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FRIDAY

- quiz today
- NO COMPUTING WORKSHOPS, FRIDAY
- EXTRA MIDTERM TA OFFICE HOURS: MON 25 FEB
(during comp workshop hours)
WATCH for CANVAS ANNOUNCEMENT

LAST DAY

- the REAL L-SQ – matrix solve version
"RANK MATTERS"

- for OVERDETERMINED ($M>N$) linear system

$$[A]_{M\times N}\,\vec{v} = \vec{y}$$

  - a L-Sq solution ($\mathcal{E}$ could be zero) exists for

$$M > N \geq \text{rank}\,[A] = \text{rank}\,[A^TA]$$

and

$$[A^TA]\,\vec{v}_{LSQ} = [A^T]\,\vec{y}$$

  - when $N = \text{rank}\,[A] = \text{rank}\,[A^TA]_{N\times N}$

then $\vec{v}_{LSQ}$ is UNIQUE

• RANK & LINEAR INDEPENDENCE: definitions

• RANK THEOREM: row rank = col rank

• L-SQ POLYNOMIALS

use Legendre polynomials on $-1 \le x \le 1$

• shift & rescale domain

$a \le x \le b$

means

$$-1 \le \underbrace{\frac{2x-a-b}{b-a}}_{z} \le 1$$

Rodrigues' formula and other explicit formulas [edit]

An especially compact expression for the Legendre polynomials is given by Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n .$$

This formula enables derivation of a large number of properties of the P_n's. Among these are explicit representations such as

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k}^2 (x-1)^{n-k}(x+1)^k,$$

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k,$$

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k}\binom{2n-2k}{n} x^{n-2k},$$

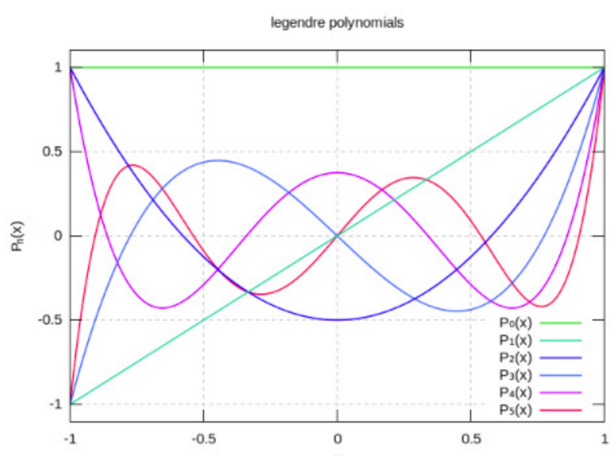
$$P_n(x) = 2^n \sum_{k=0}^{n} x^k \binom{n}{k}\binom{\frac{n+k-1}{2}}{n},$$

where the last, which is also immediate from the recursion formula, expresses the Legendre polynomials by simple monomials and involves the multiplicative formula of the binomial coefficient.

The first few Legendre polynomials are:

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$
6	$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7	$\frac{1}{16}(429x^7-693x^5+315x^3-35x)$
8	$\frac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35)$
9	$\frac{1}{128}(12155x^9-25740x^7+18018x^5-4620x^3+315x)$
10	$\frac{1}{256}(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63)$

The graphs of these polynomials (up to $n = 5$) are shown below:



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- take points $\{(x_k, y_k)\}_{k=1\to M}$

  + change to $\{(z_k, y_k)\}_{k=1\to M}$

- use $\vec{c} = [A] \backslash \vec{y}$

  and $$y(x) = \sum_{j=1}^{N} c_j P\left(\frac{2x-a-b}{b-a}\right)$$

- DEMO OBSERVATIONS
  - increasing M (more good data) is better
    - $M \geq N^2/4$ using Legendre poly is ROBUST for large N
  - increasing N leads to large cond # warnings.
- REMAINING question — what is Matlab \ doing?

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MIDTERM REVIEW

• MACM 316 COURSE STRUCTURE

TUTORIALS	LECTURES + TEXTBOOK	COMPUTING WORKSHOPS
problems + QUIZZES		computing + REPORTS
MIDTERMS + EXAMS		

INTRO TO NUMERICAL COMPUTING.

