Measure theory review:

Def. measure = a set function (real-valued) s.t.

- $\mu(\phi) = 0$
- $\mu(A) \le \mu(B)$ if $A \subset B$
- $\mu(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \mu(A_i)$.

Def. measure for $A \subset \mathbb{R}^n$ s.t. $0 < \mu(A) < \infty$ will be called a mass distribution.

Def. (book-specific)

A function $f: D \to \mathbb{R}$ is measurable if $\forall a \in \mathbb{R}$, $\{x \in D \mid f(x) \le a\}$ is measurable.
(D is a Borel set)

Def. We call a collection F of subsets of a sample space Ω an event space if $\Omega \setminus A \in F$ when $A \in F$.

Def. (probability measure)

Let F be an event space in Ω.
Then $P: F \to \mathbb{R}$ is called a probability measure if

(1) $0 \le P(A) \le 1 \quad \forall A \in F$.
(2) $P(\emptyset) = 0$, $P(\Omega) = 1$ (note: ϕ, Ω are both in F)
(3) $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ if $A_1, A_2, \ldots$ are pairwise disjoint.

Def. (probability space)

(Ω, F, P) is a probability space if F is an event space of subsets of Ω, and P is a probability measure defined on sets in F.

Def. (random variable)

Let (Ω, F, P) be a probability space.
We say X is a random variable on (Ω, F, P) if $X: \Omega \to \mathbb{R}$ st. $\forall a \in \mathbb{R}$, $\{A \in \Omega \mid X(A) \le a\} \in F$.

So basically $\forall a \in \mathbb{R}$, for any $A \in \Omega$ st. $X(A) \le a$, $A \in F$.
Does that agree with the intuition?
Suppose X is the number of heads in 10 coin flips.
Then $\Omega = \{HHH \cdots H, \ldots, T \cdots T\}$, and X maps each thing in Ω to a real number.

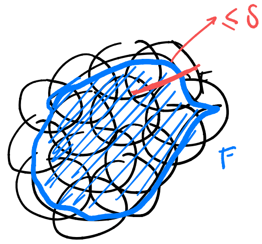
So basically a random variable is smth that maps from sample space $\longrightarrow \mathbb{R}$, with some constraints for the CDF??

Hausdorff measure. (assume $\mathbb{R}^n$)

Def. (diameter) Let nonempty $U \subset \mathbb{R}^n$. Then $|U| = \sup\{|x-y| : x, y \in U\}$.

Def. (δ-cover) Suppose $F \subset \bigcup_{i=1}^{\infty} U_i$, where $\forall i$, $|U_i| \leq \delta$.
Then $\{U_i\}_{i=1}^{\infty}$ is called a δ-cover of F ($\{U_i\}$ can be finite).

defined for any $F \subset \mathbb{R}^n$, since $\mathbb{R}^n$ is separable.



Def. ($\mathcal{H}^s_\delta(F)$) Let $s \geq 0$, and $F \subset \mathbb{R}^n$.
Then $\forall \delta > 0$, define $\mathcal{H}^s_\delta(F) = \inf\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\}$.

Denote $C_\delta(F) = \{\delta\text{-covers of } F\}$.

As δ decreases, the set $\{\delta\text{-covers of } F\}$ gets smaller and smaller.
This means that if $\delta \leq \delta'$, then since $\{\delta\text{-covers}\} \subset \{\delta'\text{-covers}\}$, $\mathcal{H}^s_\delta(F) \geq \mathcal{H}^s_{\delta'}(F)$.

lower bound.

I know that $0 \leq \mathcal{H}^s_\delta$, so since $\{\sum |U_i|^s : \{U_i\} \in C_\delta(F)\} \subset \mathbb{R}$, this means $\mathcal{H}^s_\delta(F)$ must exist.

Def. (s-dimensional Hausdorff measure)

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F).$$

As shown above, $\mathcal{H}^s_\delta(F)$ is a monotonic sequence as $\delta \to 0$, so the limit exists, if it is the case that $\{\mathcal{H}^s_\delta(F)\}$ is bounded. If not, $\mathcal{H}^s(F)$ is allowed to be ∞.

Hausdorff measure:

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \in C_\delta(F)\right\}$$

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F).$$

Prop. The s-dimensional Hausdorff measure is a measure.

(i) $\mathcal{H}^s_\delta(\phi) = 0$, so $\mathcal{H}^s(\phi) = 0$.

(ii) Clearly $\mathcal{H}^s(F) \geq 0$ for any F.

(iii) $$\mathcal{H}^s\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{\delta \to 0} \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \in C_\delta\left(\bigcup_{i=1}^{\infty} E_i\right)\right\}$$

$$\overset{(\text{WTS})}{=} \lim_{\delta \to 0} \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \in C_\delta(E_i)\right\}$$

$$+ \\ \vdots \\ +$$

$$\lim_{\delta \to 0} \inf\left\{\sum |U_i|^s : \{U_i\} \in C_\delta(E)\right\} + \cdots$$

Consider $A \sqcup B$.

Then WTS: $\mathcal{H}^s(A \sqcup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B)$.

$\inf\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \in C_\delta(A \sqcup B)\} \in \mathbb{R}^+$, so it can equal ∞.

(Need to show that $\mathcal{H}^s$ is a measure)

Prop. Scaling follows intuition for $\mathcal{H}^s$.

i.e. If $F \subset \mathbb{R}^n$ and $\lambda > 0$, then $\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$,

where $\lambda F = \{\lambda x : x \in F\}$.

Proof Let $\{U_i\} \in C_\delta(F)$.

Then $\bigcup_{i=1}^{\infty} U_i \supset F$, and $|U_i| \leq \delta \ \forall i$. This means that if $\lambda x \in \lambda F$, then since $x \in U_i$ for some i, $\lambda x \in \lambda U_i$.

