Final Review

String Basics

Theorem. If $\mathcal{A}$ is an alphabet with k symbols, there are k^n strings of size n over $\mathcal{A}$ for all $n \geq 0$.

Theorem. If $\mathcal{A}$ is an alphabet with k symbols, there are $k!$ permutations over $\mathcal{A}$.

Theorem. If n and k are positive integers with $0 \leq k \leq n$, the number of ways to choose k elements from a set of size n is equal to

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial Theorem. If x and y are variables and n is a positive integer:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Example: $(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$

$x^4 + 4x^3y + 6x^2y^2 + 4\cancel{x}y^3 + y^4$

Multinomials. Let $(n_1, n_2, \ldots, n_k)$ be a sequence of k non-negative numbers summing to n. The number of strings over an alphabet of size k with content $(n_1, n_2, \ldots, n_k)$ is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}.$$

Exercise: How many ways can the letters of the word $ACCESS$ be arranged? What if the word must contain the string SAS?

2 C's 2 S's $\binom{6}{1,2,1,2} = \frac{6!}{2 \cdot 2}$ possible

$SAS = X$ $CCEX$ $\frac{4!}{2}$ possible with SAS

Counting by inclusion-exclusion

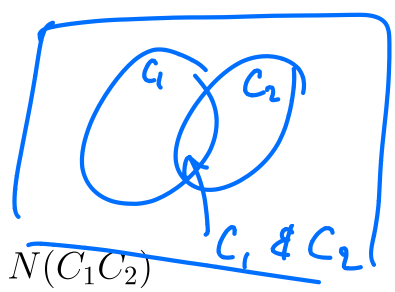
Inclusion-Exclusion

- Let $\mathcal{S}$ be the ground set (i.e. all possible elements) and assume $|\mathcal{S}| = N$.
- Let $C_1, \ldots, C_t$ be properties that elements of $\mathcal{S}$ may or may not satisfy.
- We let $\overline{N}$ denote the number of elements of $\mathcal{S}$ satisfying none of the properties $C_1, C_2, \ldots, C_t$
- For a subset of properties, say $\{C_1, C_3, C_6\}$ we let $N(C_1C_3C_6)$ denote the number of elements of $\mathcal{S}$ that satisfy C_1 and C_3 and C_6.

$$\textbf{Theorem: } \overline{N} = \sum_{k=0}^{t} (-1)^k S_k = S_0 - S_1 + S_2 - S_3 \ldots + (-1)^t S_t$$

Here each S_k denotes the sum over all possible sets of k conditions of the number of elements satisfying all k of these conditions. So for instance $S_3 = N(C_1C_2C_3) + N(C_1C_2C_4) + \ldots$.

Examples:



- For two conditions C_1, C_2 we have

$$\begin{aligned} \overline{N} &= S_0 - S_1 + S_2 \\ &= N - \Big(N(C_1) + N(C_2)\Big) + N(C_1C_2) \end{aligned}$$

- For three conditions C_1, C_2, C_3 we have

$$\begin{aligned} \overline{N} &= S_0 - S_1 + S_2 - S_3 \\ &= N - \Big(N(C_1) + N(C_2) + N(C_3)\Big) \\ &\quad + \Big(N(C_1C_2) + N(C_1C_3) + N(C_2C_3)\Big) - N(C_1C_2C_3) \end{aligned}$$

Note: inclusion-exclusion problems come in two varieties:

1. There are few conditions and you must evaluate each term individually.
2. The terms contributing to each S_k are equal.

Example of type 1: There are 100 students taking an intro course in three subjects: Math, History, and Economics. 30 of these students get an A in math, 40 get an A in history, and 35 get an A in econ. There are 20 who get an A in math and econ, 25 who get an A in history and econ, and 15 who get an A in math and History. Finally, assume that there are 10 people who get an A in all three classes. How many of the 100 students did not get an A?

