

Measure theory review:

Def. measure = a set function (real-valued) s.t.

- $\mu(\emptyset) = 0$
- $\mu(A) \leq \mu(B)$ if $A \subset B$
- $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$.

Def. measure for $A \subset \mathbb{R}^n$ s.t. $0 < \mu(A) < \infty$ will be called a mass distribution.

Def. (book-specific)

A function $f: D \rightarrow \mathbb{R}$ is measurable if $\forall a \in \mathbb{R}$, $\{x \in D \mid f(x) \leq a\}$ is measurable.
(D is a borel set)

Def. We call a collection F of subsets of a sample space Ω an event space if $\Omega \setminus A \in F$ whenever $A \in F$.

Def. (probability measure)

Let F be an event space in Ω .

Then $P: F \rightarrow \mathbb{R}$ is called a probability measure if

- (1) $0 \leq P(A) \leq 1 \quad \forall A \in F$.
- (2) $P(\emptyset) = 0, P(\Omega) = 1$ (note: \emptyset, Ω are both in F)
- (3) $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ if A_1, A_2, \dots are pairwise disjoint.

Def. (probability space)

(Ω, F, P) is a probability space if F is an event space of subsets of Ω , and P is a probability measure defined on sets in F .

Def. (random variable)

Let (Ω, F, P) be a probability space.

We say X is a random variable on (Ω, F, P) if $X: \Omega \rightarrow \mathbb{R}$ s.t. $\forall a \in \mathbb{R}$, $\{A \in \Omega \mid X(A) \leq a\} \in F$.

So basically $\forall a \in \mathbb{R}$, for any $A \in \Omega$ s.t. $X(A) \leq a$, $A \in F$.

Does that agree with the intuition?

Suppose X is the number of heads in 10 coin flips.

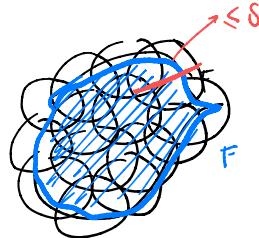
Then $\Omega = \{HHTT\ldots, TTTT\}$, and X maps each thing in Ω to a real number.

So basically a random variable is something that maps from sample space $\rightarrow \mathbb{R}$, with some constraints for the CDF??

Hausdorff measure. (assume \mathbb{R}^n)

Def. (diameter) Let nonempty $U \subset \mathbb{R}^n$. Then $|U| = \sup \{|x-y| : x, y \in U\}$.

Def. (δ -cover) Suppose $F \subset \bigcup_{i=1}^{\infty} U_i$, where $\forall i, |U_i| \leq \delta$.
 then $\{U_i\}_{i=1}^{\infty}$ is called a δ -cover of F ($\{U_i\}$ can be finite). defined for any $F \subset \mathbb{R}^n$,
since \mathbb{R}^n is separable.



Def. ($\mathcal{H}_\delta^s(F)$) Let $s \geq 0$, and $F \subset \mathbb{R}^n$.

Then $\forall \delta > 0$, define $\mathcal{H}_\delta^s(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}$.

Denote $C_\delta(F) = \{\delta\text{-covers of } F\}$.

As δ decreases, the set $\{\delta\text{-covers of } F\}$ gets smaller and smaller.

This means that if $\delta \leq \delta'$, then since $\{\delta\text{-covers}\} \subset \{\delta'\text{-covers}\}$, $\mathcal{H}_\delta^s(F) \geq \mathcal{H}_{\delta'}^s(F)$. ↓ lower bound.

I know that $0 \in \mathcal{H}_\delta^s$, so since $\{\sum |U_i|^s : \{U_i\} \in C_\delta(F)\} \subset \mathbb{R}$, this means $\mathcal{H}_\delta^s(F)$ must exist.

Def. (s -dimensional Hausdorff measure)

$$\mathcal{H}^s(F) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(F).$$

As shown above, $\mathcal{H}_\delta^s(F)$ is a monotonic sequence as $\delta \rightarrow 0$, so the limit exists, if it is the case that $\{\mathcal{H}_\delta^s(F)\}$ is bounded. If not, $\mathcal{H}^s(F)$ is allowed to be ∞ .

Hausdorff measure:

$$\mathcal{H}_\delta^s(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \in C_\delta(F) \right\}$$

$$\mathcal{H}^s(F) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(F).$$

Prop. The s -dimensional Hausdorff measure is a measure.

(i) $\mathcal{H}_\delta^s(\emptyset) = 0$, so $\mathcal{H}^s(\emptyset) = 0$.

