HW6 Solution

1) a)

$$x_1[n] = 4 + \underbrace{\cos\left(\frac{\pi}{6}n + \frac{\pi}{8}\right)}_{T_1\,(N_1)} + \underbrace{2\sin\left(\frac{\pi}{4}n\right)}_{T_2\,(N_2)}$$

متناوب است ← سری فوریه

$$N_1 = 12,\ N_2 = 8 \Rightarrow N = 24,\ \omega_0 = \frac{2\pi}{24} = \frac{\pi}{12}$$

$$x_1[n] = 3 + \frac{1}{2}\left(e^{j(\frac{\pi}{6}n+\frac{\pi}{8})} + e^{-j(\frac{\pi}{6}n+\frac{\pi}{8})}\right) + 2\frac{1}{2j}\left(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right)$$

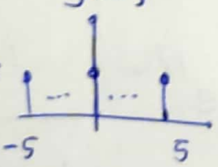
$$\Rightarrow \begin{cases} a_0 = 3 \\ a_2 = \frac{1}{2}e^{j\frac{\pi}{8}} \\ a_{-2} = \frac{1}{2}e^{-j\frac{\pi}{8}} \\ a_3 = \frac{1}{j} \\ a_{-3} = \frac{-1}{j} \end{cases}$$

$$\hat{X}_1(e^{j\omega}) = 2\pi \sum_{k=0}^{23} a_k\,\delta\left(\omega - k\frac{\pi}{12}\right) = \underbrace{6\pi\delta(\omega) + \cdots}$$

پنج جمله دارد چرا که تنها پنج ضریب سری فوریه دارد و بقیه ضرایب صفر هستند

b) $x_2[n] = u[n-2] - u[n-13]$

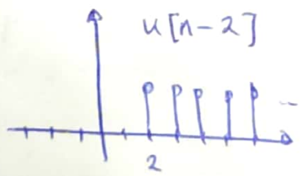
y داریم :



$$\longleftrightarrow \frac{\sin(\frac{11}{2}\omega)}{\sin(\frac{\omega}{2})} = Y(e^{j\omega})$$

$$\rightarrow x_2[n] = y[n-7]$$

$$\Rightarrow X_2(e^{j\omega}) = e^{-j\omega 7}\,Y(e^{j\omega})$$



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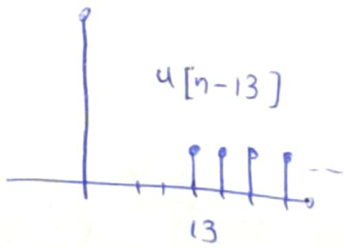

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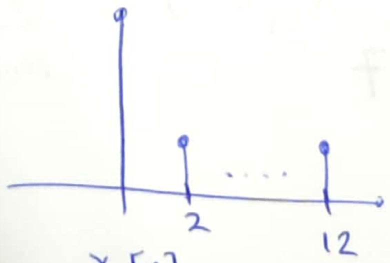
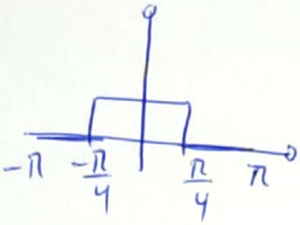


$\underbrace{x_2[n]}_{\text{impulse } 11}$

c) $x_3[n] = \dfrac{\sin(\frac{\pi}{4}n)}{\pi n}$

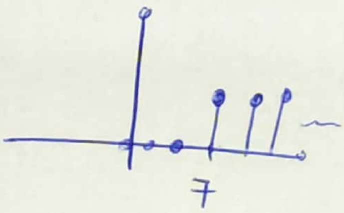
می دانیم $\quad \dfrac{\sin(Wn)}{\pi n} \longleftrightarrow \hat{X}(e^{j\omega}) = \begin{cases} 1 & 0 \le |\omega| \le W \\ 0 & W \le |\omega| \le \pi \end{cases}$

