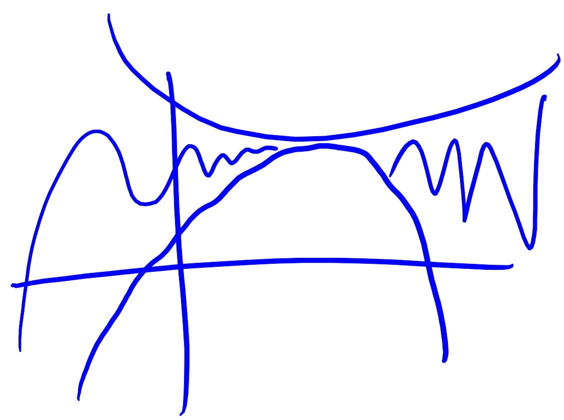
Midterm 1 - Review

1. State the following definitions / theorems:
   (a) function ← relationship that passes VLT.
   (b) graph of a relation   Set of pts that satisfy the relation
   (c) limit of a function; LHL and RHL $\lim_{x\to a} f(x)$ exists if
      i) $\lim_{x\to a^-} f(x)$ exists
      ii) $\lim_{x\to a^+} f(x)$, ..
      iii) they're equal.
   (d) Squeeze Theorem
   (e) vertical asymptote
   (f) horizontal asymptote
   (g) continuity of a function (at a point and on an interval)
   (h) Intermediate Value Theorem
   (i) derivative of a function at a point
   (j) derivative of a function
   (k) Power Rule
   (l) Product Rule
   (m) Quotient Rule
   (n) the two limit lemmas
   (o) the **most** important trig identity

   You might need to know other ones...

d) Squeeze thm: Consider



$$\ell(x) \le f(x) \le u(x)$$

$$\lim_{x\to a} \ell(x) = \lim_{x\to a} u(x) = L.$$

Then

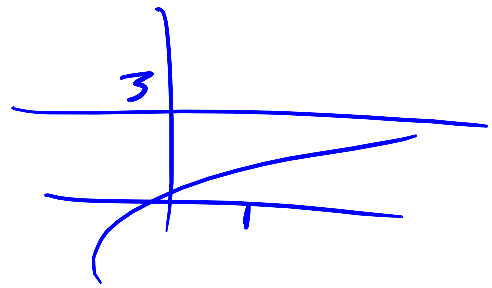
$$\lim_{x\to a} f(x) = L$$

e) V.A. at $x = a$ if

$$\lim_{x \to a^{\pm}} f(x) = \pm\infty$$

f) H.A. @ $y = L$

$$\lim_{x \to +\infty} f(x) = L$$



or $$\lim_{x \to -\infty} f(x) = L$$

g) $f(x)$ is cts @ $x = a$ if

i) $\lim_{x \to a} f(x)$ exist

ii) $f(a)$ exists

iii) they're equal.

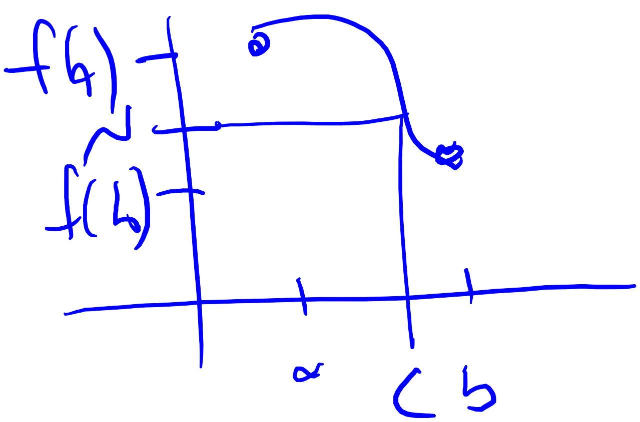
$f(x)$ is cts on (x_0, x_1) if

$\forall a \in (x_0, x_1)$, $f(x)$

is cts at $x = a$

"for all"

b) IVT: Consider $f(x)$ that



is continuous on

$[a,b]$ & N s.t.

$f(a) < N < f(b)$

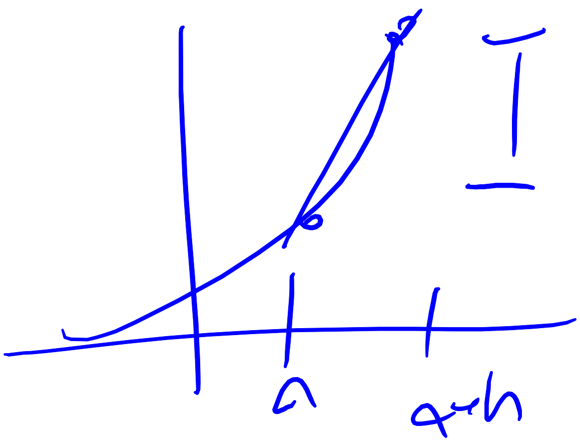
or $f(b) < N < f(a)$

Then there exist $c \in (a,b)$ s.t.

$$f(c) = N.$$

j) The der. @ $x=a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



k) Power rule:

$$\frac{d}{dx}\left[x^n\right] = n x^{n-1}$$

m) Qustient rule:

Consider $h = \dfrac{f(x)}{g(x)}$, $g(x) \neq 0$, then

$$h' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

n) $\displaystyle\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

$\displaystyle\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$

o) $\sin^2\theta + \cos^2\theta = 1$.

