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MONDAY

• COMPUTING REPORT due on CROWDMARK

LAST DAY

• condition number as warning for low accuracy linear solve

$$\text{worst case err} \approx K([A]) \cdot \varepsilon \longrightarrow \text{machine epsilon} \approx 10^{-16}$$

• GE & the LU FACTORIZATION

• successful GE implies the matrix [A] can be written as a matrix product: $[L][U] = [P][A]$

from row

• facts & ideas

fact #1 : row operations are matrix multiplies (on left)

idea #1 : row reduction is a long string of matrix multiplies

fact #2 : transformation matrices are invertible.

idea #2 : inverse matrix arithmetic factors [A]

$$[A] = [T_1^{-1} \dots T_M^{-1}][U] = [L][U]$$

fact #3 : product of lower Δ is also lower Δ.

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LU discussion

- Matlab has LU command with

```
>> A = round(5*randn(3,3))

A =

     7    -6     2
     7     4     5
     3     8     4

>> [mL,mU] = lu(A)

mL =

    1.0000         0         0
    1.0000    0.9459    1.0000
    0.4286    1.0000         0


mU =

    7.0000   -6.0000    2.0000
         0   10.5714    3.1429
         0         0    0.0270
```

[ ] =

- means (-)

$[\ ][A] = [L][U]$.

is GE where pivoting occurs!

- or

$[A] = [P^{-1}L][U]$

[ ] = row- lower Δ.

($[P]^{-1} = [\ ]$ is also permutation mtx)

- so permuted [ ] means row exchanges

```
>> [mL,mU,mP] = lu(A)

mL =

    1.0000         0         0
    0.4286    1.0000         0
    1.0000    0.9459    1.0000


mU =

    7.0000   -6.0000    2.0000
         0   10.5714    3.1429
         0         0    0.0270


mP =

     1     0     0
     0     0     1
     0     1     0

>> inv(mP)*mL

ans =

    1.0000         0         0
    1.0000    0.9459    1.0000
    0.4286    1.0000         0
```

= [ ]

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- textbook details (p468)
  - [U] is same as GE

Theorem 6.19 If Gaussian elimination can be performed on the linear system $A\mathbf{x} = \mathbf{b}$ without row interchanges, then the matrix A can be factored into the product of a lower-triangular matrix L and an upper-triangular matrix U, that is, $A = LU$, where $m_{ji} = a_{ji}^{(i)}/a_{ii}^{(i)}$,

$$U = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1,n}^{(n-1)} \\ 0 & \cdots & 0 & a_{nn}^{(n)} \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ m_{n1} & \cdots & m_{n,n-1} & 1 \end{bmatrix}.$$

[L] has on diagonal
& off-diagonal elements are
!

- LU linear solve
  - if we know LU factorization, linear solve

$[\ \ ][\ \ ]\vec{x} = [\ \ ][A]\vec{x} = [\ \ ]\vec{b}$

uses 2 !

$[L]\vec{\ } = [\ \ ]\vec{b}$

$[U]\vec{x} = \vec{\ }$

- operation count is $O(\ \ )$ for & $[\ \ ]\vec{b}$

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- uses of LU
  - solves of $[A]\vec{x} = \vec{b}$ → for many $\vec{b}_k$
    - do LU once, $O(\quad)$
    - then to get $\vec{x}$ for every $\vec{b}$
  - pre-compute lfled. for - solves.
  - calculate matrix
    - cofactor method needs $O(\quad)$ operations

      but $n! = O(\qquad)$ grows
    - LU method

$$\det[A] = \det[\qquad]$$

$$= \det[\quad]\,\det[\quad]\,\det[\quad]$$

$$(\pi\qquad)$$

det of Δ mtx = of elements

- IDEA for FUTURE
  - other matrix factorizations lead to !

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NON-GE SOLVES & SPECIAL MATRICES

- DIAGONALLY-DOMINANT
  - for all rows j,

$$|a_{jj}| \geqslant \sum_{\substack{k=1 \\ k \neq j}}^{n} |a_{jk}| = \text{sum of abs vals of all elements in row } j$$

strictly diag. dominant for $>$

Theorem 6.21 A strictly diagonally dominant matrix A is nonsingular. Moreover, in this case, Gaussian elimination can be performed on any linear system of the form $A\mathbf{x} = \mathbf{b}$ to obtain its unique solution without row or column interchanges, and the computations will be stable with respect to the growth of round-off errors. ■

(proof p 417-418)

- SYMMETRIC & POSITIVE-DEFINITE
  - symmetric means = $\Rightarrow$ all eigenvalues
  - positive-definite means all λ $\Rightarrow$ -singular

    (for all non-zero $\vec{x}$, $\vec{x}^T \underline{A} \vec{x} >$)

Theorem 6.26 The symmetric matrix A is positive definite if and only if Gaussian elimination without row interchanges can be performed on the linear system $A\mathbf{x} = \mathbf{b}$ with all pivot elements positive. Moreover, in this case, the computations are stable with respect to the growth of round-off errors. ■

- two new factorizations : & LDL^T fact.

