

# The Role of Entanglement in Quantum Reservoir Computing with Coupled Kerr Nonlinear Oscillators

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Quantum Reservoir Computing (QRC) uses quantum dynamics to efficiently process temporal data. In this work, we investigate a QRC framework based on two coupled Kerr nonlinear oscillators, a system well-suited for time-series prediction tasks due to its complex nonlinear interactions and potentially high-dimensional state space. We explore how its performance in time-series prediction depends on key physical parameters: input drive strength, Kerr nonlinearity, and oscillator coupling, and analyze the role of entanglement in improving the reservoir's computational performance, focusing on its effect on predicting non-trivial time series. Using logarithmic negativity to quantify entanglement and normalized root mean square error (NRMSE) to evaluate predictive accuracy, our results suggest that entanglement provides a computational advantage on average—up to a threshold in the input frequency—that persists under some levels of dissipation and dephasing. In particular, we find that higher dissipation rates can enhance performance. While the entanglement advantage manifests as improvements in both average and worst-case performance, it does not lead to improvements in the best-case error. These findings contribute to the broader understanding of quantum reservoirs for high performance, efficient quantum machine learning and time-series forecasting.

## I. INTRODUCTION

Modern machine learning has achieved remarkable success in processing complex datasets, yet handling temporal or sequential data—such as speech, sensor signals, or chaotic time series—remains challenging. Recurrent neural networks were designed to capture temporal dependencies, but their training is often difficult and resource-intensive due to issues like vanishing and exploding gradients [1] and large parameter counts [2]. This has motivated alternative approaches to temporal processing that reduce training costs. One such framework is reservoir computing (RC), which employs a fixed nonlinear dynamical system—the “reservoir”—to project inputs into a high-dimensional space, requiring training only a simple linear readout [3, 4]. RC has proven effective for tasks like time-series prediction and chaos modeling, and has been physically realized in hardware ranging from electronic circuits to optical and photonic systems [5–11].

In parallel, the past decades have seen rapid advances in quantum information science, with the promise of quantum computers solving certain problems exponentially faster than classical ones [12, 13]. However, current quantum devices in the noisy intermediate-scale quantum (NISQ) era face severe noise and decoherence limitations [4]. An intriguing strategy is to combine ideas from noise-tolerant classical computing with quantum hardware [4]. Quantum Reservoir Computing (QRC) was proposed as a way to exploit the natural dynamics of quantum systems for machine learning [14], analogous to classical RC but potentially leveraging quantum effects.

Fujii and Nakajima first introduced the concept of

QRC in 2017 [14], demonstrating that an ensemble of just 5–7 qubits could emulate the performance of a classical recurrent neural network with hundreds of neurons on chaotic time-series tasks. This result suggested that the large state space of even small quantum systems could be harnessed for computation. Subsequent works have explored QRC implementations on various platforms: for example, networks of qubits (spin-1/2 systems) have been used to process temporal data with high memory capacity [15], and even real quantum processors with inherent noise have been treated as reservoirs, where the unavoidable decoherence can serve as a useful computational resource rather than an obstacle [16]. QRC has also been extended to quantum tasks; for instance, a quantum reservoir processor was shown to recognize entanglement in quantum input states and estimate properties like entropy or logarithmic negativity, tasks that ordinarily would require multiple quantum measurements [17]. It has been argued that QRC can naturally achieve a form of nonlinear computation even when the underlying quantum evolution is linear, either through measurement-induced nonlinearities [4] or through clever input encodings [18].

Quantum reservoir computing with bosonic modes has attracted increasing attention. Prior studies have shown that Kerr nonlinearities can endow a quantum reservoir with rich dynamics. The intrinsic nonlinearity in this kind of systems ensures complex, nonlinear transformations of inputs [19–21], which is crucial for reservoir computing. Moreover, Kerr oscillator systems are experimentally feasible [22]. Govia et al. [4] proposed QRC using a single Kerr oscillator and demonstrated that the enlarged

bosonic Hilbert space can improve time-series prediction performance. Kalfus et al. [23] similarly showed that increasing the Hilbert-space dimension of a single-mode reservoir yields clear performance gains over classical counterparts. Networks of coupled oscillators have also been explored: Dudas et al. [24] implemented QRC using two coherently coupled superconducting resonators, demonstrating that a pair of Fock-populated oscillators can process temporal data. Cheamsawat and Chotibut [25] analyzed how different drive and coupling regimes in a pair of coupled Kerr cavities affect information encoding.

A key question driving QRC research is what intrinsic quantum features can provide computational advantages over classical reservoirs. Quantum systems offer two notable features: (i) a state space that grows exponentially with the number of constituents, and (ii) uniquely quantum phenomena such as superposition and entanglement. The former means that a moderately sized quantum reservoir can explore a vast space of configurations, which in principle could embed very complex mappings of the input data [26]. The latter – entanglement and related non-classical correlations – might endow the reservoir with richer dynamics and memory than any classical counterpart [26].

While some studies show that quantum reservoir computing can function without strongly non-classical states[27], others suggest that genuine quantum resources—such as entanglement, coherence, or quantum chaos—may offer enhanced computational power. For example, memory capacity in qubit-based reservoirs has been found to correlate with entanglement, and recent work showed that this advantage depends on the input frequency [26, 28]. However, more “quantumness” is not always better: studies have shown that in certain tasks, such as NARMA, introducing squeezing (a form of non-classicality) can reduce performance [29]. These findings underscore that while quantum effects can be beneficial, their impact on reservoir computing remains subtle and task-dependent—especially the role of entanglement, which is still not fully understood.

This paper aims to expand this area of investigation. In contrast to the previous studies mentioned above, our work explicitly focuses on entanglement in a reservoir formed by two coupled Kerr oscillators and its relationship with predictive performance. By varying the system parameters such as coupling, input (driving) strength, and nonlinearity, we can tune the reservoir’s entanglement, which we quantify by means of logarithmic negativity, while evaluating the prediction performance on the target time series by calculating normalized root mean square error (NRMSE). We also explore how the relationship is affected by dissipation and dephasing. Notably, as we focus on low-excitation-number dynamics, our analysis is performed in the fully quantum regime.

Our results identify Kerr system operating regimes that optimize prediction performance and clarify that there is a nuanced relation between entanglement and op-

timized performance in the studied continuous-variable system.