2. Compute the following limits.

(a) $\lim_{x\to-\infty} \frac{3x^2-1}{2x+5}$ $\frac{+}{-}$ $= -\infty$

(b) $\lim_{x\to 0} \frac{e^{x^2-1}+\sin x}{x+1}$ $= \boxed{\frac{e^{0-1}+\sin(0)}{0+1}}$

$= \frac{\frac{1}{e}+0}{1} = \frac{1}{e}.$

(c) $\lim\limits_{x\to 3^-} \dfrac{x^2-9}{|x-3|} = \lim\limits_{x\to 3^-} \dfrac{(x+3)\cancel{(x-3)}}{-\cancel{(x-3)}}$

$= \dfrac{3+3}{-1} = -6.$

$\lim\limits_{x\to 3^+} \dfrac{x^2-9}{|x-3|} = \lim\limits_{x\to 3^+} \dfrac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} = 6.$

when you plug in $x=3$, limit disappears.

(d) $\lim\limits_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x} \cdot \dfrac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}}$

$= \lim\limits_{x\to 0} \dfrac{1-(1-x^2)}{x(1+\sqrt{1-x^2})}$

$= \lim\limits_{x\to 0} \dfrac{x^{\cancel{2}}}{\cancel{x}(1+\sqrt{1-x^2})} = \dfrac{0}{1+\sqrt{1-0}} = 0.$

(e) $\lim_{h\to 0} \dfrac{f(4+h)-f(4)}{h}$ where $f(x)=\dfrac{1}{\sqrt{x}}$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{4+h}} - \frac{1}{\sqrt{4}}}{h}$$

(f) $\lim_{x\to -4} \dfrac{\frac{1}{4}+\frac{1}{x}}{x+4}$ ← C.D.

$\lim_{x\to 0} \frac{1}{x} \cdot \frac{\cos x - 1}{x}$ won't work

(g) $\lim_{x\to 0} \frac{\cos(x)-1}{x^2} \cdot \frac{\cos x + 1}{\cos x + 1}$

$= \lim_{x\to 0} \frac{\cos^2 - 1}{x^2(\cos x + 1)}$

$= \lim_{x\to 0} \frac{-\sin^2 x}{x^2} \cdot \frac{1}{\cos x + 1}$

$= \lim_{x\to 0} (-1) \cdot \frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \frac{1}{\cos x + 1}$

$= -1 \cdot 1 \cdot 1 \cdot \frac{1}{\cos 0 + 1} = \frac{-1}{2}$

(h) $\lim_{x\to 3} (2x + |x-3|)$

$= 6$

$\square$.

change question!

(i) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{|x|}\right)$ split

LHL:

$$\lim_{x\to 0^-} \left(\frac{1}{x} - \frac{1}{|x|}\right)$$

$|x| = -x$

$$= \lim_{x\to 0^-} \left(\frac{1}{x} + \frac{1}{x}\right)$$

$$= \lim_{x\to 0^-} \frac{2}{x} = -\infty \text{ so DNE.}$$

RHL:

$$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right)$$

$\vdots$

$$= 0.$$

So entire limit DNE

(j) $\lim_{x\to 1} 7^{\frac{x^2-1}{x-1}}$

$$= \lim_{x\to 1} 7^{\frac{(x-1)(x+1)}{x-1}}$$

$$= 7^{\lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}}$$

$$= 7^2 = 49$$

3. Prove $\frac{d}{dx}\sin x = \cos x$.
Hint: $\sin(a+b) = \sin a \cos b + \cos a \sin b$.

Recall

$$\lim_{u \to 0} \frac{\sin u}{u} = 1 \qquad \lim_{u \to 0} \frac{\cos u - 1}{u} = 0$$

4. **True or False.** Justify your answer: if the answer is false, provide a counter example (or explain why it is false); if the answer is true, write a couple words or a sentence saying why it is true.

As an exercise, if the answer is false ~~true~~, state how the question could be changed to make the statement true.

(a) If $f(s) = f(t)$ then $s = t$.

False:

to make it true, f : one-to-one.

$f(x) = x^2$ $f(-2) = f(2)$.

