

Measure theory review:

Def. measure = a set function (real-valued) s.t.

- $\mu(\emptyset) = 0$
- $\mu(A) \leq \mu(B)$  if  $A \subset B$
- $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$ .

Def. measure for  $A \subset \mathbb{R}^n$  s.t.  $0 < \mu(A) < \infty$  will be called a mass distribution.

Def. (book-specific)

A function  $f: D \rightarrow \mathbb{R}$  is measurable if  $\forall a \in \mathbb{R}$ ,  $\{x \in D \mid f(x) \leq a\}$  is measurable.  
( $D$  is a borel set)

Def. We call a collection  $F$  of subsets of a sample space  $\Omega$  an event space if  $\Omega \setminus A \in F$  whenever  $A \in F$ .

Def. (probability measure)

Let  $F$  be an event space in  $\Omega$ .

Then  $P: F \rightarrow \mathbb{R}$  is called a probability measure if

- (1)  $0 \leq P(A) \leq 1 \quad \forall A \in F$ .
- (2)  $P(\emptyset) = 0, P(\Omega) = 1$  (note:  $\emptyset, \Omega$  are both in  $F$ )
- (3)  $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$  if  $A_1, A_2, \dots$  are pairwise disjoint.

Def. (probability space)

$(\Omega, F, P)$  is a probability space if  $F$  is an event space of subsets of  $\Omega$ , and  $P$  is a probability measure defined on sets in  $F$ .

Def. (random variable)

Let  $(\Omega, F, P)$  be a probability space.

We say  $X$  is a random variable on  $(\Omega, F, P)$  if  $X: \Omega \rightarrow \mathbb{R}$  s.t.  $\forall a \in \mathbb{R}$ ,  $\{A \in \Omega \mid X(A) \leq a\} \in F$ .

So basically  $\forall a \in \mathbb{R}$ , for any  $A \in \Omega$  s.t.  $X(A) \leq a$ ,  $A \in F$ .

Does that agree with the intuition?

Suppose  $X$  is the number of heads in 10 coin flips.

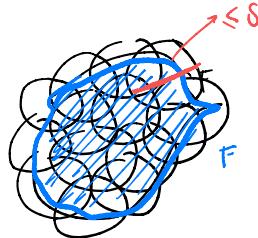
Then  $\Omega = \{HHTT\ldots, TTTT\}$ , and  $X$  maps each thing in  $\Omega$  to a real number.

So basically a random variable is something that maps from sample space  $\rightarrow \mathbb{R}$ , with some constraints for the CDF??

Hausdorff measure. (assume  $\mathbb{R}^n$ )

Def. (diameter) Let nonempty  $U \subset \mathbb{R}^n$ . Then  $|U| = \sup \{|x-y| : x, y \in U\}$ .

Def. ( $\delta$ -cover) Suppose  $F \subset \bigcup_{i=1}^{\infty} U_i$ , where  $\forall i, |U_i| \leq \delta$ .  
 then  $\{U_i\}_{i=1}^{\infty}$  is called a  $\delta$ -cover of  $F$  ( $\{U_i\}$  can be finite).  $\left\{ \begin{array}{l} \text{defined for any } F \subset \mathbb{R}^n, \\ \text{since } \mathbb{R}^n \text{ is separable.} \end{array} \right.$



Def. ( $\mathcal{H}_\delta^s(F)$ ) Let  $s \geq 0$ , and  $F \subset \mathbb{R}^n$ .

Then  $\forall \delta > 0$ , define  $\mathcal{H}_\delta^s(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}$ .

Denote  $C_\delta(F) = \{\delta\text{-covers of } F\}$ .

As  $\delta$  decreases, the set  $\{\delta\text{-covers of } F\}$  gets smaller and smaller.

This means that if  $\delta \leq \delta'$ , then since  $\{\delta\text{-covers}\} \subset \{\delta'\text{-covers}\}$ ,  $\mathcal{H}_\delta^s(F) \geq \mathcal{H}_{\delta'}^s(F)$ . (lower bound)

I know that  $0 \in \mathcal{H}_\delta^s$ , so since  $\{\sum |U_i|^s : \{U_i\} \in C_\delta(F)\} \subset \mathbb{R}$ , this means  $\mathcal{H}_\delta^s(F)$  must exist.

Def. ( $s$ -dimensional Hausdorff measure)

$$\mathcal{H}^s(F) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(F).$$

As shown above,  $\mathcal{H}_\delta^s(F)$  is a monotonic sequence as  $\delta \rightarrow 0$ , so the limit exists, if it is the case that  $\{\mathcal{H}_\delta^s(F)\}$  is bounded. If not,  $\mathcal{H}^s(F)$  is allowed to be  $\infty$ .

Hausdorff measure:

$$\mathcal{H}_\delta^s(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \in C_\delta(F) \right\}$$

$$\mathcal{H}^s(F) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(F).$$

Prop. The  $s$ -dimensional Hausdorff measure is a measure.

(i)  $\mathcal{H}_\delta^s(\emptyset) = 0$ , so  $\mathcal{H}^s(\emptyset) = 0$ .

