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2.1-3 - Consider the ***searching problem***:

Input: A sequence of n numbers $A = \langle a_1, a_2, \ldots, a_n \rangle$ and a value v.
Output: An index i such that $v = A[i]$ or the special value NIL if v does not appear in A.

Write pseudocode for LINEAR SEARCH, which scans the sequence, looking for v. Using a loop invariant, prove that your algorithm is correct. Make sure that your loop invariant fulfills the necessary three properties.

i Loop Invariants: Used for showing why an algorithm is correct. Has the following properties:
- **Initialization:** It is true prior to the first iteration of the loop.
- **Maintenance:** If it is true before an iteration of the loop, it remains true before the next iteration.
- **Termination:** When the loop terminates, the invariant gives us a useful property that helps show that the algorithm is correct.

```
LINEAR SEARCH(A, v):
for i = 0 to A.length:
    if A[i] == v:
        return cur
return NIL
```

Initialization: The index is not found prior to the first iteration since nothing has been searched yet.
Maintenance: At each step i is incremented and $A[i]$ is checked against v and the loop terminates if $v = A[i]$.
Termination: The loop terminates when either v is found, returning the index, or when all of A is searched, returning NIL.

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2.2-1 - Express the function $\frac{n^3}{1000} - 100n^2 - 100n + 3$ in terms of Θ notation.

$\Theta(n^3)$

2.2-2 - Consider sorting n numbers stored in array A by first finding the smallest element of A and exchanging it with the element in $A[0]$. Then finding the second smallest element of A, and exchange it with $A[1]$. Continue in this manner for the first $n - 1$ elements of A. This algorithm is known as SELECTION SORT.

A) Write pseudocode for SELECTION SORT.
B) What loop invariants does this algorithm maintain?
C) Why does it need to run for only the first $n - 1$ elements, rather than for all n elements?
D) Give the best-case and worst-case running times of SELECTION SORT in Θ-notation.

```
SELECTION SORT(A):
for cur = 0 to A.length:
    small = A[cur]
    index = cur
    for k = cur + 1 to A.length:
        if A[k] < small:
            small = A[k]
            index = k
    temp = A[cur]                    // Swap the values
    A[cur] = small
    A[index] = temp
```

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2.3-3 - Use mathematical induction to show that when n is an exact power of 2, the solution of the recurrance

$$T(n) = \begin{cases} 2 & \textbf{if } n = 2 \\ 2T\left(\frac{n}{2}\right) + n & \textbf{if } n = 2^k \textbf{ for } k > 1 \end{cases}$$

is $T(n) = n \log_2 n$.

We show that T holds for $n = 2$

$$T(2) = 2 \log_2 2 = 2 * 1 = 2$$

Assuming $T\left(\frac{n}{2}\right) = \frac{n}{2} \log_2 \frac{n}{2}$

$$\begin{aligned} T(n) &= 2\left(\frac{n}{2} \log_2 \frac{n}{2}\right) + n \\ &= 2\left(\frac{n}{2}(\log_2 n - \log_2 2)\right) + n \\ &= 2\left(\frac{n}{2}(\log_2 n - 1)\right) + n \\ &= (n \log_2 n - n) + n \\ &= n \log_2 n \end{aligned}$$

Therefore $T(n) = n \log_2 n$.

2.3-4 - We can express insertion sort as a recursive procedure as follows. In order to sort $A[1..n]$, we recursively sort $A[1..n-1]$ and then insert $A[n]$ into the sorted array $A[1..n-1]$. Write a recurrance for the running time of this recursive version of insertion sort.

Undefined

Induction Theorem:

$$P(n_0) \wedge (\forall n_i > n + o. P(n_i) \rightarrow P(n_{i+1})) \rightarrow \forall n > n_o P(n)$$

1.1-1 - Use mathematical induction to show that $T(n) = \frac{n^2+n}{2}$ for

$$T(n) = \begin{cases} 1 & \textbf{if } n = 1 \\ T(n-1) + n & \textbf{if } n > 1 \end{cases}$$

We show that T holds for $n_0 = 1$

$$T(1) = \frac{1^2 + 1}{2} = \frac{2}{2} = 1$$

Assuming $T(n-1) = \frac{(n-1)^2+(n-1)}{2}$ and $n > 1$

$$\begin{aligned} T(n-1) &= \frac{(n-1)^2 + n - 1}{2} \\ &= \frac{n^2 - 2n + 1 + n - 1}{2} \\ &= \frac{n^2 - n}{2} \end{aligned}$$

By the definition of $T(n)$

$$\begin{aligned} T(n) &= T(n-1) + n \\ &= \frac{n^2 - n}{2} + n \\ &= \frac{n^2 - n + 2n}{2} \\ &= \frac{n^2 + n}{2} \end{aligned}$$

Therefore

$$T(n) = \frac{n^2 + n}{2}$$

By the induction theorem:

$$T(n) = \frac{n^2 + n}{2} \ \forall n \in \mathbb{N}, n \neq 0$$

■

1.1-2 - Proof that $x * 2 = x + x$ using only the following identities:

$$2 = 1 - 1 \quad 1.$$

$$a * 1 = a \quad 2.$$

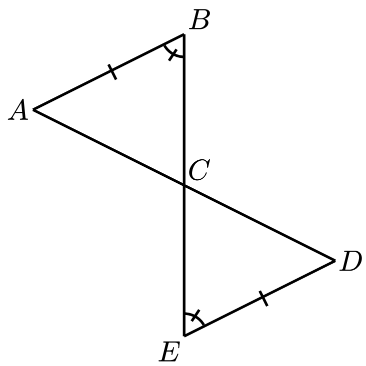
$$a * b = (a * (b - 1)) + a \quad 3.$$

$$\begin{aligned} x * 2 &= x + x \\ &\Leftrightarrow \text{(id 3)} \\ (x * (2 - 1)) + x &= x + x \\ &\Leftrightarrow \text{(id 1)} \\ (x * 1) + x &= x + x \\ &\Leftrightarrow \text{(id 2)} \\ x + x &= x + x \end{aligned}$$

Tautology, therefore we're done. ■

1.1-3 - Proof that ΔABC is congruent to ΔCDE using the following identities:

- If two triangles have two congruent angles and one congruent side in the same order relative to each other they are congruent. (Usuallt called "AAS Congruent").
- When two straight lines cross, their opposite angles are congruent to each other.



When $\overline{\mathrm{AD}}$ crosses with $\overline{\mathrm{BE}}$ the following angles become congruent:

$$\angle \mathrm{ACB} \cong \angle \mathrm{DCE}$$
$$\angle \mathrm{BCD} \cong \angle \mathrm{ACE}$$

For an Angle-Angle-Side congruent triangle we need 2 congruent angles and one congruent side.

$$\overline{\mathrm{DE}} \cong \overline{\mathrm{AB}}$$
$$\angle \mathrm{CED} \cong \angle \mathrm{ABC}$$
$$\angle \mathrm{ACB} \cong \angle \mathrm{DCE}$$

By AAS Congruency:

$$\Delta \mathrm{ABC} \cong \Delta \mathrm{CDE}$$

1.1-4 - Proof that the inversion of $\neg A \wedge ((B < C) \vee D)$ is $A \vee ((B \geq C) \wedge \neg D)$.
List all identities you used.