(b) If f and g are functions then $f \circ g = g \circ f$.

False! $f(x) = \cos x$ $g(x) = x^2$

(c) $\lim_{x \to 4} \left(\frac{2x}{x-4} - \frac{8}{x-4} \right) = \lim_{x \to 4} \left(\frac{2x}{x-4} \right) - \lim_{x \to 4} \left(\frac{8}{x-4} \right)$.

False

$\lim_{x \to 4} \frac{2x-8}{x-4}$

$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x} \right) = \lim_{x \to 0} (0) = 0.$

(d) If $\lim_{x \to 5} f(x) = 2$ and $\lim_{x \to 5} g(x) = 0$, then $\lim_{x \to 5} \frac{f(x)}{g(x)}$ does not exist.

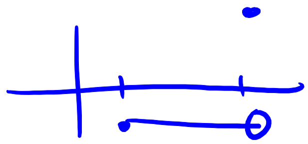
$\to \frac{2}{0}$

True

(e) If $\lim_{x \to 5} f(x) = 0$ and $\lim_{x \to 5} g(x) = 0$, then $\lim_{x \to 5} \frac{f(x)}{g(x)}$ does not exist.

False. $f(x) = x - 5 = g(x)$. $\lim_{x \to 5} \frac{f(x)}{g(x)} = 1$

False. (f) If $g(1) = -1$ and $g(2) = 5$ then there exists a number c between 1 and 2 such that $g(c) = 0$.



Fix: cts.

(g) If $1 \leq f(x) \leq x^2 + 2x + 2$ for all x near -1, then $\lim_{x\to -1} f(x) = 1$.

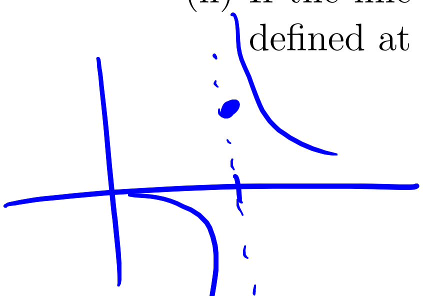
True, by Squeeze.

1

$\lim_{x \to -1} (x^2 + 2x + 2) = (-1)^2 - 2 + 2 = 1$

(h) If the line $x = 1$ is a vertical asymptote of $y = f(x)$, then f is not defined at 1 .

False!



$$f(x) = \begin{cases} \frac{1}{x-1} & \text{if } x \neq 1 \\ 7 & \text{if } x = 1 \end{cases}$$

(i) The equation $x + \ln(x+1) = x^4 - 1$ has a root in the interval $(0, 2)$.

$$\underbrace{x + \ln(x+1) - x^4 + 1}_{f(x)} = 0$$

$f(0) = 0 + \ln(1) - 0 + 1 > 0$

$f(2) = 2 + \ln(3) - 2^4 + 1 < 0$

$f(x)$ cts, so IVT says TRUE

If $x>2$, $f(x)$ is cts clearly.

5. Determine the constant c that makes f continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} c^2 + \sin(x\pi) & \text{if } x < 2 \\ cx^2 - 4 & \text{if } x \geq 2 \end{cases}.$$

If $x<2$, also cts.

Consider $x=2$.

write def of cont at $x=a$...

Check:

i) $f(2) = c \cdot 2^2 - 4$

ii) $\lim_{x \to 2} f(x)$

LHL

$$\lim_{x \to 2^-} f(x)$$

$$= \lim_{x \to 2^-} \left[c^2 + \sin(\pi x)\right]$$

$$= c^2 + \sin(2\pi)$$

$$= c^2 + 0$$

RHL:

$$\lim_{x \to 2^+} f(x)$$

$$= \lim_{x \to 2^+} \left[cx^2 - 4\right]$$

$$= 4c - 4$$

$\Rightarrow$ LHL = RHL

$\Rightarrow \quad c^2 = 4c - 4$

$\Rightarrow \quad c^2 - 4c + 4 = 0$

$\Rightarrow \quad (c-2)^2 = 0$

$\Rightarrow \quad c = 2.$

iii) if $c = 2$,

$\lim_{x \to 2} f(x) = f(2).$

6. Is there a number b such that

$$\lim_{x\to -2} \frac{3x^2+bx+b+3}{x^2+x-2}$$

$\leftarrow f(x)$

$\leftarrow g(x)$

exists? If so, find the value of b and the value of the limit.

notice

$$x^2+x-2\Big|_{x=-2} = 0.$$

$\boxed{\text{Need top} = 0.}$

$$f(-2) = 0$$

$$\Rightarrow \quad 3(-2)^2 + b(-2) + b + 3 = 0$$

$$\Rightarrow \quad 12 - 2b + b + 3 = 0$$

$$\Rightarrow \quad -b = -15 \quad \Rightarrow \quad b = 15$$

Try $b=15$

$$\lim_{x \to -2} \frac{3x^2+15x+18}{x^2+x-2}$$

$$= \lim_{x \to -2} \frac{3(x^2+5x+6)}{(x+2)(x-1)}$$

$$= \lim_{x \to -2} \frac{3\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x-1)}$$

7. Consider the function

$$f(x) = \frac{-3e^{2x} + 7e^x + 1}{e^{2x} - 2e^x}$$

Find all vertical asymptotes. Then find all horizontal asymptotes.