$\therefore \lambda F \subset \bigcup_{i=1}^{\infty} \lambda U_i$. Also WTS: $|\lambda U_i| = \lambda |U_i|$.

$$\begin{aligned}
|\lambda U_i| &= \sup\{d(x,y) : x,y \in \lambda U_i\} \\
&= \sup\{d(\lambda a, \lambda b) : \lambda a, \lambda b \in \lambda U_i\} \\
&= \sup\{\lambda d(a,b) : \lambda a, \lambda b \in \lambda U_i\} \\
&= \sup\{\lambda d(a,b) : a,b \in U_i\} = \lambda \sup\{d(a,b) : a,b \in U_i\} = \lambda |U_i|.
\end{aligned}$$

$\propto d(\lambda a, \lambda b) = \lambda d(a,b)$, (assuming Euclidean d)

$\therefore |\lambda U_i| = \lambda |U_i| \leq \lambda \delta$, so $\{\lambda U_i\} \in C_{\lambda\delta}(\lambda F)$.

$\therefore \mathcal{H}^s_{\lambda\delta}(\lambda F) \leq \sum_{i=1}^{\infty} |\lambda U_i|^s = \lambda^s \sum_{i=1}^{\infty} |U_i|^s$, where $\{U_i\} \in C_\delta(F)$.

$\therefore \mathcal{H}^s_{\lambda\delta}(\lambda F) \leq \lambda^s \mathcal{H}^s_\delta(F)$, since $\mathcal{H}^s_{\lambda\delta}(\lambda F)$ is a lower bound for $\{\lambda^s \sum |U_i|^s\}$.

Taking $\delta \to 0$, $\mathcal{H}^s(\lambda F) \leq \lambda^s \mathcal{H}^s(F)$.

By similar logic, $\mathcal{H}^s_{\frac{\delta}{\lambda}}(F) \leq \left(\frac{1}{\lambda}\right)^s \mathcal{H}^s_\delta(\lambda F) \Rightarrow \mathcal{H}^s_\delta(\lambda F) \geq \lambda^s \mathcal{H}^s_{\frac{\delta}{\lambda}}(F)$

$\therefore \mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$.

$\square$

Prop. Let $F \subset \mathbb{R}^n$, and suppose $f : F \to \mathbb{R}^m$ s.t. $|f(x) - f(y)| \leq c|x-y|^\alpha$ for all $x,y \in F$, and $\alpha, c > 0$.

Then $\mathcal{H}^{\frac{s}{\alpha}}(f(F)) \leq c^{\frac{s}{\alpha}} \mathcal{H}^s(F)$.

Prof. Let $\{U_i\} \in C_\delta(F)$.

Then $|U_i| \leq \delta \ \forall i$. Hence is $\{f(U_i)\} \in C_{\delta/\ldots}(f(F))$?

If $F \subset \bigcup_{i=1}^{\infty} U_i$, then $f(F) \subset \bigcup_{i=1}^{\infty} f(U_i)$.

Also $|f(x)-f(y)| \leq c|x-y|^\alpha \leq c\delta^\alpha$, which means that $\{f(U_i)\} \in C_{c\delta^\alpha}(f(F))$.

$$\therefore \mathcal{H}^{\frac{s}{\alpha}}_{c\delta^\alpha}(f(F)) \leq \sum |f(U_i)|^{\frac{s}{\alpha}} \leq \sum c^{s/\alpha}|U_i|^s = c^{s/\alpha}\sum |U_i|^s$$
$$\leq c^{s/\alpha}\mathcal{H}^s_\delta(F).$$

$\therefore \delta \to 0$ means $\mathcal{H}^{\frac{s}{\alpha}}(f(F)) \leq c^{\frac{s}{\alpha}}\mathcal{H}^s(F)$.

This makes intuitive sense.

$|f(x)-f(y)| \leq c|x-y|^\alpha$ means that the distance b/n $f(x)$ and $f(y)$ is less than a multiple of $|x-y|^\alpha$, so the size of F $\left(\mathcal{H}^{\frac{s}{\alpha}}(f(F))\right)$ should be less than the scaled size of F $\left(c^{\frac{s}{\alpha}}\mathcal{H}^s(F)\right)$.

Def. (Hausdorff dimension)

Let $\dim_H(F) = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$ s.t.

$$\mathcal{H}^s(F) = \begin{cases} 0 & \text{if } s > \dim_H F \\ \infty & \text{if } s < \dim_H F. \end{cases}$$

$\dim_H(F)$ can be $0, \infty$, or between 0 and ∞.

The motivation is that for $\delta < 1$, as $s \to \infty$, $\mathcal{H}^s_\delta(F)$ must be non-increasing.

$\therefore \mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F)$ must also be non-increasing as $s \to \infty$.

In fact, if $t > s$, and $\delta < 1$, then for any δ-cover $\{U_i\}$,

$$\sum_i |U_i|^t \leq \delta^{t-s}\sum_i |U_i|^s, \quad \text{so} \quad \mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F)$$

This means that as $\delta \to 0$, if $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$.

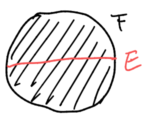
Prop. (i) $E \subset F \Rightarrow \dim_H E \leq \dim_H F$.

(ii) $\dim_H\left(\bigcup_{i=1}^{\infty} F_i\right) = \sup_i \{\dim_H F_i\}$.

(iii) If F is countable, then $\dim_H F = 0$.

Proof. (i) for any s, $\mathcal{H}^s(E) \leq \mathcal{H}^s(F)$.

$\therefore$ since $\dim_H(E) = \inf\{s : \mathcal{H}^s(E) = 0\} \leq \inf\{s : \mathcal{H}^s(F) = 0\} = \dim_H(F)$.

Also : 

$\Big\}\ \dim_H(E) = 1, \quad \dim_H(F) = 2.$

(ii) By (i), $\dim_H F_i \leq \dim_H(\bigcup F_i)$ $\forall i$.

Suppose $\exists s$ such that $\dim_H F_i \leq s$ $\forall i$ and $s < \dim_H(\bigcup F_i)$.

Then $\mathcal{H}^s(F_i) = 0$ $\forall i$, so $\mathcal{H}^s(\bigcup F_i) \leq \sum_i \mathcal{H}^s(F_i) = 0$, so $\mathcal{H}^s(\bigcup F_i) = 0$.

$\therefore \dim_H(\bigcup F_i) \leq s = \inf\{s : \mathcal{H}^s(\bigcup F_i) = 0\}$.

