19. **Example.** Investigate the sequence $\{a_n\}$ that is defined recursively by

$$a_1 = \sqrt{6},\ a_{n+1} = \sqrt{6 + a_n}\ , \quad \textbf{for } n \geq 1.$$

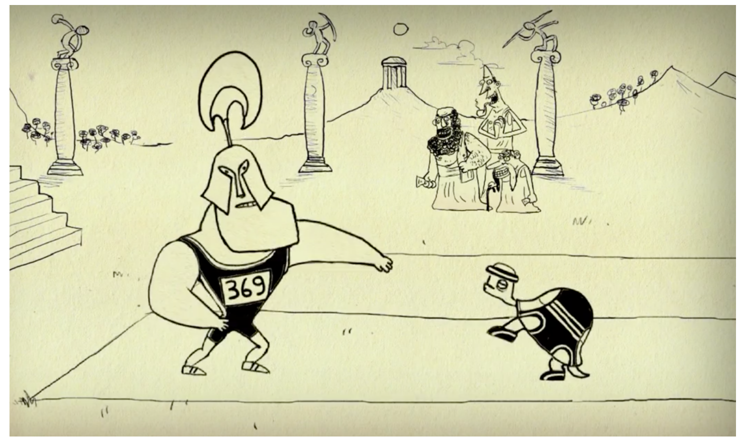
Series

1. **Quote.** “There are things which seem incredible to most men who have not studied mathematics.”

   (Aristotle, Ancient Greek Philosopher, Scientist and Physician, 384 BC-322 BC)

2. **Paradox.** “Zeno’s Paradox: Achilles and the Tortoise.”
   (Zeno of Elea, Ancient Greek Philosopher, 495BC - 430BC)

   Achilles is in a footrace with the tortoise. Achilles allows the tortoise a head start of 100 metres, for example. Supposing that each racer starts running at some constant speed (one very fast and one very slow), then after some finite time, Achilles will have run 100 metres, bringing him to the tortoise’s starting point.



   During this time, the tortoise has run a much shorter distance, say, 10 metres. It will then take Achilles some further time to run that distance, by which time the tortoise will have advanced farther; and then more time still to reach this third point, while the tortoise moves ahead.

   Thus, whenever Achilles arrives somewhere the tortoise has been, he still has some distance to go before he can even reach the tortoise.

3. **Series.**

Suppose $\{a_n\}$ is a sequence of numbers. An expression of the form

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

is called an **infinite series** and it is denoted by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \textbf{or} \quad \sum a_n.$$

4. **Partial Sum.**

If $\sum_{i=1}^{\infty} a_i$ is a series then

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

is called its nth **partial sum**.

But does it make sense to ”add infinitely many numbers”?

Not directly, so we imagine adding finitely many terms, but more and more terms each time, and look at what happens to these cumulative sums.

5. **Definition.**

Given the series $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \ldots$, let s_n denote its n^{th} partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n.$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \ldots = s$$

The number s is called the **sum** of the series.

If the limit above does not exist, then the series is called **divergent**.

6. **Example.** The series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ has partial sums $s_1 = 1$, $s_2 = 1.5$, $s_3 = 1.75$,

$s_4 = 1.875\ldots$ and in general it turns out that $s_n = 2 - \frac{1}{2^{n-1}}$.
Since $s_n \to 2$ as $n \to \infty$, the series is convergent and has sum 2.

7. **Example.** Show that the **geometric series** $\sum_{i=1}^{\infty} ar^{i-1} = a + ar + ar^2 + \ldots$ is convergent if $|r| < 1$ and its sum is

$$\sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r}, \quad |r| < 1$$

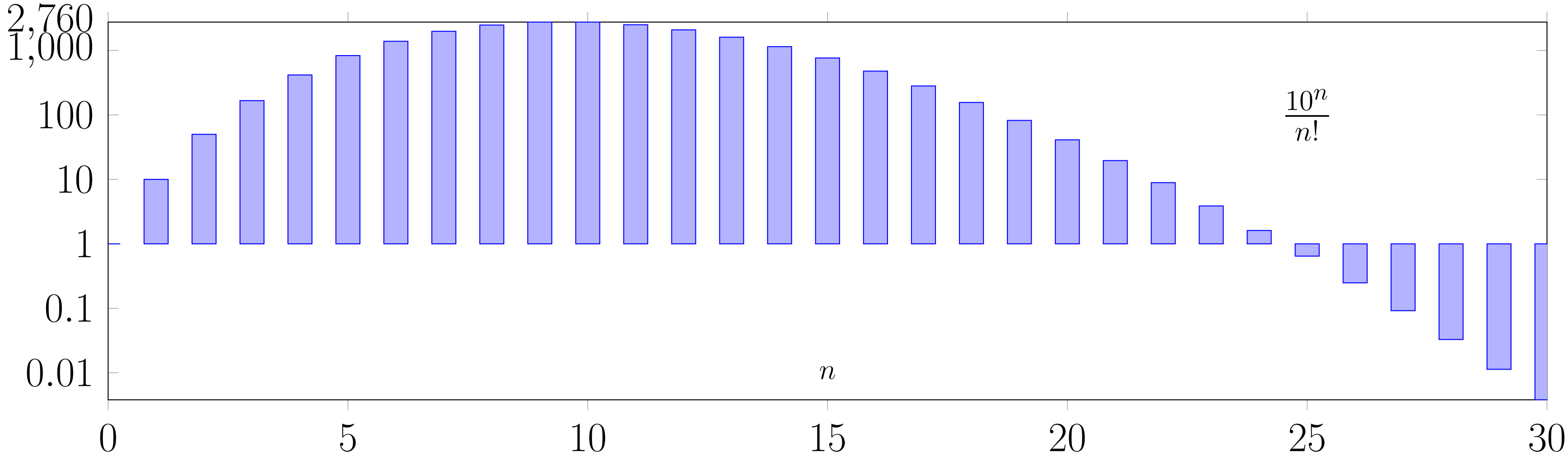
If $|r| \geq 1$, the geometric series is divergent.
(Here we are assuming $a \neq 0$, otherwise the series converges to 0 regardless of the value of r.)

8. **Examples.** Determine whether the given series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{e}{10}\right)^n$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{3}{e}\right)^n$

9. **Example:** **[**e^{10}**]** Does $\sum_{n=0}^{\infty} \frac{10^n}{n!}$ converge?



10. **Example.** Express $0.5555\ldots$ as a rational number.

11. **Example.** Show that the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$ is convergent and find its sum.

12. **Example.** Show that the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

is divergent.

13. **Two Useful Results.**

Theorem.

(a) If the series $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

(b) If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

14. **Example.** Show that

$$\sum_{n=1}^{\infty} n \sin(1/n)$$

is divergent.

15. **Theorem.**

If $\sum a_n$ and $\sum b_n$ are convergent series and c is a constant, then $\sum ca_n$, $\sum(a_n + b_n)$, $\sum(a_n - b_n)$ are also convergent, and

(a) $\sum ca_n = c\sum a_n$

(b) $\sum(a_n + b_n) = \sum a_n + \sum b_n$

(c) $\sum(a_n - b_n) = \sum a_n - \sum b_n$

16. **Example.** If $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5}{2^n} - \frac{6}{(n+1)(n+2)}\right)$ is convergent, find its sum.

From Examples 7 and 11 , we know that the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$ are convergent, with sums 1 and $\frac{1}{2}$, respectively.

The given series is convergent, since it can be written as

$$5\sum_{n=1}^{\infty}\frac{1}{2^n} - 6\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)} = 5(1) - 6(\tfrac{1}{2}) = 2.$$

Notes.