(ii) Clearly $\mathcal{H}^s(F) \geq 0$ for any F .

$$(iii) \mathcal{H}^s\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{\delta \rightarrow 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \in C_\delta\left(\bigcup_{i=1}^{\infty} E_i\right) \right\}$$

$$= \lim_{\delta \rightarrow 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \in C_\delta(E_i) \right\}$$

+

:

$$\lim_{\delta \rightarrow 0} \inf \left\{ \sum |U_i|^s : \{U_i\} \in C_\delta(E) \right\} + \dots$$

Consider $A \cup B$.

Then WTS: $\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B)$.

$\inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \in \mathcal{C}_s(A \cup B) \right\} \in \mathbb{R}^+$, so it can equal ∞ .

(Need to show that \mathcal{H}^s is a measure)

Prop. Scaling follows intuition for \mathcal{H}^s .

i.e. If $F \subset \mathbb{R}^n$ and $\lambda > 0$, then $\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$,

where $\lambda F = \{\lambda x : x \in F\}$.

Proof Let $\{U_i\} \in \mathcal{C}_s(F)$.

Then $\bigcup_{i=1}^{\infty} U_i \supset F$, and $|U_i| \leq s \ \forall i$. This means that if $\lambda x \in \lambda F$, then since $x \in U_i$ for some i , $\lambda x \in \lambda U_i$.

$\therefore \lambda F \subset \bigcup_{i=1}^{\infty} \lambda U_i$. Also WTS: $|\lambda U_i| = \lambda |U_i|$.

$$\begin{aligned} |\lambda U_i| &= \sup \{d(x, y) : x, y \in \lambda U_i\} \\ &= \sup \{d(\lambda a, \lambda b) : \lambda a, \lambda b \in \lambda U_i\} \\ &= \sup \{\lambda d(a, b) : \lambda a, \lambda b \in \lambda U_i\} \\ &= \sup \{\lambda d(a, b) : a, b \in U_i\} = \lambda \sup \{d(a, b) : a, b \in U_i\} = \lambda |U_i|. \end{aligned}$$

$\star d(\lambda a, \lambda b) = \lambda d(a, b)$, (assuming Euclidean d)

$\therefore |\lambda U_i| = \lambda |U_i| \leq \lambda s$, so $\{\lambda U_i\} \in \mathcal{C}_{\lambda s}(\lambda F)$.

$\therefore \mathcal{H}_{\lambda s}^s(\lambda F) \leq \sum_{i=1}^{\infty} |\lambda U_i|^s = \lambda^s \sum_{i=1}^{\infty} |U_i|^s$, where $\{U_i\} \in \mathcal{C}_s(F)$.

$\therefore \mathcal{H}_{\lambda s}^s(\lambda F) \leq \lambda^s \mathcal{H}_s^s(F)$, since $\mathcal{H}_{\lambda s}^s(\lambda F)$ is a lower bound for $\{\lambda^s \sum |U_i|^s\}$.

Taking $s \rightarrow 0$, $\mathcal{H}^s(\lambda F) \leq \lambda^s \mathcal{H}^s(F)$.

By similar logic, $\mathcal{H}_{\frac{s}{\lambda}}^s(F) \leq \left(\frac{1}{\lambda}\right)^s \mathcal{H}_s^s(F) \Rightarrow \mathcal{H}_s^s(\lambda F) \geq \lambda^s \mathcal{H}_{\frac{s}{\lambda}}^s(F)$

$\therefore \mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$.

□

Prop Let $F \subset \mathbb{R}^n$, and suppose $f: F \rightarrow \mathbb{R}^m$ st. $|f(x) - f(y)| \leq c|x-y|^\alpha$ for all $x, y \in F$, and $\alpha, c > 0$.

Then $\mathcal{H}^{\frac{s}{\alpha}}(f(F)) \leq c^{\frac{s}{\alpha}} \mathcal{H}^s(F)$.

Proof. Let $\{U_i\} \in C_\delta(\bar{F})$.

Then $|U_i| < \delta$ $\forall i$. Hence is $\{f(U_i)\} \in C_{c\delta^\alpha}(f(\bar{F}))$?

If $F \subset \bigcup_{i=1}^{\infty} U_i$, then $f(F) \subset \bigcup_{i=1}^{\infty} f(U_i)$.

Also $|f(x)-f(y)| \leq c|x-y|^\alpha \leq c\delta^\alpha$, which means that $\{f(U_i)\} \in C_{c\delta^\alpha}(f(\bar{F}))$.

$$\begin{aligned}\mathcal{H}_{c\delta^\alpha}^s(f(\bar{F})) &\leq \sum |f(U_i)|^s \leq \sum c^{s\alpha} |U_i|^s = c^{s\alpha} \sum |U_i|^s \\ &\leq c^{s\alpha} \mathcal{H}_\delta^s(\bar{F}).\end{aligned}$$

$$\therefore \delta \rightarrow 0 \text{ means } \mathcal{H}^s(f(\bar{F})) \leq c^{s\alpha} \mathcal{H}^s(\bar{F}).$$

This makes intuitive sense.