(ii) Clearly  $\mathcal{H}^s(F) \geq 0$  for any  $F$ .

$$(iii) \mathcal{H}^s\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{\delta \rightarrow 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \in C_\delta\left(\bigcup_{i=1}^{\infty} E_i\right) \right\}$$

$$= \lim_{\delta \rightarrow 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \in C_\delta(E_i) \right\}$$

+

:

$$\lim_{\delta \rightarrow 0} \inf \left\{ \sum |U_i|^s : \{U_i\} \in C_\delta(E) \right\} + \dots$$

Consider  $A \cup B$ .

Then WTS:  $\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B)$ .

$\inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \in \mathcal{C}_s(A \cup B) \right\} \in \mathbb{R}^+$ , so it can equal  $\infty$ .

(Need to show that  $\mathcal{H}^s$  is a measure)

Prop. Scaling follows intuition for  $\mathcal{H}^s$ .

i.e. If  $F \subset \mathbb{R}^n$  and  $\lambda > 0$ , then  $\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$ ,

where  $\lambda F = \{\lambda x : x \in F\}$ .

Proof Let  $\{U_i\} \in \mathcal{C}_s(F)$ .

Then  $\bigcup_{i=1}^{\infty} U_i \supset F$ , and  $|U_i| \leq s \ \forall i$ . This means that if  $\lambda x \in \lambda F$ , then since  $x \in U_i$  for some  $i$ ,  $\lambda x \in \lambda U_i$ .

$\therefore \lambda F \subset \bigcup_{i=1}^{\infty} \lambda U_i$ . Also WTS:  $|\lambda U_i| = \lambda |U_i|$ .

$$\begin{aligned} |\lambda U_i| &= \sup \{d(x, y) : x, y \in \lambda U_i\} \\ &= \sup \{d(\lambda a, \lambda b) : \lambda a, \lambda b \in \lambda U_i\} \\ &= \sup \{\lambda d(a, b) : \lambda a, \lambda b \in \lambda U_i\} \\ &= \sup \{\lambda d(a, b) : a, b \in U_i\} = \lambda \sup \{d(a, b) : a, b \in U_i\} = \lambda |U_i|. \end{aligned}$$

$\star d(\lambda a, \lambda b) = \lambda d(a, b)$ , (assuming Euclidean d)

$\therefore |\lambda U_i| = \lambda |U_i| \leq \lambda s$ , so  $\{\lambda U_i\} \in \mathcal{C}_{\lambda s}(\lambda F)$ .

$\therefore \mathcal{H}_{\lambda s}^s(\lambda F) \leq \sum_{i=1}^{\infty} |\lambda U_i|^s = \lambda^s \sum_{i=1}^{\infty} |U_i|^s$ , where  $\{U_i\} \in \mathcal{C}_s(F)$ .

$\therefore \mathcal{H}_{\lambda s}^s(\lambda F) \leq \lambda^s \mathcal{H}_s^s(F)$ , since  $\mathcal{H}_{\lambda s}^s(\lambda F)$  is a lower bound for  $\{\lambda^s \sum |U_i|^s\}$ .

Taking  $s \rightarrow 0$ ,  $\mathcal{H}^s(\lambda F) \leq \lambda^s \mathcal{H}^s(F)$ .

By similar logic,  $\mathcal{H}_{\frac{s}{\lambda}}^s(F) \leq \left(\frac{1}{\lambda}\right)^s \mathcal{H}_s^s(F) \Rightarrow \mathcal{H}_s^s(\lambda F) \geq \lambda^s \mathcal{H}_{\frac{s}{\lambda}}^s(F)$

$\therefore \mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$ .

□

Prop Let  $F \subset \mathbb{R}^n$ , and suppose  $f: F \rightarrow \mathbb{R}^m$  st.  $|f(x) - f(y)| \leq C|x-y|^\alpha$  for all  $x, y \in F$ , and  $\alpha, C > 0$ .

Then  $\mathcal{H}^{\frac{s}{\alpha}}(f(F)) \leq C^{\frac{s}{\alpha}} \mathcal{H}^s(F)$ .

Proof. Let  $\{U_i\} \in C_\delta(\bar{F})$ .

Then  $|U_i| < \delta$   $\forall i$ . Hence is  $\{f(U_i)\} \in C_{c\delta^\alpha}(f(\bar{F}))$ ?

If  $F \subset \bigcup_{i=1}^{\infty} U_i$ , then  $f(F) \subset \bigcup_{i=1}^{\infty} f(U_i)$ .

Also  $|f(x)-f(y)| \leq c|x-y|^\alpha \leq c\delta^\alpha$ , which means that  $\{f(U_i)\} \in C_{c\delta^\alpha}(f(\bar{F}))$ .

$$\begin{aligned}\mathcal{H}_{c\delta^\alpha}^s(f(\bar{F})) &\leq \sum |f(U_i)|^s \leq \sum c^{s\alpha} |U_i|^s = c^{s\alpha} \sum |U_i|^s \\ &\leq c^{s\alpha} \mathcal{H}_\delta^s(\bar{F}).\end{aligned}$$

$$\therefore \delta \rightarrow 0 \text{ means } \mathcal{H}^s(f(\bar{F})) \leq c^{s\alpha} \mathcal{H}^s(\bar{F}).$$

This makes intuitive sense.

$|f(x)-f(y)| \leq c|x-y|^\alpha$  means that the distance b/w  $f(x)$  and  $f(y)$  is less than a multiple of  $|x-y|^\alpha$ , so the size of  $F$  ( $\mathcal{H}^s(f(\bar{F}))$ ) should be less than the scaled size of  $\bar{F}$  ( $c^{s\alpha} \mathcal{H}^s(\bar{F})$ ).

