Math 1171 Final Exam Practice Questions

To prepare for the exam you should: (i) read all sections of the textbook again; (ii) go through all the homework questions again and make sure you can do every single one of them, (iii) work through more problems from the textbook. Only once you have done all this should you attempt the following questions.

The following is a list of practice exam questions. This will give you an idea of the types of questions you will be asked on the exam.

Questions 1-3 are on the statements of **definitions and theorems** and on your ability to give **examples** of functions with specified properties..

1. Define the following terms.
   (a) Limit (for this course, intuition will suffice since we didn't see $\varepsilon - \delta$ proofs)
   (b) Function continuous at a number
   (c) Function continuous on an interval
   (d) Tangent line
   (e) The derivative of a function at a number
   (f) Function differentiable on a set
   (g) The number e
   (h) Differential
   (i) Absolute maximum and absolute minimum
   (j) Local maximum and local minimum
   (k) Critical number
   (l) Function concave upward and concave downward
   (m) Inflection point
   (n) Antiderivative of a function
2. State the following theorems.
   (a) The Squeeze Theorem
   (b) The Intermediate Value Theorem
   (c) Fermat's Theorem
   (d) Extreme Value Theorem
   (e) The Mean Value Theorem
   (f) L'Hospital's Rule
   (g) The First Derivative Test
   (h) The Second Derivative Test
3. Give an example for each of the following.
   (a) Function with an infinite number of vertical asymptotes.

1) a) $\lim_{x \to a} f(x) = L$ means

As x gets close to a, $f(x) \to L$.

or:

$\lim_{x \to a^-} f(x) = L$ & $\lim_{x \to a^+} f(x) = L$

b) $f(x)$ is cts at $x = a$ if

i) $f(a)$ exists

ii) $\lim_{x \to a} f(x)$ exists

iii) they're equal

c) $f(x)$ is cts on $[a,b]$ if for all $c \in [a,b]$, $f(c)$ is cts

(end pts taken to mean one-sided cts).

e) Der. of $f(x)$ at $x = a$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

f) $f(x)$ is differentiable on (x_0, x_1) if $\forall\, a \in (x_0, x_1)$, $f'(a)$ exists

g) $e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

h) $dy = f'(x) \cdot dx$

k) crit number at $x=c$ if

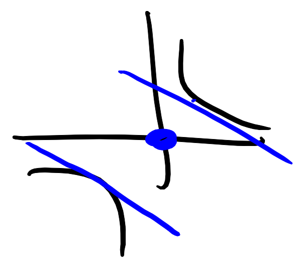
$f'(c) = 0$ or DNE.

l) ~~$f(x)$ is CU on (x_0, x_1) if $f''(x) > 0$ there.~~

$f(x)$ is CU if $f(x)$ is above its tangent line.

m) An inflection pt at $x=c$ if there is a cts change in concavity at $x=c$.

caution: $f''(x)=0$ doesn't neccessitate an inflection pt.



eg) $f(x) = x+1$.

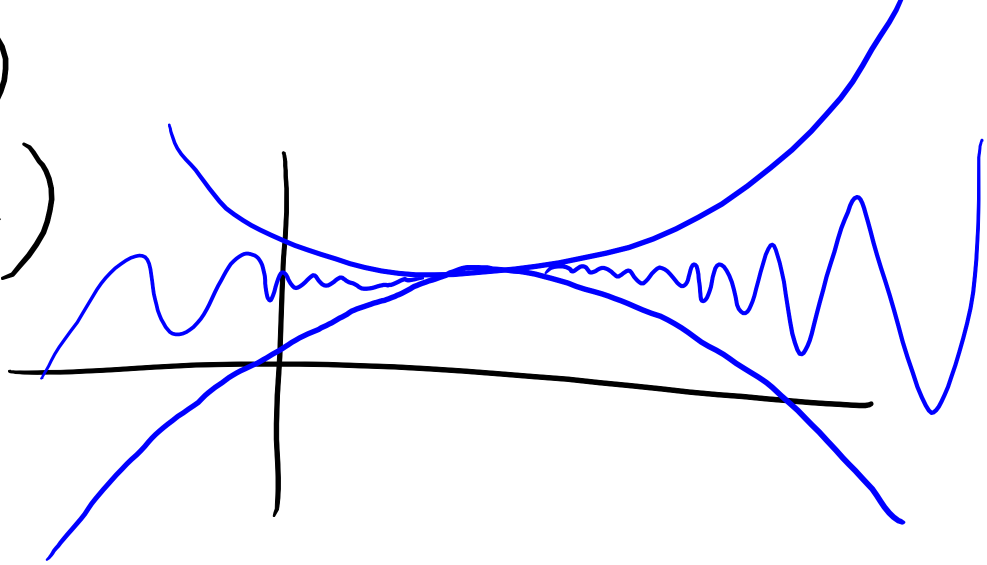
$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

n) if $g(x)$ is an antiderivative of $f(x)$,

then $g'(x) = f(x)$.

