\text{but}}$ I cannot write it explicitly

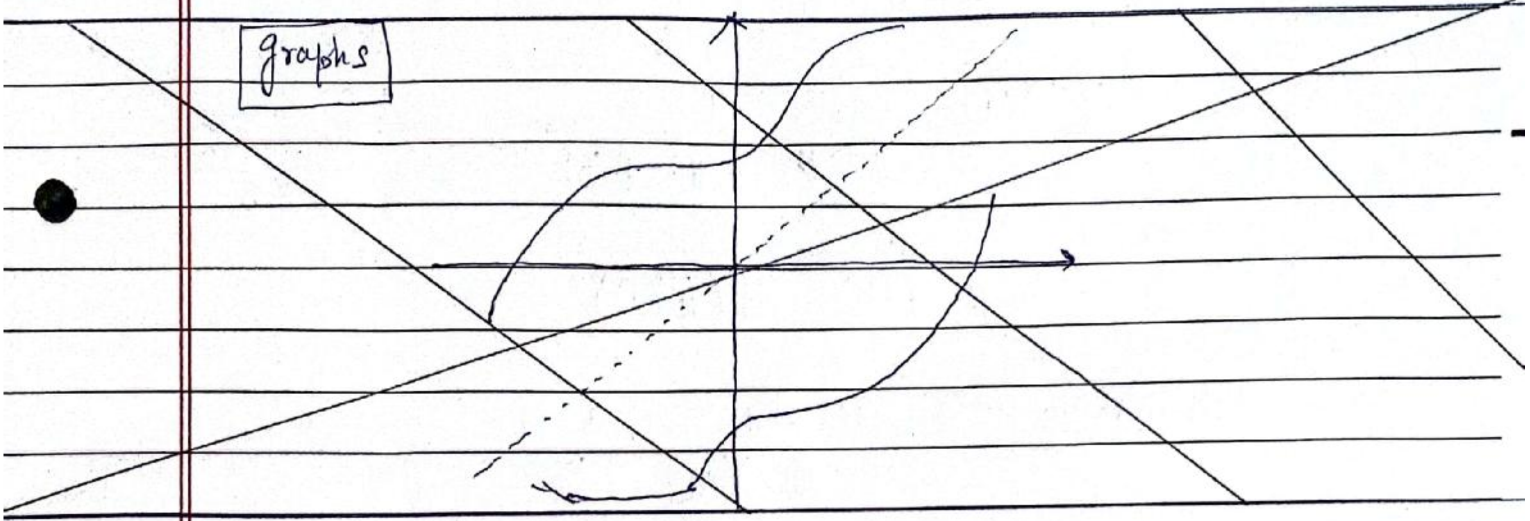
Date:

<u>Transcendental Functions</u>: are functions that can NOT be written with a finite number of algebraic operations (+, *, power, roots)

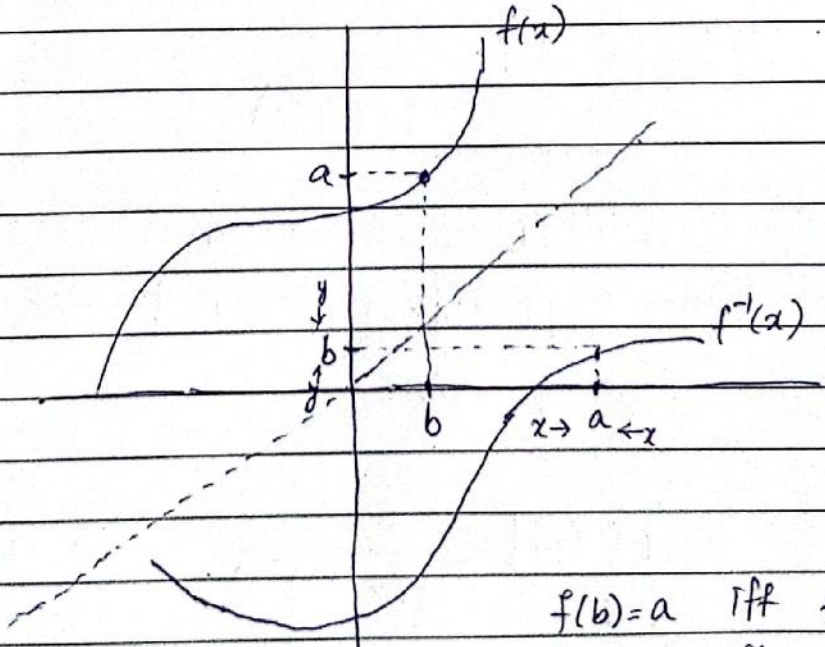
$\boxed{\text{Derivatives of } f^{-1}}$

$\boxed{Th^m}$ f is one-to-one (f^{-1} exists) and continuous, then f^{-1} is also continuous.

$\boxed{\text{graphs}}$



$\boxed{\text{graphs}}$



$f(b) = a$ iff $f^{-1}(a) = b$

$f(x) = y$ iff $f^{-1}(y) = x$

$f(\square) = \Delta$ iff $f^{-1}(\Delta) = \square$

Date:

$D_x[f^{-1}(x)]$

slope of $f^{-1}(x)$ @ $x=a$ $\quad (f^{-1})'(a)$

$$(f^{-1})'(a) = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x-a}$$

$\boxed{\text{Know}}$ $f^{-1}(x) = y$ iff $f(y) = x$. as $x \to a$

$f^{-1}(a) = b$ iff $f(b) = a$ $\quad y \to b$

$$(f^{-1})'(a) = \lim_{y \to b} \frac{y-b}{f(y)-f(b)} = \frac{1}{\lim_{y \to b} \frac{f(y)-f(b)}{y-b}}$$

$$= \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}$$

<u>Different Notation</u> ① slope @ $x=a$ of f^{-1}

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

② $D_x[f^{-1}(x)] = \dfrac{1}{f'(f^{-1}(x))}$

ex $\quad f(x) = x^{1/3}+1$, $f^{-1}(x) = (x-1)^3$

obvious $D_x[f^{-1}(x)] = D_x[(x-1)^3] = 3(x-1)^2$

also $\quad f'(x) = \frac{1}{3} x^{-2/3}$

$$D_x[f^{-1}(x)] = \frac{1}{\frac{1}{3}((x-1)^3)^{-2/3}} = 3(x-1)^2$$

A new transcendental Function:
(exponential $f(x) = b^x$)