Def. (Hausdorff dimension)

Let  $\dim_H(\bar{F}) = \inf \{s : \mathcal{H}^s(\bar{F}) = 0\} = \sup \{s : \mathcal{H}^s(\bar{F}) = \infty\}$  s.t.

$$\mathcal{H}^s(\bar{F}) = \begin{cases} 0 & \text{if } s > \dim_H \bar{F} \\ \infty & \text{if } s < \dim_H \bar{F}. \end{cases}$$

$\dim_H(\bar{F})$  can be  $0, \infty$ , or between  $0$  and  $\infty$ .

The motivation is that for  $s < 1$ , as  $\delta \rightarrow 0$ ,  $\mathcal{H}_\delta^s(\bar{F})$  must be non-increasing.

$\therefore \mathcal{H}^s(\bar{F}) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(\bar{F})$  must also be non-increasing as  $s \rightarrow \infty$ .

In fact, if  $t > s$ , and  $\delta < 1$ , then for any  $\delta$ -car  $\{U_i\}$ ,

$$\sum_i |U_i|^t \leq \delta^t \sum_i |U_i|^s, \text{ so } \mathcal{H}_\delta^t(\bar{F}) \leq \delta^{ts} \mathcal{H}_\delta^s(\bar{F})$$

This means that as  $\delta \rightarrow 0$ , if  $\mathcal{H}^s(\bar{F}) < \infty$ , then  $\mathcal{H}^t(\bar{F}) = 0$ .

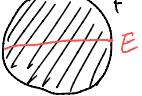
Prop. (i)  $E \subset F \Rightarrow \dim_H E \leq \dim_H F$ .

(ii)  $\dim_H(\bigcup_{i=1}^{\infty} F_i) = \sup \{ \dim_H F_i \}$ .

(iii) If  $F$  is countable, then  $\dim_H F = 0$ .

Prof. (i) for any  $s$ ,  $\mathcal{H}^s(E) \leq \mathcal{H}^s(F)$ .

$$\therefore \dim_H(E) = \inf \{s : \mathcal{H}^s(E) = 0\} \leq \inf \{s : \mathcal{H}^s(F) = 0\} = \dim_H(F).$$

Also:   $\left. \begin{array}{l} \dim_H(E) = 1, \\ \dim_H(F) = 2. \end{array} \right\}$

(ii) By (i),  $\dim_H F_i \leq \dim_H(UF_i) \quad \forall i$ .

Suppose  $\exists s$  such that  $\dim_H F_i \leq s \quad \forall i$  and  $s < \dim_H(UF_i)$ .

Then  $\mathcal{H}^s(F_i) = 0 \quad \forall i$ , so  $\mathcal{H}^s(UF_i) \leq \sum_i \mathcal{H}^s(F_i) = 0$ , so  $\mathcal{H}^s(UF_i) = 0$ .

$$\therefore \dim_H(UF_i) \leq s = \inf \{s : \mathcal{H}^s(UF_i) = 0\}.$$

(iii) Suppose  $F = \bigcup F_i$ , where  $F_i$  are single points.

Then  $\mathcal{H}^0(F_i) = 1$ , and  $\mathcal{H}^t(F_i) = 0$  for  $t > 0$ , so  $\dim_H(F_i) = 0$ .

$$\therefore \dim_H(UF_i) = 0 \text{ by (i).}$$

Prop. Let  $F \subset \mathbb{R}^n$ , and  $f: F \rightarrow \mathbb{R}^m$  satisfy  $|f(x) - f(y)| \leq c|x-y|^\alpha$  for  $\alpha > 0$ .

$$\text{Then } \dim_H f(F) \leq \frac{\dim_H F}{\alpha}.$$

Prof We know  $\mathcal{H}^{\frac{s}{\alpha}}(f(F)) \leq c^{\frac{s}{\alpha}} \mathcal{H}^s(F)$ .

Want to show that  $\mathcal{H}^{\frac{\dim_H F}{\alpha}}(f(F)) = 0$ .  $\mathcal{H}^{\frac{s}{\alpha}}(f(F)) \leq c^{\frac{s}{\alpha}} \mathcal{H}^s(F)$   
 $\therefore \mathcal{H}^{\frac{s}{\alpha}}(f(F)) \leq c^{\frac{s}{\alpha}} \mathcal{H}^s(F)$

If  $s > \dim_H(F)$ , then  $\mathcal{H}^{\frac{s}{\alpha}}(f(F)) = 0$  also.

$$\therefore t > \frac{\dim_H(F)}{\alpha} \Rightarrow \dim_H(f(F)) \leq \frac{\dim_H(F)}{\alpha}.$$

Corollary (i) if  $f: F \rightarrow \mathbb{R}^m$  is Lipschitz (*i.e.*  $|f(x) - f(y)| \leq c|x-y|$ ),  
then  $\dim_H(f(F)) \leq \dim_H(F)$

(ii) if  $f: F \rightarrow \mathbb{R}^m$  is bi-Lipschitz (*i.e.*  $c_1|x-y| \leq |f(x) - f(y)| \leq c_2|x-y|$ , where  $0 < c_1 \leq c_2 < \infty$ ),  
then  $\dim_H(f(F)) = \dim_H(F)$ .

