<u>Math 1171 Midterm 2 Practice Questions</u>

1. Compute the following derivatives. You do not need to simplify your answers.

   (a) $f'(x)$ if $f(x) = (2x^6 - 4x + 3)^4$.

   $f'(x) = 4(2x^6 - 4x + 3)^3 \cdot (12x^5 - 4 + 0)$

   (b) $g'(x)$ if $g(x) = \dfrac{\sec x}{xe^x}$.

   $\log_a(x) = \dfrac{\ln x}{\ln a}$

   (c) $h'(x)$ if $h(x) = \dfrac{3 + 2\sin x}{x^3 + 1}$

   recall $\dfrac{d}{dx}\left[\log_a(x)\right] = \dfrac{1}{x \cdot \ln a}$

   (d) y' if $y = x^2 \log_3 (x^{2/3})$

   $y' = 2x \cdot \log_3(x^{\frac{2}{3}}) + x^2 \cdot \dfrac{d}{dx}\left[\log_3(x^{\frac{2}{3}})\right]$

$s = 2^{t^2}$

Think: func. is in exponent, hard. So use logs

$\Rightarrow \ln s = \ln(2^{t^2})$

$= t^2 \cdot \ln 2$

(e) $\dfrac{ds}{dt}$ if $s = 2^{t^2}$

$\Rightarrow \frac{d}{dt}[\ln s] = \frac{d}{dt}[t^2 \ln 2]$

$\Rightarrow \frac{1}{s} \cdot \frac{ds}{dt} = 2t \ln 2$

$\Rightarrow \frac{ds}{dt} = s \cdot 2t \ln 2 = 2^{t^2} \, 2t \ln 2$

(f) $h^{(51)}(t)$ if $h(t) = \ln(t^2)$. (Compute the first few derivatives to find a pattern.)

$h(t) = 2 \ln t$

$h'(t) = 2 \cdot \frac{1}{t}$

$h''(t) = -2 \cdot \frac{1}{t^2}$

$h'''(t) = (-2) \cdot (-2) \frac{1}{t^3} = 2 \cdot \frac{2}{t^3} = 2 \cdot 2 t^{-3}$

$\Rightarrow h^{(51)}(t) = (-1)^{51+1} \cdot 2 \cdot 50! \cdot t^{-51}$

(g) $\left.\dfrac{dy}{dx}\right|_{x=0}$ if $2\left(\dfrac{x}{y}\right) - \ln(x+y) = 0$

$h^{(iv)}(t) = -2 \cdot (2 \cdot 3) t^{-4}$

$h^{(v)}(t) = +2 \cdot (2 \cdot 3 \cdot 4) t^{-5}$

$h^{17}(t) = +2 \cdot (2 \cdot 3 \cdot 4 \cdots 16) t^{-17}$

(h) y' if $y = x^{\cos x}$.

$h^{(k)}(t) = (-1)^{k+1} \cdot 2 \cdot (k-1)! \cdot t^{-k}$

$h^{(5)}(t) = (-1)^{5+1} \cdot 2 \cdot (5-1)! \; t^{-5}$

$= 1 \cdot 2 \cdot 4! \cdot t^{-5}$

9) $\left.\dfrac{dy}{dx}\right|_{x=0}$ if $2\left(\dfrac{x}{y}\right) - \ln(x+y) = 0$

$$\Rightarrow 2\left(\frac{x}{y}\right) = \ln(x+y)$$

So, diff:

$$\frac{d}{dx}\left[2\cdot\left(\frac{x}{y}\right)\right] = \frac{d}{dx}\left[\ln(x+y)\right]$$

$$\Rightarrow 2\cdot\frac{1\cdot y - x\cdot\frac{dy}{dx}}{y^2} = \frac{1}{x+y}\cdot\left(1+\frac{dy}{dx}\right)$$

solve for $\dfrac{dy}{dx}$

$$y - x\frac{dy}{dx} = \frac{y^2}{2(x+y)}\left(1+\frac{dy}{dx}\right)$$

$$= \frac{y^2}{2(x+y)} + \frac{y^2}{2(x+y)}\frac{dy}{dx}$$

$$y - \frac{y^2}{2(x+y)} = \left(x + \frac{y^2}{2(x+y)} \right) \frac{dy}{dx}$$

$$\Rightarrow \frac{y - \frac{y^2}{2(x+y)}}{x + \frac{y^2}{2(x+y)}} = \frac{dy}{dx}$$