The remainder of this paper is organized as follows: In section II, we introduce the physical model of the coupled Kerr oscillators and we define how input signals are fed into the oscillators, how outputs are read out, and how performance is evaluated. We also discuss the measures of entanglement and performance. Following that, section III presents and discusses the results of our simulations. Finally, we outline our conclusions in section IV.

## II. MODEL AND METHODOLOGY

Here we consider a reservoir composed of two coupled Kerr-nonlinear oscillators,  $a$  and  $b$  (Fig. 1). In this case, the reservoir “neurons” are given by the Fock basis states  $|n_a, n_b\rangle$  and the reservoir output by their occupation probabilities [24].

### A. Physical Model

The system can be described by the following Hamiltonian:

$$\hat{H}(t) = \hat{H}_{nl} + \hat{H}_{int} + \hat{H}_{drive} \quad (1)$$

where

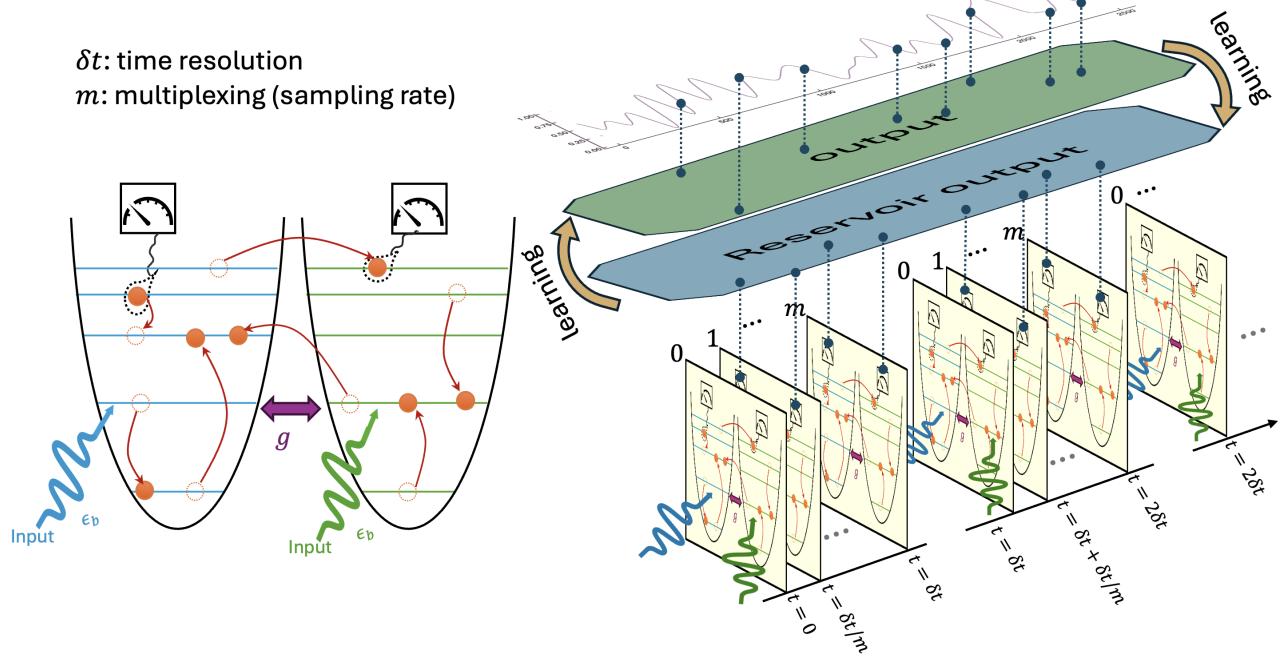
$$\hat{H}_{nl} = \hat{H}_{nl}^{(a)} + \hat{H}_{nl}^{(b)} = K_a \frac{\hat{N}_a^2}{2} + K_b \frac{\hat{N}_b^2}{2}, \quad (2)$$

$$\hat{H}_{int} = g(\hat{a}\hat{b}^\dagger + \hat{a}^\dagger\hat{b}), \quad (3)$$

$$\hat{H}_{drive} = i\epsilon_a \sqrt{2\kappa_a}(\hat{a} - \hat{a}^\dagger) + i\epsilon_b \sqrt{2\kappa_b}(\hat{b} - \hat{b}^\dagger) \quad (4)$$

$\hat{H}_{nl}$  describes Kerr nonlinearity where  $K_a$  and  $K_b$  are Kerr coefficients (strength of nonlinearity) for individual oscillators where  $\hat{N}_a = \hat{a}^\dagger\hat{a}$  and  $\hat{N}_b = \hat{b}^\dagger\hat{b}$  are number operators.  $\hat{H}_{int}$  corresponds to an internal coherent coupling with coupling strength of  $g$  which can be implemented in several ways [30–32]. Finally,  $\hat{H}_{drive}$  represents a resonant time-dependent drive with  $\kappa_a$  and  $\kappa_b$  being associated dissipation rates. This term feeds in the external input, where  $\epsilon_a$  and  $\epsilon_b$  are input strengths (drive amplitudes) that encode the input data  $X_i$  by setting  $\epsilon_a = \epsilon_b = \epsilon_0 X_i$  [24].

We model the coupled Kerr oscillator system as an open quantum system with Markovian dynamics. The evolution is governed by a Lindblad master equation for the density operator  $\rho(t)$ :



**FIG. 1. Overview of the coupled Kerr Oscillator QRC.** The oscillators are driven by the input signal with amplitudes  $\epsilon_a$  and  $\epsilon_b$  at a coupling strength of  $g$ . The reservoir state is measured in the Fock basis (read-out) at moments of  $t = \delta t/m$ . In response, the quantum state of the system evolves, and the computational nodes evolve due to the natural dynamics of the system and input signal. The state of the system is measured in the Fock basis at different times at a constant rate (sampling via multiplexing). The output of the reservoir is fed to the ridge regression to generate weights. The weights are then used to generate the output time series, which is a prediction of the target.

$$\frac{d\hat{\rho}(t)}{dt} = i[\hat{H}(t), \hat{\rho}(t)] + \sum_{i=a,b} (\hat{c}_i \hat{\rho}(t) \hat{c}_i^\dagger - \frac{1}{2} \{ \hat{c}_i^\dagger \hat{c}_i, \hat{\rho}(t) \}) \quad (5)$$

where the collapse operators describing decay in modes  $a$  and  $b$  are given by  $c_a = \sqrt{2\kappa_a}\hat{a} + \sqrt{\kappa_\phi}\hat{N}_a$  and  $c_b = \sqrt{2\kappa_b}\hat{b} + \sqrt{\kappa_\phi}\hat{N}_b$  with  $\kappa_\phi$  being associated dephasing rate. Throughout this work, we set the parameters to be equal for both oscillators and, for convenience, we work in a rotating frame at the drive frequencies (taken equal to  $\omega_a$  and  $\omega_b$  for each mode); in this frame the free oscillation terms are eliminated (effectively setting  $\omega_a = \omega_b = 0$ ). It is worth noting that  $\hat{H}_{int}$  does not include nonlinear coupling proportional to  $\hat{N}_a\hat{N}_b$ , but only the linear one, described by Eq. 3. The same Hamiltonian  $\hat{H}_{nl} + \hat{H}_{int}$  was used, e.g., by Bernstein [33] and Chefles and Barnett [34] to describe the nonlinear coupler. Note that we are using dimensionless units throughout this study.