C_1 = A in math

C_2 = A in hist

C_3 = A in econ

$$\overline{N} = N - (N(C_1) + N(C_2) + N(C_3)) + (N(C_1C_2) + N(C_1C_3) + N(C_2C_3)) - N(C_1C_2C_3)$$

$$= 100 - (30 + 40 + 35) + (25 + 15 + 20) - 10$$

$$=$$

Example of type 2: How many functions from $\{1, 2, \ldots, 10\}$ to $\{A, B, C, D, E\}$ are onto (Recall: a function f from X to Y is onto if every $y \in Y$ has some $x \in X$ with $f(x) = y$)

C_1 A not in range

C_2 B

C

D

C_5 E "

$\overline{N}$ = # onto funct.

$N(C_1) = 4^{10}$

$N(C_1C_3) = 3^{10}$

$N(C_1C_3C_4) = 2^{10}$

$N(C_1C_2C_3C_4) = 1$

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + S_4 - S_5$$

$$= 5^{10} - \binom{5}{1}4^{10} + \binom{5}{2}3^{10} - \binom{5}{3}2^{10} + \binom{5}{4}1$$

Recurrences

Definition. An infinite sequence $a_0, a_1, a_2, \ldots$ of integers satisfies a **recurrence relation** of **order** k if there exist functions f and g so that the following equation holds for all $n \geq k$

$$f(a_n, a_{n-1}, \ldots, a_{n-k}) = g(n)$$

(we will always assume that f is a polynomial)

Notation:

- **linear** means that f is a polynomial of degree 1
- **homogeneous** means the $g(n) = 0$
- **constant coefficients** means that the coefficients of the a_i's in the function f are constants not depending on n.

Finding recurrence relations. This is usually a matter of understanding how to recursively assemble (or disassemble) the objects of interest.

Ex: For every nonnegative integer n, let $\mathcal{S}_n$ be the set of strings over the alphabet $\{A, B, C\}$ with the property that every A and every B is immediately followed by a C. If $s_n = |\mathcal{S}_n|$ find a recurrence relation for s_n.

AC $\underline{(n-2)}$

BC $\underline{(n-2)}$

C $\underline{(n-1)}$

$S_0 = 1$

$S_1 = 1$ 'C'

$S_2 = 3$ AC, BC, CC

gen^l rule $S_n = S_{n-1} + 2S_{n-2}$

Solving recurrence relations. We have only solved recurrence relations with order ≤ 2 that are linear with constant coefficients. Here there are three skills of interest:

1. **Find the general solution** to a homogeneous equation.

   Ex: Find the general solution to

$$2a_n - 5a_{n-1} + 3a_{n-2} = 0$$

char poly $\quad 2r^2 - 5r + 3 = 0$

$$(2r-3)(r-1) = 0$$

$$r = 1 \qquad r = \frac{3}{2}$$

$$\boxed{a_n = C + D\left(\frac{3}{2}\right)^n}$$

2. **Find a particular solution** to a nonhomogeneous equation.

   Ex: Find a particular solution to

$$a_n - 5a_{n-1} + a_{n-2} = 2^n$$

guess $\quad a_n = C \cdot 2^n$

$$C2^n - 5C2^{n-1} + C2^{n-2} = 2^n$$

div by 2^{n-2}

$$4C - 10C + C = 4$$

$$-5C = 4$$

$$C = -\frac{4}{5}$$

3. Solve for initial conditions to **find unique solutions**.

Ex: Consider the recurrence relation

$$a_0 = 2 \qquad a_1 = 8$$

$$a_n - 4a_{n-1} + 3a_{n-2} = 4 \qquad \text{for } n \geq 2$$

You are given:

- the associated homogeneous equation $a_n - 4a_{n-1} + 3a_{n-2} = 0$ has general solution $C + D3^n$
- the recursive equation $a_n - 4a_{n-1} + 3a_{n-2} = 4$ has a particular solution $a_n = -2n$

$$(-2n) - 4(-2(n-1)) + 3(-2(n-2)) = 4 \quad ?$$

Find the unique solution to the given recurrence relation.

Sol has form $a_n = C + D3^n + (-2n)$

① $2 = a_0 = C + D$

② $8 = a_1 = C + 3D - 2$

② - ① $6 = 2D - 2$ $\quad D = 4$

$C = -2$

$$\boxed{a_n = -2 + 4 \cdot 3^n - 2n}$$

Generating Functions

Recall: A generating function $A(x) = a_0 + a_1x + a_2x^2 + \ldots = \sum_{n=0}^{\infty} a_n x^n$ is **not** a function. It is a *formal power series*, a convenient way of working with an infinite sequence of numbers.

