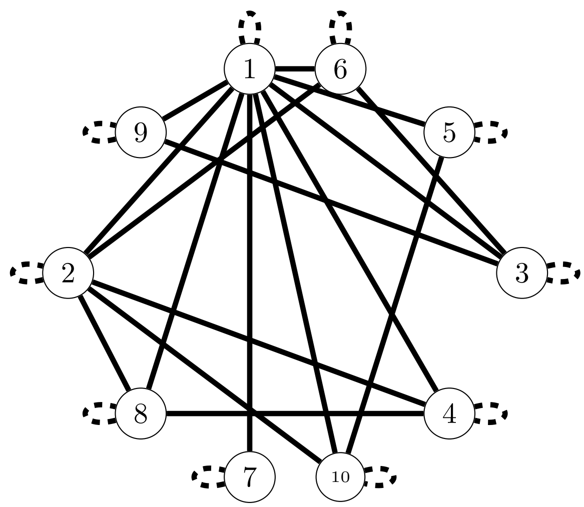
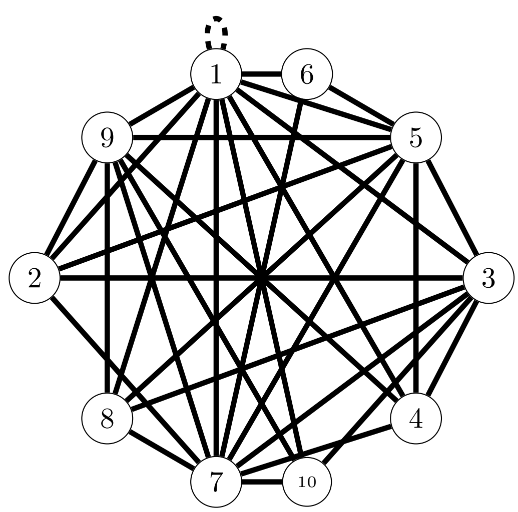
Math 314: Discrete Mathematics
Homework 5 Solutions

1. Problem 7.1.4.:

   (a) Draw a graph with nodes representing the numbers $1, 2, \ldots, 10$ in which two nodes are connected by an edge if and only if one is a divisor of the other.
   Solution:



   (b) Draw a graph with nodes representing the numbers $1, 2, \ldots, 10$ in which two nodes are connected by an edge if and only if they have no common divisor larger than 1.
   Solution:

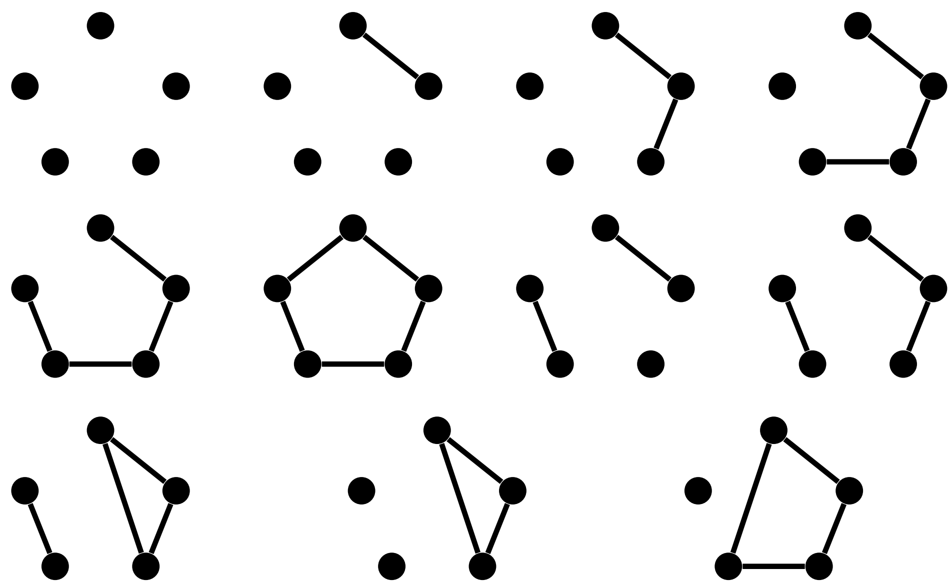


(c) Find the number of edges and the degrees in these graphs, and check that Theorem 7.1.1 holds.

Solution: The graph in a) has 27 edges (if you count a loop at every vertex, which I would), and degrees (listed in numerical order of the corresponding vertex) $(11, 7, 5, 5, 4, 5, 3, 5, 4, 5)$ (the loops, counted twice, increase all degrees by 2). Theorem 7.1.1., which says that the number of vertices of odd degree must be even, holds. The graph in b) has 32 edges (if you count the loop, which I would), and degrees $(11, 5, 7, 5, 8, 3, 9, 5, 7, 4)$ (the loop, counted twice, makes the first value 11 rather than 9). The theorem again holds.

2. Problem 7.3.4.: Draw all graphs on 5 nodes in which every node has degree at most 2 (you only need to draw those which are not "essentially the same").

   Solution:



3. a) Prove that any (simple) graph G with at least two vertices always has two vertices of the same degree.