## B. Learning and Simulation

An input signal  $s_k$  with a frequency scale  $f$  is generated as follows:

$$s_k = \sum_i^{20} \sin(2\pi f_i t_k + 2\pi\zeta) \quad (6)$$

20 frequencies  $f_i$  are evenly spaced in the interval  $[f/5000, f/50]$ . Here  $t_k$  denotes the time elapsed after  $k$  time steps, and  $\zeta$  is a random number uniformly distributed within the interval  $[0, 1]$ . All series are normalized to ensure a range between 0 and 1 [28]. We use this task to have a way of tuning the level of complexity. Examples of such series at different frequency scales, denoted by  $f$ , are provided in section III.

We partition the input signal  $s_k$  into data points based on the time resolution  $\delta t$  and as mentioned earlier these data points ( $X_i$  in Eq. 4) are injected by means of a drive Hamiltonian a widely utilized form of input encoding [4, 23, 24]: we initialize the system in vacuum state  $(|0, 0\rangle)$ , we then drive it by the first input data. The master equation then describes how the state evolves under the influence of the drive and the internal Hamiltonian. We employ time multiplexing, where the reservoir state is measured in Fock basis (readout) at moments of  $\delta t/m$ , thus creating  $m$  virtual nodes. These readouts happen in between time steps, a time duration called time resolution (shown by  $\delta t$  in Fig. 1). Then, we repeat this process for each time step until the end of the time interval for the whole process. The signals extracted from the virtual nodes, which are the expectation values of the Fock basis measurements, are trained in a linear regression to predict the target, which is the input time series after a certain delay in the future. This process is illustrated in Fig. 1. Throughout this work, we choose  $\delta t = 100$  and

$m = 10$ .

The first half of the input time series is used for training and the other half for testing. For training, Tikhonov regularization is utilized with a regularization parameter that is chosen to get maximum performance. For quantifying the performance of the reservoir, we use the normalized root mean square error (NRMSE), which is calculated as

$$NRMSE = \frac{1}{y_{max} - y_{min}} \sqrt{\frac{\sum_i^N (y_i - \tilde{y}_i)^2}{N}} \quad (7)$$

where  $\tilde{y}$  indicates the test target and  $y$  the prediction.

### C. Entanglement

For quantifying the entanglement we use logarithmic negativity [35, 36], which can be calculated as

$$E_N(\rho) = \log_2 \|\rho^{\Gamma_A}\|_1 \quad (8)$$

where  $\rho^{\Gamma_A}$  is the partial transpose with respect to the subsystem A. Here,  $\|\cdot\|_1$  denotes the trace norm, which is defined as  $\|X\|_1 = \text{Tr}(\sqrt{X^\dagger X})$ . Examples of the behavior of this quantity are given in section III. for some frequency scales.

### D. Cutoff Dimension

To numerically simulate the coupled Kerr oscillator system, we truncate the infinite-dimensional Hilbert space by imposing an excitation number cutoff  $N_{cut}$ . For this study, we put  $N_{cut} = 3$ . To ensure physical validity and convergence across different cutoffs, we constrain the mean excitation numbers  $\langle \hat{n}_a \rangle$  and  $\langle \hat{n}_b \rangle$  to remain below one. This ensures negligible occupation of higher Fock states to avoid truncation artifacts.

## III. RESULTS

In this section, we present the results of our numerical investigation into the relationship between entanglement and computational performance in a QRC framework based on coupled Kerr nonlinear oscillators. We examine the system's predictive performance on nontrivial time-series tasks, quantify entanglement using logarithmic negativity, and analyze the impact of system nonlinearity, as well as dissipation and dephasing.

To ground the subsequent analysis, Fig. 2 provides an overview of the reservoir's behavior across different input frequencies. The top row shows the time evolution of entanglement, measured via logarithmic negativity, during training and testing. To avoid transient effects, only the portion after a set initialization period (indicated by the dashed vertical line) is used. This removed interval

(washout phase) is chosen based on the lowest dissipation rate to ensure a right choice for all the other dissipation rates of interest. The bottom row displays examples of the input time series (or target) alongside the reservoir's predictions at selected frequencies, with fixed parameters.

The corresponding NRMSE values reveal that prediction quality declines with increasing input frequency.

Building on these observations, we analyze the effect of each physical system parameter on performance by plotting the NRMSE against the specific parameter and simultaneously assessing the entanglement to identify potential relationships. Subsequently, we synthesize these findings through binned analysis to visualize the underlying relation more effectively.

### A. Input Strength

We randomly selected 30 input strengths ( $\epsilon$ ) from the interval  $[0, 10]$  and evaluated the normalized root mean square error (NRMSE) for input time series at various input frequency scales while computing the logarithmic negativity of the system in each instance. It is important to note that we utilize the same 30 values of input strengths for all frequencies. Furthermore, we ensure that the average excitation number remains below one as discussed in section IID and appendix B. The outcome is presented in Fig. 3.

Increasing input strength leads to slightly different behaviors for different frequencies. For  $f = 0.05$  (purple points in Fig. 3), NRMSE continuously decreases up to  $\epsilon \simeq 4$  and then slightly increases. For  $f = 0.8$  (green points in Fig. 3), NRMSE decreases up to  $\epsilon \simeq 2$ , then it slightly increases and again degrades up to  $\epsilon \simeq 5$ , and it almost saturates afterwards. For  $f = 1.5$  (orange points in Fig. 3), a similar behavior to  $f = 0.8$  is observed except for after  $\epsilon \simeq 5$ , where NRMSE increases rather than saturating. Finally, for  $f = 2.5$  (orange points in Fig. 3), no improvement is observed. This is because the task becomes too difficult for our system, which has a limited cutoff dimension and average excitation number, a phenomenon that might not occur in a practical setting.