Note: We work with generating functions $A(x)$ in two forms:

(i) $A(x) = \sum_{n=0}^{\infty} a_n x^n$ (an infinite sequence of coefficients)

(ii) $A(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials (as rational functions)

Know:

- $1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$1 + \square + \square^2 + \ldots = \frac{1}{1-\square}$

- $1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}$

Problems.

1. Express the GF associated with the sequence $(-2, 2, -2, 2, -2, 2, \ldots)$ as a rational function.

$$\frac{-2}{1+x}$$

2. What is the coefficient of x^5 in $\frac{5x^2}{3-x}$?

$$\frac{5x^2}{3-x} = \frac{5}{3}x^2\left(\frac{1}{1-\frac{x}{3}}\right)$$

$$= \frac{5}{3}x^2\left(1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \ldots\right)$$

Coeff of x^5 is $\frac{5}{81}$

3. Apply partial fractions to $\frac{1}{(x-2)(x-4)}$

$$\frac{1}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4}$$

$$1 = (x-4)A + (x-2)B$$

$1 = 2B \quad B = \frac{1}{2}$

$1 = -2A \quad A = -\frac{1}{2}$

$$\frac{1}{(x-2)(x-4)} = \frac{-1/2}{x-2} + \frac{1/2}{x-4}$$

4. Assume that $A(x) = \sum_{n=0}^{\infty} a_n x^n$ where the sequence $a_0, a_1, \ldots$ satisfies the recurrence relation

$$a_0 = 2 \qquad a_1 = 9$$

$$\boxed{a_n - 3a_{n-1} + 7a_{n-2} = 0}$$

Express $A(x)$ as a rational function.

$n \geq 2$

$$[x^n]\left(A(x) - 3xA(x) + 7x^2A(x)\right)$$

$$= a_n - 3a_{n-1} + 7a_{n-2}$$

$$= 0$$

$$[x^0]\left(A(x) - 3xA(x) + 7x^2A(x)\right) = a_0 = 2$$

$$[x^1]\left(A(x) - 3xA(x) + 7x^2A(x)\right) = a_1 - 3a_0 = 3$$

$$\underbrace{A(x) - 3xA(x) + 7x^2A(x)}_{\parallel} = 2 + 3x$$

$$A(x)\left(1 - 3x + 7x^2\right)$$

$$A(x) = \frac{2+3x}{1-3x+7x^2}$$

Trees and Rooted Trees

Definition. A graph is a **tree** if it is connected and has no cycle.

Theorem. Let $G = (V, E)$ be a graph and consider the following properties:

1. G is connected
2. G has no cycle
3. $|V| = |E| + 1$

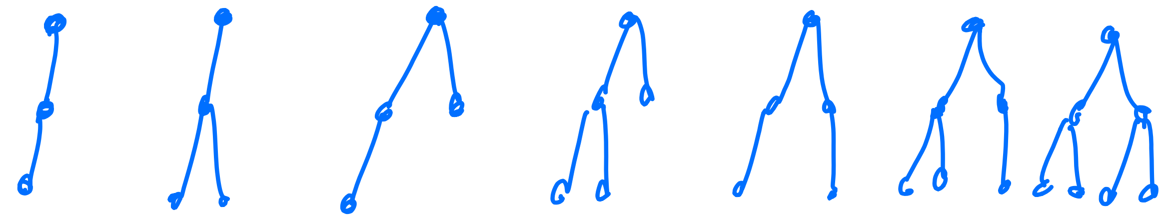
If G has any two of these properties, then it has the third (and G is a tree)

Definition. A **rooted tree** is a pair (T, r) where T is a tree and $r \in V(T)$ is a distinguished vertex called the **root**.

Vocabulary for rooted trees.

- **parent**, **children**, **ancestor**, **descendant**
- The **level** of a vertex is the distance from the root.
- A rooted tree is **ordered** if the children of each vertex are equipped with a linear order.
- A rooted tree is m-**ary** if every vertex has at most m children. (2-ary also known as **binary**
- An m-ary tree is **complete** if every vertex has 0 or m children.
- An m-ary tree is **balanced** if every leaf is on the last or second-to-last level.

