Necklaces, Cycles, and Counting

A Journey through Pólya's Theorem

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How many necklaces can be made by n distinct beads?

With n beads and k colors, how many different necklaces can we make?

Group

Definition

A group is a set G together with a binary operation $*$ such that the following axioms hold:

- $\forall$a,b,c$\in$G a$*$(b$*$c)=(a$*$b)$*$c
- $\exists$e$\in$G $\forall$a$\in$G a$*$e=e$*$a=a
- $\forall$a$\in$G $\exists a^{-1}\in$G a$*a^{-1}=a^{-1}*$a=e

Group Action

Definition

Let G be a group and S a set. An *action* of G on S is a map

$$* : G \times S \longrightarrow S$$

such that for every $s \in S$ and $a, b \in G$, the following properties hold:

1. $a * (b * s) = (ab) * s$.
2. $e * s = s$, where e is the identity element of the group G.

Group Action

Definition

Let G be a group acting on a set S. We define a relation "$\sim$" on S by

$$s_1 \sim s_2 \iff \exists\, g \in G \text{ such that } g * s_1 = s_2.$$

Equivalently, $s_1 \sim s_2$ if and only if there exists $g \in G$ with $g * s_1 = s_2$.

Lemma

The relation "$\sim$" defined above is an equivalence relation on S. Hence the elements of S are partitioned into equivalence classes.

Group Action

Definition

Let G be a group acting on a set S. For each $x \in S$, define the *orbit* of x, denoted $\bar{x}$, by

$$\bar{x} \;=\; \{\, s \in S \mid \exists\, g \in G:\; g * x = s\} \;=\; \{\, g * x \mid g \in G\}.$$

Definition

Let G be a group acting on a set S. For each $x \in S$, define the *stabilizer* of x, denoted G_x, by

$$G_x \;=\; \{\, g \in G \mid g * x = x\}.$$

Group Action

Lemma

Let G be a group acting on a set S. For each $x \in S$, G_x is a subgroup of G.

Theorem

If G be a group acting on a set S and $x \in S$,
then $|\bar{x}| = [G : G_x]$.

Group Action

Corollary

Let G be a finite group acting on a set S. Choose a set of representatives $S' \subseteq S$ with exactly one element from each orbit. Then the class equation (orbit sum) is:

$$|S| = \sum_{t \in S'} [G : G_t],$$

where $G_t = \{ g \in G \mid g * t = t \}$ is the stabilizer of t, and $[G : G_t]$ is the index of the stabilizer in G.

Burnside's Lemma

Definition

Let G be a group acting on a set S. For each $g \in G$, define:

$$S_g = \{ s \in S \mid g * s = s \}.$$

Theorem (Burnside)

Let G be a finite group acting on a non-empty, finite set S. Suppose the action partitions S into m distinct orbits. Then Burnside's Lemma states that the number of orbits m is given by:

$$m = \frac{1}{|G|} \sum_{g \in G} |S_g|.$$

Application of Burnside's Lemma

Question

With 5 beads and 2 colors, how many different necklaces can we make?

Answer

Let $G = D_5$ (the dihedral group of order 10), and let

$$S = \{\text{all colorings of 5 beads with 2 colors}\}.$$

We wish to count the number of orbits of S under the action of G, i.e. the number of distinct necklaces up to rotation and reflection.
By Burnside's Lemma:

$$\#\text{necklaces} = \frac{1}{|G|} \sum_{g \in G} |S_g| .$$

Application of Burnside's Lemma

Answer

We partition the group elements:

1. Identity e: fixes every coloring, so $|S_e| = 2^5 = 32$.
2. Rotations by $72^\circ, 144^\circ, 216^\circ, 288^\circ$: A coloring is fixed by a nontrivial rotation only if all beads are the same color.

   $$|S_g| = 2 \quad \text{for each such rotation.}$$

   There are 4 such rotations, contributing $4 \times 2 = 8$.
3. Reflections (5 axes): Each reflection fixes the bead lying on the axis, and pairs the remaining 4 beads into 2 mirrored pairs. Thus:

   $$|S_g| = 2^3 = 8 \quad \text{for each reflection.}$$

   Total from reflections is $5 \times 8 = 40$.