\* Hausdorff dimension is invariant under bi-Lipschitz transformations.

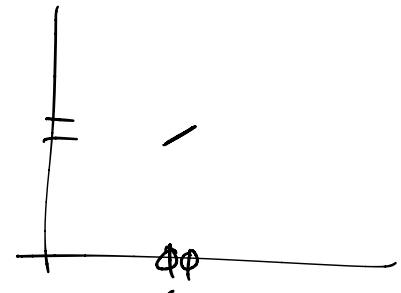
\* If 2 sets have diff dimension, then  $\nexists$  bi-Lipschitz transform.

Proof. (i) clear from  $\dim_H f(F) \leq \frac{\dim_H F}{\alpha=1} = \dim_H F$ .

(ii)  $\dim_H f(F) \leq \dim_H F$  by the 2nd half of inequality. }  
 $\dim_H f^{-1}(f(F)) = \dim_H F \leq \dim_H f(F)$  by 1st half. }  $\dim_H f(F) = \dim_H F$ .

Def. (homeomorphism) continuous, 1-1 mapping with continuous inverse.

we can view two sets as being the same if  $\exists$  bi-Lipschitz mapping b/w them.  
 Is it true that if  $\dim_H F = \dim_H E$ , then bi-Lipschitz mapping  $E \rightarrow F$ ??

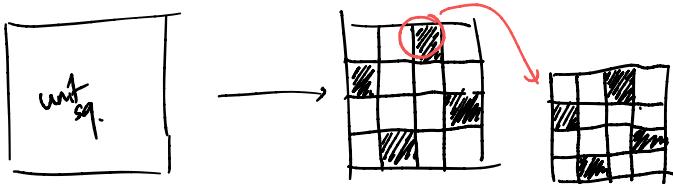


Prop. A set  $F \subset \mathbb{R}^n$  with  $\dim_H F < 1$  is totally disconnected, i.e. no two points lie in the same connected component.

(Proof skipped) Imagine if  $F$  is only a couple of points.

Then if  $\dim_H F < 1$ , then this means (from intuition) that you cannot use the points of  $F$  to draw a line.  
 $\therefore$  no points can lie on a connected component of  $F$  i.e. it's totally disconnected.

Example. (Cantor dust)



Call  $F = \text{Cantor dust set. } \dim_H F = ?$

Claim:  $1 \leq \mathcal{H}^1(F) < \sqrt{2}$ , so  $\dim_H F = 1$ .

Consider  $\delta > 0$ . Then  $\mathcal{H}_\delta^1(F) = \left\{ \sum |U_i| : \{U_i\} \in C_\delta(F) \right\}$ .

If  $\delta = \sqrt{2}$ , then the unit square covers, right?

$$\therefore \mathcal{H}_\delta^1(F) \leq \sqrt{2}$$

If  $\delta = \frac{\sqrt{2}}{4}$ , then  $\mathcal{H}_\delta^1(F) \leq \sqrt{2}$  also.

Basically as  $\delta \rightarrow 0$ ,  $\mathcal{H}_\delta^1(F) \leq \sqrt{2}$ , so  $\mathcal{H}^1(F) \leq \sqrt{2}$  also.

For the other side, consider the projection to the  $x$ -axis (i.e.  $(x,y) \rightarrow x$ ).

Then since  $|\text{proj}(x) - \text{proj}(y)| \leq |x-y|$ ,  $\text{proj}$  is Lipschitz  $\Rightarrow$  i.e.  $\dim_H \text{proj}(F) \leq \dim_H F$ .

$\therefore \text{proj}(F) = [0,1]$ , so  $\mathcal{H}^1([0,1]) = \mathcal{H}^1(\text{proj}(F)) = 1 \leq \mathcal{H}^1(F)$ .

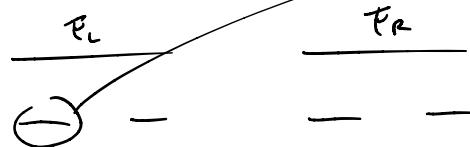
$$\therefore 1 \leq \mathcal{H}^1(F) \leq \sqrt{2}.$$

$$\therefore \dim_H F = 1.$$

Example: Cantor set  $F$ .

$$\dim_H F = \frac{\log 2}{\log 3}, \text{ and } \frac{1}{2} \leq \mathcal{H}^{\frac{\log 2}{\log 3}}(F) \leq 1. \quad \Rightarrow \text{scaled version of } F_L.$$

Heuristic calculation.



$$\therefore \mathcal{H}^s(F) = \frac{1}{3} \mathcal{H}^s(F) + \frac{1}{3} \mathcal{H}^s(F)$$

\* this heuristic method usually gives the right answer for a lot of self-similar sets.

↳ applies b/c of the scaling property.

$$\text{Assuming } 0 < \mathcal{H}^s(F) < \infty, \text{ we have } 1 = 2 \left(\frac{1}{3}\right)^s \Rightarrow \frac{1}{2} = \left(\frac{1}{3}\right)^s \Rightarrow 3^s = 2 \Rightarrow s = \log_3 2.$$

$$\therefore s = \frac{\log 2}{\log 3} \text{ by change of base.}$$

Rigorous calculation

Suppose  $s = \frac{\log 2}{\log 3}$ . Then we can cover  $F$  with  $2^k$  intervals of length  $3^{-k}$ .

$$\text{This means that } \mathcal{H}_{3^{-k}}^s(F) \leq 2^k 3^{-ks} = 2^k 3^{-k} = 1.$$

To show that  $\mathcal{H}^s(F) \geq \frac{1}{2}$ , we show that  $\sum |U_i|^s \geq \frac{1}{2}$  for any cover  $\{U_i\}$ .