$\Rightarrow \hat{X}_3(e^{j\omega}) =$

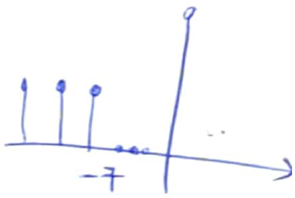


d) $x_4[n] = \left(\frac{1}{4}\right)^{|n|} u[-n-7]$

می دانیم $\quad u[n-7] =$



$\Longrightarrow u[-n-7] =$



$$\hat{X}_4(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_4[n] e^{-jn\omega} = \sum_{n=-\infty}^{+\infty} \left(\frac{1}{4}\right)^{|n|} u[-n-7]\, e^{-jn\omega}$$

$$= \sum_{n=-\infty}^{-7} \left(\frac{1}{4}\right)^{|n|} e^{-jn\omega} = \sum_{n=-\infty}^{-7} \left(\frac{1}{4}\right)^{-n} e^{-jn\omega} = \sum_{n=-\infty}^{-7} \left(\frac{1}{4}e^{j\omega}\right)^{-n}$$

$$= \sum_{n=7}^{+\infty} \left(\frac{1}{4}e^{j\omega}\right)^{n} = \sum_{m=0}^{+\infty} \left(\frac{1}{4}e^{j\omega}\right)^{m+7} = \frac{1}{4^7} e^{j7\omega} \frac{1}{1-\frac{1}{4}e^{j\omega}}$$

e) $x_5[n] = 2^n \sin(\frac{\pi}{4}n) u[-n]$

می دانیم $\quad \alpha^n u[n] \quad |\alpha|<1 \longleftrightarrow \dfrac{1}{1-\alpha e^{-j\omega}}$

در این سؤال $\alpha = 3$ است و $|\alpha| \not< 1$ ،

برای حل این سؤال، ابتدا تبدیل فوریه $x_5[-n]$ را محاسبه می‌کنیم

$$y[n] = x_5[-n] = 2^{-n} \left(\sin(\frac{-\pi}{4}n)\right) u[n]$$

$$= \underbrace{-\left(\frac{1}{2}\right)^n u[n]}_{r[n]} \; \underbrace{\sin\left(\frac{\pi}{4}n\right)}_{s[n]}$$

$$r[n] = -\left(\frac{1}{2}\right)^n u[n] \Rightarrow R(e^{j\omega}) = \frac{-1}{1-\frac{1}{2}e^{-j\omega}}$$

$$s[n] = \sin\left(\frac{\pi}{4}n\right) \Rightarrow \hat{S}(e^{j\omega}) = \frac{\pi}{j}\left(\delta(\omega-\frac{\pi}{4}) - \delta(\omega+\frac{\pi}{4})\right)$$

$$y[n] = x_5[-n] = r[n]s[n] \Rightarrow Y(e^{j\omega}) = \underbrace{\frac{1}{2\pi}\left(R(e^{j\omega}) * \hat{S}(e^{j\omega})\right)}_{= X_5(e^{-j\omega})}$$

← حالا به جای همه ω ها $-\omega$ می‌گذاریم تا $X_5(e^{j\omega})$ محاسبه شود

② a) $\underbrace{x[5-n]}_{A} + \underbrace{x[-2-n]}_{B}$

A $\quad x[n] \leftrightarrow X(e^{j\omega})$

$x[n+5] \leftrightarrow e^{j\omega 5} X(e^{j\omega})$

$x[-n+5] \leftrightarrow e^{-j\omega 5} X(e^{-j\omega})$

B $\quad x[n-2] \leftrightarrow e^{-j\omega 2} X(e^{j\omega})$

$x[-n-2] \leftrightarrow e^{j\omega 2} X(e^{-j\omega})$

$\Rightarrow (e^{-j\omega 5} + e^{j\omega 2}) X(e^{-j\omega})$

b) $x^*[-n]$

$x[n] \leftrightarrow X(e^{j\omega})$

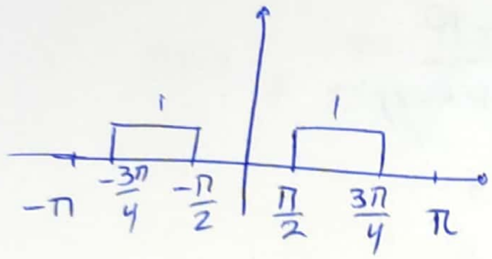
$x^*[n] \leftrightarrow X^*(e^{-j\omega})$

$x^*[-n] \leftrightarrow X^*(e^{j\omega})$

$x[n]$ real $\Rightarrow x[n] = x^*[n] \Rightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$

$\xRightarrow[\text{both sides}]{\text{conjugate}} \boxed{X^*(e^{j\omega}) = X(e^{-j\omega})}$