2)

a)



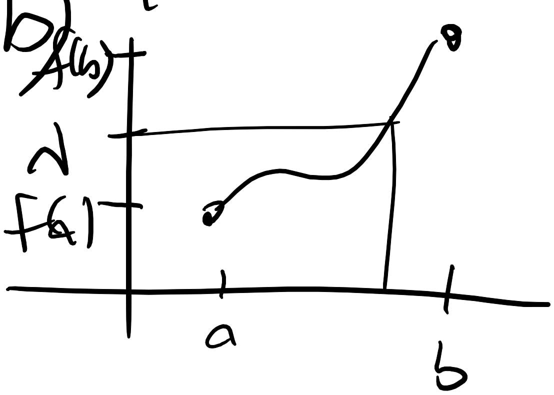
Consider

- $\ell(x) \leq f(x) \leq u(x)$
- $\lim_{x \to a} \ell(x) = L = \lim_{x \to a} u(x)$

Then

$$\lim_{x \to a} f(x) = L.$$

b) IVT:



Say $f(x)$ is cts & d is between $f(a)$ & $f(b)$.

$\Rightarrow \exists c \in (a,b)$

$f(c) = d$

c) Fermat's Thm:

If $f(x)$ is differentiable at $x=c$

& $f(c)$ is a local extrema,

then $f'(c)=0$.

d) Extreme Value Thm:

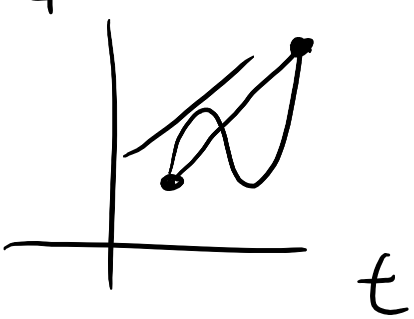
If $f(x)$ is cts on $[a,b]$, then

$f(x)$ attains its abs. max & abs min.

in $[a,b]$.

e) MVT: (mean) value thm.

Like the IVT, except it says "you're ugly":



Suppose $f(x)$ is cts on $[a,b]$ & diff. on (a,b). Then $\exists c \in (a,b)$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

f) L'Hôpital's Rule

Consider $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ where

$$\lim_{x\to a} f(x) = 0 \quad \& \quad \lim_{x\to a} g(x) = 0.$$

$$\left(\text{or} \quad \lim_{x\to a} f(x) = \pm\infty \quad \& \quad \lim_{x\to a} g(x) = \pm\infty \right)$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \underbrace{\lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

provided this exist (or is $\pm\infty$)

In practice:

- $\frac{\infty}{\infty}$ or $\frac{0}{0}$ Use L'H.
- $\infty \cdot 0 \longrightarrow \dfrac{\infty}{\frac{1}{0}}$ or $\dfrac{0}{\frac{1}{\infty}}$

$\cdot\ \infty^0,\ 0^\infty,\ \text{or}\ 1^\infty$ use ln.

a) $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = L.$

$\Rightarrow \ln\left(\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}\right) = \ln L$

LHS:

$$\lim_{x\to\infty} \ln\left(1+\frac{a}{x}\right)^{bx}$$

$$= \lim_{x\to\infty} \underset{\infty\cdot}{bx}\cdot \underset{0}{\ln\left(1+\frac{a}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{bx}}$$

$\frac{0}{0}$ LH

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{a}{x}}\cdot\left(-\frac{a}{x^2}\right)}{\frac{-1}{bx^2}}$$

$$= \lim_{x\to\infty} \frac{(-bx^2)\cdot(-a)}{x^2\cdot\left(1+\frac{a}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{ab\,\cancel{x^2}}{\cancel{x^2}\left(1+\frac{a}{x}\right)} = ab.$$

$$\ln L = ab$$

$$\Rightarrow L = e^{ab}.$$

check: if $a=b=1$, $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = e$.