① why? geometric Growth.

a) Rs 10,000

b) $1 + 2 + 2^2 + 2^3 + \cdots + 2^{60} = 2^{61} - 1$

② Compounding Interest: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

cont. compounding $n \to +\infty$

$$A = P \lim_{n \to +\infty} \left(1 + \frac{r}{n}\right)^{nt}$$

Let $h = \frac{n}{r}$

$$A = P\Big[\underbrace{\lim_{h \to \infty} \left(1 + \frac{1}{n}\right)^{h}}_{\text{const} = e}\Big]^{rt}$$

$$\boxed{A = Pe^{rt}}$$

$f(x) = b^x$ need $b > 0$

Properties

① $b^{n} = \underbrace{b \cdot b \cdot b \cdots b}_{n \text{ factors}}$ → positive integer

② $b^{-n} = \frac{1}{b^n} = \underbrace{\frac{1}{b} \cdot \frac{1}{b} \cdots \frac{1}{b}}_{n \text{ factors}}$

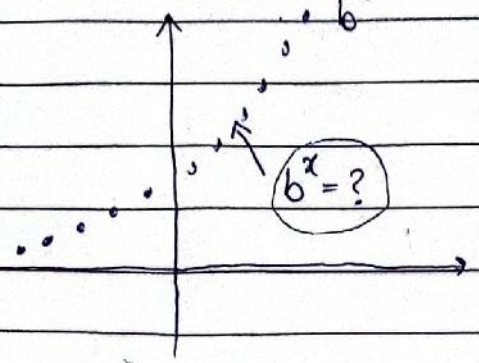
③ b^r where $r = \frac{p}{q}$ a rational number

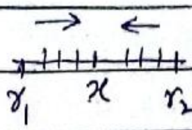
$$b^{p/q} = \left(b^{1/q}\right)^p$$

where $b^{1/q} = a$ where $a^q = b$

④ $x \in \mathbb{R}$

Rationals





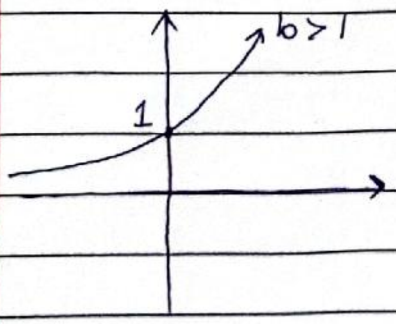
$$b^x = \lim_{r \to x} b^r$$

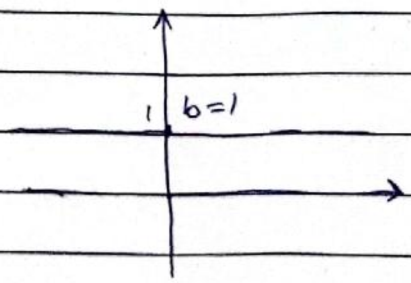
⑤ $b^x b^y = b^{x+y}$

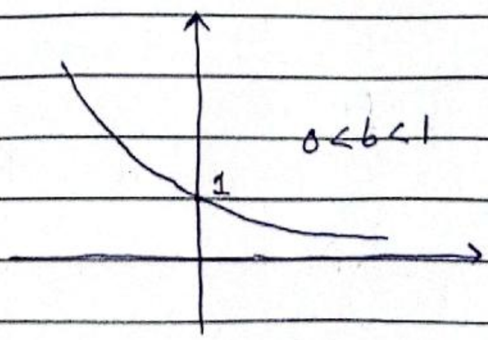
⑥ $\dfrac{b^x}{b^y} = b^{x-y}$

⑦ $(b^x)^y = b^{xy}$

⑧ $(ab)^x = a^x b^x$

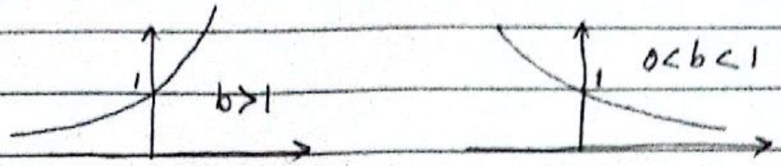






Lecture 35

$f(x) = b^x$



Domain: all reals

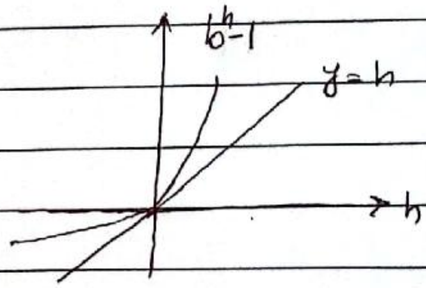
Range: $\boxed{0 < b^x}$ $(0, +\infty)$

Slope of b^x

$$D_x[b^x] = \lim_{h \to 0} \frac{\overset{b^x b^h}{b^{x+h}} - b^x}{h}$$

$$D_x[b^x] = b^x \boxed{\lim_{h \to 0} \frac{b^h - 1}{h}}$$

$$\lim_{h \to 0} \frac{b^h - 1}{h} = \text{constant}$$



If $b = e$

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

$\boxed{\text{so}}$ $\boxed{f(x) = e^x \qquad D_x[e^x] = e^x}$

$$\boxed{\int e^x \, dx = e^x + c}$$

ex $D_x\left[\sin(x^2) + e^{x^2}\right] = \cos(x^2)(2x) + e^{x^2}(2x)$

$= 2x\cos(x^2) + 2x\,e^{x^2}$

ex $\displaystyle\int \frac{e^{2x}}{\sqrt{1+e^{2x}}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\,du = u^{1/2} + c$

