and all remaining irreducibles are of degree 2 (see, e.g.,[358, pp. 65-66]). However, $I_2(m)$ always has exactly four left cells, see Exercise 1.

It is easy to see that $\{e\}$ and $\{w_0\}$ are left (and right) cells and that they give the *trivial representation*

$$KL_{\{e\}} = 1$$
 (6.8)

and the alternating representation

$$KL_{\{w_0\}} = \varepsilon, \tag{6.9}$$

defined by $\varepsilon(w) = (-1)^{\ell(w)}$ (cf. Lemma 1.4.1). We will frequently write ε_w instead of $\varepsilon(w)$.

The w_0 -induced symmetries have the following effect on Kazhdan-Lusztig representations.

Proposition 6.3.5 Let C be a left cell. Then, the following hold:

- (i) $KL_{\mathcal{C}w_0} \cong \varepsilon KL_{\mathcal{C}}$.
- (ii) $KL_{w_0C} \cong \varepsilon KL_C$.
- (iii) $KL_{w_0Cw_0} \cong KL_C$.

Proof. We begin with part (i). The key fact is that $\widetilde{\Gamma}_{\mathcal{C}w_0}$ is obtained from $\widetilde{\Gamma}_{\mathcal{C}}$ by reversing the direction of all arrows, keeping their color s and weight $\overline{\mu}$, except that the ± 1 weights on loops are switched (cf. Proposition 6.2.9(ii)). This implies on the character level that

$$KL_{\mathcal{C}w_0}(x) = \varepsilon_x KL_{\mathcal{C}}(x^{-1}),$$

for all $x \in W$; namely the trace of $A_{\mathcal{C}w_0}(x)$ is the sum of the weights of all directed circuits in $\widetilde{\Gamma}_{\mathcal{C}w_0}$ beginning and ending in some $y \in \mathcal{C}w_0$ and whose color sequence is (s_1, s_2, \ldots, s_k) for some fixed expression $x = s_1 s_2 \ldots s_k$. Similarly, $KL_{\mathcal{C}}(x^{-1})$ is the sum of the weights of directed circuits in $\widetilde{\Gamma}_{\mathcal{C}}$ whose color sequence is $(s_k, s_{k-1}, \ldots, s_1)$. By the previous remark, these quantities are equal, except possibly for the sign. Whether the sign will change depends on the distribution of the number of (+1)-labeled and (-1)-labeled loops traversed. However, since $\widetilde{\Gamma}_{(W,S)}$ without its loops is a bipartite graph (edges connect elements of even length with elements of odd length), and hence every circuit with its loops removed is of even length, a change of sign will take place for each individual path if and only if k is odd (i.e., if $\varepsilon_x = -1$).

Now use that

$$KL_{\mathcal{C}}(x^{-1}) = \overline{KL_{\mathcal{C}}(x)} = KL_{\mathcal{C}}(x),$$

where the last equality is true because the matrices, and hence character values, are real. Consequently, the characters agree, and part (i) is proved.

For part (iii), one observes (using Proposition 6.2.9(iii)) that $\widetilde{\Gamma}_{w_0Cw_0}$ is obtained from $\widetilde{\Gamma}_C$ by applying the operator $x \mapsto w_0xw_0$ to all nodes