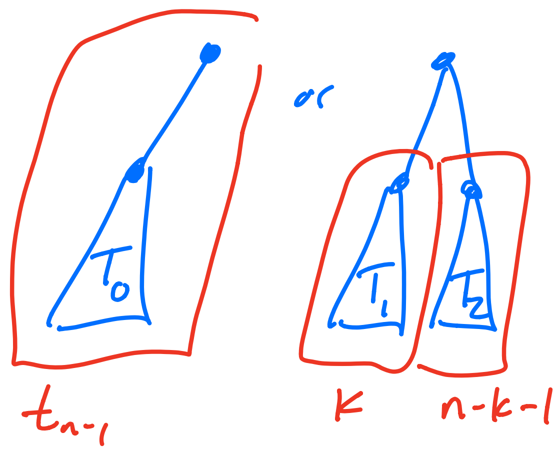
Ex: Draw all rooted binary trees with height 2.



Ex: Let t_n denote the number of rooted ordered binary trees on n vertices. Find a recursive formula for t_n.

$t_1 = 1$

$n \geq 2$



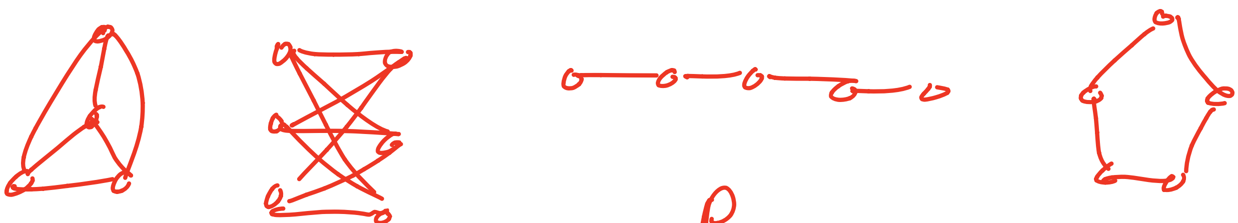
$$\text{For } n \geq 2 \quad t_n = t_{n-1} + \sum_{k=1}^{n-2} t_k t_{n-k-1}$$

Graphs

Note:

- Know basic terminology
- Know basic graphs (complete, complete bipartite, paths, cycles)

Ex: Draw K_4, $K_{3,3}$, P_5 and C_5.



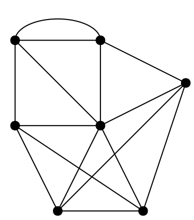
K_4 $K_{3,3}$ P_5 C_5

Theorem. For every graph $G = (V, E)$
$$2|E| = \sum_{v \in V} \deg(v).$$

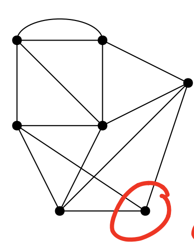
Corollary. Every graph has an even number of vertices with odd degree.

Theorem (Euler). A connected graph G has an Euler circuit if and only if every vertex of G has even degree.

Ex: Do the following graphs have Euler circuits?



all even degs
so Eul cct
exists



degree 3
no Euler cct

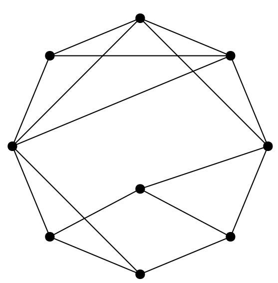
Terminology.

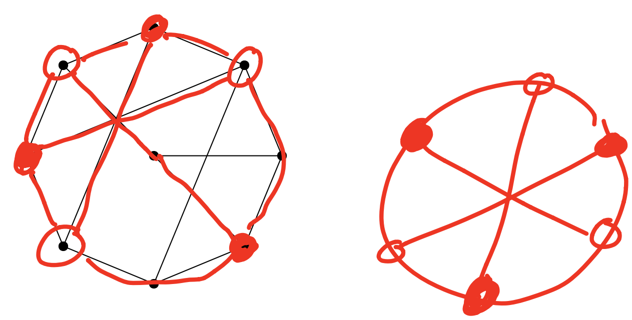
(i) A *drawing* of a graph in the plane is an **embedding** if there are no crossings, so the edges tough other edges and vertices only at their ends