$= \sqrt{1+e^{2x}} + c$

$u = 1 + e^{2x}$

$du = 2e^{2x}\,dx$

(ex) $\int (e^x e^{e^x})\,dx = \int e^u\,du = e^u + c = e^{e^x} + c$

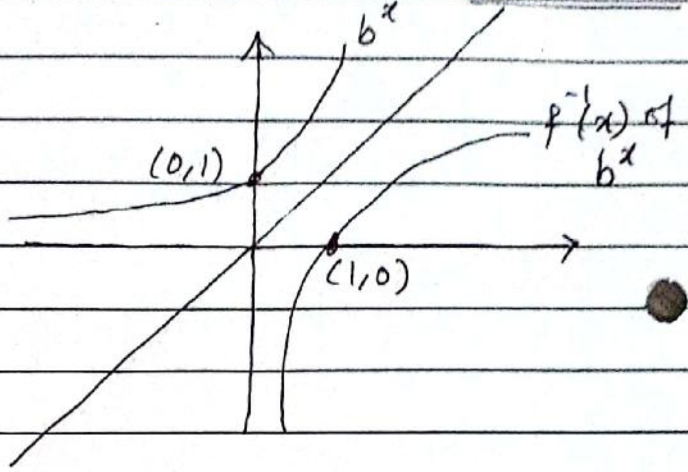
$u = e^x$

$du = e^x\,dx$

$f(x) = b^x$ it is a one-to-one function $\rightarrow$ $f^{-1}(x)$ exists

b/c

	$f(x) = b^x$	f^{-1}
Domain	$(-\infty, \infty)$	$(0, \infty)$
Range	$(0, \infty)$	$(-\infty, \infty)$



Find $f^{-1}(x)$?

(1) $y = b^x$

(2) $x = b^y$

(3) $f^{-1}(x)$ $\rightarrow$ no explicit relation by algebra

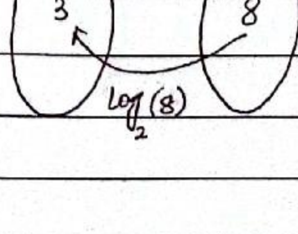
$\downarrow$

$\log_b(x)$

$\boxed{\text{Def}}$ The inverse function of $f(x) = b^x$ is: $f^{-1}(x) = \log_b(x)$

where: $\log_b(x) = y$ iff $x = b^y$

$\frac{4}{2} = 2$



(ex) $\log_2 16 = y$ iff $16 = 2^y$

$\boxed{y = 4}$ b/c $2^4 = 16$

(ex) Solve $\log_3(2x+1) = 2$ iff $2x+1 = 3^2$

$2x+1 = 9$

$2x = 8$

$\boxed{x = 4}$

Date:

$\boxed{\text{Inverse Properties}}$ $f(f^{-1}(x)) = x$

$f^{-1}(f(x)) = x$

$\boxed{\text{So}}$ $\log_b(b^x) = x$

$b^{\log_b(x)} = x$

ex $e^{7-4x} = 6$

$\log_e e^{7-4x} = \log_e 6$

$7-4x = \log_e 6$

$\boxed{x = \dfrac{7-\log_e 6}{4}}$

ex $2^{3x^2-1} = 8$, $2^{3x^2-1} = 2^3$

$\log_2(2^{3x^2-1}) = \log_2(8)$

$3x^2-1 = 3$

$x = \pm\sqrt{\dfrac{4}{3}}$

ex $4^{3x^2-1} = 8$

$\log_4(4^{3x^2-1}) = \log_4(8)$ $\quad 4^{?} = 8$

$3x^2-1 = 3/2$ $\quad 4^{3/2} = 8$

$3x^2 = 5/2$

$x = \pm\sqrt{\dfrac{5}{6}}$

$b^x b^y = b^{x+y}$ $\quad \log_b(xy) = \log_b x + \log_b y$

$\dfrac{b^x}{b^y} = b^{x-y}$ $\quad \log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

$(b^x)^r = b^{xr}$ $\quad \log(x^r) = r\log_b x$

$\boxed{\text{Special bases}}$

$b=e$ $\quad b=10$

(natural number) $\quad$ (common)

e^x vs $\log_e(x)$ $\quad 10^x$ vs $\log_{10}(x)$

$\to \ln(x)$ $\quad \to \log(x)$

$\boxed{\text{Apps}}$ ① $2\ln x + 3\ln y - \ln z$

$= \ln x^2 + \ln y^3 - \ln z = \ln\left(\dfrac{x^2 y^3}{z}\right)$

Date:

(2) Geometric Mean $(x_1 \cdot x_2 \cdot x_3 \cdot x_4)^{1/4}$

$\boxed{\text{use}}$ $e^{\ln x} = x$

$$(x_1 \cdot x_2 \cdot x_3 \cdot x_4)^{1/4} = e^{\ln(x_1 \cdot x_2 \cdot x_3 \cdot x_4)^{1/4}} = e^{\frac{1}{4}\ln(x_1 \cdot x_2 \cdot x_3 \cdot x_4)}$$

$$= e^{\frac{\ln(x_1)+\ln(x_2)+\ln(x_3)+\ln(x_4)}{4}}$$

$\boxed{\text{Change of base}}$

$$b^x = \log_b(x) \qquad b \neq e,\ b \neq 10$$

change of base b into b = e

(1) $\log_b x = \frac{1}{\ln(b)} \times \ln(x)$

(2) $b^x = e^{\ln(b^x)} = e^{x\ln(b)}$

(so) $2^{x+1} = e^{(x+1)\ln(2)}$, $\log_2(3x-5) = \frac{1}{\ln(2)}\ln(3x-5)$

$\boxed{\text{Derivatives}}$ (1) $D_x[e^x] = e^x$

(2) $D_x[b^x] = D_x[e^{x\ln b}] = e^{x\ln b} \cdot \ln b = b^x[\ln b]$

$$D_x[b^x] = b^x \lim_{h\to 0} \frac{b^h-1}{h} = \ln(b)$$

$\boxed{f^{-1}?}$ $D_x[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$

(3) $D_x[\ln x] = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$

$f(x) = e^x \rightarrow f'(x) = e^x$

$f^{-1}(x) = \ln(x)$

$\boxed{\text{Note:}}$ on (1) $D_x[e^x] = e^x$ Domain: $(-\infty, +\infty)$

on (3) $D_x[\ln(x)] = \frac{1}{x}$ Domain: $(0, +\infty)$

(4) $D_x[\log_b(x)] = D_x\left[\frac{1}{\ln(b)} \cdot \ln(x)\right] = \frac{1}{\ln(b)} \cdot \frac{1}{x}$ Domain: $(0, +\infty)$