(iii) Suppose $F = \bigcup F_i$, where F_i are single points.

Then $\mathcal{H}^0(F_i) = 1$, and $\mathcal{H}^t(F_i) = 0$ for $t > 0$, so $\dim_H(F_i) = 0$.

$\therefore \dim_H(\bigcup F_i) = 0$ by (ii).

Prop. Let $F \subset \mathbb{R}^n$, and $f : F \to \mathbb{R}^m$ satisfy $|f(x) - f(y)| \leq c|x-y|^\alpha$ for $\alpha > 0$.

Then $\dim_H f(F) \leq \dfrac{\dim_H F}{\alpha}$.

Proof We know $\mathcal{H}^{\frac{s}{\alpha}}(f(F)) \leq c^{\frac{s}{\alpha}} \mathcal{H}^s(F)$.

Want to show that $\mathcal{H}^{\frac{\dim_H F}{\alpha}}(f(F)) = 0$. $\quad \mathcal{H}^{\frac{s}{\alpha}}(f(F)) \leq c^{\frac{s}{\alpha}} \mathcal{H}^s(F)$

$\therefore \mathcal{H}^{\frac{s}{\alpha}}(f(F)) \leq c^{\frac{s}{\alpha}} \mathcal{H}^s(F)$

if $s > \dim_H(F)$, then $\mathcal{H}^{\frac{s}{\alpha}}(f(E)) = 0$ also.

$\therefore t > \dfrac{\dim_H(F)}{\alpha} \Rightarrow \dim_H(f(F)) \leq \dfrac{\dim_H(F)}{\alpha}$.

Corollary (i) if $f : F \to \mathbb{R}^m$ is Lipschitz (i.e. $|f(x) - f(y)| \leq c|x-y|$),

then $\dim_H(f(F)) \leq \dim_H(F)$

(ii) if $f : F \to \mathbb{R}^m$ is bi-Lipschitz (i.e. $c_1|x-y| \leq |f(x) - f(y)| \leq c_2|x-y|$, where $0 < c_1 \leq c_2 < \infty$),

then $\dim_H f(F) = \dim_H F$.

* Hausdorff dimension is invariant under bi-Lipschitz transformations.

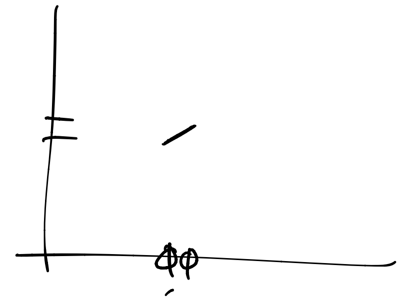
* if 2 sets have diff dimension, then $\nexists$ bi-Lipschitz transformation.

Proof. (i) clear from $\dim_H f(F) \leq \frac{\dim_H F}{\alpha = 1} = \dim_H F$.

(ii) $\dim_H f(F) \leq \dim_H F$ by the 2nd half of inequality.
$\dim_H f^{-1}(f(F)) = \dim_H F \leq \dim_H f(F)$ by 1st half. } $\dim_H f(F) = \dim_H F$.

Def. (homeomorphism) continuous, 1-1 mapping with continuous inverse.

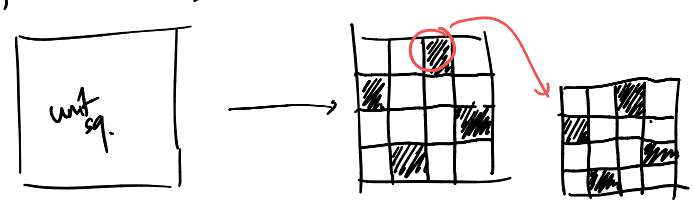
* can view two sets as being the same if $\exists$ bi-Lipschitz mapping b/w them.
is it true that if $\dim_H F = \dim_H E$, then $\exists$ bi-Lipschitz mapping $E \to F$??



Prop. A set $F \subset \mathbb{R}^n$ with $\dim_H F < 1$ is totally disconnected, i.e. no two points lie in the same connected component.

(Proof skipped) imagine if F is only a couple of points.
Then if $\dim_H F < 1$, then this means (from intuition) that you cannot use the points of F to draw a line.
$\therefore$ no points can lie on a connected component of F i.e. it's totally disconnected.

Example. (Cantor dust)



Call F = Cantor dust set. $\dim_H F = ?$

Claim: $1 \leq \mathcal{H}^1(F) < \sqrt{2}$, so $\dim_H F = 1$.

Consider $\delta > 0$. Then $\mathcal{H}^1_\delta(F) = \{ \sum |U_i| : \{U_i\} \in C_\delta(F)\}$.

If $\delta = \sqrt{2}$, then the unit square suffices, right?

$\therefore \mathcal{H}^1_\delta(F) \leq \sqrt{2}$

If $\delta = \frac{\sqrt{2}}{4}$, then $\mathcal{H}^1_\delta(F) \leq \sqrt{2}$ also.

Basically as $\delta \to 0$, $\mathcal{H}^1_\delta(F) \leq \sqrt{2}$, so $\mathcal{H}^1(F) \leq \sqrt{2}$ also.

For the other side, consider the projection to the x-axis (i.e. $(x,y) \to x$).
Then since $|\mathrm{proj}(x) - \mathrm{proj}(y)| \leq |x-y|$, proj is Lipschitz i.e. $\dim_H \mathrm{proj}(F) \leq \dim_H F$.
$\therefore \mathrm{proj}(F) = [0,1]$, so $\mathcal{H}^1([0,1]) = \mathcal{H}^1(\mathrm{proj}(F)) = 1 \leq \mathcal{H}^1(F)$.

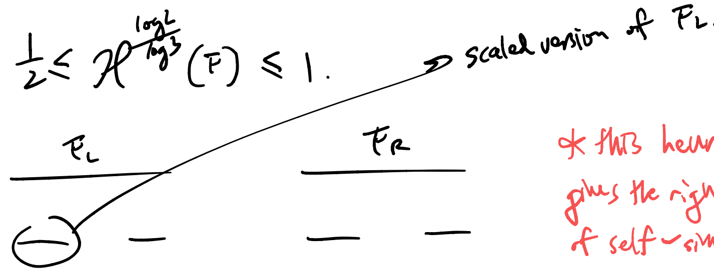
$\therefore \; 1 \leq \mathcal{H}^1(F) \leq \sqrt{2}.$

$\therefore \dim_H F = 1.$

Example. Cantor set $= F$.