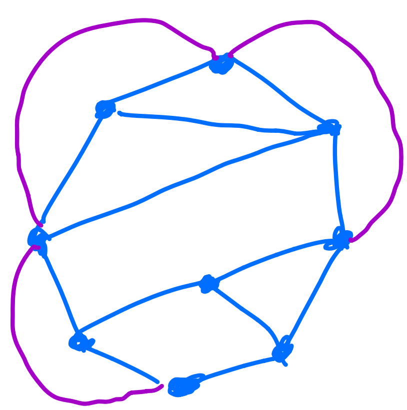
(ii) A *graph* is **planar** if there exists an embedding of it in the plane.

> **Theorem (Kuratowski-Wagner)** A graph G is planar if and only if G does not contain a subdivision of $K_{3,3}$ or a subdivision of K_5 as a subgraph.

Ex: Determine if the following graphs are planar. To do so, either find an embedding of the graph in the plane, or find a subdivision of $K_{3,3}$ or K_5.







Theorem (Euler) If $G = (V, E)$ is a connected planar graph, embedded in the plane with face set F, then

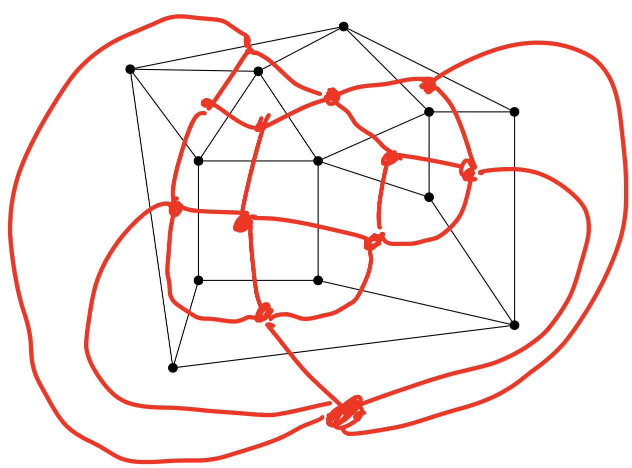
$$|V| - |E| + |F| = 2.$$

Theorem. If $G = (V, E)$ is a multigraph embedded in the plane with faces $f_1, \ldots, f_k$, then

$$\sum_{i=1}^{k} \deg(f_i) = 2|E|.$$

Definition. Let G be a multigraph embedded in the plane. To construct a **dual** multigraph G^*, put one vertex of G^* in each face of G, then for each edge $e \in E(G)$, if e lies on the boundary of faces f and f' (in the embedding of G), make an edge e^* in the dual graph G^* between the vertices corresponding to f and f' (this may be done so that e^* crosses e and G^* also ends up embedded in the plane).

Ex: Draw the planar dual of the following graph:



Ex: If G is a planar graph with 20 vertices and 36 edges, what is the average degree of a dual graph G^* of G?

$$2 = |V| - |E| + |F|$$

$$= 20 - 36 + |F|$$

so $|F| = 18$

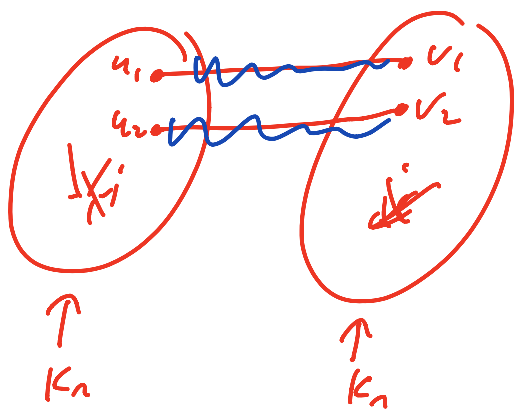
dual has 18 vert 36 edges avg deg $= \frac{2|E|}{|V|} = 4$

Definition. Let G be a graph. A path of G is **Hamiltonian** if it contains every vertex of G. Similarly, a cycle of G is **Hamiltonian** if it contains every vertex of G.

Ex: Let $n \geq 2$ and let G_n be a graph with vertex set $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ and edges:

- $\{u_i, u_j\}$ for all $1 \leq i < j \leq n$,
- $\{v_i, v_j\}$ for all $1 \leq i < j \leq n$,
- $\{u_1, v_1\}$ and $\{u_2, v_2\}$.