Application of Burnside's Lemma

Answer

Combining:

$$\begin{aligned}\#\text{necklaces} &= \frac{1}{10}\left(32 + 4 \cdot 2 + 5 \cdot 8\right) \\ &= \frac{1}{10}\left(32 + 8 + 40\right) \\ &= \frac{80}{10} = 8.\end{aligned}$$

Therefore, there are $\boxed{8}$ distinct necklaces with 5 beads and 2 colors under D_5.

Application of Burnside's Lemma

Another Insight

We analyze each type of permutation via its cycle structure:

- Identity e: cycle structure (1)(2)(3)(4)(5), i.e. five 1-cycles.

$$|S_g| = 2^5 = 32.$$

- Nontrivial rotations:
  - Rotation by 1 step r: cycle structure (1 2 3 4 5), one 5-cycle.
  - Rotation by 2 steps r^2: also a 5-cycle, similarly for r^3 and r^4.

  A coloring is fixed iff all beads in the cycle share the same color, so:

$$|S_{r^k}| = 2, \quad k = 1, 2, 3, 4.$$

  Total from rotations: $4 \times 2 = 8$.

Application of Burnside's Lemma

Another Insight

- Reflections: each has cycle structure $(i)(ab)(cd)$, i.e. one fixed point and two 2-cycles (since 5 is odd). To be fixed, the bead in the 1-cycle is free (2 choices), and each 2-cycle must be monochromatic:

$$|S_g| = 2^3 = 8, \quad \text{for each of the 5 reflections.}$$

Total from reflections: $5 \times 8 = 40$.

Application of Burnside's Lemma

Another Insight

Cycle structures summary:

$$\begin{array}{lll} e: & (1)(2)(3)(4)(5) & 5 \times 1\text{-cycles} \\ r^k\ (k = 1,2,3,4): & (1\ 2\ 3\ 4\ 5) & 1 \times 5\text{-cycle} \\ s_i\ (i = 1,\dots,5): & (i)\,(a\,b)\,(c\,d) & 1 \times 1\text{-cycle} + 2 \times 2\text{-cycles} \end{array}$$

Fixed colourings count:

$$2^5 = 32, \quad 4 \times 2 = 8, \quad 5 \times 2^3 = 40.$$

$$\begin{aligned} \#\text{necklaces} &= \frac{1}{10}(32 \ + \ 8 \ + \ 40) \\ &= \frac{80}{10} = 8. \end{aligned}$$

Thus, there are $\boxed{8}$ distinct necklaces with 5 beads and 2 colors under the action of D_5.

Cycle Index Polynomial

For each $g \in G$, let $c_k(g)$ be the number of k-cycles in the permutation of D induced by g. The cycle index of G acting on D is defined by

$$P_G(X_1, X_2, \ldots, X_n) = \frac{1}{|G|} \sum_{g \in G} \prod_{k=1}^{n} X_k^{c_k(g)}.$$

Cycle Index for D_6 on 6 Beads

Cycle–Type Summary

$$
\begin{aligned}
&r^0 : (1)(2)(3)(4)(5)(6),\\
&r^{\pm 1} : (1\,2\,3\,4\,5\,6),\ (1\,6\,5\,4\,3\,2),\\
&r^{\pm 2} : (1\,3\,5)(2\,4\,6),\ (1\,5\,3)(2\,6\,4),\\
&r^3 : (1\,4)(2\,5)(3\,6),\\
&\text{vertex–reflections} : 3\ \big((1)(a\,b)(c\,d)\big),\\
&\text{edge–reflections} : 3\ \big((a\,b)(c\,d)(e\,f)\big).
\end{aligned}
$$

Cycle Index for D_6 on 6 Beads

Cycle Index Polynomial

$$P_{D_6} = \frac{1}{12}\left(X_1^6 + 2X_6 + 2X_3^2 + 4X_2^3 + 3X_1X_2^2\right)$$