Let  $\{U_i\}$  be a cover of  $F$ . We can assume WLOG that  $\{U_i\}$  consists of bounded intervals.

$\therefore$  by extending  $\{U_i\}$ , we obtain an open cover of  $F$ .

$\therefore$  since  $F$  is closed & bounded, it is compact.

$\therefore$  we only need be concerned with finite covers (b/c of compactness).

Suppose  $k \in \mathbb{N}$  such that  $3^{-(kn)} \leq |U_i| < 3^{-k}$ .

Then if  $j \geq k$ , then  $U_i$  intersects at most  $2^{j-k} = 2^j 3^{-sk} \leq 2^j 3^{-s} |U_i|^s$  basic intervals (basic intervals being intervals of length  $3^{-m}$ ).

$\therefore$  For  $j$  sufficiently large, we have  $2^j \leq 2^j 3^s \sum_i |U_i|^s$ , so  $1 \leq 3^s \sum_i |U_i|^s \Rightarrow \sum_i |U_i|^s \geq 3^{-s} = \frac{1}{2}$ .

Remark.

Suppose  $B_\delta^s(F) = \inf \left\{ \sum_i |B_i|^s : \{B_i\} \text{ is a } \delta\text{-cover of } F \text{ by balls} \right\}$ .

Clearly  $\mathcal{H}_\delta^s(F) \leq B_\delta^s(F)$ .  $\rightarrow \mathcal{H}^s(F) \leq B^s(F)$

Also if  $\{V_i\} \in C_s(E)$ , then  $\{B_i\}$  such that  $B_i \supset V_i$  gives  $\sum |B_i|^s \leq \sum |2V_i|^s \leq 2^s \sum |V_i|^s$ .

$\therefore B_{2\delta}^s(F) \leq 2^s \mathcal{H}_\delta^s(F)$ .

$\therefore B^s(F) \leq 2^s \mathcal{H}^s(E) \rightarrow \mathcal{H}^s(F) \leq B^s(F) \leq 2^s \mathcal{H}^s(E)$ .

This means that if  $s = \dim_H F = \inf \{s : \mathcal{H}^s(F) = 0\} = \sup \{s : \mathcal{H}^s(F) = \infty\}$ ,

then using  $B^s(F)$  gives the same  $s$ .

i.e.  $\inf \{s : \mathcal{H}^s(F) = 0\} = \inf \{s : B^s(F) = 0\} = \sup \{s : \mathcal{H}^s(F) = \infty\} = \sup \{s : B^s(F) = \infty\}$ .

Finer notions of dimension.

Suppose  $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an increasing,  $cts$  function. (called a dimension function)

Define  $\mathcal{H}_\delta^h(F) = \inf \left\{ \sum_i h(|V_i|) : \{V_i\} \in C_s(E) \right\}$ . (normally,  $h(t) \sim t^s$ ).

If  $h, g$  are dimension functions such that  $\frac{h}{g} \rightarrow 0$  as  $t \rightarrow 0$ ,

then it can be shown that  $\mathcal{H}^h(F) = 0$  if  $\mathcal{H}^g(F) < \infty$ .

(analogue: if  $t > s$ , and  $\mathcal{H}^t(F) < \infty$ , then  $\mathcal{H}^s(F) = 0$ )

Exercises

(1) Show that the value of  $\mathcal{H}^s(F)$  is unchanged if we consider only  $\delta$ -covers of closed sets.

WTS: Let  $F$  be any set of finite diameter. Then  $|F| = |\bar{F}|$ .

$$|F| = \sup \{d(x, y) : x, y \in F\}.$$

Because  $F \subset \bar{F}$ ,  $|F| \leq |\bar{F}|$ . Hence we need only show that  $|F| \geq |\bar{F}|$ .

Assume  $|F| < |\bar{F}|$ . Then  $\exists x, y \in \bar{F}$  such that  $|F| < d(x, y) \leq |\bar{F}|$ .

• suppose  $x, y$  are both limit points of  $F$ .

Then if  $\epsilon = d(x, y) - |F|$ , then  $\exists x', y' \in F$  such that  $0 < d(x, x') < \frac{\epsilon}{2}$  and  $0 < d(y, y') < \frac{\epsilon}{2}$ .

$$\therefore d(x, y) \leq d(x, x') + d(x', y') + d(y', y) < d(x', y') + \varepsilon$$

$$\therefore \underbrace{d(x, y)}_{=|F|} - \varepsilon < d(x', y'), \text{ so } \cancel{\cancel{x}}.$$

Now, since  $\{\text{closed } \delta\text{-conv}\} \subset C_\delta(\mathbb{E})$ ,  $\inf \{\sum |C_i|^s : \{C_i\} \in \{\text{closed}\}\} \geq \mathcal{H}_\delta^s(\mathbb{E})$ .