$\dim_H F = \frac{\log 2}{\log 3}$, and $\frac{1}{2} \leq \mathcal{H}^{\frac{\log 2}{\log 3}}(F) \leq 1.$

Heuristic calculation.



* this heuristic method usually gives the right answer for a lot of self-similar sets.
↪ applies b/c of the scaling property.

$$\therefore \mathcal{H}^s(F) = \frac{1}{3}^s \mathcal{H}^s(F) + \frac{1}{3}^s \mathcal{H}^s(F)$$

Assuming $0 < \mathcal{H}^s(F) < \infty$, we have $1 = 2\left(\frac{1}{3}\right)^s \Rightarrow \frac{1}{2} = \left(\frac{1}{3}\right)^s \Rightarrow 3^s = 2 \Rightarrow s = \log_3 2.$

$\therefore s = \frac{\log 2}{\log 3}$ by change of base.

Rigorous calculation

Suppose $s = \frac{\log 2}{\log 3}$. Then we can cover F with 2^k intervals of length 3^{-k}.

This means that $\mathcal{H}^s_{3^{-k}}(F) \leq 2^k 3^{-ks} = 2^k 3^{-k} = 1.$

To show that $\mathcal{H}^s(F) \geq \frac{1}{2}$, we show that $\sum |U_i|^s \geq \frac{1}{2}$ for any cover $\{U_i\}$.

Let $\{U_i\}$ be a cover of F. We can assume WLOG that $\{U_i\}$ consists of bounded intervals.

$\therefore$ by extending $\{U_i\}$, we obtain an open cover of F.

$\therefore$ since F is closed & bounded, it is compact.

$\therefore$ we only need be concerned with finite covers (b/c of compactness).

Suppose k is such that $3^{-(k+1)} \leq |U_i| < 3^{k}$.

Then if $j \geq k$, then U_i intersects at most $2^{j-k} = 2^j 3^{-sk} \leq 2^j 3^s |U_i|^s$ basic intervals (basic intervals being intervals of length 3^{-m}).

$\therefore$ For j sufficiently large, we have $2^j \leq 2^j 3^s \sum_i |U_i|^s$, so $1 \leq 3^s \sum_i |U_i|^s \Rightarrow \sum_i |U_i|^s \geq 3^{-s} = \frac{1}{2}.$

Remark.

Suppose $B^s_\delta(F) = \inf\{\sum_i |B_i|^s : \{B_i\} \text{ is a } \delta\text{-cover of } F \text{ by balls}\}$.

Clearly $\mathcal{H}^s_\delta(F) \leq B^s_\delta(F)$. $\longrightarrow$ $\mathcal{H}^s(F) \leq B^s(F)$

Also if $\{U_i\} \in C_\delta(F)$, then $\{B_i\}$ such that $B_i \supset U_i$ gives $\sum |B_i|^s \leq \sum |2U_i|^s \leq 2^s \sum |U_i|^s$.

$\therefore$ $B^s_{2\delta}(F) \leq 2^s \mathcal{H}^s_\delta(F)$.

$\therefore$ $B^s(F) \leq 2^s \mathcal{H}^s(F)$ $\longrightarrow$ $\mathcal{H}^s(F) \leq B^s(F) \leq 2^s \mathcal{H}^s(F)$.

This means that if $s = \dim_H F = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$, then using $B^s(F)$ gives the same s.

i.e. $\inf\{s : \mathcal{H}^s(F) = 0\} = \inf\{s : B^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\} = \sup\{s : B^s(F) = \infty\}$.

Finer notions of dimension.

Suppose $h : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing, cts function. (called a dimension function)

Define $\mathcal{H}^h_\delta(F) = \inf\{\sum_i h(|U_i|) : \{U_i\} \in C_\delta(F)\}$. (normally, $h(t) = t^s$).

If h, g are dimension functions such that $\frac{h}{g} \to 0$ as $t \to 0$, then it can be shown that $\mathcal{H}^h(F) = 0$ if $\mathcal{H}^g(F) < \infty$.

(analogue: if $t > s$, and $\mathcal{H}^t(F) < \infty$, then $\mathcal{H}^s(F) = 0$)

Exercises

(1) Show that the value of $\mathcal{H}^s(F)$ is unchanged if we consider only δ-covers of closed sets.

WTS: Let F be any set of finite diameter. Then $|F| = |\bar{F}|$.

$|\bar{F}| = \sup\{d(x,y) : x, y \in \bar{F}\}$.

Because $F \subset \bar{F}$, $|F| \leq |\bar{F}|$. Hence we need only show that $|F| \not< |\bar{F}|$.

Assume $|F| < |\bar{F}|$. Then $\exists x, y \in \bar{F}$ such that $|F| < d(x,y) \leq |\bar{F}|$.

- suppose x, y are both limit points of F.

Then if $\epsilon = d(x,y) - |F|$, then $\exists x', y' \in F$ such that $0 < d(x,x') < \frac{\epsilon}{2}$ and $0 < d(y,y') < \frac{\epsilon}{2}$.

$\therefore\ d(x,y) \leq d(x,x') + d(x',y') + d(y',y) < d(x',y') + \varepsilon$

$\therefore\ \underbrace{d(x,y)}_{=|F|} - \varepsilon < d(x',y')$, so $\times$.

Now, since $\{\text{closed } \delta\text{-cover}\} \subset C_\delta(F)$, $\inf\{\sum |C_i|^s : \{C_i\} \in \{\text{closed}\}\} \geq \mathcal{H}^s_\delta(F)$.

But since $\{\overline{U_i}\} \in \{\text{closed}\}$, it must be that altered $\mathcal{H}^s_\delta(F) = \mathcal{H}^s_\delta(F)$.