Setting limit $= L$ could be cheating

$$1 + z + z^2 + z^3 + z^4 + \ldots$$

$$S_n = \sum_{i=0}^{n} z^i = z^0 + z^1 + z^2 + \ldots + z^n$$

$$\lim_{n\to\infty} S_n = \infty \text{, clearly}$$

let

$$\boxed{L = \lim_{n\to\infty} S_n}$$

$$= \lim_{n\to\infty} \left[1 + z + z^2 + z^3 + z^4 + \ldots \right]$$

$$= \lim_{n\to\infty} \left[1 + z \left(1 + z^1 + z^2 + z^3 \ldots \right) \right]$$

$$= 1 + zL$$

$\Rightarrow \quad L = 1 + 2L$

$\Rightarrow \quad -L = 1$

$\Rightarrow L = -1.$

g) First Der. Test   Say $f(x)$ is cts at $x=c$.

Suppose $f'(c)=0$ or dne. Then if at $x=c$

- $f'(x)$ goes from $\oplus$ to $\ominus$ $\Rightarrow$ local max @ $x=c$
- " " " " $\ominus$ to $\oplus$ $\Rightarrow$ local min @ $x=c$.

h) 2nd Der. Test: Say $f'(c)=0$.

If $f''(c)>0 \Rightarrow$ local min at $x=c$.

If $f''(c)<0 \Rightarrow$ local max at $x=c$

If $f''(c)=0 \Rightarrow$ inconclusive

Product rule: Let $h(x) = f(x) \cdot g(x)$,

then $h' = f'g + f \cdot g'$

Chain rule: Let $h(x) = f(g(x))$. Then

$$h'(x) = f'(g(x)) \cdot g'(x)$$

Lin Approx: Linearization of $f(x)$ at $x = a$ is
$L(x) = f(a) + f'(a) \cdot (x - a)$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

Final Exam Breakdown

pre MT1 ~ 20%

MT1 – MT2 ~ 40%

post MT2 ~ 40%

(b) Function $F = f \cdot g$ so that the limits of F and f at a exist and the limit of g at a does not exist.

(c) Function with a removable discontinuity.

(d) The most general form of a function with the property that its second derivative is the zero function.

(e) Function that is continuous but not differentiable at a point.

(f) Function with a critical number but no maximum or minimum.

(g) Function with a local minimum at which the second derivative equals 0.

Questions 4-16 are **short answer** questions. The questions are given in no particular order.

4. Find the derivative $y' = \dfrac{dy}{dx}$ of each of the following:

   (a) $y = \cos^{-1}(x^2) - \ln(1 + x^3)$ [Note: Another notation for $\cos^{-1}$ is arccos.]

   (b) $y = x^{\sin(x)}$.

   (c) $\arctan\left(\dfrac{y}{x}\right) = \frac{1}{2}\ln(x^2 + y^2)$.

5. Let $f(x) = \tan x$. Find $f''(x)$, the second derivative of f.

6. Find the tangent line to the curve $y + x\ln y - 2x = 0$ at the point $(1/2, 1)$.

7. Evaluate $\displaystyle\lim_{x\to\infty} \frac{2x^3 + 3x - 1}{1 - 2x^2 + 5x^3}$. L C.

8. Evaluate $\displaystyle\lim_{x\to 1} \frac{x^2 - 1}{x^2 - 3x + 2}$. factor

9. Let $f(x) = \dfrac{2x}{x^2 + 3}$. $y = m(x - x_0) + y_0$, $x_0 = 1$, $y_0 = f(1)$

   (a) Find the equation of the tangent line to the curve $y = f(x)$ at $x = 1$. $m = f'(1)$

   (b) Use linear approximation to give an approximate value for $f(1.2)$.

10. Prove there is a solution to the equation $2^x = x + 5$. IVT.

11. The curve $y = xe^{-x}$ has one inflection point. Find the x-coordinate of this point. Find y''. set $y'' = 0$. Solve. check for change in sign.

12. Suppose $f(x)$ and its derivatives are

$$f(x) = \frac{x^2 + x + 2}{x + 2}, \quad f'(x) = \frac{x(x+4)}{(x+2)^2}, \quad f''(x) = \frac{8}{(x+2)^3}.$$

$f' > 0$

   (a) Find its intervals of increase/decrease. Then find and classify all of its local extrema. ← 1st der. test.

   (b) Find where $f(x)$ is concave up / down. Then find all of its inflection points.