But since  $\{\overline{U_i}\} \in \{\text{closed}\}$ , it must be that  $\text{altered } \mathcal{H}_\delta^s(\mathbb{E}) = \mathcal{H}_\delta^s(\mathbb{E})$ .

$$\therefore \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(\mathbb{E}) = \lim_{\delta \rightarrow 0} \text{altered } \mathcal{H}_\delta^s(\mathbb{E}).$$

(2) Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be  $C^1$ . Show that  $\forall F \subset \mathbb{R}$ ,  $\dim_H f(F) \leq \dim_H(F)$ .

Suppose  $F$  is bounded (i.e. suppose  $|x| \leq M \quad \forall x \in F$ ).

$$\forall \varepsilon > 0, \exists \delta \text{ s.t. } |x-y| < \delta \Rightarrow |f'(x) - f'(y)| < \varepsilon.$$

WTS:  $\forall s$  such that  $\mathcal{H}_\delta^s(F) = 0$ , then  $\dim_H f(F) \leq s$ .

$$\dim_H f(F) = \inf \{s : \mathcal{H}_\delta^s(f(F)) = 0\}.$$

## Alternative definitions of dimension.

Most definitions of dimension use "measurement at scale  $\delta$ "

↪ measure at  $\delta$  in such a way that ignores irregularities of size  $< \delta$ .

Then see how measurement behaves as  $\delta \rightarrow 0$ .

\* natural scaling behavior for dimension.

Desirable properties for dimension.

(1) Monotonicity

(2)  $\dim_H(E \cup F) = \max(\dim_H E, \dim_H F)$ , or  $\dim_H(\bigcup F_i) = \sup \dim_H(F_i)$

(3)  $\dim_H f(E) = \dim_H E$  if  $f$  is geometrically invariant (?) } like translation, rotation, scaling, etc.

(4)  $\dim_H f(E) = \dim_H E$  if  $f$  is bi-Lipschitz. } Lipschitz invariance  $\Rightarrow$  geometric invariance

(5)  $\dim_H F = 0$  if  $F$  is countable.

(6)  $\dim_H F = n$  if  $F \subset \mathbb{R}^n$  is open.

(7)  $\dim_H F = m$  if  $F$  is smooth  $m$ -dimensional manifold.

## Box-counting dimension

Def. Let  $\phi \neq F \subset \mathbb{R}^n$  be bounded. Let  $N_\delta(F) =$  smallest # of sets of diameter at most  $\delta$  that can cover  $F$ .

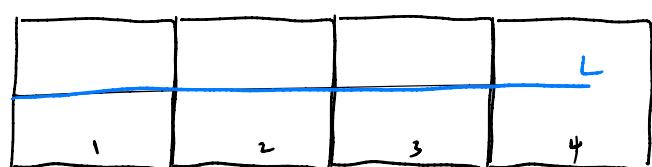
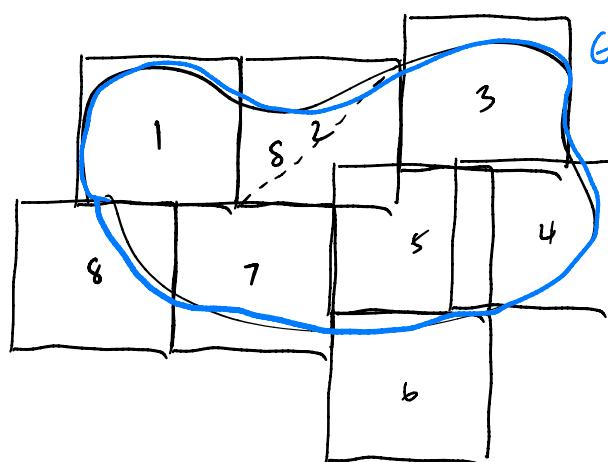
Then lower, upper box-counting dimensions of  $F$  are

$$\underline{\dim}_B F = \liminf_{\delta \rightarrow 0} \frac{\log N_\delta(F)}{-\log \delta}, \quad \overline{\dim}_B F = \limsup_{\delta \rightarrow 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

some kind of scaling law  
with  $N_\delta(F)$ ? As  $\delta$  gets  
smaller,  $N_\delta(F)$  could increase  
with power...

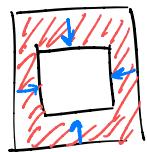
If  $\underline{\dim}_B F = \overline{\dim}_B F$ , then box-dimension of  $F$  is  $\dim_B F = \underline{\dim}_B F = \overline{\dim}_B F$ .

Comparing how measurements scale as  $\delta \rightarrow 0$ .

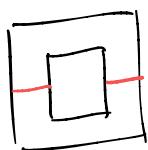


Intuitively, if you have line  $L \subset \mathbb{R}^2$  and open set  $G \subset \mathbb{R}^2$ , then  $N_\delta(G)$  should increase much faster than  $N_\delta(L)$  since decreasing the size of the boxes for  $G$  means you are losing more area.

e.g.



$\square$  = loss in area.



$\square$  = loss in length

} you are losing "more" ground for top  
than bottom by reducing size.

$\therefore$  makes sense that dimension of  $G$  greater than  $L$ 's dimension.

Remark Equivalent definitions of box dimension.

