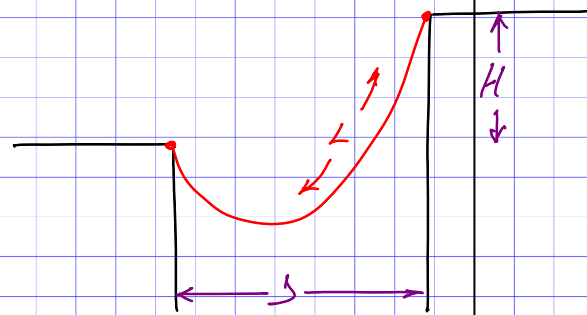
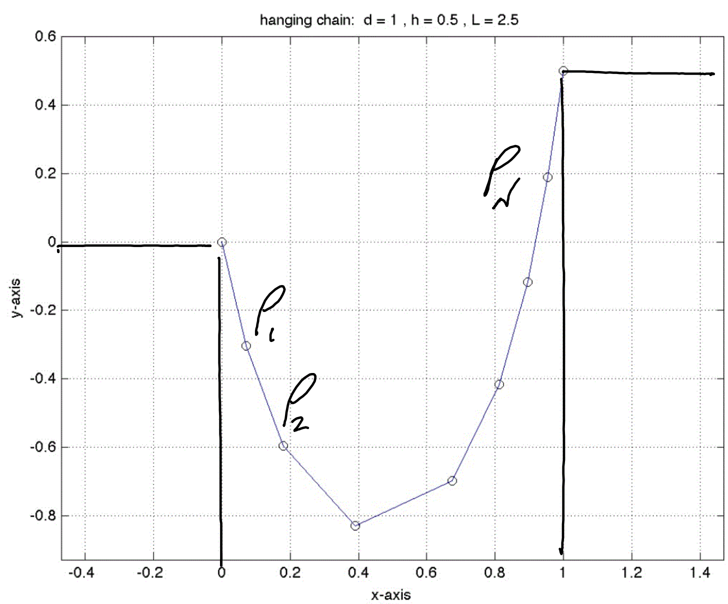
# Model of a Hanging Chain

- Upscaling Newton's method
- Physical problem

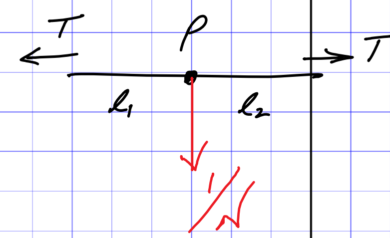



- Chain is uniform in mass along length & gravity pulls down

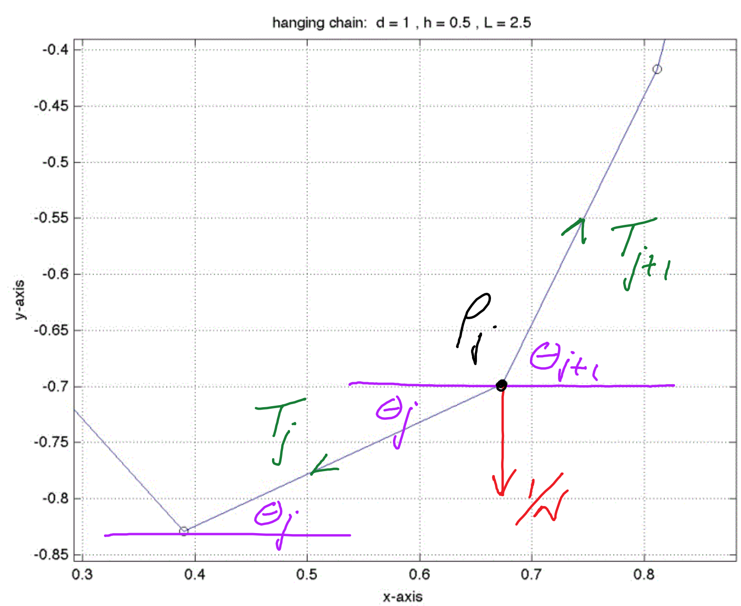
- Model chain as N+1 straight links with N joints (+ 2 ends)