First draw G_4. Then determine how many Hamiltonian cycles and paths are in G_n for every $n \geq 2$.



Ham path in K_n on $\{u_1 .. u_n\}$ from u_1 to u_2

$u_1 \quad \underset{n-2}{\uparrow} \quad \underset{n-3}{\uparrow} \quad - \quad - \quad - \quad .. \quad u_2$ $\quad (n-2)!$

$\boxed{\text{total } ((n-2)!)^2}$

Theorem. Let $G = (V, E)$ be a graph with a Hamiltonian cycle.

1. Then $G - v$ is connected for every $v \in V$.
2. If G is bipartite, with bipartition $\{V_1, V_2\}$, then $|V_1| = |V_2|$.

Proofs Techniques

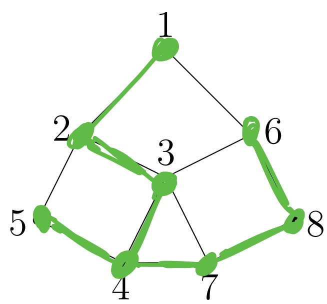
- induction
- contradiction
- extreme choice
- construction

Ex: Prove that every graph with all vertices of degree k and no cycle of length 3 has at least $2k$ vertices. Hint: consider a longest path.

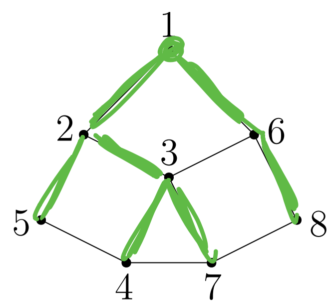
Optimization

Breadth-first and depth-first search trees

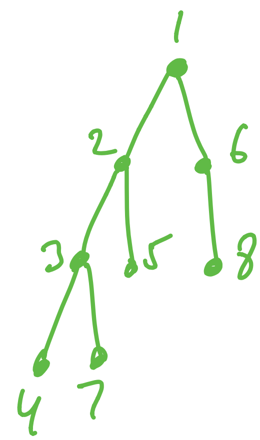
Ex: For the following graphs, indicate the breadth-first and depth-first search trees by shading the appropriate edges.



depth-first

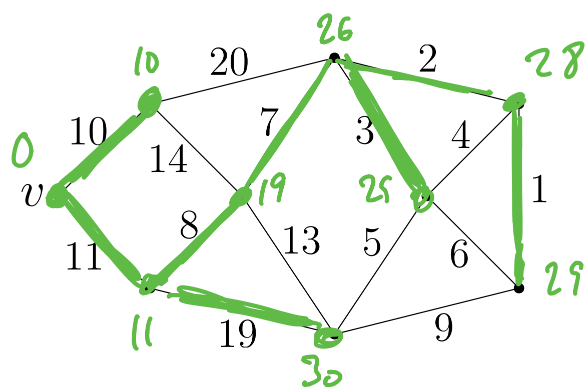


breadth-first

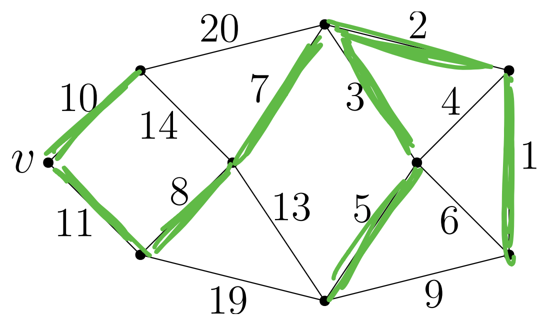


Dijkstra, Kruskal, and Prim. Kruskal and Prim's Algorithm compute min-weight spanning trees. Dijkstra's algorithm determines the distance of every vertex in the graph from an initial vertex v. However, the extended version of Dijkstra's algorithm also computes a shortest path tree for v.

Ex: Execute Dijkstra's algorithm to determine the shortest path three from v and the distance of every vertex from v in the graph below.