By focusing on the entanglement (right panel of Fig. 3), we observe that it increases up to  $\epsilon \simeq 2$  for all frequencies. Beyond this point, entanglement decreases, leading to a peak in entanglement that corresponds to the first NRMSE dip, especially for  $f = 0.8$  and  $f = 1.5$ . It should be noted that while the best performance is not achieved with maximum entanglement, it does occur at a moderate degree of entanglement. Therefore, a certain amount of entanglement appears to be advantageous.

### B. Kerr Nonlinearity

We randomly selected 30 Kerr nonlinearity values ( $K$ ) from the interval  $[0, 10]$  and evaluated the NRMSE for

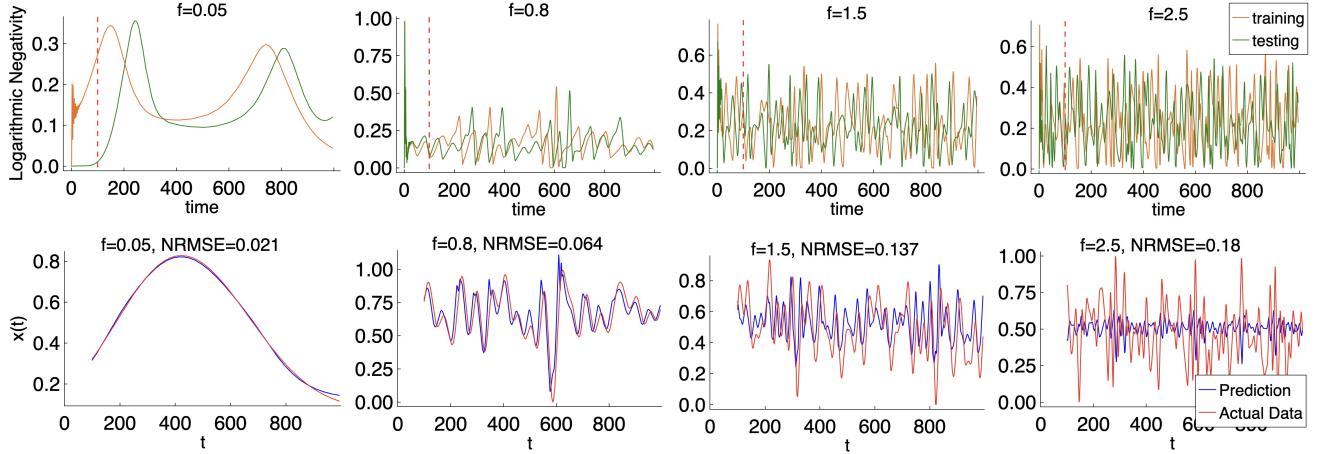


FIG. 2. Time evolution of entanglement (top row) and corresponding time-series prediction performance (bottom row) at different input frequencies  $f$ . Top: Behavior of logarithmic negativity in time during testing and training. To ensure the reservoir reflects stable dynamics rather than initial transients, we use the portion of the evolution after the vertical dashed line for training and testing. Bottom: Examples of input sequences at three different frequency scales and The corresponding predictions of the reservoir at coupling rate of  $g = 0.9$ , input strengths of  $\epsilon_a = \epsilon_b = 3$ , nonlinearities of  $K_a = K_b = 1$  and dissipation rates of  $\kappa_a = \kappa_b = 0.1$  in a rotating frame. The performance gets worse as the input frequency increases.

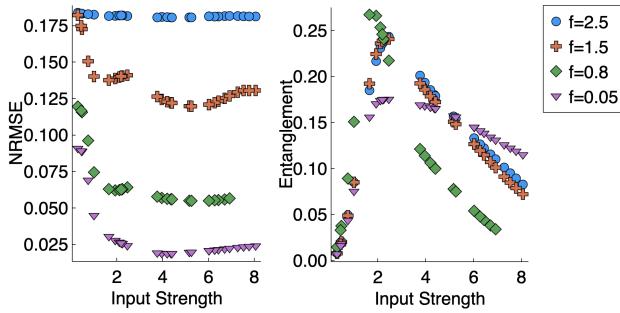


FIG. 3. NRMSE (left) and entanglement (right) versus input strength  $\epsilon$  for different input frequencies, with fixed parameters  $K = 1$ ,  $\kappa_a = \kappa_b = 0.1$ ,  $g = 0.9$ , and  $\kappa_\phi = 0$ . Left: While the first dips are observed at  $\epsilon \approx 2$ , NRMSE decreases with increasing input strength up to  $\epsilon \approx 5$ , after which it saturates or worsens depending on the input frequency. Right: Logarithmic negativity increases with  $\epsilon$  until a peak near  $\epsilon \approx 2$ , beyond which it declines with a lower slope. Best-case errors correspond to moderate entanglement for frequencies below the threshold (e.g.,  $f = 2.5$ ).

input time series at different input frequency scales while calculating the logarithmic negativity of the system in each case. Note that we use the same 30 values of  $K$  for all frequencies. The results are shown in Fig. 4.

The relation is somewhat more subtle here. The NRMSE for  $f = 0.8$  and  $f = 1.5$  (green and orange points in Fig. 4) fluctuates up to  $K \approx 2$ . Beyond this point, the NRMSE starts increasing. For  $f = 0.05$ , it increases monotonically, and for  $f = 2.5$ , no clear trend is observed (left panel of Fig. 4).

By focusing on the entanglement (right panel of Fig. 4), we observe that it fluctuates up to  $K \approx 1$ , which makes it

hard to infer any advantage up to this point. Beyond this point, entanglement decreases while the error increases, which could be interpreted as a hint of entanglement advantage for all frequencies below the threshold, i.e.,  $f = 2.5$ . Overall, we can say that best-case errors correspond to moderate entanglement for frequencies below the threshold (e.g.,  $f = 2.5$ ).