At $x=0$, need $y|_{x=0}$

$$2 \cdot \frac{x}{y} - \ln(x+y) = 0$$

if $x=0$,

$$2 \cdot \frac{0}{y} - \ln(0+y) = 0$$

$$\ln(y) = 0 \Rightarrow y = 1$$

So

$$\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1 - \frac{1}{2}}{0 + \frac{1}{2}}$$

h)

$$y = x^{\cos x}$$

Just... don't... please..

$$\frac{dy}{dx} = \cos x \cdot x^{\cos x - 1}$$

$$\ln y = \ln(x^{\cos x}) = \cos x \ln x$$

$$\Rightarrow \frac{d}{dx}[\ln y] = \frac{d}{dx}[\cos x \ln x]$$

$$\Rightarrow \frac{dy}{dx} = y\left(-\sin x \ln x + \cos x \frac{1}{x}\right)$$

2. **True or False.** Justify your answers.

(a) If f and g are differentiable then the derivative of $f(x)g(x)$ is $f'(x)g'(x)$.

False. $\exists$ product rule

(b) The function $f(x) = |x|$ is differentiable for all real numbers.

False. corner at $x=0$.

(c) If f is differentiable, then $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{x}}$.

False:

$$\frac{d}{dx}\left[\sqrt{f(x)}\right] = \frac{1}{2\sqrt{f(x)}} \cdot f'(x)$$

(d) $\frac{d}{dx}(10^x) = x10^{x-1}$.

H.W.

Hint: False.

3. Without computing any derivatives, demonstrate that $f(x) = 2^x + x$ has a point $c \in (0, 3)$ such that $f'(c) = \frac{10}{3}$.

MVT, not in exam.

$-100,000$

4. Find the linearization of $f(x) = \sqrt[3]{1+3x}$ about $x = 0$. Use it to approximate $\sqrt[3]{1.03}$.

$$=(1+3x)^{\frac{1}{3}}.$$

Recall the linearization of $f(x)$ @ $x=a$ is

$$L(x) = f(a) + f'(a)(x-a)$$

here:

$$f(x) = (1+3x)^{\frac{1}{3}} \qquad a=0$$

$$\Rightarrow \quad f'(x) = \frac{1}{3}(1+3x)^{-\frac{2}{3}} \cdot 3$$

$$= (1+3x)^{-\frac{2}{3}} \qquad f(0)=1$$

$$\Rightarrow \quad f'(a) = 1$$

So

$$L(x) = 1 + 1\cdot(x-0) = 1+x$$

Now, want approx. to $\sqrt[3]{1.03} = \cancel{f(1.03)}$

$$\sqrt[3]{1.03} = \sqrt[3]{1+3\frac{1}{100}} = f\left(\frac{1}{100}\right)$$

$$\approx L\left(\frac{1}{100}\right)$$

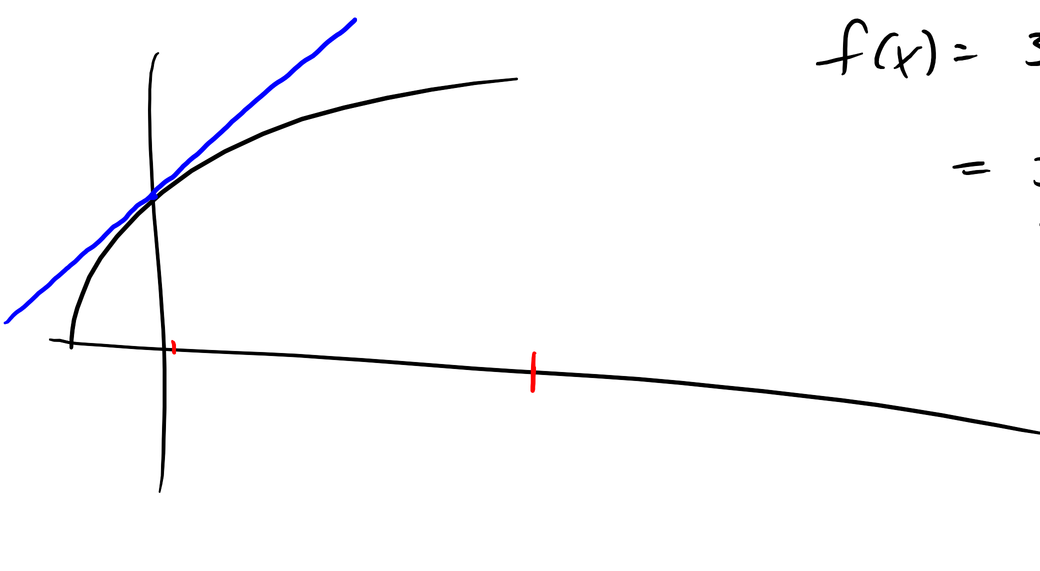
$f(x) = \sqrt[3]{1+3x}$

Notice:

$f(1.03) = \sqrt[3]{1+3\cdot 1.03} = \sqrt[3]{4.09}$

we want an approx to

$$\sqrt[3]{1.03} = f\left(\frac{1}{100}\right) \approx L\left(\frac{1}{100}\right)$$



$f(x) = \sqrt[3]{1+3x}$

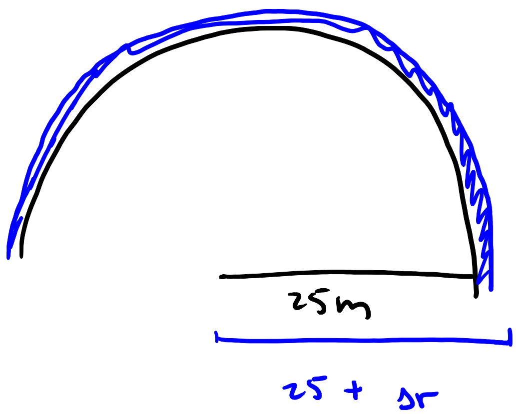
$= \sqrt[3]{3\left(\frac{1}{3}+x\right)}$

Hint: $V = \frac{4}{3}\pi r^3$ hemisphere: $H = \frac{2}{3}\pi r^3$

$0.05 \text{ cm} = 0.05 \text{ cm} \cdot \frac{1\text{m}}{100\text{cm}} = \frac{5}{10\,000} \text{ m}$

5. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05cm thick to a hemispherical dome with radius 25m.

Let
H : volume of paint



$\Delta r = 0.0005$ m

The amount of paint needed is

$\Delta H = H(25+\Delta r) - H(25) \approx dH.$

recall: $dy = f'(x)\,dx$

$\Rightarrow \quad dH = H'(25) \cdot dr \qquad H(r) = \frac{2}{3}\pi r^3$

$\Rightarrow H'(r) = 2\pi r^2$

$\Rightarrow \boxed{dH = 2\pi \cdot (25)^2 \cdot \frac{5}{100^2}} \qquad dr = \Delta r = \frac{5}{100^2}$

6. Consider the function $g(x) = (x^2 - 1)^3$. Find the absolute max and absolute min of $g(x)$ on $x \in [-2, 3]$.

Steps:

1) Find crit. pts. , ie

$f'(c) = 0$ & $f'(c)$ DNE

2) Plug crit pts / end pts into $f(x)$

biggest is abs max

smallest " " min.

