Technology Exercises

T1. (a) Some technology utilities provide a command for finding the rank of a matrix. Determine whether your utility has this capability; if so, use that command to find the rank of the matrix in Example 1.

(b) Confirm that the rank obtained in part (a) is consistent with the rank obtained by using your utility to find the number of nonzero rows in the reduced row echelon form of the matrix.

T2. Most technology utilities do not provide a direct command for finding the nullity of a matrix since the nullity can be computed using the rank command and Formula (2). Use that method to find the nullity of the matrix in Exercise T1 of Section 7.3, and confirm that the result obtained is consistent with the number of basis vectors obtained in that exercise.

T3. Confirm Formula (2) for some 5×7 matrices of your choice.

T4. ***Sylvester's rank inequalities*** (whose proofs are somewhat detailed) state that if A is a matrix with n columns and B is a matrix with n rows, then

$$\operatorname{rank}(A) + \operatorname{rank}(B) - n \le \operatorname{rank}(AB) \le \operatorname{rank}(A)$$
$$\operatorname{rank}(A) + \operatorname{rank}(B) - n \le \operatorname{rank}(AB) \le \operatorname{rank}(B)$$

Confirm these inequalities for some matrices of your choice.

T5. (a) Consider the matrices

$$A = \begin{bmatrix} 7 & 4 & -2 & 4 \\ 2 & -3 & 7 & -6 \\ 5 & 6 & 2 & -5 \\ 3 & 3 & -5 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 7.1 & 4 & -2 & 4 \\ 2 & -3 & 7 & -6 \\ 5 & 6 & 2 & -5 \\ 3 & 3 & -5 & 8 \end{bmatrix}$$

which differ only in one entry. Compute A^{-1} and use the result in Exercise P6 to compute B^{-1}.

(b) Check your result by computing B^{-1} directly.

T6. It can be proved that the rank of a matrix A is the order of the largest square submatrix of A (formed by deleting rows and columns of A) whose determinant is nonzero. Use this result to find the rank of the matrix

$$A = \begin{bmatrix} 3 & -1 & 3 & 2 & 5 \\ 5 & -3 & 2 & 3 & 4 \\ 1 & -3 & -5 & 0 & -7 \\ 7 & -5 & 1 & 4 & 1 \end{bmatrix}$$

and check your answer by using a different method to find the rank.

Section 7.5 The Rank Theorem and Its Implications

In this section we will prove that the row space and column space have the same dimension, and we will discuss some of the implications of this result.

THE RANK THEOREM

The following theorem, which is proved at the end of this section, is one of the most important in linear algebra.

Theorem 7.5.1 (***The Rank Theorem***) *The row space and column space of a matrix have the same dimension.*

EXAMPLE 1 Row Space and Column Space Have the Same Dimension

In Example 4 of Section 7.3 we showed that the row space of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & -4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix} \tag{1}$$

is three-dimensional, so the rank theorem implies that the column space is also three-dimensional. Let us confirm this by finding a basis for the column space. One way to do this is to transpose A (which converts columns to rows) and then find a basis for the row space of A^T by reducing it to row echelon form and extracting the nonzero row vectors. Proceeding in this way, we first underdetermined system governs a robot for which certain actions can be achieved in infinitely many ways, which may not be desirable. ■

MATRICES OF THE FORM A^TA AND AA^T

Matrices of the form A^TA and AA^T play an important role in many applications, so we will now focus our attention on matrices of this form.

To start, recall from Formula (9) of Section 3.6 that if A is an $m \times n$ matrix with column vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, then

$$A^TA = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{a}_1 & \mathbf{a}_1 \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_1 \cdot \mathbf{a}_n \\ \mathbf{a}_2 \cdot \mathbf{a}_1 & \mathbf{a}_2 \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_2 \cdot \mathbf{a}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_n \cdot \mathbf{a}_1 & \mathbf{a}_n \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_n \cdot \mathbf{a}_n \end{bmatrix} \tag{8}$$

Since transposing a matrix converts columns to rows and rows to columns, it follows from (8) that if $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ are the row vectors of A, then

$$AA^T = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{r}_1 & \mathbf{r}_1 \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_1 \cdot \mathbf{r}_m \\ \mathbf{r}_2 \cdot \mathbf{r}_1 & \mathbf{r}_2 \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_2 \cdot \mathbf{r}_m \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_m \cdot \mathbf{r}_1 & \mathbf{r}_m \cdot \mathbf{r}_2 & \cdots & \mathbf{r}_m \cdot \mathbf{r}_m \end{bmatrix} \tag{9}$$

The next theorem provides some important links between properties of a general matrix A, its transpose A^T, and the square symmetric matrix A^TA.

Theorem 7.5.8 *If A is an $m \times n$ matrix, then:*

(a) A and A^TA have the same null space.

(b) A and A^TA have the same row space.

(c) A^T and A^TA have the same column space.

(d) A and A^TA have the same rank.

We will prove part (*a*) and leave the remaining proofs for the exercises.

Proof (a) We must show that every solution of $A\mathbf{x} = \mathbf{0}$ is a solution of $A^TA\mathbf{x} = \mathbf{0}$, and conversely. If $\mathbf{x}_0$ is any solution of $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}_0$ is also a solution of $A^TA\mathbf{x} = \mathbf{0}$ since

$$A^TA\mathbf{x}_0 = A^T(A\mathbf{x}_0) = A^T\mathbf{0} = \mathbf{0}$$

Conversely, if $\mathbf{x}_0$ is any solution of $A^TA\mathbf{x} = \mathbf{0}$, then $\mathbf{x}_0$ is in the null space of A^TA and hence is orthogonal to every vector in the row space of A^TA by Theorem 3.5.6. However, A^TA is symmetric, so $\mathbf{x}_0$ is also orthogonal to every vector in the column space of A^TA. In particular, $\mathbf{x}_0$ must be orthogonal to the vector $A^TA\mathbf{x}_0$; that is, $\mathbf{x}_0 \cdot (A^TA\mathbf{x}_0) = 0$. From Formula (23) of Section 3.1, we can write this as

$$\mathbf{x}_0^T(A^TA\mathbf{x}_0) = 0 \quad \text{or, equivalently, as} \quad (A\mathbf{x}_0)^T(A\mathbf{x}_0) = 0$$

This implies that $A\mathbf{x}_0 \cdot A\mathbf{x}_0 = 0$, so $A\mathbf{x}_0 = \mathbf{0}$ by part (*d*) of Theorem 1.2.6. This proves that $\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{0}$. ■

The following companion to Theorem 7.5.8 follows on replacing A by A^T in that theorem and using the fact that A and A^T have the same rank for part (*d*).

Theorem 7.5.9 *If A is an $m \times n$ matrix, then:*

(a) A^T and AA^T have the same null space.

(b) A^T and AA^T have the same row space.

(c) A and AA^T have the same column space.

(d) A and AA^T have the same rank.