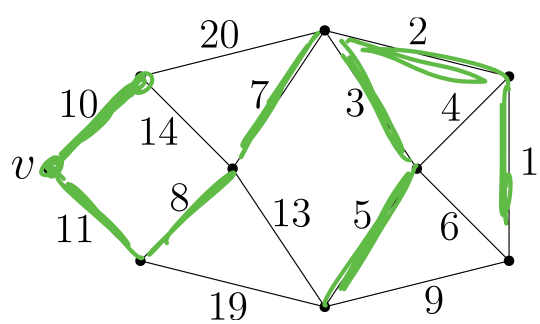


Ex: Execute Kruskal's algorithm on the graph below. In what order are the edges added?



1, 2, 3, 5, 7, 8, 10, 11 ← in order!

Ex: Execute Prim's algorithm on the graph below. starting from v In what order are the edges added?



10, 11, 8, 7, 2, 1, 3, 5

Colouring

Colouring. Let $G = (V, E)$ be a graph. A proper k-colouring of G is a function $f : V \to \{1, 2, \ldots, k\}$ with the property that every edge $\{u, v\}$ satisfies $f(u) \neq f(v)$. The **Chromatic number** of a graph G, denoted $\chi(G)$, is the smallest k for which a proper k-colouring exists.

Chromatic Polynomial. For every graph $G = (V, E)$ there is a polynomial, denoted $P(G, \lambda)$ and called the chromatic polynomial with the property that $P(G, k)$ is the number of proper k-colourings of G for every integer $k \geq 1$.

The chromatic polynomial can be computed recursively using the equation

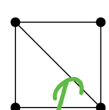
$$P(G, \lambda) = P(G - e, \lambda) - P(G/e, \lambda)$$

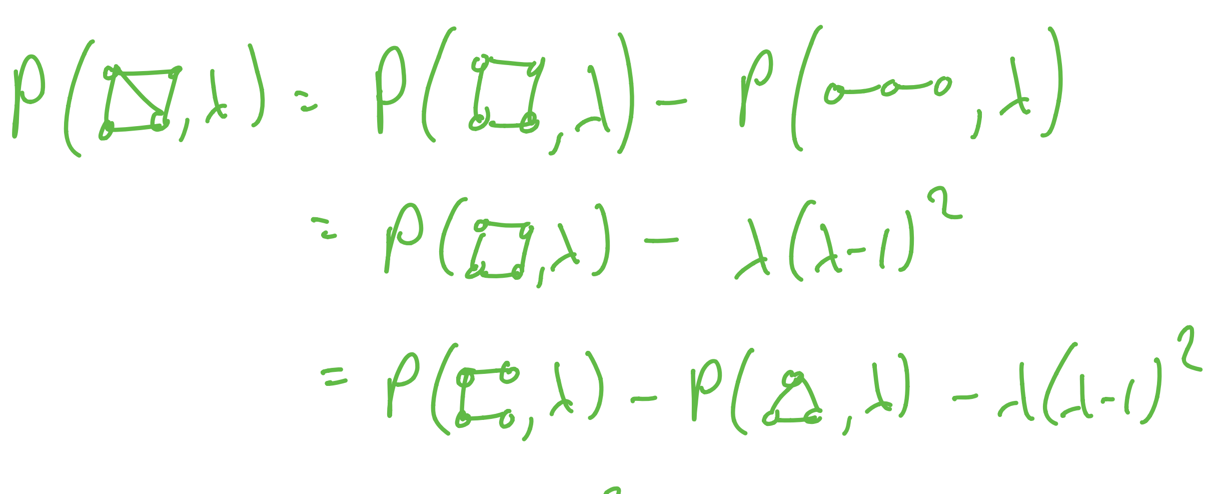
(here e is an edge of G, and $G - e$ is the graph obtained from G by deleting the edge e while G/e is the graph obtained from G by contracting e.

Computing Help:

- $P(K_n, \lambda) = \lambda(\lambda - 1)(\lambda - 2)\ldots(\lambda - n + 1)$
- $P(P_n, \lambda) = \lambda(\lambda - 1)^{n-1}$
- If G consists of G' plus an isolated vertex, $P(G, \lambda) = \lambda P(G', \lambda)$

Ex: Compute the chromatic polynomial of the following graph





$$= \lambda(\lambda-1)^3 - \lambda(\lambda-1)(\lambda-2) - \lambda(\lambda-1)^2$$