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- CHOLESKY FACTORIZATION (p423-25)

$$[\underline{A}] = [\underline{\ }][\underline{\ }]^T$$

lower Δ → (lower Δ) = Δ.

- operation count is / of usual LU $\approx \frac{}{3} N^3$ (& p.piv.)
- but pos def advantage uses N . ( )

- TRIDIAGONAL (p426)

$$[\underline{A}] = \begin{bmatrix} x & x & & & \\ x & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & x \\ & & & x & x \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & u_{23} & 0 \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & 0 & 0 \\ l_{21} & l_{22} + l_{21}u_{12} & l_{22}u_{23} & 0 \\ 0 & l_{32} & l_{33} + l_{32}u_{23} & l_{33}u_{34} \\ 0 & 0 & l_{43} & l_{44} + l_{43}u_{34} \end{bmatrix}.$$

- special CROUT FACTORIZATION

(LU)

- more solves later in the term...

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SPARSE MATRIX TOOLS

- matrices normally stored as arrays
- sparse matrices are : store ONLY non-zero elements + position.
- special sparse matrix +
- Matlab commands: sparse, full, speye, spy ...

WHERE DO MATRICES COME FROM ?

- unstructured (like from for example): need generic routines
- structured (from questions): some have special routines specifically addressing: accur, effic + robust.

TRANSITION: APPLICATION of DIRECT LINEAR SOLVE

- nonlinear system of equations : Newton's method.

- calculus example

$$u(x,y) = 3x^2 + xy - 4y^2 + 1 = 0 \quad \text{hyperbola}$$
$$v(x,y) = x^2 + y^2 - 1 = 0 \quad \text{circle}$$

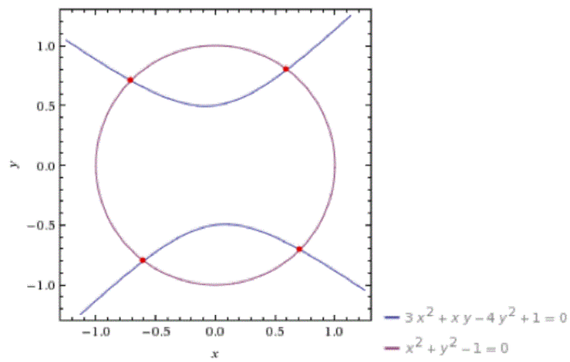
3*x^2 + x*y - 4*y^2 +1 = 0 and x^2 + y^2 -1 = 0

Examples Random

Input:

$$\{3x^2 + xy - 4y^2 + 1 = 0,\ x^2 + y^2 - 1 = 0\}$$

Plot of solution set:



Enable interactivity

Alternate forms:

$$\{x(3x + y) + 1 = 4y^2,\ x^2 + y^2 = 1\}$$

$$\{3x^2 + xy + 1 = 4y^2,\ x^2 + y^2 = 1\}$$

$$\{3x^2 + xy - 4y^2 = -1,\ x^2 + y^2 = 1\}$$

Solutions: Approximate forms

$$x = -\frac{3}{5},\quad y = -\frac{4}{5}$$

$$x = \frac{3}{5},\quad y = \frac{4}{5}$$

$$x = -\frac{1}{\sqrt{2}},\quad y = \frac{1}{\sqrt{2}}$$

$$x = \frac{1}{\sqrt{2}},\quad y = -\frac{1}{\sqrt{2}}$$

- vector function of a vector variable

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \vec{F}(\vec{x}) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

$\rightarrow$ find $\vec{x}_0$ such that $\vec{F}(\vec{x}_0) = \vec{0}$

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- multiple-variable Taylor expansion
  OR linear approx of $\vec{F}(\vec{x})$

$$\underbrace{\vec{F}(\vec{x}+\vec{\delta x})}_{\substack{\text{small vector} \\ (|\vec{\delta x}| \text{ suff small})}} \simeq \vec{F}(\vec{x}) + \underbrace{\left[J(\vec{x})\right]}_{\text{Jacobian matrix}}\vec{\delta x}$$

$$\left[J(\vec{x})\right] = \underbrace{\begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix}_{\vec{x}}}_{\text{eval at } \vec{x}} \quad \text{in 2D.}$$

- for our problem: $\left[J(\vec{x})\right] = \begin{bmatrix} 6x+y & x-8y \\ 2x & 2y \end{bmatrix}$

- NEXT: 2 variable NEWTON'S METHOD.

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