Date:

Antiderivatives

① $\int e^x dx = e^x + c$

② $\int b^x dx = \frac{1}{\ln(b)} b^x + c$

③ $\int \frac{1}{x} dx = \ln|x| + c$

Consider $\ln|x| = \begin{cases} \ln(x), & x > 0 \\ \ln(-x), & x < 0 \end{cases}$

$D_x[\ln|x|] = \begin{cases} D_x[\ln(x)] \\ D_x[\ln(-x)] \end{cases} = \begin{cases} \frac{1}{x} \\ \frac{1}{-x} \cdot (-1) = \frac{1}{x} \end{cases}$

ex $D_x[x^5 + 5^x] = D_x[x^5] + D_x[5^x] = 5x^4 + 5^x \ln(5)$

ex $D_x\left[\ln\left(\frac{\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}}\right)\right]$

$= \frac{1}{\sqrt{\frac{a^2-x^2}{a^2+x^2}}} \cdot D_x\left[\sqrt{\frac{a^2-x^2}{a^2+x^2}}\right]$

$= \sqrt{\frac{a^2+x^2}{a^2-x^2}} \cdot \frac{1}{2}\left(\frac{a^2-x^2}{a^2+x^2}\right)^{-1/2} \left[\frac{-2x(a^2+x^2) - (a^2-x^2)(2x)}{(a^2+x^2)^2}\right]$

vs

$D_x\left[\ln\left(\frac{\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}}\right)\right]$

$\ln\left(\sqrt{\frac{a^2-x^2}{a^2+x^2}}\right) = \frac{1}{2}\ln\frac{a^2-x^2}{a^2+x^2} = \frac{1}{2}\left[\ln(a^2-x^2) - \ln(a^2+x^2)\right]$

$$= D_x\left[\frac{1}{2}\ln(a^2-x^2) - \frac{1}{2}\ln(a^2+x^2)\right]$$

$$= \frac{1}{2}\,\frac{1}{a^2-x^2}\cdot(-2x) - \frac{1}{2}\,\frac{1}{a^2+x^2}\cdot(2x)$$

$$= \frac{x}{x^2-a^2} - \frac{x}{x^2+a^2}$$

Logarithmic Differentiation

ex $\quad D_x\left[x^x\right] \xleftarrow{\text{Nope}}$ $\boxed{\begin{array}{l} x^{\text{const}} \to x^2 \\ \text{const}^x \to e^x \end{array}}$ Know These

Note

$$y = x^x$$
$$D_x[y] = D_x\left[x^x\right]$$
$$y' = D_x\left[x^x\right]$$

consider $\ln(y) = \ln(x^x) \Rightarrow \ln(y) = x\ln(x)$

Now Implicit Derivatives

$$\frac{1}{y}y' = (1)\ln(x) + x\left(\frac{1}{x}\right)$$

$$y' = y\left[\ln(x)+1\right]$$
$$y' = x^x\left[\ln(x)+1\right]$$

$$D_x\left[x^x\right] = x^x\left[\ln(x)+1\right]$$

Know $\quad f(x) = e^x \qquad f(x) = \ln(x)$

1. Derivatives
2. Antiderivatives

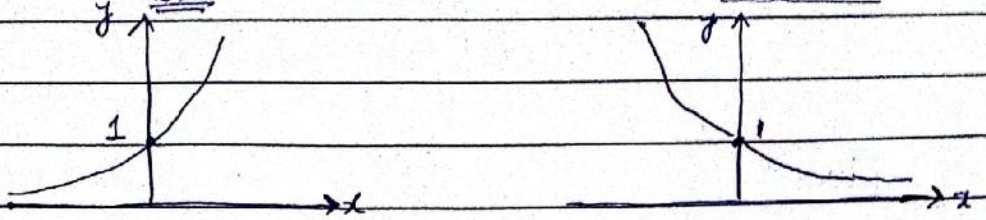
Date:

Lecture 36

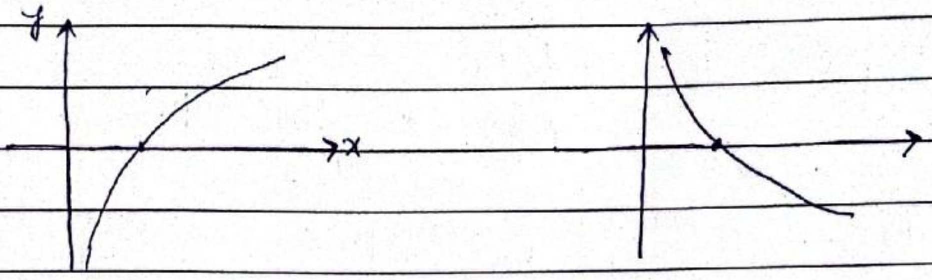
b^x	$\log_b(x)$	
$(-\infty,\infty)$	$(0,+\infty)$	$\leftarrow$ Domain
$(0,+\infty)$	$(-\infty,\infty)$	$\leftarrow$ Range