# Physics

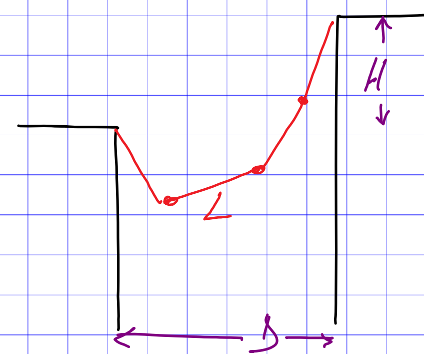



- Chain is under tension (force)
- Link $\ell_1$ exerts a tension force of $\leftarrow T$ at P
- link $\ell_2$ exerts a tension force of $T \rightarrow$ at P
- Chain has weight due to gravity (force)
- Distribute weight evenly at each node point $P_j$
- at each node point, $P_j$, Force vectors must balance (sum to zero)




$$\left.\begin{aligned} \text{HORIZ:} \quad & T_{j+1}\cos\theta_{j+1} - T_j\cos\theta_j = 0 \\ \text{VERT:} \quad & T_{j+1}\sin\theta_{j+1} - T_j\sin\theta_j - 1/N = 0 \end{aligned}\right\} \quad j = 1 \to N$$

- so far, $2N$ equations for $\{T_j, \theta_j \mid j=1 \to N+1\}$ → $2N+2$ unknowns
- Final 2 equations from geometry
- Each link has length L/(N+1)




HORIZ. $\frac{L}{N+1}\left\{\cos\theta_1 + \cos\theta_2 + \dots + \cos\theta_{j+1}\right\} = D$

VERT $\frac{L}{N+1}\left\{\sin\theta_1 + \quad \dots \quad \sin\theta_{j+1}\right\} = H$

- $2(N+1)$ nonlinear equations for $\{T_j, \theta_j\}_{j=1 \to N+1}$
  given $D$, $H$ & $L$.

## Newton's Method

$$\vec{x} = \begin{pmatrix} T_1 \\ \vdots \\ T_{N+1} \\ \theta_1 \\ \vdots \\ \theta_{N+1} \end{pmatrix} \qquad \vec{F}(\vec{x}) = \begin{pmatrix} \vdots \\ T_{j+1}\cos\theta_{j+1} - T_j\cos\theta_j \\ \vdots \\ T_{j+1}\sin\theta_{j+1} - T_j\sin\theta_j - 1/N \\ \vdots \\ \sum_1^{N+1}\cos\theta_j - \frac{D}{L}(N+1) \\ \sum_1^{N+1}\sin\theta_j - \frac{H}{L}(N+1) \end{pmatrix}$$

Jacobian matrix (p433)

$$J = \left[\begin{array}{ccccc|ccccc}
-\cos\theta_1 & \cos\theta_2 & & & \emptyset & T_1\sin\theta_1 & -T_2\sin\theta_2 & & & \emptyset \\
& \ddots & \ddots & & & & \ddots & \ddots & & \\
\emptyset & & & -\cos\theta_N & \cos\theta_{N+1} & \emptyset & & & T_N\sin\theta_N & -T_{N+1}\sin\theta_{N+1} \\
\hline
-\sin\theta_1 & \sin\theta_2 & & & \emptyset & -T_1\cos\theta_1 & T_2\cos\theta_2 & & & \emptyset \\
& \ddots & \ddots & & & & \ddots & \ddots & & \\
\emptyset & & & -\sin\theta_N & \sin\theta_{N+1} & \emptyset & & & -T_N\cos\theta_N & T_{N+1}\cos\theta_{N+1} \\
\hline
0 & - & - & - & 0 & -\sin\theta_1 & - & - & - & -\sin\theta_{N+1} \\
0 & - & - & - & 0 & \cos\theta_1 & - & - & - & \cos\theta_{N+1}
\end{array}\right]$$

• for $N = 3$

Jacob =

$$\left[\begin{array}{cccc|cccc}
-0.2338 & 0.9825 & 0 & 0 & -0.3492 & 0.0159 & 0 & 0 \\
0 & -0.9825 & 0.2557 & 0 & 0 & -0.0159 & -0.3174 & 0 \\
0 & 0 & -0.2557 & 0.1280 & 0 & 0 & 0.3174 & -0.6508 \\
\hline
0.9723 & -0.1862 & 0 & 0 & -0.0840 & 0.0840 & 0 & 0 \\
0 & 0.1862 & 0.9667 & 0 & 0 & -0.0840 & 0.0840 & 0 \\
0 & 0 & -0.9667 & 0.9918 & 0 & 0 & -0.0840 & 0.0840 \\
\hline
0 & 0 & 0 & 0 & 0.9723 & 0.1862 & -0.9667 & -0.9918 \\
0 & 0 & 0 & 0 & 0.2338 & 0.9825 & 0.2557 & 0.1280
\end{array}\right]$$