$f'' > 0$

13. Find a number x_0 between 0 and π such that the tangent line to the curve $y = \sin x$ at $x = x_0$ is parallel to the line $y = -x/2$. want x_0 s.t. $\left.\dfrac{dy}{dx}\right|_{x_0} = -\frac{1}{2}$

14. Evaluate $\displaystyle\lim_{x\to 0} x \sin\frac{1}{x}$.

Squeeze. $u = \frac{1}{x}$. RHL / LHL.

15. Evaluate $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$. L H

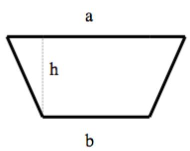
16. Evaluate $\lim_{x\to 0} x^{\sin x}$. $\lim e^{\ln(x^{\sin x})}$

Questions 17-33 are **full-solution** problems. Justify your answers and show all your work. The questions are given in no particular order.

17. Let $f(x) = \frac{5}{3x-1}$. Calculate $f'(2)$ directly from the *definition* of derivative.

18. A water-trough is 10m long and has a cross-section which is the shape of an isosceles trapezoid that is 30cm wide at the bottom, 80cm wide at the top, and has height 50cm. If the trough is being filled with water at the rate of 0.2 m^3/min, how fast is the water level rising when the water is 30cm deep?

    [Recall: The area of an isosceles trapezoid as shown in the diagram is $A = \frac{1}{2}(a+b)h$.]



$V = \frac{1}{2}\,\frac{4}{3}\pi r^3$

$25 + \Delta r$

$\Delta r = \frac{0.05}{100}$

19. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05cm thick to a hemispherical dome with diameter 50m. $\Rightarrow$ rad = 25m.

Vol of paint = $V(r+\Delta r) - V(r) = \Delta V \approx dV = V'(r)\Delta r$

20. At 2:00 p.m.a car's speedometer reads 30 mi/h. At 2:10 p.m. it reads 50 mi/h. Show that at some time between 2:00 and 2:10 the acceleration is exactly 120 mi/h^2.

MVT.

21. A piece of wire 10m long is cut into two pieces. One piece is bent into a square and the other is bent into an circle. How should the wire be cut so that the total area enclosed is minimum.

22. Find the linearization of $f(x) = \sqrt{1-5x}$ at $x = -3$.

· univariate func $f(x)$ on $x \in [a,b)$
· find crit pts, $f'(x) = 0$ or dne
· plug crit pts into $f(x)$.
· handle end pts. $f(a)$ & $\lim_{x\to b^-} f(x)$

23. Prove that $e^x = x + 5$ has a solution. IVT

24. Let $f(x) = 2x^3 - 6x^2 + 3x + 1$.

    IVT

    (a) First show that f has at least on zero in the interval $[2,3]$ and then use the first derivative of f to show that there is exactly one root of f between 2 and 3.

    (b) Use Newton's method to approximate the root of f in the interval $[2,3]$ by starting with $x_1 = 5/2$ and finding x_2.

    $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

25. Find the dimensions of the largest rectangle that can be inscribed inside a semicircular region of radius 5 such that one side of the rectangle is parallel to the base of the semicircular region.

26. (a) A metal storage tank with fixed volume V is to be constructed in the shape of a right circular cylinder surmounted by a hemisphere. What dimensions will require the least amount of metal?

    (b) Suppose the metal for the hemisphere costs twice as much as the metal for the lateral sides. What are the dimensions for the tank that minimizes cost?

    (Recall: The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ and the surface area is $4\pi r^2$.)

27. (a) Show that Newton's Method applied to the equation $x^2 - a = 0$ yields the iterative formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

and thus provides a method for approximating the square root $\sqrt{a}$ which uses only addition and multiplication.

(b) Approximate $\sqrt{3}$ by taking $x_1 = 3/2$ and calculating x_2.

$f''(x) = 2 + \cos x$

28. Find f if $f''(x) = 2 + \cos x$, $f(0) = -1$ and $f(\pi/2) = 0$.

$f'(x) = 2x + \sin x + C_1$

$f(x) = x^2 - \cos x + C_1 \cdot x + C_2$

29. Sketch the curve which is given by the parametric equations

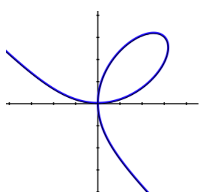
$$x = \cos(\pi t), \quad y = \sin(\pi t), \quad 1 \le t \le 2.$$

Clearly label the initial and terminal points and describe the motion of the point $(x(t), y(t))$ as t varies in the given interval (i.e. indicate the direction the point is traveling).