$b \geq 1$ $\qquad\qquad\qquad\qquad$ $0 < b < 1$

b^x



$\log_b(x)$



① $D_x[e^x] = e^x$, $\int e^x\,dx = e^x + c$

② $D_x[\ln x] = \frac{1}{x}$, $\int \frac{1}{x}\,dx = \ln|x| + c$

any b $\quad \log_b(x) = \frac{1}{\ln b} \cdot \ln x$ b/c $y = \log_b(x)$ iff $b^y = x$

$$\ln b^y = \ln x$$
$$y \ln b = \ln x$$
$$y = \frac{\ln x}{\ln b}$$
$$\log_b(x) = \frac{1}{\ln b} \cdot \ln x$$

$$b^x = e^{x\ln(b)}$$

① $D_x[b^x] = \ln(b)\, b^x$, $\int b^x\,dx = \frac{1}{\ln b} \cdot b^x + c$

② $D_x[\log_b(x)] = \frac{1}{\ln(b)} \cdot \frac{1}{x}$

ex $f(t) = \sin\left(e^{t^2+1}\right)$

$f'(t) = \cos\left(e^{t^2+1}\right) \frac{d}{dt}\left(e^{t^2+1}\right)$

$= \cos\left(e^{t^2+1}\right) e^{t^2+1} \frac{d}{dt}\left[t^2+1\right]$

$= \cos\left(e^{t^2+1}\right) e^{t^2+1}(2t)$

$= 2t\, e^{t^2+1} \cos\left(e^{t^2+1}\right)$

ex $\int \tan\theta \, d\theta$ vs $\int \sec^2\theta \, d\theta = \tan\theta + c$

b/c $D_\theta[\tan\theta] = \sec^2\theta$

$= \int \frac{\sin\theta}{\cos\theta} d\theta = -\int \frac{1}{u} du$

$= -\ln|u| + c = -\ln|\cos\theta| + c$

$u = \cos\theta$ $\quad = -\ln|\cos\theta|^{-1} + c$

$du = -\sin\theta \, d\theta$ $\quad = \ln|\sec\theta| + c$

$\int \tan\theta \, d\theta = \ln|\sec\theta| + c = -\ln|\cos\theta| + c$

ex $f(x) = \frac{x}{1-\ln(x-1)}$

Domain ① $x-1 > 0 \rightarrow x > 1$

and

$1-\ln(x-1) \neq 0$

$1-\ln(x-1) = 0$

$\ln(x-1) = 1$

$\log_e(x-1) = 1$

$e^1 = x-1$

$x = 1+e$

Domain: $x > 1$, $x \neq e+1$

$(1, e+1) \cup (e+1, +\infty)$

Intercepts

① no y-intercept (b/c $x > 1$)

② x-intercept $y=0$

$0 = \frac{x}{1-\ln(x-1)}$

$\rightarrow x \neq 0$ not in domain

No x-intercept

Asymptotes

$\boxed{\text{Vertical}}$ Zeros of denominator: $x = e+1$

① $\lim_{x \to (e+1)^+} \frac{x}{1-\ln(x-1)} = \lim_{x \to (e+1)^+} \frac{e+1}{1-\ln(x-1)} = -\infty$

② $\lim_{x \to (e+1)^-} \frac{x}{1-\ln(x-1)} = +\infty$

$\boxed{\text{horizontal}}$ $\lim_{x \to +\infty} \frac{x}{1-\ln(x-1)} = -\infty$

$$f(x) = \frac{x}{1-\ln(x-1)}$$

$$f'(x) = \frac{(1)(1-\ln(x-1)) - (x)(-\frac{1}{x-1} \cdot 1)}{(1-\ln(x-1))^2}$$

$$f'(x) = \frac{(1-\ln(x-1))(x-1) + x}{(1-\ln(x-1))^2(x-1)}$$

ex $\int \frac{\cos(\ln x)}{x} dx = \int \cos u \, du = \sin u + c$

$= \sin(\ln(x)) + c$

$u = \ln x$

$du = \frac{1}{x} dx$

ex $\int \frac{\sin 2x}{1+\cos^2 x} dx$ Know: $\sin 2x = 2\sin x \cos x$

Let $u = 1+\cos^2 x$

$du = -2\cos x \sin x \, dx$

$= -\int \frac{1}{u} du = -\ln|u| + c = -\ln|1+\cos^2 x| + c$

ex $\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \int \frac{1}{u} du = \ln|u| + c$

$= \ln|\sin x| + c$

Let $u = \sin x$

$du = \cos x \, dx$

vs $\int \tan x \, dx = -\ln|\cos x| + c = \ln|\sec x| + c$

Date:

Logarithmic Differentiation

use $\ln(a^b) = b\ln(a)$

$\ln(ab) = \ln(a) + \ln(b)$

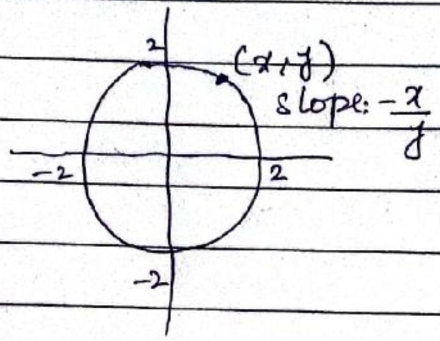
$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

with implicit differentiation

ex Implicit Derivatives: $x^2 + y^2 = 4$

$2x + 2yy' = 0$

$y' = -\frac{x}{y}$



ex $y = \frac{e^{-x}\cos^2 x}{x^2+x+1}$ explicit Derivation: $y' = D_x\left[\frac{e^{x}\cos^2 x}{x^2+x+1}\right]$

need: quotient Rule, Product Rule, chain Rule, exponents, Trig Polynomials.