$N_S(F)$  based on sets:

- (i) closed balls of radius  $S$ ,
- (ii) arbitrary cubes (rectangles) of side length  $S$
- (iii) mesh cubes of side length  $S$
- (iv) sets of diameter  $S$  (main def.)
- (v) disjoint balls of radius  $S$  with centers in  $F$ .

Proof (iii) = (iv),

Suppose we have a mesh of size  $S$  for  $\mathbb{R}^n$ .

Then let  $N_S^1(F) = \text{smallest } \# \text{ of mesh } S\text{-cubes that intersect } F$ .

$\therefore N_{S^n}(F) \leq N_S^1(F)$  (using def. that  $N_{S^n}(F) = \text{smallest } \# \text{ of sets of diam. } S^n \text{ that cover } F$ )

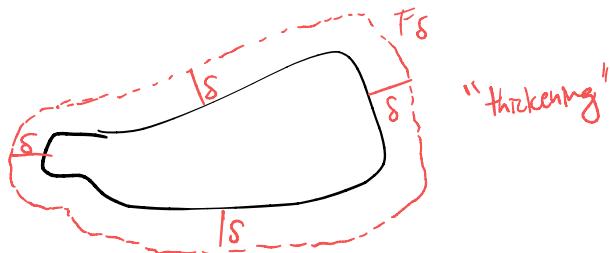
$$\therefore \text{if } S^n < 1, \text{ then } \underbrace{\frac{\log N_{S^n}(F)}{-\log(S^n)}}_A \leq \underbrace{\frac{\log N_S^1(F)}{-\log(S^n)}}_B.$$

$$\therefore \varliminf_{S \rightarrow 0} A = \varliminf_{S \rightarrow 0} B \stackrel{(1)}{\leq} \varlimsup_{S \rightarrow 0} A = \varlimsup_{S \rightarrow 0} F \stackrel{(2)}{\geq} \varlimsup_{S \rightarrow 0} B.$$

Also,  $N_S^1(F) \leq 3^n N_S(F)$ , so this gives opposite ineq. to (1), (2).  $\Rightarrow$  (iii) = (iv).

Def. The  $S$ -parallel body of  $F$  is defined as

$$F_S = \{x : |x-y| \leq S \text{ for some } y \in F\}.$$



Prop. If  $F \subset \mathbb{R}^n$ , then

$$\underline{\dim}_B F = n - \overline{\lim}_{\delta \rightarrow 0} \frac{\log \text{vol}^n(F_\delta)}{-\log \delta}, \quad \overline{\dim}_B F = n - \lim_{\delta \rightarrow 0} \frac{\log \text{vol}^n(F_\delta)}{-\log \delta}.$$

Proof: The intuition is that we observe the rate at which the  $n$ -dimensional volume ( $\text{vol}^n$ ) of  $F_\delta$  shrinks as  $\delta \rightarrow 0$ .

If  $F$  can be covered by  $N_\delta(F)$  balls of radius  $\delta$ , then  $F_\delta$  can be covered by the concentric balls of radius  $2\delta$ .

$$\therefore \text{vol}^n(F_\delta) \leq N_\delta(F) \cdot C_n \cdot (2\delta)^n.$$

$$\because \text{Given } \delta < 1, \quad \frac{\log \text{vol}^n(F_\delta)}{-\log \delta} \leq \frac{\log N_\delta(F) + \log 2^n C_n + n \log \delta}{-\log \delta}.$$

$$\therefore \lim_{\delta \rightarrow 0} \frac{\log \text{vol}^n(F_\delta)}{-\log \delta} \leq -n + \underline{\dim}_B F, \text{ since } \lim_{\delta \rightarrow 0} \frac{\log 2^n C_n}{-\log \delta} = 0.$$

If you instead switch out the definition of  $N_\delta(\bar{F})$  for disjoint balls, then  $N_\delta(F) \cdot C_n \cdot (2\delta)^n \leq \text{vol}^n(F_\delta)$ . (this is because of the new overlap) between balls.

Prop. (Relationship between Box & Hausdorff)

$$\dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F \text{ for any } F \subset \mathbb{R}^n.$$

Proof. By definition,  $\mathcal{H}_\delta^s(F) \leq N_\delta(F) \cdot \delta^s$  for any  $\delta > 0$ .

Since  $\mathcal{H}^s(F) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(F)$ , we have the following 2 cases:

If  $\mathcal{H}^s(F) > 1$ , then  $\exists r$  such that if  $\delta < r$ , then  $\mathcal{H}_\delta^s(F) > 1$  also.

$$\therefore 1 < \mathcal{H}_\delta^s(F) \leq N_\delta(F) \cdot \delta^s \text{ for } \delta < r, \Rightarrow 0 < \log N_\delta(F) + s \log \delta.$$

$$\therefore s < \lim_{\delta \rightarrow 0} \frac{\log N_\delta(F)}{-\log \delta} = \dim_H F \quad (\star)$$

Note that when  $\delta < r$ ,  $(\star)$  follows for any  $s$  such that  $\mathcal{H}^s(F) > 1$ .

$$\therefore \text{Given that } \dim_H F = \sup \{s : \mathcal{H}^s(F) = \infty\} = \inf \{s : \mathcal{H}^s(F) = 0\},$$

and  $\dim_B(F)$  is an upperbound of  $\{s : \mathcal{H}^s(F) = \infty\}$  by  $(\star)$ , we have  $\dim_H F \leq \dim_B F \leq \overline{\dim}_B F$ .