Cycle Index for D_6 on 6 Beads

Pattern Insights

- X_1^6: identity symmetry.
- $2X_6$: rotations by $\pm 60^\circ$ — full 6-cycles.
- $2X_3^2$: rotations by $\pm 120^\circ$ — two 3-cycles.
- X_2^3: 180° rotation — three 2-cycles.
- $3X_1X_2^2$: vertex reflections — fix one bead + two swaps.
- $3X_2^3$: edge reflections — three swaps, no fixed points.

With n beads and two colors (blue and red), how many necklaces can be formed that have exactly p blue beads and $n-p$ red beads?

Group Action on Functions

Let G act on a finite set $D = \{1, 2, \ldots, n\}$. This induces an action on C^D, the set of functions $f \colon D \to C$, by

$$\big(g \cdot f\big)(d) = f\big(g^{-1}d\big), \quad \forall\, g \in G,\ d \in D.$$

Orbits of this action correspond to colorings of D up to symmetry via G.

Coloring and Equivalence

Each function $f: D \to C$ is a "coloring" of D. Two colorings f_1, f_2 are equivalent iff they lie in the same G-orbit:

$$f_2 = g \cdot f_1 \quad \text{for some } g \in G.$$

We aim to count or weight these orbits.

Weights and Generating Functions

Define a weight function $w : C \to R(commutative\ ring)$, and extend it to functions by

$$W(f) \; = \; \sum_{d \in D} w\big(f(d)\big).$$

Suppose $f_1, f_2 \in C^D$ lie in the same G-orbit:

$$f_2 = g \cdot f_1 \quad \text{for some } g \in G.$$

Then for each $d \in D$, $f_2(d) = f_1(g^{-1}d)$. Hence

$$W(f_2) = \sum_{d \in D} w\big(f_2(d)\big) = \sum_{d \in D} w\big(f_1(g^{-1}d)\big) = \sum_{d' \in D} w\big(f_1(d')\big) = W(f_1),$$

showing that all functions in an orbit share the same weight.

Weights and Generating Functions

Thus if F denotes a pattern (an orbit under the action of G on C^D), instead of considering weights f in F, it is enough to consider the weight $W(F)$ of F, which is then $W(f)$ for any choice of $f \in F$.

Pólya Enumeration Theorem

Theorem

Let D, C be two finite sets, and let G be a finite group acting on C^D. We assign a weight $w(c)$ to each element $c \in C$. The patterns F have induced weights $W(F)$. Then the pattern inventory is

$$\sum_F W(F) \;=\; P_G\left(\sum_{c\in C} w(c),\ \sum_{c\in C} w(c)^2,\ \sum_{c\in C} w(c)^3,\ \ldots\right),$$

where P_G is the cycle-index polynomial of G.

Pólya Enumeration Theorem

Corollary

If all the weights are chosen to be equal to 1, then the number of patterns (or orbits of G on C^D) is given by

$$\sum_F W(F) = P_G(|C|, |C|, \ldots, |C|),$$

Counting 4-Bead Necklaces with 2 Colors

Goal: Count distinct arrangements of 4 beads (red/blue), with exactly 2 red and 2 blue, up to *rotations and reflections* (D_4).

1. Cycle index for the dihedral group D_4 (8 symmetries):

$$P_{D_4}(X_1, X_2, X_3, X_4) = \frac{1}{8}\left(X_1^4 + 3X_2^2 + 2X_4 + 2X_1^2X_2\right)$$

2. Substitute $X_k = R^k + B^k$:

$$P = \frac{1}{8}\left[(R+B)^4+3(R^2+B^2)^2+2(R^4+B^4)+2(R+B)^2(R^2+B^2)\right]$$