vs $y = \frac{e^{-x}\cos^2 x}{x^2+x+1}$

$\ln(y) = \ln\left[\frac{e^{-x}\cos^2 x}{x^2+x+1}\right]$

$\ln(y) = (-x) + 2\ln(\cos x) - \ln(x^2+x+1)$

use implicit derivation

$\frac{1}{y}y' = -1 + -\frac{2\sin x}{\cos x} - \frac{2x+1}{x^2+x+1}$

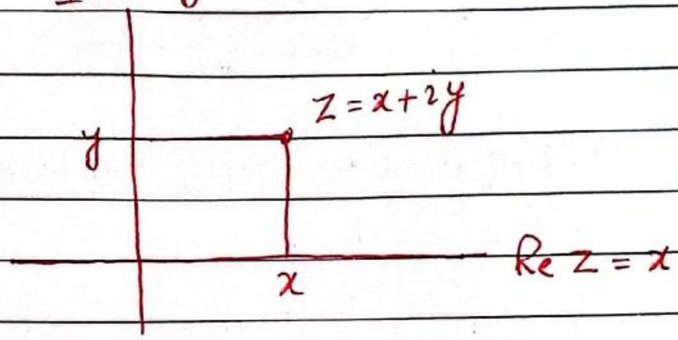
$y' = y\left[-1 - 2\tan x - \frac{2x+1}{x^2+x+1}\right]$

Date:

Lecture 37

The complex Plane

- complex numbers: expressions of the form $z = x + iy$, where
  - x is called the real part of z; $\mathrm{Re}\, z = x$
  - y is called the imaginary part of z; $y = \mathrm{Im}\, z$
- Set of complex numbers: $\mathbb{C}$ (the complex plane)
- Real numbers: subset of the complex numbers (those whose imaginary part is zero). $z = x + 0i$
- The complex ~~com~~plane can be identified with $\mathbb{R}^2$



Adding Complex Numbers

$$z = x + iy$$
$$w = u + iv$$
$$z + w = \underbrace{(x+u)}_{\mathrm{Re}(z+w)} + i\underbrace{(y+v)}_{\mathrm{Im}(z+w)}$$

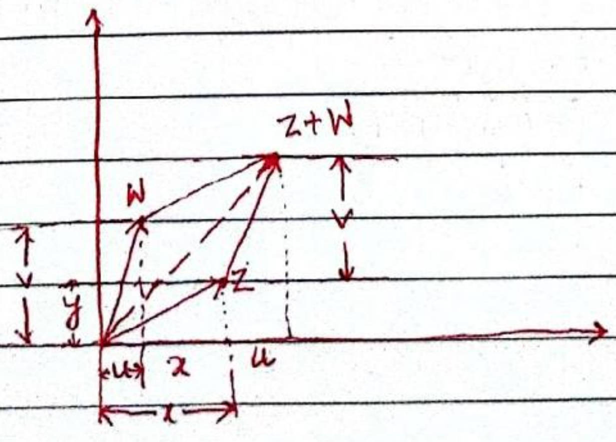
$(3+5i)+(-1+2i) = 2+7i$

Thus $\mathrm{Re}(z+w) = \mathrm{Re}\, z + \mathrm{Re}\, w$

~~Im~~

$$\mathrm{Im}(z+w) = \mathrm{Im}\, z + \mathrm{Im}\, w$$

Graphically, this corresponds to vector addition

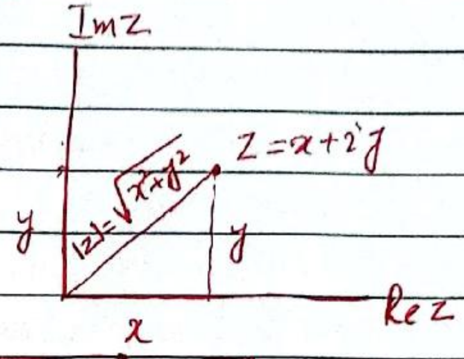


Date:

The Modulus of a Complex Number

The modulus of the complex number $z = x + iy$ is the length of the vector z:

$$|z| = \sqrt{x^2 + y^2}$$



~~What is i? $i = 0 + 1i$~~
~~so $i^2 = (0+1i)(0+1i) = 0 \cdot 0 + 0 \cdot 1i + 1i \cdot 0 +$~~

Multiplication of complex Numbers

$i = \sqrt{-1}$, $i^2 = -1$

- Motivation: $(x+iy)(u+iv) = xu + ixv + iyu + i^2 yv$
$$= (xu - yv) + i(xv + yu)$$

$$(3+4i)(-1+7i) = (-3-28) + i(21-4) = (-31+17i)$$

$(z_1 z_2) z_3 = z_1 (z_2 z_3)$ (Associative)

$z_1 z_2 = z_2 z_1$ (commutative)

$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$ (Distributive)

What is i $\quad i = 0 + 1i$

$$i^2 = (0+1i)(0+1i) = (0 \cdot 0 - 1 \cdot 1) + i(0 \cdot 1 + 1 \cdot 0) = -1$$

$i^3 = i^2 \cdot i = -1 \cdot i = -i$

$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

$i^5 = i^4 \cdot i = i$

$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i$

Date:

Complex Number Division

$z = x+iy \qquad w = u+iv$

$\frac{z}{w} \quad (w \neq 0)$

$$\frac{z}{w} = \frac{x+iy}{u+iv}$$

$$= \frac{(x+iy)(u-iv)}{(u+iv)(u-iv)} = \frac{(xu+yv)+i(-xv+yu)}{(u^2+v^2)+i(-uv+vu)}$$

$$= \frac{xu+yv}{u^2+v^2} + i\,\frac{yu-xv}{u^2+v^2}$$

$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$, as long as $z \neq 0$

$$\frac{3+4i}{-1+7i} = \frac{(3+4i)(-1-7i)}{(-1+7i)(-1-7i)} = \frac{(-3+28)+i(-21-4)}{(1+49)+i(7-7)} = \frac{25+i(-25)}{50}$$

$$= \frac{1}{2} - \frac{1}{2}i$$

The Complex Conjugate

If $z = x+iy$ then $\bar{z} = x-iy$ is the complex conjugate of z.

properties:

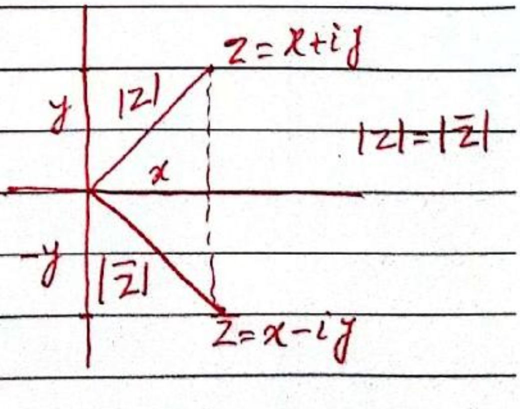
$\bar{\bar{z}} = z$

$\overline{z+w} = \bar{z}+\bar{w}$

$|z| = |\bar{z}|$

$z\bar{z} = (x+iy)(x-iy) = x^2+y^2 = |z|^2$

$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$



$z = \bar{z}$ if $z \in \mathbb{R}$

$z+\bar{z} = (x+iy)+(x-iy) = 2x$ so $\operatorname{Re} z = \frac{z+\bar{z}}{2}$, similarly $\operatorname{Im} z = \frac{z-\bar{z}}{2}$

$|z \cdot w| = |z|\,|w|$

$\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}, \ (w \neq 0)$, $\quad |z| = 0$ if and only if $z = 0$

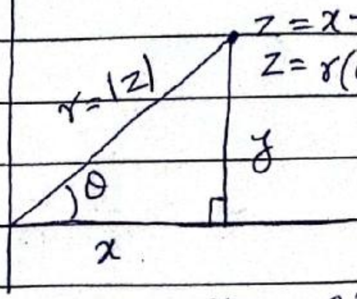
Date:

Lecture 38

- Consider $z = x + iy \in \mathbb{C}$, $z \neq 0$
- z can also be described by the distance 'r' from the origin ($r = |z|$) and the angle θ between the positive x-axis and the line segment from 0 to z.

- (r, θ) are the polar coordinates of z.
- Relation between Cartesian (x, y) and polar coordinates (r, θ):

$$x = r\cos\theta$$
$$y = r\sin\theta$$



So
$$z = x + iy$$
$$= r\cos\theta + i\, r\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

$\cos\theta = \dfrac{x}{r}$, $\sin\theta = \dfrac{y}{r}$

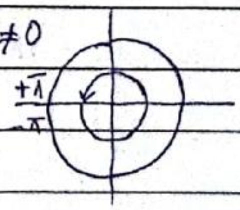
This is called the polar representation of z.

The Argument of a complex Number

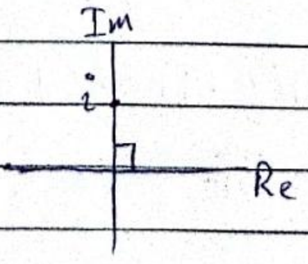
$r = |z|$ is easy to find, but how to find θ? Note θ is not unique!

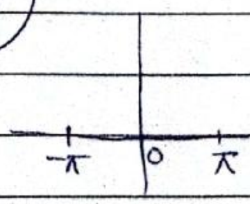
The principal argument of z, called $\operatorname{Arg} z$, is the value of θ for which $-\pi < \theta \leq \pi$

$\arg z = \{\operatorname{Arg} z + 2\pi k : k = 0, \pm 1, \pm 2, \cdots\}$, $z \neq 0$

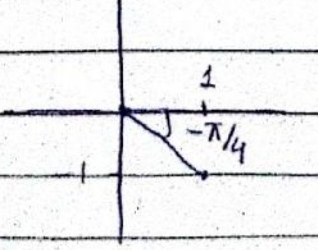


$\operatorname{Arg} i$





$\operatorname{Arg}(1 - i)$

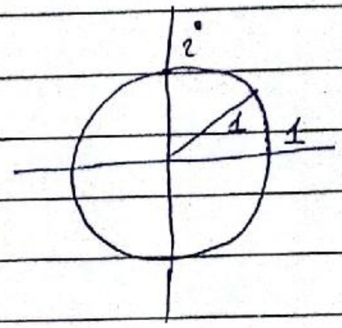


Exponential Notation

- Convenient notation: $e^{i\theta} = \cos\theta + i\sin\theta$
- So $z = r(\cos\theta + i\sin\theta)$ becomes $z = re^{i\theta}$, the polar form of z.
- Note: $e^{i(\theta+2\pi)} = e^{i\theta} = e^{i(\theta+4\pi)} = \cdots = e^{i(\theta+2k\pi)}$, $k \in \mathbb{Z}$

$$e^{i\pi/2} = i \qquad 1e^{i\pi/2} = 1\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = i$$

$$e^{j2\pi} = 1 \qquad 1e^{j2\pi} = 1\left(\cos(2\pi) + i\sin(2\pi)\right) = 1$$



Properties of the exponential Notation

$$|e^{i\theta}| = 1, \qquad |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta}$$
$$= 1$$

$$\overline{e^{i\theta}} = e^{-i\theta}$$

$$\overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$$
$$\because \cos(-\theta) = \cos(\theta), \quad \sin(-\theta) = -\sin(\theta)$$
$$= \cos(-\theta) + i\sin(-\theta)$$
$$= e^{-i\theta}$$

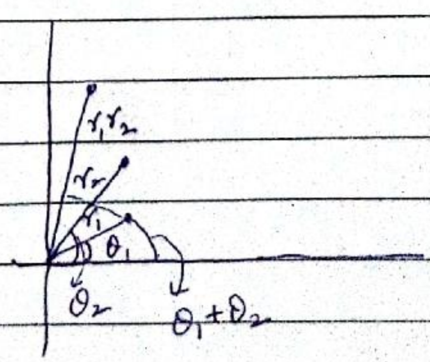
$$\frac{1}{e^{i\theta}} = e^{-i\theta}$$

$$e^{i(\theta+\varphi)} = e^{i\theta} \cdot e^{i\varphi}$$

Multiplication In polar Form

- Consider $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, $\quad z_1 z_2 = ?$

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = (r_1 r_2)\, e^{i(\theta_1+\theta_2)}$$



Date:

Lecture 39

De Moivre's Formula

Basis $$e^{i\theta}\cdot e^{i\theta} = e^{i(\theta+\theta)} = e^{i2\theta}$$

$$e^{i2\theta}\cdot e^{i\theta} = e^{i(2\theta+\theta)} = e^{i3\theta}$$

$$\left(e^{i\theta}\right)^2 = e^{i2\theta}$$
$$\left(e^{i\theta}\right)^3 = e^{i3\theta}$$

$$\left(e^{i\theta}\right)^n = e^{in\theta}$$
$$\therefore e^{i\theta} = \cos\theta + i\sin\theta$$
$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

This can be used to derive equations for the sine and cosine.