$\therefore \lim_{\delta \to 0} \mathcal{H}^s_\delta(F) = \lim_{\delta \to 0}$ altered $\mathcal{H}^s_\delta(F)$.

(2) Let $f: \mathbb{R} \to \mathbb{R}$ be C^1. Show that $\forall F \subset \mathbb{R}$, $\dim_H f(F) \leq \dim_H(F)$.

Suppose F is bounded (i.e. suppose $|x| \leq M \quad \forall x \in F$).

$\forall \varepsilon > 0$, $\exists \delta$ s.t. $|x-y| < \delta \Rightarrow |f'(x) - f'(y)| < \varepsilon$.

WTS: $\forall s$ such that $\mathcal{H}^s(F) = 0$, then $\dim_H f(F) \leq s$.

$\dim_H f(F) = \inf\{s : \mathcal{H}^s(f(F)) = 0\}$.

Alternative definitions of dimension.

Most definition of dimension use "measurement at scale δ"

↳ measure at in such a way that ignores irregularities of size $< \delta$.

Then see how measurement behaves as $\delta \to 0$.

* natural scaling behavior for dimension.

Desirable properties for dimension.

(1) Monotonicity

(2) $\dim_H(E \cup F) = \max(\dim_H E, \dim_H F)$, or $\dim_H(\bigcup F_i) = \sup \dim_H(F_i)$

(3) $\dim_H f(E) = \dim_H F$ if f is geometrically invariant (?) → like translations, rotations, scaling, etc.

(4) $\dim_H f(F) = \dim_H F$ if f is bi-Lipschitz.

Lipschitz invariance ⇒ geometric invariance

(5) $\dim_H F = 0$ if F is countable.

(6) $\dim_H F = n$ if $F \subset \mathbb{R}^n$ is open.

(7) $\dim_H F = m$ if F is smooth m-dimensional manifold.

Box-counting dimension

Def. Let $\phi \neq F \subset \mathbb{R}^n$ be bounded. Let $N_\delta(F)$ = smallest # of sets of diameter at most δ that can cover F.

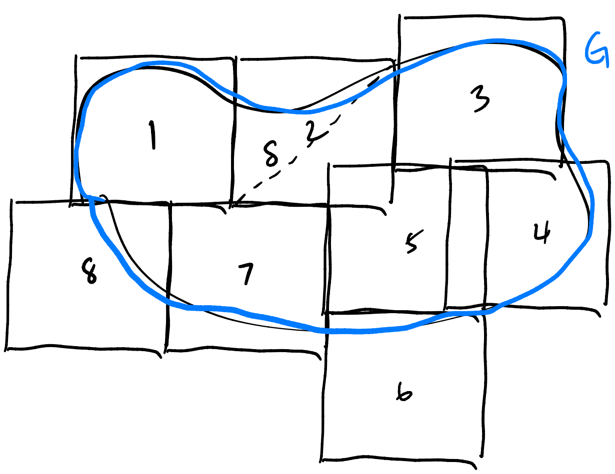
Then lower, upper box-counting dimensions of F are

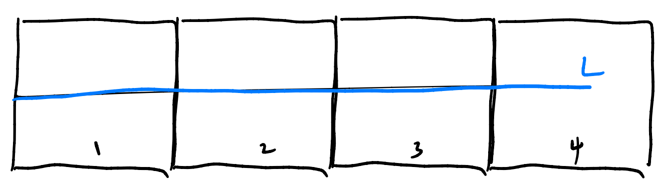
$$\underline{\dim}_B F = \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}, \quad \overline{\dim}_B F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

Some kind of scaling law with $N_\delta(F)$? As δ gets smaller, $N_\delta(F)$ could increase with power...

If $\underline{\dim}_B F = \overline{\dim}_B F$, then box-dimension of F is $\dim_B F = \underline{\dim}_B F = \overline{\dim}_B F$.

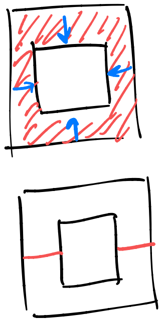
Comparing how measurements scale as $\delta \to 0$.





Intuitively, if you have line $L \subset \mathbb{R}^2$ and open set $G \subset \mathbb{R}^2$, then $N_\delta(G)$ should increase must faster than $N_\delta(L)$ since decreasing the size of the boxes for G means you are losing more area.

e.g.



▨ = loss in area.

⊟ = loss in length

you are losing "more" ground for top than bottom by reducing size.

$\therefore$ makes sense that dimension of G greater than L's dimension.

Remark Equivalent definitions of box dimension.

$N_\delta(F)$ based on sets:

(i) closed balls of radius δ,

(ii) arbitrary cubes (rectangles) of side length δ

(iii) mesh cubes of side length δ

(iv) sets of diameter δ (main def)

(v) disjoint balls of radius δ with centers in F.

Proof (iii) = (iv),

Suppose we have a mesh of size δ for $\mathbb{R}^n$.

Then let $N'_\delta(F)$ = smallest # of mesh δ-cubes that intersect F.

$\therefore N_{\delta\sqrt{n}}(F) \leq N'_\delta(F)$ (using def. that $N_{\delta\sqrt{n}}(F)$ = smallest # of sets of diam. $\delta\sqrt{n}$ that cover F)

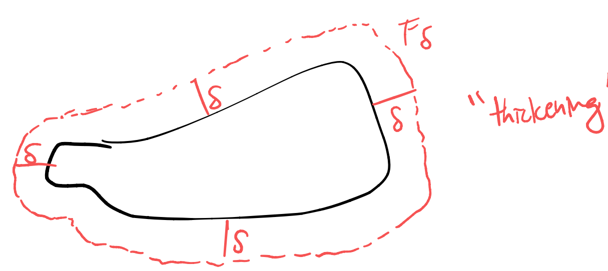
$\therefore$ if $\delta\sqrt{n} < 1$, then
$$\underbrace{\frac{\log N_{\delta\sqrt{n}}(F)}{-\log(\delta\sqrt{n})}}_{A} \leq \underbrace{\frac{\log N'_\delta(F)}{-\log(\delta\sqrt{n})}}_{B}.$$

$$\therefore \underline{\lim}_{\delta\to 0} A = \underline{\dim}_B F \overset{(1)}{\leq} \underline{\lim}_{\delta\to 0} B, \quad \overline{\lim}_{\delta\to 0} A = \overline{\dim}_B F \overset{(2)}{\geq} \overline{\lim}_{\delta\to 0} B.$$

Also, $N'_\delta(F) \leq 3^n N_\delta(F)$, so this gives opposite ineq. to (1), (2). $\Rightarrow$ (iii) = (iv).

