HW3 Solution

1

a) $x(t) = 2\sin\left(\frac{2\pi}{6}t + \frac{\pi}{6}\right) + 5\cos\left(\frac{2\pi}{12}t\right)$

$$\omega_1 = \frac{2\pi}{6} \Rightarrow T_1 = 6 \qquad \omega_2 = \frac{2\pi}{12} \Rightarrow T_2 = 12$$

$$T = \mathrm{lcm}(6,12) = 12 \Rightarrow \omega = \frac{2\pi}{12} = \frac{\pi}{6}$$

$$\rightarrow x(t) = \frac{2}{2j}\left(e^{j\left(\frac{2\pi}{6}t+\frac{\pi}{6}\right)} - e^{-j\left(\frac{2\pi}{6}t+\frac{\pi}{6}\right)}\right) + \frac{5}{2}\left(e^{j\frac{2\pi}{12}t} + e^{-j\frac{2\pi}{12}t}\right)$$

$$= \sum a_k e^{jk\omega_0 t} = \sum a_k e^{jk\frac{\pi}{6}t}$$

$$\Rightarrow \frac{2}{2j}e^{j\frac{\pi}{6}} = a_2 \qquad \frac{-2}{2j}e^{-j\frac{\pi}{6}} = a_{-2}$$

$$\frac{5}{2} = a_1 \qquad \frac{5}{2} = a_{-1}$$

b) $x(t) = 2\cos\left(\frac{2\pi}{3}t + \frac{\pi}{6}\right)$ $\rightarrow \omega = \frac{2\pi}{3}$

$$= \frac{2}{2}\left(e^{j\left(\frac{2\pi}{3}t+\frac{\pi}{6}\right)} + e^{-j\left(\frac{2\pi}{3}t+\frac{\pi}{6}\right)}\right)$$

$$\Rightarrow a_1 = e^{j\frac{\pi}{6}} \qquad a_{-1} = e^{-j\frac{\pi}{6}}$$

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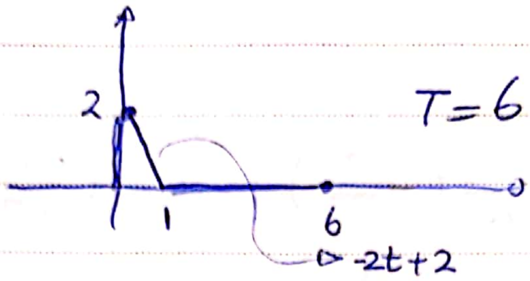
c) one Period :



$T=6 \Rightarrow \omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$

$$a_0 = \frac{1}{6}\int_0^6 x(t)\,dt = \frac{1}{6}$$

$$a_k = \frac{1}{6}\int_0^T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{6}\int_0^1 (-2t+2)e^{-jk\frac{\pi}{3}t}\,dt$$

$$= \frac{-1}{3}\int_0^1 te^{-jk\frac{\pi}{3}t}\,dt + \frac{1}{3}\int_0^1 e^{-jk\frac{\pi}{3}t}\,dt$$

$$= -\frac{1}{3}I_1 + \frac{1}{3}I_2$$

$\boxed{I_1}$ $\int_0^1 u\,dv = uv\Big|_0^1 - \int_0^1 v\,du$

$u=t \Rightarrow du=dt \qquad dv = e^{-jk\frac{\pi}{3}t}dt \Rightarrow v = \frac{-3}{jk\pi}e^{-jk\frac{\pi}{3}t}$

$$uv\Big|_0^1 = \frac{-3t}{jk\pi}e^{-jk\frac{\pi}{3}t}\Big|_0^1 = \frac{-3}{jk\pi}e^{-jk\frac{\pi}{3}}$$

$$\int_0^1 v\,du = \frac{-3}{jk\pi}\left(\frac{-3}{jk\pi}e^{-jk\frac{\pi}{3}t}\Big|_0^1\right) = \frac{-9}{k^2\pi^2}\left(e^{-jk\frac{\pi}{3}}-1\right)$$

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$$\Rightarrow I_1 = \frac{-3}{jk\pi}e^{-jk\frac{\pi}{3}} + \frac{9}{k^2\pi^2}\left(e^{-jk\frac{\pi}{3}}-1\right)$$

1) c) ادامه

$$\boxed{I_2} \qquad I_2 = \int_0^1 e^{-jk\frac{\pi}{3}t}\,dt = \frac{-3}{jk\pi}\left(e^{-jk\frac{\pi}{3}} - 1\right)$$

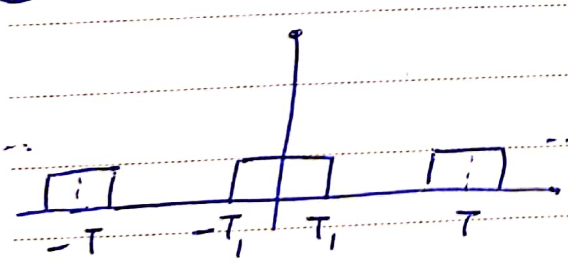
$$\Rightarrow a_k = \frac{-1}{3} I_1 + \frac{1}{3} I_2 = \frac{1}{3}\left(I_2 - I_1\right)$$

$$e^{-jk\frac{\pi}{3}} = m$$ تغییر متغیر برای سادگی نوشتار

$$\Rightarrow a_k = \frac{1}{3}\left(\frac{-3}{jk\pi}(m-1) + \frac{3}{jk\pi}m - \frac{9}{k^2\pi^2}(m-1)\right)$$

$$= \frac{1}{3}\left(\frac{3}{jk\pi} + \frac{9}{k^2\pi^2} - \frac{9m}{k^2\pi^2}\right) = \frac{1}{jk\pi} + \frac{3}{k^2\pi^2} - \frac{3m}{k^2\pi^2}$$