[3]

7. Use logarithmic differentiation to find the derivative y' of the following function (you do not need to simplify your answer)

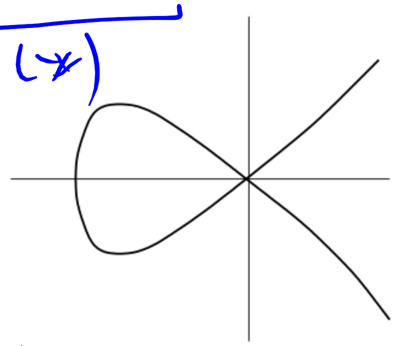
$$y = \frac{\sqrt{x^2+1}\,(3-4x)^5}{2(3x-1)^{1/4}\,(x-2)^4}$$

$\Rightarrow \ln y = \frac{1}{2}\ln(x^2+1) + 5\ln(3-4x) - \ln 2$

$- \frac{1}{4}\ln(3x-1) - 4\ln(x-2)$

$\Rightarrow \frac{1}{y} \cdot y' =$ ⋮

8. Consider the curve defined by $\underbrace{y^2 = x^3 + 5x^2}_{(*)}$. The graph of the curve is shown below.



(a) Show that the point $(-1, 2)$ is on the curve.

LHS: $y^2 = 2^2 = 4$

RHS: $x^3 + 5x^2 = (-1)^3 + 5\cdot(-1)^2 = -1 + 5 = 4$

$(-1,2)$ satisfies $(*)$ & thus is on curve.

(b) Use implicit differentiation to find $\frac{dy}{dx}$.

(c) Find the equation of the tangent line to the curve at the point $(-1, 2)$.

$$y = m(x - x_0) + y_0$$

$$(x_0, y_0) = (-1, 2)$$

$$m = \left.\frac{dy}{dx}\right|_{(-1,2)}.$$

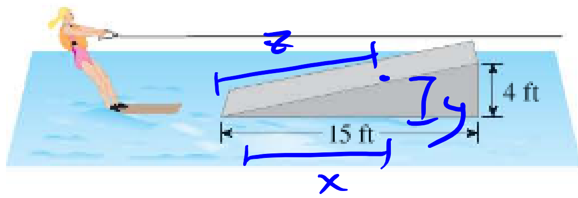
9. A girl facing North is standing next to a river which flows East. She tosses a stick into the water exactly 4 meters North of where she stands. The river carries the stick East at the constant rate of 3 m/s. How fast is the stick moving away from the girl after 2 seconds?
Note - solution is $\frac{9}{\sqrt{13}} \approx 2.5\ m/s$

10. A waterskier skis over the ramp shown in the figure at a speed of 30 ft/s. How fast is she rising as she leaves the ramp?

Note: Solution is $\frac{120}{\sqrt{241}}$ *ft/s*

Given

$$\frac{dz}{dt} = 30$$



Let:

z = distance travelled by skier [ft]

x = horiz. dist " " "

y = vert " " " "

want:

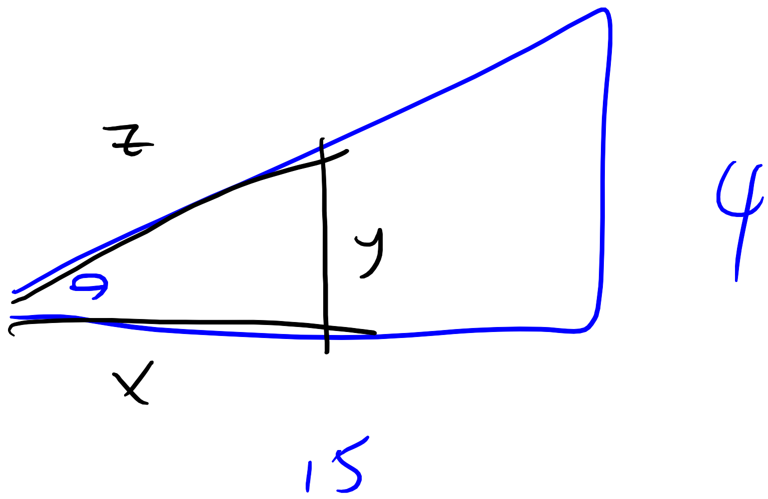
$$\left.\frac{dy}{dt}\right|_{x=15} \quad \text{or} \quad \Big|_{y=4}$$

Notice:

$$z^2 = x^2 + y^2$$

could differentiate now, but life is hard..

eliminate x. (because given $\frac{dz}{dt}$)