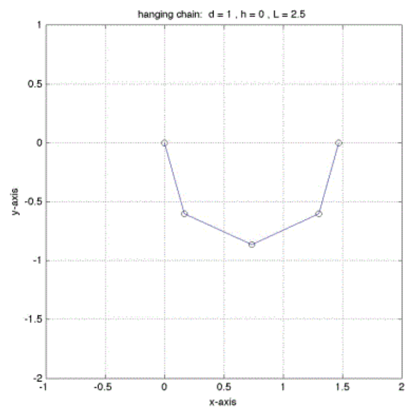
• newton iteration for $\vec{F}(\vec{x}) = \vec{\emptyset}$

$$\vec{x}_{k+1} = \vec{x}_k + J \backslash \left(-\vec{F}(\vec{x}_k)\right)$$

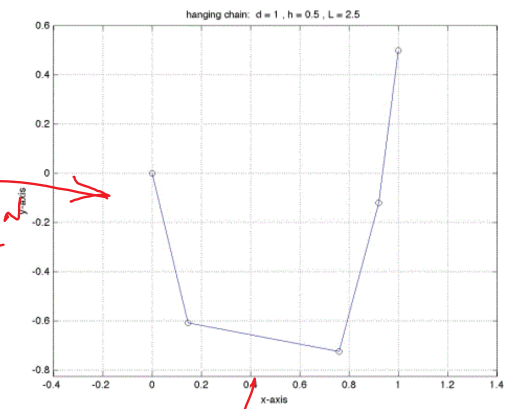
• how to find $\vec{x}_0$ ?

# Running & upscaling

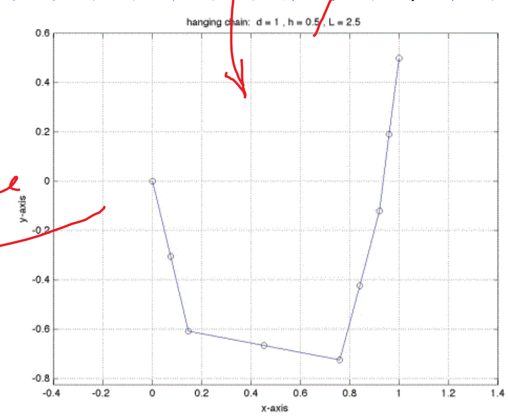
• $N = 3$ , guess rough parabola shape



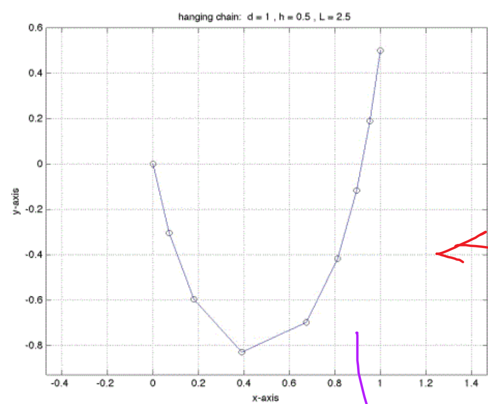

7 NEWTON iter



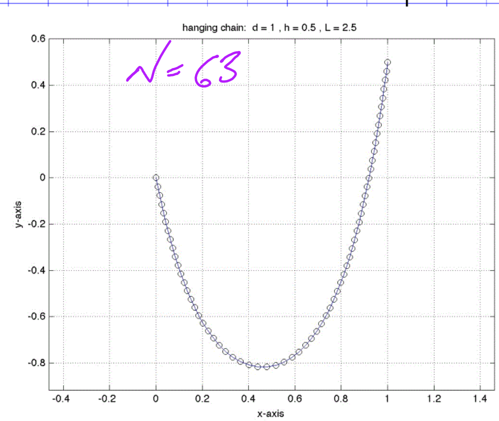

twin points




6 more iter.




more twinning + iterations




Newton's method in 128 dimensions

muraki