2) a) Oppenheim Pages 193-194



$$\xleftrightarrow{FS} \quad \begin{cases} \dfrac{\sin(k\omega_0 T_1)}{k\pi} & k \neq 0 \\[2ex] \dfrac{2T_1}{T} & k = 0 \end{cases}$$

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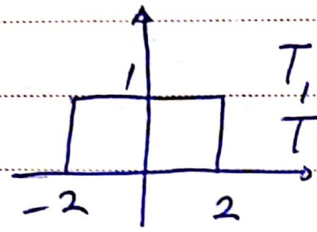
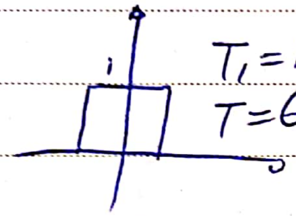
2) b)



$T_1 = 2$

$T = 6$

$$\xleftrightarrow{FS} \quad a_k = \frac{\sin\left(k\frac{\pi}{3}2\right)}{k\pi}$$

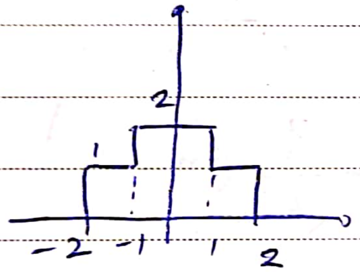


$T_1 = 1$

$T = 6$

$$\xleftrightarrow{FS} \quad b_k = \frac{\sin\left(k\frac{\pi}{3}\right)}{k\pi}$$

$\xrightarrow{\text{Add}}$

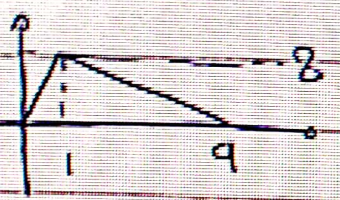


$$\xleftrightarrow{FS} \quad c_k = a_k + b_k$$

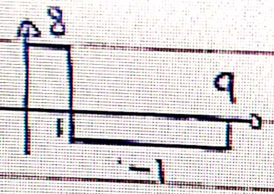
$\xrightarrow[(2.5)]{\text{Shift Right}}$

$$\boxed{d_k = e^{-jk\frac{\pi}{3}(5)} c_k}$$

(c) one period:

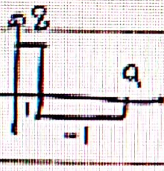


derivative



FS properties; $x(t) \longleftrightarrow a_k$

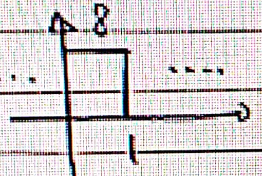
$$\int_{-\infty}^{t} x(t)dt \longleftrightarrow \frac{1}{jk\omega_0} a_k \quad (\text{if } a_0 = 0)$$

in this case, $x(t) =$  $\overset{FS}{\longleftrightarrow} a_k$

and we want to compute FS coeffs for $\int_{-\infty}^{t} x(t)dt$.

since $a_0 = \frac{1}{9}(8-8) = 0$, it is possible to find a_k

and then compute $\frac{1}{jk\omega_0} a_k$ $\quad (\omega_0 = \frac{2\pi}{9})$

$x(t) =$  $+$ 

$T_1 = \frac{1}{2}$, $T = 9$ $\qquad$ $T_1 = 4$, $T = 9$

$$\Rightarrow a_k = 8 e^{-jk\frac{2\pi}{9}(\frac{1}{2})} \frac{\sin(k\frac{2\pi}{9}\frac{1}{2})}{k\pi} \qquad e^{-jk\frac{2\pi}{9}(5)} \frac{\sin(k\frac{2\pi}{9}(4))}{k\pi}$$

$$\Rightarrow ANS = \frac{1}{jk\,2\pi/9} a_k$$

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(3) $T = 2 \Rightarrow \omega_0 = \pi$

$$x(t) = \sum_{k=-2}^{2} a_k e^{jk\pi t} = a_{-2} e^{j(-2)\pi t} + a_{-1} e^{-j(-1)\pi t} + a_1 e^{j\pi t} + a_2 e^{j2\pi t}$$

(a) $$y(t) = \sum_{k=-2}^{2} \underbrace{a_k H(jk\omega_0)}_{= b_k} e^{jk\omega_0 t}$$

$$\rightarrow b_{-2} = a_{-2} H(j(-2\pi)) = a_{-2} \left| H(j(-2\pi)) \right| e^{j\angle H(j(-2\pi))}$$

$b_{-1} = \ldots$

$b_1 = \ldots$

$b_2 = \ldots$

(b) $$\text{Avg Power of } y(t) = \sum_{k=-2}^{2} |b_k|^2$$