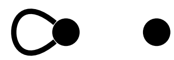
Proof: Let G be a graph on $n \geq 2$ vertices (it's false when $n = 0$ or 1). The minimum degree of a vertex in G is 0, and the maximum is $n - 1$, for a total of n options. However, a degree of 0 and a degree of $n - 1$ cannot both occur in the same graph. Then the set of degrees for each of the n vertices is a subset of either

$$\{0, 1, \ldots, n-3, n-2\} \text{ OR } \{1, 2, \ldots, n-2, n-1\}.$$

Both sets have size $n - 1$, so by the pigeonhole principle, there are two vertices of the same degree.

b) Draw a non-simple graph with at least two vertices where this is false.

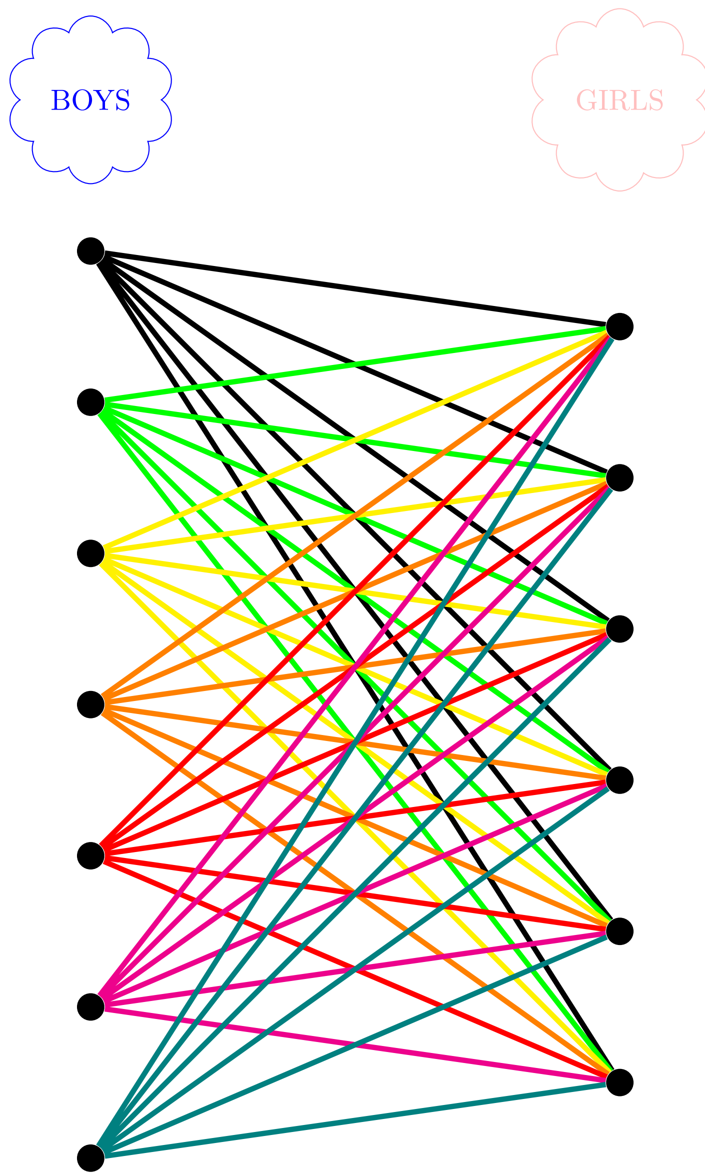
Solution:



c) Believe it or not, but you've been asked this question before (in disguise). Which question from a previous homework is this? Describe the graph associated to that question which makes it the same as this one.

Solution: On HW #2, problem #8 asks: Prove that at any party with $n \geq 2$ people, there are two people who know the same number of people at the party. If we make a graph of acquaintances at the party, where vertices are people, and edges represent acquaintance, this is the same problem.

4. Problem 7.3.7: At a party there were 7 boys and 6 girls. Every boy danced with every girl. Draw the graph representing the dancing. How many edges does it have? What are its degrees?
   Solution: Every vertex corresponding to a boy has degree 6, and every vertex corresponding to a girl has degree 7. This graph has 42 edges.



5. Problem 7.3.9: Prove that at least one of G and $\overline{G}$ is connected.
Proof: Let G be a graph. If G is connected, we're done. So assume G is not connected. Then there are at least two components. Consider two vertices $x, y \in V(G)$. If x and y are in different components of G, there is no path between them, so certainly there is no edge between them, in which case they are adjacent in $\overline{G}$. If x and y are in the same component of G, but are not adjacent, they are adjacent in $\overline{G}$. Finally, if x and y are adjacent in G, they are in the same component of G, and because G is not connected, there is necessarily another component, with at least one additional vertex z. And, because z is adjacent to neither x nor y, xzy is a path of length 2 in $\overline{G}$. We have shown that, if G is not connected, then any two arbitrary vertices have a path between them (of length 1 or 2, in fact) in $\overline{G}$, so $\overline{G}$ is connected.

REMINDER: These represent possible solutions to each problem. The solution methods are not necessarily unique, and there are likely other correct solutions.