Notice that we need not be concerned with the case  $\mathcal{H}^s(E) \leq 1$  since we have abstracted out to the dimension of  $F$ . □

Intuition.  $\mathcal{H}$  takes into account covering sets of various sizes.

But for box dimension, you would only use covering sets of maximal size  $\delta$  since you want the smallest # of sets required to cover  $F$ .

You have substantially more flexibility w/  $\mathcal{H}$ , which is evident in the definition

$$N_\delta(F) \cdot \delta^s = \inf \left\{ \sum \delta^s : \{U_i\} \text{ is a finite } \delta\text{-cover} \right\}.$$

$$\mathcal{H}_\delta(F) = \inf \left\{ \sum |U_i|^s : \{U_i\} \in C_\delta(F) \right\}.$$

Example. Set  $C = \text{Cantor set.}$

$$\text{Then } \underline{\dim}_B C = \overline{\dim}_B C = \frac{\log 2}{\log 3}.$$

Proof Can be covered by  $2^k$  intervals of size  $3^k < \delta \leq 3^{k+1}$ .

$$\therefore \overline{\dim}_B C = \overline{\lim}_{\delta \rightarrow 0} \frac{\log N_\delta(C)}{-\log \delta} \leq \overline{\lim}_{k \rightarrow \infty} \frac{k \log 2}{(k+1) \log 3} = \frac{\log 2}{\log 3}.$$

Also for  $3^{k+1} \leq \delta < 3^k$ , any  $\delta$ -interval intersects with basic interval of length  $3^{-k}$ .

$$\therefore 2^k \leq N_\delta(C), \text{ so } \underline{\lim}_{\delta \rightarrow 0} \frac{\log N_\delta(C)}{-\log \delta} \geq \underline{\lim}_{k \rightarrow \infty} \frac{k \log 2}{(k+1) \log 3} = \frac{\log 2}{\log 3}.$$

$$\therefore \frac{\log 2}{\log 3} \leq \underline{\dim}_B C \leq \overline{\dim}_B C \leq \frac{\log 2}{\log 3}, \text{ so } \underline{\dim}_B C = \frac{\log 2}{\log 3}.$$

Prop. (Nice stuff abt. Box-dim.)

- (i) If  $F$  is a smooth  $m$ -dimensional manifold in  $\mathbb{R}^n$ , then  $\underline{\dim}_B F = m$ .
- (ii)  $\underline{\dim}_B F$ ,  $\overline{\dim}_B F$  are both monotonic (i.e.  $E \subset F \Rightarrow \underline{\dim}_B E \leq \underline{\dim}_B F$ ).
- (iii)  $\overline{\dim}_B(E \cup F) = \max \{ \overline{\dim}_B E, \overline{\dim}_B F \}$ . (but  $\underline{\dim}_B$  is NOT finitely stable)
- (iv)  $\underline{\dim}_B$ ,  $\overline{\dim}_B$  are both Lipschitz invariant.  
i.e.  $\underline{\dim}_B f(E) = \underline{\dim}_B F$  if  $c_1|x-y| \leq |f(x) - f(y)| \leq c_2|x-y| \quad \forall x, y \in F$ .

Proof. (only (iii), (iv))

(iii) Suppose  $\max\{\dim_B E, \dim_B F\} = \dim_B E$ .

Clearly  $N_\delta(E \cup F) \geq N_\delta(E)$ ,  $\forall \delta > 0$ .

$$\therefore \frac{\log N_\delta(E)}{-\log \delta} \leq \frac{\log N_\delta(E \cup F)}{-\log \delta} \implies \overline{\dim}_B(E) \leq \overline{\dim}_B(E \cup F)$$

Also  $N_\delta(E \cup F) \leq N_\delta(E) + N_\delta(F) \leq 2N_\delta(E)$

$$\therefore \frac{\log N_\delta(E \cup F)}{-\log \delta} \leq \frac{\log 2 + \log N_\delta(E)}{-\log \delta}$$

$$\therefore \lim_{\delta \rightarrow 0} \frac{\log N_\delta(E \cup F)}{-\log \delta} \leq 0 + \lim_{\delta \rightarrow 0} \frac{\log N_\delta(E)}{-\log \delta} \implies \overline{\dim}_B(E \cup F) \leq \overline{\dim}_B(E).$$

$$\therefore \overline{\dim}_B(E \cup F) = \dim_B(E) = \max\{\dim_B E, \dim_B F\}.$$

(iv). If  $c_1|x-y| \leq |f(x)-f(y)| \leq c_2|x-y|$ , then we know that for any finite  $\delta$ -cover  $\{U_i\}$ ,  $\{cU_i\}$  is sufficient to cover  $f(F)$ , given  $c_1 \leq c \leq c_2$ .

$$\therefore N_\delta(F) \geq N_{c\delta}(f(F)).$$

$$\begin{aligned} \therefore \log N_\delta(F) &\geq \log N_{c\delta}(f(F)) \implies \frac{\log N_\delta(F)}{-\log \delta} \geq \frac{\log N_{c\delta}(f(F))}{-\log \delta} \\ &\implies \overline{\dim}_B(F) \geq \overline{\dim}_B(f(F)). \end{aligned}$$

Applying the same proof for  $f^{-1}$  gives the other inequality.