$|f(x)-f(y)| \leq c|x-y|^\alpha$ means that the distance b/w $f(x)$ and $f(y)$ is less than a multiple of $|x-y|^\alpha$, so the size of F ($\mathcal{H}^s(f(\bar{F}))$) should be less than the scaled size of \bar{F} ($c^{s\alpha} \mathcal{H}^s(\bar{F})$).

Def. (Hausdorff dimension)

Let $\dim_H(\bar{F}) = \inf \{s : \mathcal{H}^s(\bar{F}) = 0\} = \sup \{s : \mathcal{H}^s(\bar{F}) = \infty\}$ s.t.

$$\mathcal{H}^s(\bar{F}) = \begin{cases} 0 & \text{if } s > \dim_H \bar{F} \\ \infty & \text{if } s < \dim_H \bar{F}. \end{cases}$$

$\dim_H(\bar{F})$ can be $0, \infty$, or between 0 and ∞ .

The motivation is that for $s < 1$, as $\delta \rightarrow 0$, $\mathcal{H}_\delta^s(\bar{F})$ must be non-increasing.

$\therefore \mathcal{H}^s(\bar{F}) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(\bar{F})$ must also be non-increasing as $s \rightarrow \infty$.

In fact, if $t > s$, and $\delta < 1$, then for any δ -car $\{U_i\}$,

$$\sum_i |U_i|^t \leq \delta^t \sum_i |U_i|^s, \text{ so } \mathcal{H}_\delta^t(\bar{F}) \leq \delta^{ts} \mathcal{H}_\delta^s(\bar{F})$$

This means that as $\delta \rightarrow 0$, if $\mathcal{H}^s(\bar{F}) < \infty$, then $\mathcal{H}^t(\bar{F}) = 0$.

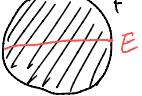
Prop. (i) $E \subset F \Rightarrow \dim_H E \leq \dim_H F$.

(ii) $\dim_H(\bigcup_{i=1}^{\infty} F_i) = \sup \{ \dim_H F_i \}$.

(iii) If F is countable, then $\dim_H F = 0$.

Prof. (i) for any s , $\mathcal{H}^s(E) \leq \mathcal{H}^s(F)$.

$$\therefore \dim_H(E) = \inf \{s : \mathcal{H}^s(E) = 0\} \leq \inf \{s : \mathcal{H}^s(F) = 0\} = \dim_H(F).$$

Also:  $\left. \begin{array}{l} \dim_H(E) = 1, \\ \dim_H(F) = 2. \end{array} \right\}$

(ii) By (i), $\dim_H F_i \leq \dim_H(UF_i) \quad \forall i$.

Suppose $\exists s$ such that $\dim_H F_i \leq s \quad \forall i$ and $s < \dim_H(UF_i)$.

Then $\mathcal{H}^s(F_i) = 0 \quad \forall i$, so $\mathcal{H}^s(UF_i) \leq \sum_i \mathcal{H}^s(F_i) = 0$, so $\mathcal{H}^s(UF_i) = 0$.

$$\therefore \dim_H(UF_i) \leq s = \inf \{s : \mathcal{H}^s(UF_i) = 0\}.$$

(iii) Suppose $F = \bigcup F_i$, where F_i are single points.

Then $\mathcal{H}^0(F_i) = 1$, and $\mathcal{H}^t(F_i) = 0$ for $t > 0$, so $\dim_H(F_i) = 0$.

$$\therefore \dim_H(UF_i) = 0 \text{ by (i).}$$

Prop. Let $F \subset \mathbb{R}^n$, and $f: F \rightarrow \mathbb{R}^m$ satisfy $|f(x) - f(y)| \leq c|x-y|^\alpha$ for $\alpha > 0$.

$$\text{Then } \dim_H f(F) \leq \frac{\dim_H F}{\alpha}.$$

Prof We know $\mathcal{H}^{\frac{s}{\alpha}}(f(F)) \leq c^{\frac{s}{\alpha}} \mathcal{H}^s(F)$.