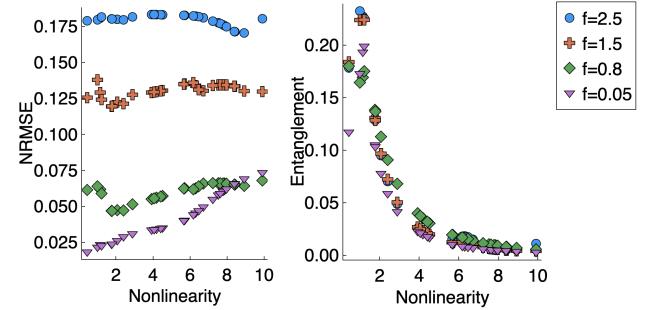


FIG. 4. NRMSE (left) and Entanglement (right) versus Kerr nonlinearity  $K$  for different input frequencies, with fixed parameters  $\epsilon = 3$ ,  $\kappa_a = \kappa_b = 0.1$ ,  $g = 0.9$ , and  $\kappa_\phi = 0$ . Left: NRMSE fluctuates up to  $K \approx 2$  for  $f = 0.8$  and  $f = 1.5$ , and increases thereafter. For  $f = 0.05$ , it increases monotonically, while no clear trend is observed for  $f = 2.5$ . Right: Entanglement fluctuates up to  $K \approx 1$ , beyond which it decreases. Best-case errors correspond to moderate entanglement for frequencies below the threshold (e.g.,  $f = 2.5$ ).

### C. Coupling Constant

We randomly selected 30 coupling constant values ( $g$ ) from the interval  $[0, 10]$  and evaluated the NRMSE on

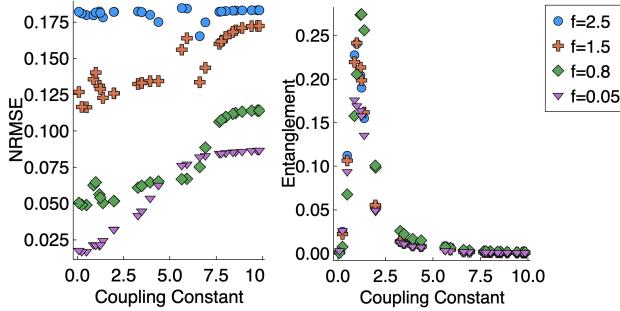


FIG. 5. NRMSE (left) and Entanglement (right) versus coupling constant  $g$  for different input frequencies, with fixed parameters  $\epsilon = 3$ ,  $\kappa_a = \kappa_b = 0.1$ ,  $K = 1$ , and  $\kappa_\phi = 0$ . Left: NRMSE for  $f = 0.8$  and  $f = 1.5$  fluctuates up to  $g \simeq 1.4$ , beyond which it increases. For  $f = 0.05$ , NRMSE increases monotonically, and no clear trend is observed for  $f = 2.5$ . Right: Entanglement peaks near  $g \simeq 0.9$ . Beyond this point, entanglement decreases. Best-case errors (third points from the left on the left panel) correspond to moderate entanglement (the same points for the right panel) for frequencies below the threshold (e.g.,  $f = 2.5$ ).

input time series at different input frequency scales while calculating the logarithmic negativity of the system in each case. Note that we use the same 30 values of  $g$  for all frequencies. The results are shown in Fig. 5.

The relation is very similar to the case of nonlinearity. The NRMSE for  $f = 0.8$  and  $f = 1.5$  (green and orange points in Fig. 5) fluctuates up to  $g \simeq 1.4$ . Beyond this point, The NRMSE starts increasing. For  $f = 0.05$ , it increases monotonically, and for  $f = 2.5$ , no clear trend is observed (left panel of Fig. 5).

By focusing on the entanglement (right panel of Fig. 5), we observe that it peaks at  $g \simeq 0.9$ , which suggests no clear correlation with NRMSE. Beyond this point, entanglement decreases while the error increases, which could be interpreted as a hint of entanglement advantage for all frequencies below the threshold. Best-case errors correspond to moderate entanglement for frequencies below the threshold (e.g.,  $f = 2.5$ ).

#### D. Entanglement Advantage

We now consolidate the results from the sections III A, III B, and III C by performing a binned analysis: the data are grouped into multiple bins with equal sizes based on entanglement levels, and a summary statistic of the NRMSE is computed for each bin. This allows us to observe the overall relationship between entanglement and predictive performance.

Our analysis reveals a non-monotonic relationship between entanglement and predictive performance. As shown in Fig. 6, we first consider the minimum NRMSE in each bin (left panel), which shows no discernible pattern as entanglement varies. Next, we observe the average predictive performance: moderate levels of entan-

glement are associated with improved average predictive performance, with NRMSE reaching a minimum within a specific frequency regime of the input. However, for frequency scale  $f = 0.05$  (purple curve in Fig. 6), error saturates for higher entanglement, whereas a dip is observed for other cases. Beyond this regime, increasing the input frequency leads to a breakdown in reservoir memory and a decline in average performance, despite sustained or even increasing levels of entanglement, as the task becomes too difficult for the predictor in this setup. A potential reason for this could be the low excitation number limit that was implemented to reduce the artifacts caused by the cutoff dimension (IID and appendix B). Finally, the maximum NRMSE in each bin (right panel) exhibits a similar but slightly stronger relation to entanglement compared to the mean NRMSE.

#### E. Role of Dissipation

To examine the impact of dissipation, we repeat the data generation process and binned analysis described earlier in the presence of different dissipation rates,  $\kappa_a = \kappa_b = \kappa$  and compute the average entanglement and NRMSE to produce the same type of plot as the middle panel of Fig. 6, now for different  $\kappa$  (see Fig. 9 in Appendix A). It is worth mentioning that our system does not reach a steady state in the absence of dissipation and therefore does not exhibit steady-state entanglement; hence, we do not consider the zero-dissipation case. To better analyze the effect of dissipation, we extract the optimal points (those with minimum error) and plot their corresponding NRMSE and entanglement as a function of dissipation for different input frequencies, resulting in Fig. 7.

As shown in Fig. 7, a higher dissipation level is better for frequencies below the frequency threshold up to a point where it saturates. By looking at Fig. 9 and Appendix A one can see that realizing an entanglement advantage requires sufficient dissipation, with slightly higher values offering a consistent advantage across all frequencies below the frequency threshold.

#### F. Role of Dephasing

To examine the impact of dephasing, we repeat the data generation process and binned analysis described earlier in the presence of different dephasing rates  $\kappa_\phi$ , and compute the average entanglement and NRMSE to produce the same type of plot as in Fig. 6, now for different  $\kappa_\phi$  (Fig. 10 in Appendix A). To more clearly assess the role of dephasing, we identify the optimal points—those corresponding to the minimum prediction error—and plot their associated NRMSE and entanglement as a function of dephasing for different input frequencies, as shown in Fig. 8 and Fig. 10 in Appendix A.