③ a) $\hat{X}(e^{jw}) = \begin{cases} 1 & \frac{\pi}{2} < |w| < \frac{3\pi}{4} \\ 0 & \text{o.w} \end{cases}$



$$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{jw}) e^{jnw} \, dw$$

$$= \frac{1}{2\pi} \left[\int_{-\frac{3\pi}{4}}^{-\frac{\pi}{2}} e^{jnw} \, dw + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} e^{jnw} \, dw \right]$$

$$= \frac{1}{2\pi jn} \left(e^{jnw} \Big|_{-\frac{3\pi}{4}}^{-\frac{\pi}{2}} + e^{jnw} \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \right)$$

$$= \frac{1}{\pi n} \left(\sin\left(\frac{3\pi}{4} n\right) - \sin\left(\frac{\pi}{2} n\right) \right)$$

b) $X(e^{jw}) = 3 + 3e^{-j2w} + 5e^{j144w}$

$\longrightarrow x[n] = 3\delta[n] + 3\delta[n-2] + 5\delta[n+144]$

c) $X(e^{j\omega}) = \dfrac{7e^{-j\omega}+10}{e^{-j2\omega}+2e^{-j\omega}-8}$

$e^{-j\omega}=k \Rightarrow X(e^{j\omega}) = \dfrac{7k+10}{k^2+2k-8} = \dfrac{7k+10}{(k+4)(k-2)}$

$= \dfrac{A}{k+4} + \dfrac{B}{k-2} \Rightarrow \begin{cases} A=3 \\ B=4 \end{cases}$

$\Rightarrow X(e^{j\omega}) = \dfrac{3}{e^{-j\omega}+4} + \dfrac{4}{e^{-j\omega}-2} = \dfrac{3/4}{1+\frac{1}{4}e^{-j\omega}} + \dfrac{-2}{1-\frac{1}{2}e^{-j\omega}}$

$\Rightarrow x[n] = \frac{3}{4}\left(\frac{-1}{4}\right)^n u[n] + (-2)\left(\frac{1}{2}\right)^n u[n]$

(4) $H(e^{j\omega}) = H_1(e^{j\omega})\, H_2(e^{j\omega})$ $\qquad e^{-j\omega}=k$

$= \dfrac{3-k^8}{1+k} \cdot \dfrac{1}{1-\frac{1}{4}k+\frac{1}{8}k^2} = \dfrac{3-k^8}{1-\frac{1}{4}k+\frac{1}{8}k^2+k-\frac{1}{4}k^2+\frac{1}{8}k^3}$

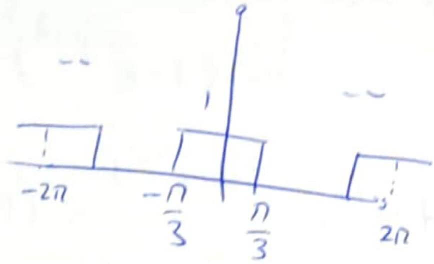
$= \dfrac{3-k^8}{1+\frac{3}{4}k-\frac{1}{8}k^2+\frac{1}{8}k^3} = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})}$

$\Rightarrow 3X(e^{j\omega}) - e^{-j8\omega}X(e^{j\omega}) = Y(e^{j\omega}) + \frac{3}{4}e^{-j\omega}Y(e^{j\omega}) - \frac{1}{8}e^{-j2\omega}Y(e^{j\omega}) + \frac{1}{8}e^{-j3\omega}Y(e^{j\omega})$

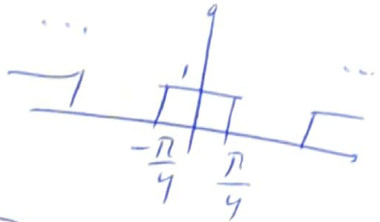
$\Rightarrow 3x[n] - x[n-8] = y[n] - \frac{3}{4}y[n-1] - \frac{1}{8}y[n-2] + \frac{1}{8}y[n-3]$

5 $$h[n]=\frac{\sin(\frac{\pi}{3}n)\,\sin(\frac{\pi}{4}n)}{(\pi n)(\pi n)}$$

$$h_1[n]=\frac{\sin(\frac{\pi}{3}n)}{\pi n}\longleftrightarrow$$



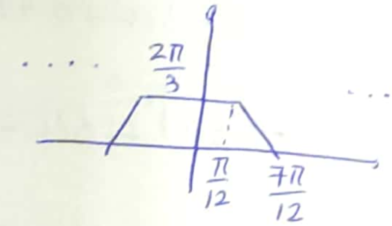
$$h_2[n]=\frac{\sin(\frac{\pi}{4}n)}{\pi n}\longleftrightarrow$$