Counting 4-Bead Necklaces with 2 Colors

3. Expand and combine coefficients of R^2B^2:
1. $(R+B)^4 = R^4 + 4R^3B + 6R^2B^2 + 4RB^3 + B^4$
2. $3(R^2+B^2)^2 = 3R^4 + 6R^2B^2 + 3B^4$
3. $2(R^4+B^4)$
4. $2(R+B)^2(R^2+B^2) = 2(R^2+2RB+B^2)(R^2+B^2) =$
$2R^4 + 4R^3B + 4R^2B^2 + 4RB^3 + 2B^4$

The coefficient of R^2B^2 becomes:

$$\frac{1}{8}(6+6+0+4) = \frac{16}{8} = 2$$

There are $\boxed{2}$ distinct bracelets with exactly 2 red and 2 blue beads (under D_4).

Counting Non-Isomorphic Graphs on 4 Vertices

Let G_4 be the set of all simple graphs on vertex set $V = \{1, 2, 3, 4\}$. We seek a generating function for the number of non-isomorphic graphs in G_4 with a specified number of edges.
Define the edge set:

$$X = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

A graph $H_1 = (V, E_1) \in G_4$ corresponds to a 2-coloring of X (each edge is colored "yes" or "no"), where edges in E_1 get color "yes" and the rest "no." Let C be the set of all such colorings.

Counting Non-Isomorphic Graphs on 4 Vertices

Two graphs H_1, H_2 are isomorphic iff their corresponding colorings are equivalent under the action of S_4 on X via vertex permutations:

$$\{i, j\} \mapsto \{f(i), f(j)\}.$$

Hence, counting non-isomorphic graphs reduces to counting non-equivalent colorings of X under S_4.

Example permutation:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

acts on the edges as:

$$\begin{pmatrix} \{1,2\} & \{1,3\} & \{1,4\} & \{2,3\} & \{2,4\} & \{3,4\} \\ \{2,3\} & \{3,4\} & \{1,3\} & \{2,4\} & \{1,2\} & \{1,4\} \end{pmatrix}$$

Let $S_4^{(2)}$ denote the induced permutation group on the 6 edges.

Conclusion: By Theorem Pólya's Counting Theorem, the number of non-isomorphic graphs on 4 vertices equals the number of inequivalent 2-colorings of X under $S_4^{(2)}$. We must compute the cycle index of $S_4^{(2)}$ — based on the cycle-structure of the 24 edge-permutations — and apply the standard substitution to generate the counting polynomial.

Counting Non-Isomorphic Graphs on 4 Vertices

Consider the induced action of S_4 on the 6 edges of K_4. Its cycle index is:

$$P_{S_4^{(2)}}(X_1, \ldots, X_6) = \frac{1}{24}\left(X_1^6 + 9X_1^2X_2^2 + 8X_3^2 + 6X_2X_4\right)$$

Apply Pólya: substitute $X_j = y^j + n^j$, where $y =$ "edge present", $n =$ "edge absent":

$$\frac{1}{24}\left((y+n)^6+9(y+n)^2(y^2+n^2)^2+8(y^3+n^3)^2+6(y^2+n^2)(y^4+n^4)\right)$$

Expanding and collecting terms yields the generating polynomial:

$$y^6 + y^5n + 2y^4n^2 + 3y^3n^3 + 2y^2n^4 + yn^5 + n^6$$

Interpretation: the coefficient of $y^k n^{6-k}$ gives the number of non-isomorphic graphs with k edges:

Edges (k)	**Non-isomorphic Graphs**
0	1
1	1
2	2
3	3
4	2
5	1
6	1

$\sum = 11$ total non-isomorphic simple graphs on 4 vertices.

Example: Coloring the corners and faces of a cube

Determine the symmetry group of a cube and the number of nonequivalent ways to color the corners and faces of a cube with a specified number of colors.

Further Reading

You have a square grid (2×2 cells). Each cell can be painted with one of 3 colors, but swapping color labels (e.g., red blue) is considered the same. Additionally, colorings that differ by rotations or reflections of the square are also considered identical. Count the number of essentially distinct colorings under these combined symmetries.

KEYWORD: de Bruijn Theorem

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