V.A. if bottom $= 0$

(and top $\neq 0$ there)

$$e^{2x} - 2e^x = 0$$

$$\Rightarrow e^x(e^x - 2) = 0$$

$\Rightarrow$

$e^x = 0$ or $e^x - 2 = 0$

no sol.

$$e^x = 2$$

$$\Rightarrow x = \ln(2)$$

check if top $= 0$ when $x = \ln 2$.

⋮

HA.

$$f(x) = \frac{-3e^{2x} + 7e^x + 1}{e^{2x} - 2e^x}$$

~~Recall:~~ Check:

$\lim_{x\to\infty} f(x)$ & $\lim_{x\to-\infty} f(x)$

$$\lim_{x\to\infty} \frac{-3e^{2x} + 7e^x + 1}{e^{2x} - 2e^x}$$

let $u = e^x$.

$x \to \infty \Rightarrow u \to \infty$.

$$= \lim_{u\to\infty} \frac{-3u^2 + 7u + 1}{u^2 - 2u}$$

$= -3$. L.C.

$$\lim_{x\to-\infty} \frac{\cancelto{0}{-3e^{2x}} + \cancelto{0}{7e^{x}} + 1}{2e^{2x} - 2e^{x}} \qquad \text{DNE.}$$

$$\frac{\oplus}{\ominus}$$

$$= -\infty.$$

8. In the theory of relativity, the Lorentz contraction formula

$$L = L_0\sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim\limits_{v\to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

$$\lim_{v\to c^-} L = \lim_{v\to c^-} L_0\sqrt{1-\frac{v^2}{c^2}}$$

$$= L_0\sqrt{\lim_{v\to c^-}\left(1-\frac{v^2}{c^2}\right)}$$

$$= L_0\sqrt{1-1} = L_0 \cdot 0$$

$$= 0.$$

LHL necessary ∵ can't pass speed of light

(or, $\nexists$ neg. #).

rationalize relentlessly.

9. Evaluate $\lim_{x \to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$ $\cdot \frac{\sqrt{6-x}+2}{\sqrt{6-x}+2} \cdot \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}$

10. Solve for x, where we know

$$\ln x + \ln(x-1) = 1$$

11. Find a formula for the inverse of the function

$$y = \frac{e^x}{1+2e^x}$$

12. Prove $\lim_{x \to -\infty} e^x \cos(x) = 0$.

Use Squeeze thm.

Recall

$$-1 \le \cos x \le 1$$

$$\Rightarrow \underbrace{-e^x}_{\ell(x)} \le e^x \cos x \le \underbrace{e^x}_{u(x)}$$

$$\lim_{x \to -\infty} \ell(x) = 0 = \lim_{x \to -\infty} u(x)$$

So by Squeeze thm,

$$\lim_{x \to -\infty} e^x \cos x = 0.$$

13. Prove

$$\log_{10} x = x - 3$$

has a solution.

use IVT.

when using IVT, be clear about

$f(x)$, N, (a,b)

Let $f(x) = \log_{10} x - x + 3$

another choice

$g(x) = \log_{10} x - x$

$M = -3$

$N = 0$

$f(x)$ cts.

$f(1) = \log_{10}(1) - 1 + 3 > N$

$f(10) = 1 - 10 + 3 < N$

$\Rightarrow$ IVT says $\exists\, c \in (1, 10)$ s.t.
$f(c) = N$

$$\Rightarrow \quad \log_{10} c - c + 3 = 0$$

$$\Rightarrow \quad \log_{10} c = c - 3$$

$\rfloor$.

14. Compute the following derivatives. You do not need to simplify your answers.

(a) $f'(x)$ if $f(x) = (2x^6 - 4x + 3)e^x$.

(b) $g'(x)$ if $g(x) = \dfrac{\sin x \cos x}{xe^x}$.

(c) $h'(x)$ if $h(x) = \dfrac{3 + 2\sin x}{x^3 + 1}$

15. Suppose the displacement function of a particle is given by $s(t) = \sqrt{t}$ for $t > 0$.

    (a) Using the limit definition of the derivative, compute the velocity function. Then check your work using the power rule.

    (b) Compute the acceleration function using the definition of the derivative. Then check your work using the power rule.