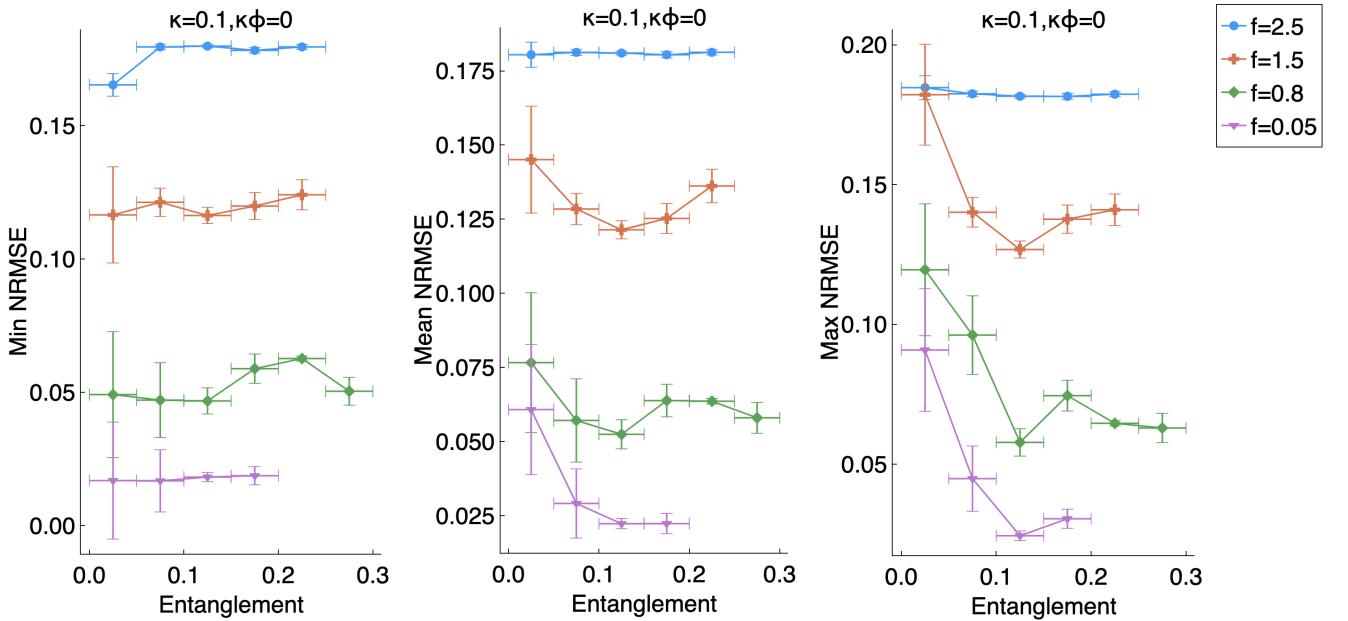


FIG. 6. Prediction error (NRMSE) vs. entanglement in the case of zero dephasing ( $\kappa_\phi = 0$ ) and a dissipation rate of  $\kappa_a = \kappa_b = 0.1$ , for different input frequencies. The data from figs. 3 to 5 are combined and binned based on entanglement levels, quantified via logarithmic negativity, with vertical error bars representing the standard deviation within each bin. Left: Minimum NRMSE within each entanglement bin. No consistent correlation is observed. Middle: Mean NRMSE across all configurations in each bin. This captures the overall performance trend with respect to entanglement. Moderate entanglement improves performance on average. There is an optimum level of entanglement. However, a threshold in the frequency is observed at  $f = 2.5$  where no trend exists. Right: Maximum NRMSE per entanglement bin. This reflects the worst-performing cases within each bin. A stronger relation is seen here: higher entanglement reduces the worst-case error more than the average error, but again, a frequency threshold is observed.

As one can see in Fig. 8 and Fig. 10 in Appendix A, zero dephasing is the optimal regime both for overall performance and the entanglement advantage (because at high dephasing we do not have much entanglement). However, focusing on Fig. 8 reveals that for very low frequency (e.g.  $f = 0.05$ ) some amount of dephasing could be beneficial.

#### IV. DISCUSSION AND CONCLUSION

In this work, we presented a systematic analysis of how the computational performance of a quantum reservoir composed of two coupled Kerr nonlinear oscillators depends on various physical parameters, including input drive strength, Kerr nonlinearity, coupling strength, dissipation, and dephasing. We identified optimum performance regimes for the system's parameters. Our goal was to investigate whether and under what conditions quantum entanglement within the reservoir contributes to an improvement in time-series prediction tasks.

Our findings indicate that within the various system configurations analyzed, entanglement is indeed linked to improvement in both average and worst-case predictive performance, albeit only within a certain range of operational parameters and up to a threshold in the in-

put frequency. It was noted that optimal average performance, measured in terms of NRMSE, generally takes place when the reservoir maintains a moderate level of entanglement rather than the highest, with the worst-case performance following a similar trend. This effect was most evident for lower input frequency scales, where the temporal structure of the input could be effectively retained and processed by the reservoir dynamics. This kind of relation between quantumness and learning capabilities is consistent with other recent findings in the literature [26, 28, 37], which suggest that excessive “quantumness” may saturate the reservoir or introduce dynamical instability, ultimately hampering learning.

Notably, our results demonstrate that the observed connection between entanglement and performance persists under realistic conditions with dissipation and dephasing (look at Appendix A). We showed that increasing dissipation can actually improve performance, up to a point that it saturates. Likewise, we found that the absence of dephasing is generally optimal for performance and entanglement (look at Appendix A); a small amount of dephasing can be tolerated (and at very low input frequency, it even provided a slight benefit).

There are several limitations to note. First, our study was limited to a two-mode Kerr system with a truncated Hilbert space—a limitation that we addressed by

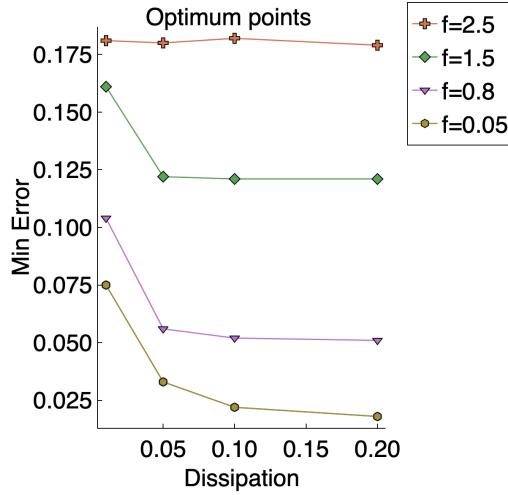


FIG. 7. Minimum NRMSE at optimal points vs. input frequency for different dissipation rates where  $\kappa_\phi = 0$ . These optimal points correspond to the minimum average NRMSE in Fig. 6 and Fig. 9 in Appendix A. Higher dissipation improves performance for all frequencies below the frequency threshold, up to a point in dissipation where it tends to saturate.