Def. The δ-parallel body of F is defined as

$$F_\delta = \{x : |x-y| \leq \delta \text{ for some } y \in F\}.$$



Prop. If $F \subset \mathbb{R}^n$, then

$$\underline{\dim}_B F = n - \overline{\lim_{\delta \to 0}} \frac{\log \mathrm{vol}^n(F_\delta)}{\log \delta}, \quad \overline{\dim}_B F = n - \underline{\lim}_{\delta \to 0} \frac{\log \mathrm{vol}^n(F_\delta)}{\log \delta}.$$

Proof. The intuition is that we observe the rate at which the n-dimensional volume (vol^n) of F_δ shrinks as $\delta \to 0$.

If F can be covered by $N_\delta(F)$ balls of radius δ, then F_δ can be covered by the concentric balls of radius 2δ.

$\therefore \mathrm{vol}^n(F_\delta) \leq N_\delta(F) \cdot c_n \cdot (2\delta)^n$.

$\therefore$ Given $\delta < 1$, $\dfrac{\log \mathrm{vol}^n(F_\delta)}{-\log \delta} \leq \dfrac{\log N_\delta(F) + \log 2^n c_n + n \log \delta}{-\log \delta}$.

$\therefore \underline{\lim}_{\delta \to 0} \dfrac{\log \mathrm{vol}^n(F_\delta)}{-\log \delta} \leq -n + \underline{\dim}_B F$, since $\lim_{\delta \to 0} \dfrac{\log 2^n c_n}{-\log \delta} = 0$.

If you instead switch out the definition of $N_\delta(F)$ for disjoint balls, then $N_\delta(F)\, c_n \cdot (2\delta)^n \leq \mathrm{vol}^n(F_\delta)$.
(this is because of the new overlap) between balls.

Prop. (Relationship between Box & Hausdorff)

$\dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F$ for any $F \subset \mathbb{R}^n$.

Proof. By definition, $\mathcal{H}^s_\delta(F) \leq N_\delta(F) \cdot \delta^s$ for any $\delta > 0$.

Since $\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F)$, we have the following 2 cases:

If $\mathcal{H}^s(F) > 1$, then $\exists \gamma$ such that if $\delta < \gamma$, then $\mathcal{H}^s_\delta(F) > 1$ also.

$\therefore 1 < \mathcal{H}^s_\delta(F) \leq N_\delta(F) \cdot \delta^s$ for $\delta < \gamma$, so $0 < \log N_\delta(F) + s \log \delta$.

$\therefore s \leq \underline{\lim}_{\delta \to 0} \dfrac{\log N_\delta(F)}{-\log \delta} = \underline{\dim}_B F \quad (*)$

Note that when $\delta < \gamma$, $(*)$ follows for any s such that $\mathcal{H}^s(F) > 1$.

$\therefore$ Given that $\dim_H F = \sup\{s : \mathcal{H}^s(F) = \infty\} = \inf\{s : \mathcal{H}^s(F) = 0\}$,

and $\underline{\dim}_B(F)$ is an upper bound of $\{s : \mathcal{H}^s(F) = \infty\}$ by $(*)$, we have $\dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F$.

Notice that we need not be concerned with the case $\mathcal{H}^s(F) \leq 1$ since we have abstracted out to the dimension of F.

$\square$

Intuition. $\mathcal{H}$ takes into account covering sets of various sizes.

But for box dimension, you would only use covering sets of maximal size δ since you want the smallest # of sets required to cover F.

You have substantially more flexibility w/ $\mathcal{H}$, which is evident in the definition

$$N_\delta(F) \cdot \delta^s = \inf \{ \sum \delta^s : \{U_i\} \text{ is a finite } \delta\text{-cover} \}$$

$$\mathcal{H}^s_\delta(F) = \inf \{ \sum |U_i|^s : \{U_i\} \in C_\delta(F) \}.$$

Example. Set $C =$ Cantor set.

Then $\underline{\dim}_B C = \overline{\dim}_B C = \frac{\log 2}{\log 3}.$

Proof. Can be covered by 2^k intervals of size $3^{-k} < \delta \leq 3^{-k+1}$.

$$\therefore \overline{\dim}_B C = \overline{\lim_{\delta \to 0}} \frac{\log N_\delta(C)}{-\log \delta} \leq \overline{\lim_{k \to \infty}} \frac{k \log 2}{(k-1) \log 3} = \frac{\log 2}{\log 3}.$$

Also for $3^{-k-1} \leq \delta < 3^{-k}$, any δ-interval intersects with basic interval of length 3^{-k}.

$$\therefore 2^k \leq N_\delta(C), \text{ so } \underline{\lim_{\delta \to 0}} \frac{\log N_\delta(C)}{-\log \delta} \geq \underline{\lim_{k \to \infty}} \frac{k \log 2}{(k+1) \log 3} = \frac{\log 2}{\log 3}.$$

$$\therefore \frac{\log 2}{\log 3} \leq \underline{\dim}_B C \leq \overline{\dim}_B C \leq \frac{\log 2}{\log 3}, \text{ so } \dim_B C = \frac{\log 2}{\log 3}.$$

Prop. (Nice stuff abt. Box-dim.)

(i) If F is a smooth m-dimensional manifold in $\mathbb{R}^n$, then $\dim_B F = m$.

(ii) $\underline{\dim}_B F$, $\overline{\dim}_B F$ are both monotonic (i.e. $E \subset F \Rightarrow \underline{\dim}_B E \leq \overline{\dim}_B F$).