• digital computation with FLOATING POINT algorithms

  • finite-precision truncations of real numbers

  • finite-operation algorithms (stopping of convergent processes)

  • finite-discretizations of continuous functions (sampling)

• THREE KEYS of ALGORITHM PERFORMANCE

  • ACCURACY

  • EFFICIENCY

  • ROBUSTNESS

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FLOATING POINT ARITHMETIC (§1.2)

- truncation errors in FP arithmetic
- finite-precision calculations
- absolute vs relative errors
- simple strategies for reducing accumulation of error

Skills: synthetic N-digit arithmetic
calculating errors

GAUSSIAN ELIMINATION + SOLVING LINEAR SYSTEMS (ch 6)

- row operations on augmented matrix
- row reduction + back-substitution algorithm steps
- failures of GE: exact + numerical

singular matrix   floating pt errors + SMALL PIVOTS

- partial pivoting + related modifications
- operation counts for RRed $(O(N^3))$ + BSub $(O(N^2))$
- condition number $K(A)$ + solution error

GE SUMMARY

- ACCURACY → due to accumulated finite-prec truncation
  - minimized by pivoting (part., scaled, full)
- EFFICIENCY → op counts are $O(N^3)$
  - only can do better for special matrices (NON-GE algs)
    (sparse, Fourier matrices, etc)
    mostly zeros in matrix
- ROBUSTNESS → pivoting avoids zero/small PIVOTS
  - still need to worry if matrix is near-singular
    or large CONDITION #

Skills: small system GE + finite-precision GE
partial pivot GE
count operations

LU FACTORIZATION

- row operations as matrix multiplies (pivoting + permutation)
- LU solution of linear system
- special matrix solves (diag dom, tri-diagonal, ...)

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Skills: identify row ops & matrices
LU solution by BSub

ZEROS of CONTINUOUS FUNCTIONS (ch 2)

- continuity, intermediate value thm & bracketting
- NM, calc 1 revisited, convergence & stopping condition
- BiS & SM algorithms
- convergence rate & order of convergence
- error analysis for BiS, SM & NM
  (order of convergence: 1, 1.62..., 2)

ROOT FINDING SUMMARY

	ACC / EFF	ROB	
BiS	Lin Conv 1 eval/iter	preserve bracket.	
NM	Quad Conv 2 eval/iter	need $\|\lambda e\, E_n\| < 1$ better to start close	need f' slow/wrong start
SM	1.62 Conv 1 eval/iter	need $\|\lambda e\, E_n\| < 1$ but No bracket start	
fzero	Bis/SM hybrid.	better for bracket start	

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Skills: iterations of BiS, SM, NM
estimate error convergence

POLYNOMIAL INTERPOLATION (ch 3+)

- polynomial basis sets + linear algebra of interpolation
- monomial basis + poor conditioning of Vandermonde matrix
- Lagrange interpolating polynomial + error term
- NLI as interpolant algorithm of choice, operation counts
- Newton interpolating polynomial
- cubic splines: equations + end conditions (nat, cl, n-a-k)

Skills: write linear systems for interpolation
create Lagrange interpolating polynomials
use NLI for interpolation
operation counts for reg Lag + NLI
write spline equations, incl end conditions
partial solves + spline interpolation.
use c_j-equation

POLYNOMIAL INTERPOLATION

	MLI	cubic spline
# polys	ONE of deg N	N, each deg 3
eval	$L_{MLI}(x)$	must find $x_k \le x < x_{k+1}$
continuous derivs	all	S, S', S'' only
large N	fails for most $\{x_k\}$	ROBUST
convergence, accuracy	OK, but not guaranteed	$(\max \Delta x)^4$, n-a-k
efficiency	$O(N^2)$ pre-comp, $O(N)$ eval	$O(N)$

LEAST-SQUARES (ch 8.1-2+)

- linear L-Sq + matrix formulation $[A]\vec{v} = \vec{y}$
- Normal Equation: $[A^TA]\vec{v}_{LSQ} = [A^T]\vec{y}$
- overdetermined + UNIQUE L-Sq: $M > N = \text{rank}(A)$

Skills: write "perfect-fit" + Normal equations
demonstrate matrix rank + linear dependencies
evaluate least-square derivatives

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MIDTERM

- 5–6 quiz-like pages with multiple parts
- focus on understanding algorithms + theory
  + implementation trade-offs
- no coding questions, but know implementation issues
  (Matlab input + output details, for instance)
- do familiar questions first, prioritize high-value parts
- BRING CALCULATOR (+ practice using it)
- know ACC/EFF/ROB narratives
- study lecture notes (+ text sections)
  problems, quizzes + computing assignments

  these are all inter-related!

- STUDY QUIZ SKILLS + PROBLEM RECOGNITION