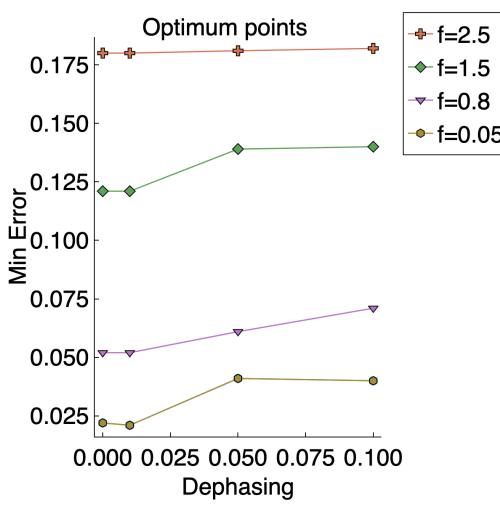


FIG. 8. Minimum NRMSE at optimal points vs. input frequency for different dephasing rates where  $\kappa_a = \kappa_b = 0.1$ . These optimal points correspond to the minimum average NRMSE in Fig. 6 and Fig. 10 in Appendix A. Zero or very small dephasing yields the best performance in most frequency regimes. However, at very low frequency ( $f = 0.05$ ), a small amount of dephasing slightly improves prediction performance.

imposing a limit on average excitation number (see Section II D, Appendix B). While this setup is well-suited for exploring foundational aspects of QRC, it restricts both the memory depth and the expressive capacity of the reservoir. Second, we focused on a specific class of input signals and employed a standard linear readout with ridge regression. Although this choice facili-

tates controlled benchmarking, our conclusions may not directly generalize to more diverse or noisy targets or to different output architectures. Additionally, our quantification of entanglement relied on logarithmic negativity, which does not capture all forms of non-classical correlation, such as Wigner negativity, quantum discord, magic/nonstabilizerness, or coherence.

Nevertheless, as described above, we were able to find a connection between entanglement and performance. This finding could potentially strengthen the broader hypothesis that quantum resources can be beneficial in machine learning frameworks, provided that system parameters and noise levels are appropriately balanced. Looking forward, extending this analysis to larger oscillator networks would allow for exploration of more complex spatial and temporal entanglement structures. It would also be valuable to consider alternative quantum resource measures and output processing techniques, including non-linear readout layers or adaptive hybrid quantum-classical schemes.

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## Appendix A: Entanglement Advantage

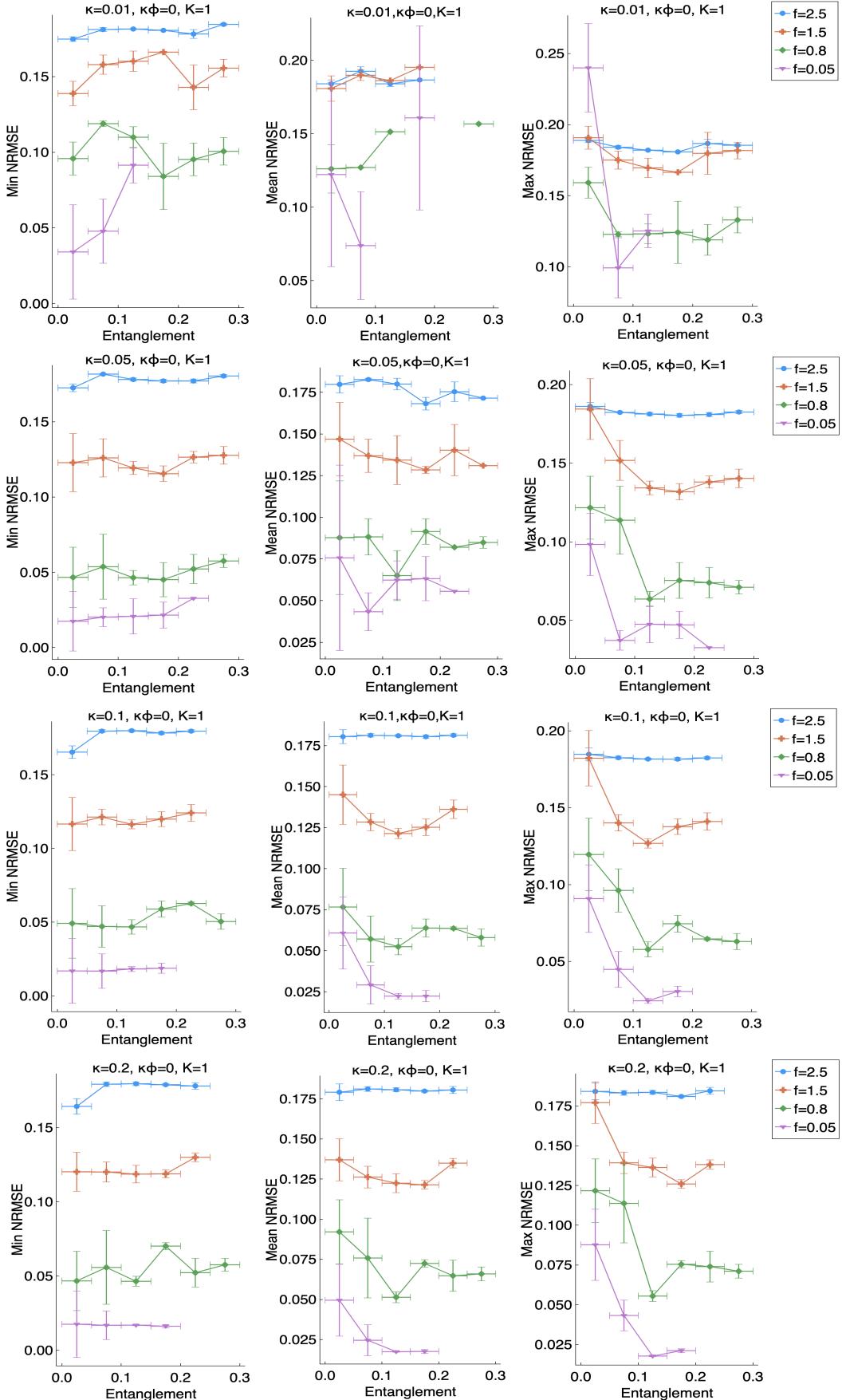
To further illustrate the entanglement advantage discussed in the main text, we provide additional figures and analysis.