Want to show that $\mathcal{H}^{\frac{\dim_H F}{\alpha}}(f(F)) = 0$. $\mathcal{H}^{\frac{s}{\alpha}}(f(F)) \leq c^{\frac{s}{\alpha}} \mathcal{H}^s(F)$
 $\therefore \mathcal{H}^{\frac{s}{\alpha}}(f(F)) \leq c^{\frac{s}{\alpha}} \mathcal{H}^s(F)$

If $s > \dim_H(F)$, then $\mathcal{H}^{\frac{s}{\alpha}}(f(F)) = 0$ also.

$$\therefore t > \frac{\dim_H(F)}{\alpha} \Rightarrow \dim_H(f(F)) \leq \frac{\dim_H(F)}{\alpha}.$$

Corollary (i) if $f: F \rightarrow \mathbb{R}^m$ is Lipschitz (*i.e.* $|f(x) - f(y)| \leq c|x-y|$),
then $\dim_H(f(F)) \leq \dim_H(F)$

(ii) if $f: F \rightarrow \mathbb{R}^m$ is bi-Lipschitz (*i.e.* $c_1|x-y| \leq |f(x) - f(y)| \leq c_2|x-y|$, where $0 < c_1 \leq c_2 < \infty$),
then $\dim_H(f(F)) = \dim_H(F)$.

* Hausdorff dimension is invariant under bi-Lipschitz transformations.

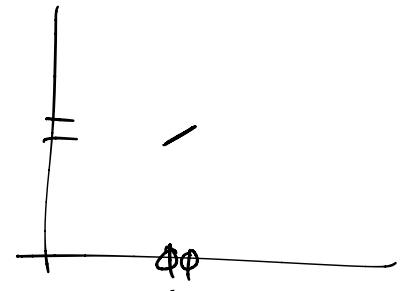
* If 2 sets have diff dimension, then \nexists bi-Lipschitz transform.

Proof. (i) clear from $\dim_H f(F) \leq \frac{\dim_H F}{\alpha=1} = \dim_H F$.

(ii) $\dim_H f(F) \leq \dim_H F$ by the 2nd half of inequality. }
 $\dim_H f^{-1}(f(F)) = \dim_H F \leq \dim_H f(F)$ by 1st half. } $\dim_H f(F) = \dim_H F$.

Def. (homeomorphism) continuous, 1-1 mapping with continuous inverse.

we can view two sets as being the same if \exists bi-Lipschitz mapping b/w them.
 Is it true that if $\dim_H F = \dim_H E$, then bi-Lipschitz mapping $E \rightarrow F$??

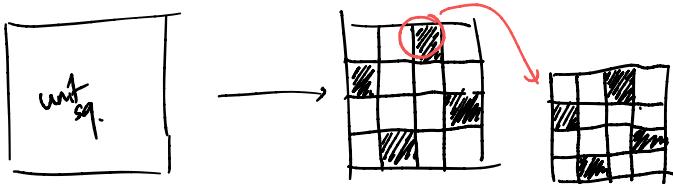


Prop. A set $F \subset \mathbb{R}^n$ with $\dim_H F < 1$ is totally disconnected, i.e. no two points lie in the same connected component.

(Proof skipped) Imagine if F is only a couple of points.

Then if $\dim_H F < 1$, then this means (from intuition) that you cannot use the points of F to draw a line.
 \therefore no points can lie on a connected component of F i.e. it's totally disconnected.

Example. (Cantor dust)



Call $F = \text{Cantor dust set. } \dim_H F = ?$

Claim: $1 \leq \mathcal{H}^1(F) < \sqrt{2}$, so $\dim_H F = 1$.

Consider $\delta > 0$. Then $\mathcal{H}_\delta^1(F) = \left\{ \sum |U_i| : \{U_i\} \in C_\delta(F) \right\}$.

If $\delta = \sqrt{2}$, then the unit square covers, right?

$$\therefore \mathcal{H}_\delta^1(F) \leq \sqrt{2}$$

If $\delta = \frac{\sqrt{2}}{4}$, then $\mathcal{H}_\delta^1(F) \leq \sqrt{2}$ also.

Basically as $\delta \rightarrow 0$, $\mathcal{H}_\delta^1(F) \leq \sqrt{2}$, so $\mathcal{H}^1(F) \leq \sqrt{2}$ also.

For the other side, consider the projection to the x -axis (i.e. $(x,y) \rightarrow x$).

Then since $|\text{proj}(x) - \text{proj}(y)| \leq |x-y|$, proj is Lipschitz \Rightarrow i.e. $\dim_H \text{proj}(F) \leq \dim_H F$.

$\therefore \text{proj}(F) = [0,1]$, so $\mathcal{H}^1([0,1]) = \mathcal{H}^1(\text{proj}(F)) = 1 \leq \mathcal{H}^1(F)$.

$$\therefore 1 \leq R^1(F) \leq 52.$$

$$\therefore \dim_H F = 1.$$