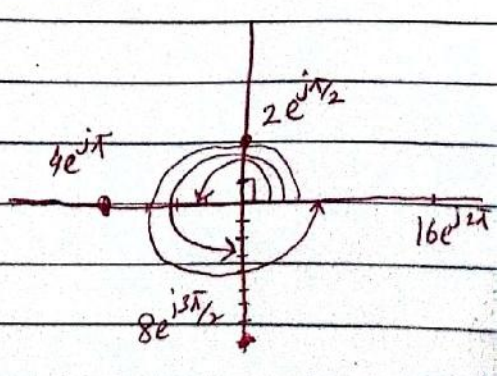
Ex: $n=3$
$$(\cos\theta + i\sin\theta)^3 \qquad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$= \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta$$
$$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$
as $i^2=-1$ and $i^3=-i$
$$= \cos(3\theta) + i\sin(3\theta)$$

Hence
$$\cos 3\theta = \cos^3\theta - 3\cos\sin^2\theta$$
$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

General $$Z = re^{j\theta}$$
$$Z^n = \left(re^{j\theta}\right)^n = r^n e^{jn\theta}$$

Ex: $Z = 2e^{j\pi/2}$
$Z^2 = 4e^{j\pi}$ $\qquad Z^4 = 16e^{j2\pi}$
$Z^3 = 8e^{j\frac{3\pi}{2}}$
⋮



Date:

Roots of Complex Numbers

Let w be a complex number. An n^{th} root of w is a complex number z such that

$$z^n = w$$

If $w \neq 0$ there are exactly n distinct n^{th} roots.

$$w = \alpha e^{i\phi} \qquad z = re^{i\theta}$$

$$z^n = w$$

$$r^n e^{in\theta} = \alpha e^{i\phi}$$

$$r^n = \alpha \quad , \quad e^{in\theta} = e^{i\phi}$$

$$r = \sqrt[n]{\alpha} = (\alpha)^{1/n}$$

$$n\theta = \phi + 2k\pi \qquad k \in \mathbb{Z}$$

$$\theta = \frac{\phi}{n} + \frac{2k\pi}{n}, \quad k = 0, 1, \ldots, n-1$$

Hence

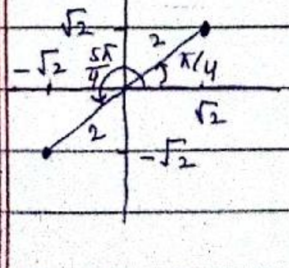
$$w^{1/n} = \sqrt[n]{\alpha}\, e^{i\left(\frac{\phi}{n} + \frac{2\pi k}{n}\right)}, \qquad k = 0, 1, \ldots, n-1$$

Ex square roots of $4i$

$$4i = 4e^{i\pi/2} \quad \text{so } \alpha = 4,\ \phi = \frac{\pi}{2},\ n = 2$$

$$(4i)^{1/2} = \sqrt{4}\, e^{j\left(\frac{\pi}{4} + \frac{2k\pi}{2}\right)} \qquad k = 0, 1$$

$$= \begin{cases} 2e^{j\pi/4} & \text{if } k = 0 \\ 2e^{j\left(\frac{\pi}{4} + \pi\right)} & \text{if } k = 1 \end{cases}$$



$$= \pm\left(\sqrt{2} + i\sqrt{2}\right)$$

Date:

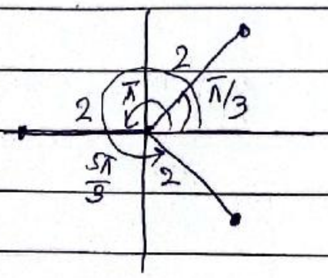
Ex Cubed roots of -8:

$$W = \alpha e^{i\phi}, \quad W^{1/n} = \sqrt[n]{\alpha}\, e^{i\left(\frac{\phi}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, 1, \cdots, n-1$$

$$-8 = 8e^{j\pi}, \text{ so } \alpha = 8,\ \phi = \pi,\ n = 3$$

$$(-8)^{1/3} = \sqrt[3]{8}\, e^{i\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)}, \quad k = 0, 1, 2$$

$$= \begin{cases} 2e^{i\pi/3} & \text{if } k = 0 \\ 2e^{i\left(\frac{\pi}{3} + \frac{2}{3}\pi\right)} = 2e^{i(\pi)} = -2 & \text{if } k = 1 \\ 2e^{i(5\pi/3)} & \text{if } k = 2 \end{cases}$$

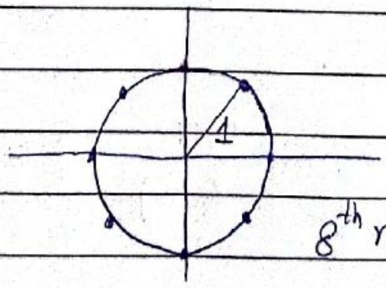


Roots of unity: the n^{th} roots of 1 are called the n^{th} roots of unity

$$1 = 1e^{j0}$$

$$1^{1/n} = \sqrt[n]{1}\, e^{i\left(\frac{0}{n} + \frac{2\pi k}{n}\right)}, \quad k = 0, 1, \cdots n-1$$

$$= e^{j\frac{2\pi k}{n}}, \quad k = 0, 1, \cdots n-1$$



$$\frac{2\pi k}{8} = \frac{\pi}{4} k$$

8^{th} roots of 1.