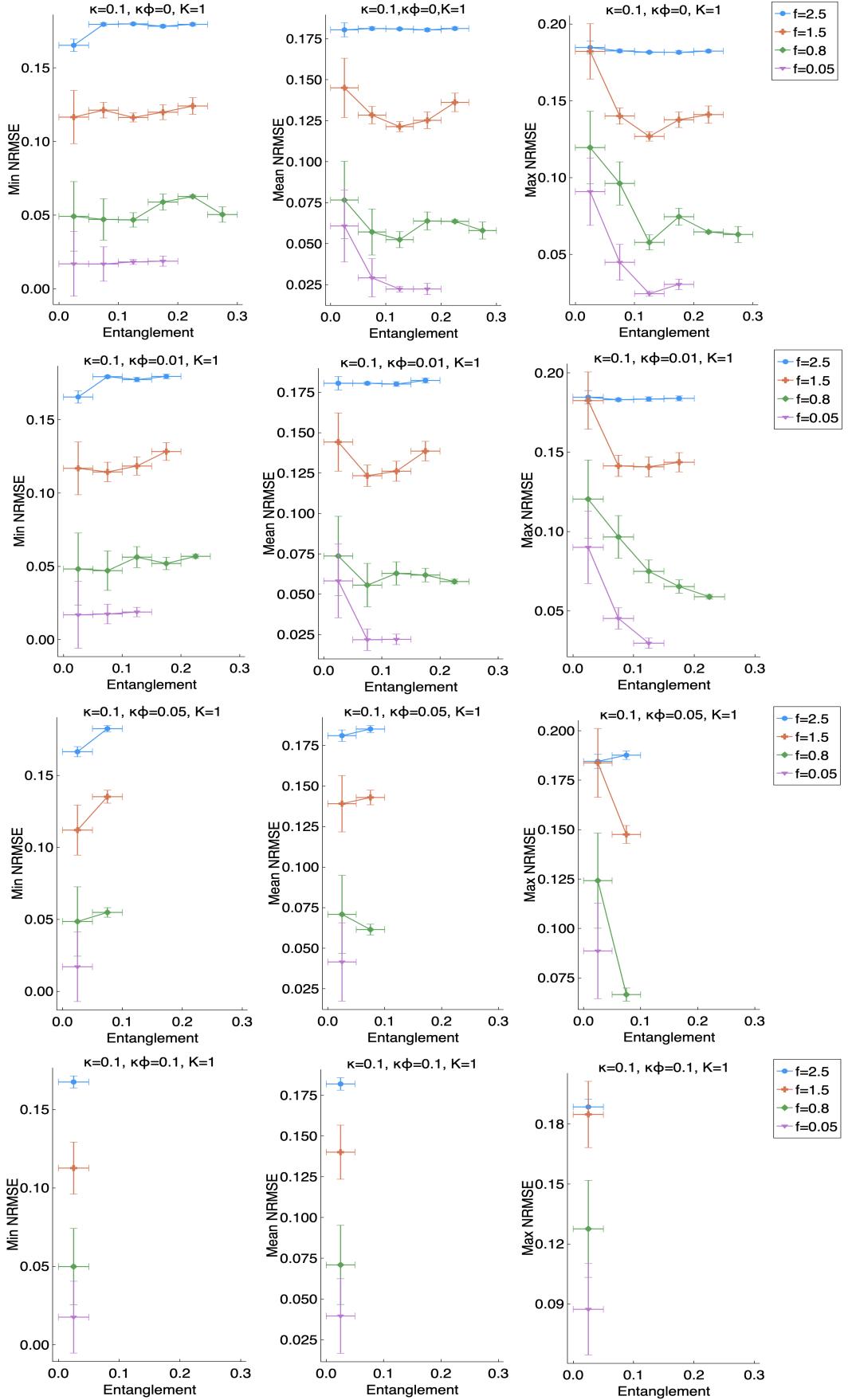
Fig. 9 shows the full binned relation between NRMSE and entanglement across multiple dissipation rates. The trend confirms that for sufficient dissipation, moderate entanglement is associated with better prediction performance, up to a saturation point. This supports our claim that optimal entanglement lies in an intermediate regime.

Fig. 10 provides a similar analysis for various dephasing rates. The best results in most cases are obtained when  $\kappa_\phi = 0$ . While small nonzero values can provide marginal benefit at low frequencies, performance generally declines as dephasing increases.

## Appendix B: Cutoff Dimension

We impose an average excitation number cutoff  $N_{\text{cut}}$  on each mode, truncating the Fock basis to states  $|n_a, n_b\rangle$  with  $n_a, n_b \leq N_{\text{cut}}$ . This cutoff is necessary for numerical feasibility but introduces an artificial boundary. To ensure the results are physically meaningful and independent of the cutoff, the average excitation number

FIG. 9. Same as Fig. 6 but for different dissipations,  $\kappa$ .

FIG. 10. Same as Fig. 6 but for different dephasings,  $\kappa_\phi$ .

populations must remain well within this truncated space specially when we increase the input strength. To that end, we constrain the mean excitation numbers  $\langle \hat{N}_a \rangle$  and  $\langle \hat{N}_b \rangle$  to remain below one (Fig. 11).

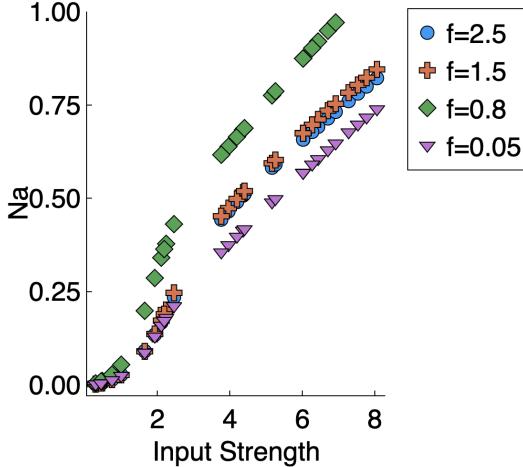


FIG. 11. Phonon number Vs different input strengths for different frequencies. The phonon number increases as the input strength increases but remains below 1.

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