use $f(\frac{\pi}{2}) = 0$ & $f(0) = -1$ to find C_1 & C_2

30. A curve called the **folium of Descartes** is defined by the parametric equations

$$x = \frac{3t^2(t+1)}{3t^2+3t+1}, \quad y = \frac{-3t(t+1)^2}{3t^2+3t+1}, \quad -\infty < t < \infty.$$



$x^3 + y^3 = 3xy$

(a) ~~Show that a Cartesian equation of this curve is $x^3 + y^3 = 3xy$.~~

(b) Find the point on the curve corresponding to $t = -1/2$. $x_0 = x(\frac{-1}{2})$ $y(\frac{-1}{2}) = y_0$

(c) Find the equation of the tangent line to the curve at the point corresponding to $t = -1/2$. $y = m(x - x_0) + y_0$

(d) Find the values of the parameter t which correspond to the point $(0, 0)$ on the curve.

(e) Find equations of the tangent lines to the curve at the point $(0, 0)$.

$m = \left.\frac{dy}{dx}\right|_{t = \frac{-1}{2}} =$

31. Consider the curve given by the parametric equations

$$x = 2\sin t, \quad y = 4 + \cos t, \quad 0 \le t \le 2\pi.$$

Determine the points on the curve which are closest to the origin and those which are furthest away.

32. Find the antiderivative of $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1}x^{n-1} + a_n x^n$.

33. Find the antiderivative of $g(x) = \sin(2x) + e^{-3x}$.

34. Suppose $f''(x) = e^{-x} + \frac{15}{4}\sqrt{x} - 4\sin(2x)$, and $f(0) = 4$. Find the most general family of functions that satisfy these conditions.

Answers:

4. (a) $y' = -\dfrac{2x}{\sqrt{1-x^4}} - \dfrac{3x^2}{1+x^3}$ (b) $y' = x^{\sin x}(\frac{\sin x}{x} + \ln x \cos x)$ (c) $y' = \dfrac{x+y}{x-y}$
5. $f''(x) = 2\sec^2(x)\tan(x)$
6. $y = \frac{4}{3}x + \frac{1}{3}$
7. 2/5
8. -2
9. (a) $y = \frac{1}{4}x + \frac{1}{4}$ (b) $f(1.2) \approx \frac{1}{4}(1.2) + \frac{1}{4} = \frac{11}{20} = 0.55$
10. (a) -4 (b) to the left (c) speed is decreasing
11. $x = 2$
12. (a) Increasing on $(-\infty, -4) \cup (0, \infty)$; decreasing on $(-4, -2) \cup (-2, 0)$; local max at $x = -4$, local min at $x = 0$.
    (b) Concave down on $(-infty, -2)$; concave up on $(-2, \infty)$; no inflection points.
13. $2\pi/3$
14. 0 (Hint: use Squeeze Theorem since $\sin(1/x)$ is bounded)
15. 1/2
16. 1
17. $-3/5$
18. $\frac{1}{3}$ cm/min
19. $\frac{5}{8}\pi \approx 2$ m^3
20. Hint: use Mean Value Theorem
21. The length of wire used to make the square should be $\frac{40}{\pi+4}$
22. $L(x) = 4 - \frac{5}{4}(x+3)$
23. Hint: use the Intermediate Value Theorem.
24.
25. base is $5\sqrt{2}$ and height is $\frac{5}{\sqrt{2}}$
26. (a) hemisphere (b) $r = \frac{1}{2}\sqrt[3]{3V/\pi}$ and $h = \sqrt[3]{3V/\pi}$.
27. (b) $\frac{7}{4}$
28. $f(x) = -\cos x + x^2 - x + \pi/2(1 - \pi/2)$
29. Consists of points on the unit semicircle in quadrants 3 and 4. Points are moving counterclockwise along curve with initial point $(-1, 0)$ and terminal point $(1, 0)$.
30. (b) $(3/2, 3/2)$ (c) $y = -x + 3$ (d) $t = -1, 0$ (e) There are two lines: the first is when $t = -1$ and the tangent is the horizontal line $y = 0$, the second occurs when $t = 0$ and is the vertical line $x = 0$.
31. closest point is $(0, 3)$, furthest point is $(0, 5)$
32. $a_0 x + \frac{a_1}{2}x^2 + \ldots + \frac{a_{n-1}}{n}x^n + \frac{a_n}{n+1}x^{n+1} + c$
33. $-\frac{1}{2}\cos(2x) + \frac{1}{3}e^{3x}$
34. $f(x) = \sin(2x) + e^{-x} + x^{5/2} - cx + 3$, c an arbitrary constant.