(iii) $\overline{\dim}_B (E \cup F) = \max\{ \overline{\dim}_B E, \overline{\dim}_B F \}$. (but $\underline{\dim}_B$ is NOT finitely stable)

(iv) $\underline{\dim}_B$, $\overline{\dim}_B$ are both Lipschitz invariant.

i.e. $\underline{\dim}_B f(E) = \underline{\dim}_B F$ if $c_1|x-y| \leq |f(x) - f(y)| \leq c_2|x-y| \quad \forall x,y \in F$.

<u>Proof.</u> (only (iii), (iv))

(iii) Suppose $\max\{\overline{\dim}_B E, \underline{\dim}_B F\} = \underline{\dim}_B E$.

Clearly $N_\delta(E \cup F) \geq N_\delta(E)$, $\forall \delta > 0$.

$$\therefore \frac{\log N_\delta(E)}{-\log \delta} \leq \frac{\log N_\delta(E\cup F)}{-\log \delta} \implies \overline{\dim}_B(E) \leq \overline{\dim}_B(E \cup F)$$

Also $N_\delta(E\cup F) \leq N_\delta(E) + N_\delta(F) \leq 2N_\delta(E)$

$$\therefore \frac{\log N_\delta(E\cup F)}{-\log\delta} \leq \frac{\log 2 + \log N_\delta(E)}{-\log \delta}$$

$$\therefore \overline{\lim_{\delta\to 0}} \frac{\log N_\delta(E\cup F)}{-\log\delta} \leq 0 + \overline{\lim_{\delta\to 0}} \frac{\log N_\delta(E)}{-\log\delta} \implies \overline{\dim}_B(E\cup F) \leq \overline{\dim}_B(E).$$

$\therefore \overline{\dim}_B(E\cup F) = \dim_B(E) = \max\{\overline{\dim}_B E, \overline{\dim}_B F\}$.

(iv). If $c_1|x-y| \leq |f(x)-f(y)| \leq c_2|x-y|$, then we know that for any finite δ-cover $\{U_i\}$, $\{cU_i\}$ is sufficient to cover $f(F)$, given $c_1 \leq c \leq c_2$.

$\therefore \quad N_\delta(F) \geq N_{c\delta}(f(F))$.

$$\therefore \quad \log N_\delta(F) \geq \log N_{c\delta}(f(F)) \implies \frac{\log N_\delta(F)}{-\log\delta} \geq \frac{\log N_{c\delta}(f(F))}{-\log \delta}$$

$$\implies \overline{\dim}_B(F) \geq \overline{\dim}_B(f(F)).$$

Applying the same proof for f^{-1} gives the other inequality.

<u>Prop.</u> $\underline{\dim}_B \overline{F} = \underline{\dim}_B F$, $\overline{\dim}_B \overline{F} = \overline{\dim}_B F$

WTS: $N_\delta(\overline{F}) = N_\delta(F)$, where N_δ is the least # of closed balls of radius δ that can cover F.

Suppose $\{U_i\}$ is a "closed δ-ball cover" of F.

Then $\{U_i\}$'s covers $\overline{F}$ also, so $N_\delta(F) = N_\delta(\overline{F})$.

Remark. The problem with this proposition is that countable sets can have nonzero dimension.

e.g. If $F = \mathbb{Q} \cap [0,1]$, then $\overline{F} = [0,1]$.

$\therefore \underline{\dim}_B F = \overline{\dim}_B = \underline{\dim}_B \overline{F} = \dim_B \overline{F} = 1$.

But F is really just a countable aggregate of singleton sets $\Rightarrow F = \bigcup_{x\in F}\{x\}$.

$\therefore$ countable stability doesn't hold i.e. $\underbrace{\dim_B F}_{=1} \neq \sup_{x\in F} \underbrace{\dim_B \{x\}}_{=0}$.

Def. Modified box dimension.

$$\underline{\dim}_{MB} F = \inf\left\{\sup_i \underline{\dim}_B F_i : F \subset \bigcup_{i=1}^{\infty} F_i\right\}, \quad \overline{\dim}_{MB} F = \inf\left\{\sup_i \overline{\dim}_B F_i : F \subset \bigcup_{i=1}^{\infty} F_i\right\}.$$

$\dim_{MB}$ = out of all possible covers of F, set the dimension as the smallest dimension of the largest dim.

Remark. Resolves the stability problem of normal box dimension.

① Observe that $\underline{\dim}_{MB} \le \underline{\dim}_B$ and $\overline{\dim}_{MB} \le \overline{\dim}_B$ (since F is a countable cover of F on its own, so $\underline{\dim}_B F \in \{\sup_i \underline{\dim}_B F_i\}$, $\overline{\dim}_B F \in \{\sup_i \overline{\dim}_B F_i\}$)

② If F is countable, then taking $F_i = \{x\}$ for $x \in F$, we re-claim the zero-dimension of countable sets.

Prop. Let $F \subset \mathbb{R}$ be compact. Suppose $\overline{\dim}_B (F \cap V) = \overline{\dim}_B F$ for all open V such that $V \cap F \neq \phi$.
Then $\overline{\dim}_B F = \overline{\dim}_{MB} F$, $\underline{\dim}_B F = \underline{\dim}_{MB} F$.

Proof. We need only show that $\overline{\dim}_{MB} \geq \overline{\dim}_B$. (since $\overline{\dim}_B \geq \overline{\dim}_{MB}$ by definition)

Let $F \subset \bigcup_{i=1}^{\infty} F_i$, where F_i are closed. By Baire, $\exists$ open V such that $F \cap V \subset F_i$ for some i.

$\therefore \overline{\dim}_B (F \cap V) = \overline{\dim}_B (F)$

$\therefore \overline{\dim}_B (F) \le \overline{\dim}_B (F_i)$ by monotonicity of $F \cap V \subset F_i$. Also $\boxed{\overline{\dim}_B (F_i) \le \overline{\dim}_B (F)}$ unclear; come back to this.

$\therefore \overline{\dim}_B (F) = \overline{\dim}_B (F_i)$

$\therefore \overline{\dim}_{MB} F = \inf\{\underbrace{\sup_i \overline{\dim}_B F_i}_{\geq \text{ that one } \overline{\dim}_B F_i = \overline{\dim}_B F} \mid \{F_i\}\text{-closed cover of } F\} \geq \overline{\dim}_B F$.