$\tan\theta = \frac{4}{15} = \frac{y}{x}$

$\Rightarrow \quad x = \frac{15}{4} y$

So

$$z^2 = \left(\frac{15}{4} y\right)^2 + y^2$$

$$\left(\frac{225}{16} + 1\right) y^2$$

$$\Rightarrow \quad z^2 = \frac{15^2}{16} y^2 + y^2$$

$$z^2 = \frac{241}{16} y^2$$

Avoid $\sqrt{\ }$

$$\Rightarrow \quad 2z \frac{dz}{dt} = \frac{241}{16} 2y \cdot \frac{dy}{dt}$$

$$\Rightarrow \left.\frac{dy}{dt}\right|_{y=4} = \frac{2z \cdot \frac{dz}{dt}}{\frac{241}{16} \cdot 2y}$$

?

Need $z\Big|_{y=4}$

$$z\Big|_{y=4} = \sqrt{15^2+4^2} = \sqrt{241}$$

$$\Rightarrow \boxed{\left.\frac{dy}{dt}\right|_{y=4} = \frac{2\sqrt{241} \cdot 30}{\frac{241}{16} \cdot 2 \cdot 4}}$$

Simplify:

$$\frac{120}{\sqrt{1241}}$$

11. Suppose we have the parameterizations $x = f(t)$ and $y = g(t)$ on $t \in [0, \infty)$, where

$$f(t) = e^{t-1}, \; g(t) = e^{2t}.$$

Sketch the parametric curve. Include features in your sketch. Then, find the equation of the tangent line at $t = 1$.

Features:
- curve shape
- direction
- initial & terminal pts.

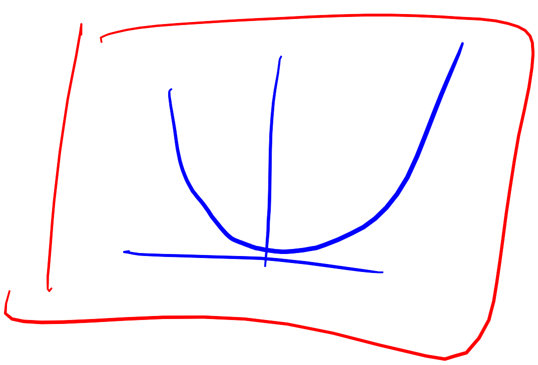
To sketch: eliminate

$$x = f(t) = e^{t-1} = \frac{e^t}{e} \qquad y = e^{2t} = (e^t)^2$$

$$\Rightarrow xe = e^t \qquad = (ex)^2$$

$$= e^2 \cdot x^2$$

So $\boxed{y = e^2 \cdot x^2}$



initial pt:

$$f(0) = e^{0-1} = \frac{1}{e}$$

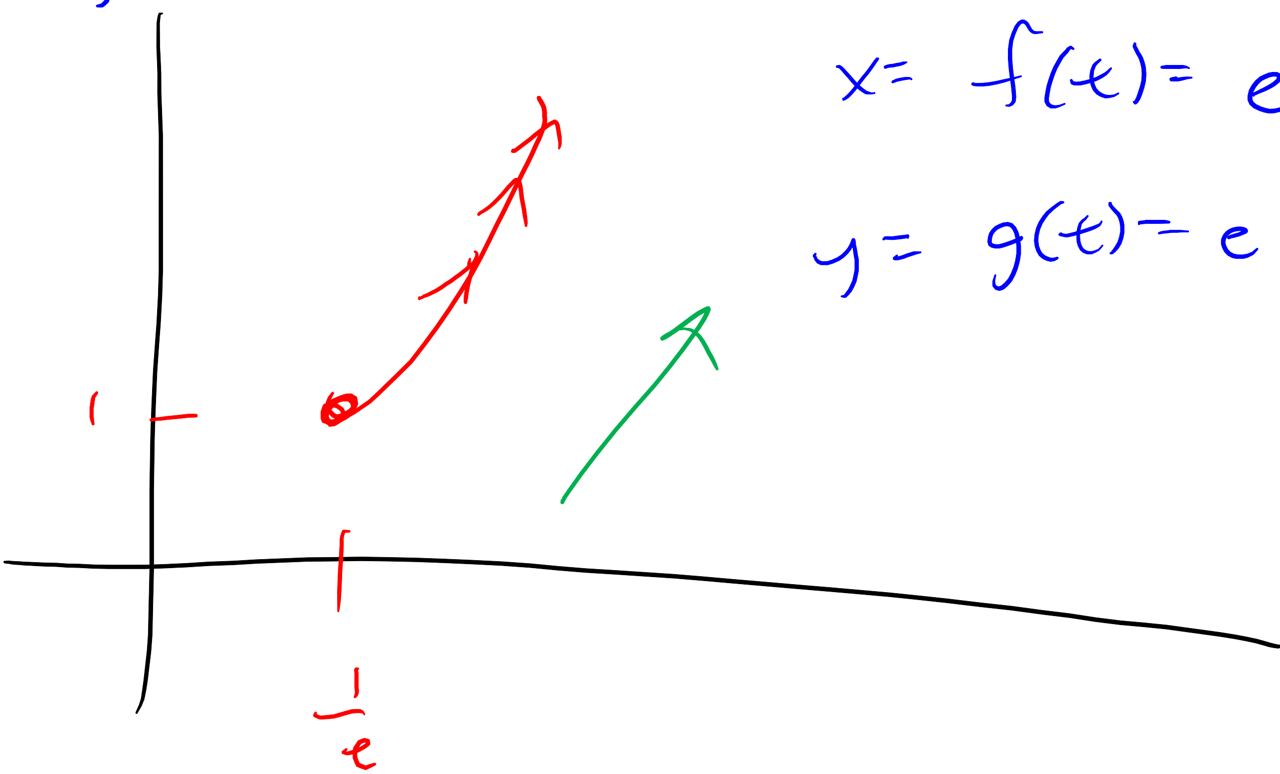
$$g(0) = e^{2\cdot 0} = 1$$

$$(x_0, y_0) = \left(\frac{1}{e}, 1\right)$$

$t \in [0, \infty)$

$$f(t) = e^{t-1} \qquad \text{ran}_f = \left[\frac{1}{e}, \infty\right).$$

$$y = e^2 \cdot x^2$$



$$x = f(t) = e^{t-1}$$

$$y = g(t) = e^{2t}$$

$$\boxed{\text{Eqn of tangent at } t = 1}$$

$$y = m(x - x_1) + y_1$$

- $x_1 = f(1) = e^{1-1} = e^0 = 1$
- $y_1 = g(1) = e^{2\cdot 1} = e^2$
- $m = \left.\frac{dy}{dx}\right|_{t=1}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2e^{2t}}{e^{t-1}}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{t=1} = \frac{2e^2}{e^0} = 2e^2$$

or $y = e^2 \cdot x^2$

$$\Rightarrow \frac{dy}{dx} = e^2 \cdot 2x \Rightarrow \left.\frac{dy}{dx}\right|_{x=x_1} = 2e^2$$

12. Consider the parameterizations $x = f(t)$ and $y = g(t)$, where

$$f(t) = t^2 - 1, \; g(t) = \cancel{t^2 + 2t + 2} \quad t^2 + 3t + 2$$

(a) Find when the function crosses the x-axis, and when it crosses the y-axis.

(b) Find when the function crosses the origin.

(c) Find the point(s) where the tangent line is horizontal, and point(s) where the tangent line is vertical.

a)

cross x-axis $\Rightarrow y=0$

$\Rightarrow g(t) = 0$

$\Rightarrow t^2+3t+2=0$

$\Rightarrow (t+2)(t+1) = 0$

$\Rightarrow t=-2$ & $t=-1$

cross y-axis: $x = f(t) = 0$

$\Rightarrow t^2 - 1 = 0 \Rightarrow t = \pm 1$.

b) At $t=-1$.

c) Horiz:

$$\frac{dy}{dx} = 0$$

Recall:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+3}{2t}$$

So $\frac{dy}{dx} = 0 \Rightarrow 2t+3 = 0$

$$\Rightarrow t = \frac{-3}{2}$$

vertical if $\frac{dx}{dt} = 0$

$$\Rightarrow t = 0.$$

if $t=0$, $x = -1$, $y = 2$