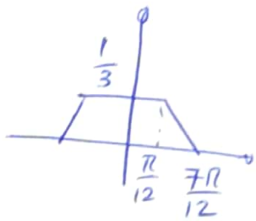
$$h[n]=h_1[n]\,h_2[n]$$

$$\Rightarrow H(e^{j\omega})=\frac{1}{2\pi}\left(H_1(e^{j\omega}) * \hat{H}_2(e^{j\omega})\right)$$

$$=\frac{1}{2\pi}\Bigg($$



$$\Bigg)=$$



$$x[n]=\sin(\frac{\pi}{12}n)\;\underline{-12\cos(\frac{2\pi}{3}n)}$$

در $\omega=\pm\frac{2\pi}{3}$ ، مقدار
H صفر است پس این
بخش فیلتر می شود

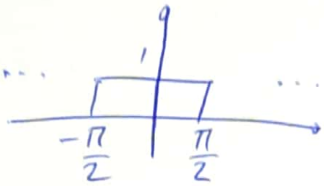
$$\Rightarrow y[n]=\frac{1}{3}\sin(\frac{\pi}{12}n)$$

6 a) $w[n] = 2x[n] \Rightarrow W(e^{j\omega}) = 2X(e^{j\omega})$

b) $H_1(e^{j\omega}) = \dfrac{W(e^{j\omega})}{X(e^{j\omega})} = 2$

c) $H_{eq}(e^{j\omega}) = H_1(e^{j\omega}) \times H_2(e^{j\omega})$

$= 2 \times$



d) $x[n] = \cos(\frac{4}{10}n\pi) + \sin(\frac{6}{10}n\pi) + 2$

$\hat{X}(e^{j\omega}) = \pi\left(\delta(\omega - \frac{4}{10}\pi) + \delta(\omega + \frac{4}{10}\pi)\right)$

$+ \frac{\pi}{j}\left(\delta(\omega - \frac{6}{10}\pi) - \delta(\omega + \frac{6}{10}\pi)\right)$

$+ 4\pi\delta(\omega)$

e) عامل sin فیلتر می‌شود چون H_{eq} در $\omega = \pm\frac{6}{10}\pi$
صفر است

دو عامل دیگر در ۲ ضرب می‌شوند

$y[n] = 2\cos(\frac{4}{10}n\pi) + 4$

⑦ $y[n] - \frac{1}{3}y[n-1] = 2x[n]$

a)

$$\longrightarrow Y(e^{j\omega}) - \frac{1}{3}e^{-j\omega}Y(e^{j\omega}) = 2X(e^{j\omega})$$

$$\Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{1-\frac{1}{3}e^{-j\omega}}$$

b)

$$x[n] = \left(\frac{1}{3}\right)^n u[n] \Rightarrow X(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

$$\Rightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}} \times \frac{2}{1-\frac{1}{3}e^{-j\omega}}$$

$$= \frac{2}{\left(1-\frac{1}{3}e^{-j\omega}\right)^2} = \frac{A}{\left(1-\frac{1}{3}e^{-j\omega}\right)^2} + \frac{B}{\left(1-\frac{1}{3}e^{-j\omega}\right)}$$

← محاسبه A و B با دو معادله دو مجهول

$$\Rightarrow y[n] = A(n+1)\left(\frac{1}{3}\right)^n u[n] + B\left(\frac{1}{3}\right)^n u[n]$$

c) مشابه b

d)

$$X(e^{j\omega}) = \frac{1-\frac{1}{4}e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}} \longrightarrow Y(e^{j\omega}) = \frac{2\left(1-\frac{1}{4}e^{-j\omega}\right)}{\left(1+\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)}$$

$$= \frac{A}{1+\frac{1}{2}e^{-j\omega}} + \frac{B}{1-\frac{1}{3}e^{-j\omega}} \xrightarrow{\text{محاسبه } B, A} y[n] = A\left(\frac{-1}{2}\right)^n u[n] + B\left(\frac{1}{3}\right)^n u[n]$$