Remark. If F is a compact, self-similar set, then the condition $\overline{\dim}_B(F \cap V) = \overline{\dim}_B(F)$ means that for any open V, $F \cap V$ must contain a geometrically similar version of F inside itself.

Def. (packing)

- $\mathcal{P}_\delta^s(F) = \sup\{\sum_i |B_i|^s : \{B_i\}$ is a collection of disjoint balls of radii at most δ with centres in $F\}$.

  (notice no mention of covering. only req. is centers in F.)

- $\mathcal{P}_0^s(F) = \lim_{\delta \to 0} \mathcal{P}_\delta^s(F)$.

  As $\delta \to 0$, $\mathcal{P}_\delta^s(F)$ decreases. (i.e. if $0 < \delta' < \delta$, then $\mathcal{P}_{\delta'}^s(F) \leq \mathcal{P}_\delta^s(F)$)

  This is because $\{\{B_i\}$ collection of disjoint balls of radii at most δ' w/ centres in $F\} = D_{\delta'}$

  $\subset$

  $\{\{B_i\}$ collection of disjoint balls of radii at most δ w/ centres in $F\} = D_\delta$

  $\therefore \{\sum |B_i|^s : \{B_i\} \in D_{\delta'}\} \subset \{\sum |B_i|^s : \{B_i\} \in D_\delta\} \Rightarrow \mathcal{P}_{\delta'}^s(F) \leq \mathcal{P}_\delta^s(F)$.

  Since $0 \leq \mathcal{P}_\delta^s(F)$ $\forall \delta > 0$, $\{\mathcal{P}_\delta^s(F)\}_{\delta > 0}$ is bounded below, so $\lim_{\delta \to 0} \mathcal{P}_\delta^s(F)$ exists.

Remark. $\mathcal{P}_0^s(F)$ is not a measure. To be a measure, countable additivity must hold. (come back to this later)

Consider a countable dense set $\underset{/\!/ F}{\mathbb{Q} \cap [0,1]} \subset \mathbb{R}$. Suppose $F = \{f_1, \dots\} = \bigcup_{i=1}^\infty \{f_i\}$. Then $\sum_i \mathcal{P}_0^s(\{f_i\}) = 0$.

Def. (s-dimensional packing measure)

$$\mathcal{P}^s(F) = \inf\left\{\sum_i \mathcal{P}_0^s(F_i) : F \subset \bigcup_{i=1}^\infty F_i\right\}.$$

Proof sketch of monotonicity: suppose $E \subset F$.

if $\{F_i\}$ is a cover of F, it's one of E as well.

$\therefore$ Let A be the set of indices that make a cover of E.

$\therefore \mathcal{P}^s(E) \leq \sum_{i \in A} \mathcal{P}_0^s(F_i) \leq \sum_{i \in A} \mathcal{P}_0^s(F_i) + \sum_{i \notin A} \mathcal{P}_0^s(F_i)$.

$\therefore \mathcal{P}^s(E)$ is a lower bound. $\Rightarrow \mathcal{P}^s(E) \leq \mathcal{P}^s(F)$.

Def. (packing dimension)

$$\dim_P F = \sup\{s : \mathcal{P}^s(F) = \infty\} = \inf\{s : \mathcal{P}^s(F) = 0\}.$$

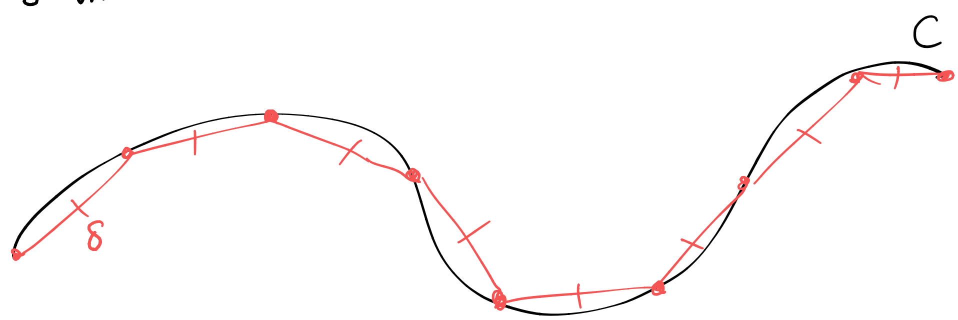
Prop. $\dim_P\left(\bigcup_{i=1}^\infty F_i\right) = \sup_i \dim_P F_i$.

Proof. if $s > \dim_P F_i$ $\forall i$, then $\mathcal{P}^s\left(\bigcup_{i=1}^\infty F_i\right) \leq \underbrace{\sum_i \mathcal{P}^s(F_i)}_{=0} = 0$, so $\mathcal{P}^s\left(\bigcup_{i=1}^\infty F_i\right) = 0$ for $s > \sup_i \dim_P F_i$.

Also if $\exists i$ such that $s < \dim_P F_i$, then $\infty = \mathcal{P}^s(F_i) \leq \mathcal{P}^s\left(\bigcup_{i=1}^\infty F_i\right)$, so $\mathcal{P}^s\left(\bigcup_{i=1}^\infty F_i\right) = \infty$ for $s < \sup_i \dim_P F_i$.

Def. Define a Jordan curve as $f([a,b])$, where $f:[a,b] \to \mathbb{R}^n$ is a continuous bijection. (so no intersections)

If C is a Jordan curve, then given $\delta > 0$, define $M_\delta(C)$ = maximum # of points $\{x_1, \dots, x_m\} \subset C$ such that $|x_n - x_{n-1}| = \delta$ $\forall n$.



$(M_\delta(C) - 1) \cdot \delta$ can be thought of as the "δ-length" of C.

Def. (divider dimension)

$$\underline{\dim}_D C = \underline{\lim}_{\delta \to 0} \frac{\log M_\delta(C)}{-\log \delta}, \quad \overline{\dim}_D C = \overline{\lim}_{\delta \to 0} \frac{\log M_\delta(C)}{-\log \delta}.$$