and so

$$\alpha = \alpha^{q^k}$$
.

Since f(x) is the minimal polynomial for α it follows that f(x) divides $x^{q^k} - x$ and so by Theorem 2 of Chapter 7 we have $d \mid k$. However, $0 \le k < d$ and so k = 0 and we are done.

It follows immediately from assertion (a) that $c_1 = \operatorname{tr}_{E/F}(\alpha)$ and that $c_d = N_{E/F}(\alpha)$.

Since $\alpha \in E = F(\alpha)$ we have $\operatorname{tr}_{K/E}(\alpha) = [K : E]\alpha = (n/d)\alpha$ and $N_{K/E}(\alpha) = \alpha^{n/d}$.

By Proposition 11.2.3,

$$\operatorname{tr}_{K/F}(\alpha) = \operatorname{tr}_{E/F}(\operatorname{tr}_{K/E}(\alpha)) = \operatorname{tr}_{E/F}\left(\frac{n}{d}\alpha\right) = \frac{n}{d}\operatorname{tr}_{E/F}(\alpha) = \frac{n}{d}c_1.$$

Similarly,

$$N_{K/F}(\alpha) = N_{E/F}(N_{K/E}(\alpha)) = N_{E/F}(\alpha^{n/d}) = N_{E/F}(\alpha)^{n/d} = c_d^{n/d}.$$

§3 The Rationality of the Zeta Function Associated to $a_0 x_0^m + a_1 x_1^m + \cdots + a_n x_n^m$

Let $f(x_0, x_1, ..., x_n)$ be the polynomial given in the title of this section [notice that this is *not* the f(x) of Section 2]. Suppose that the coefficients are in F, a finite field, with q elements and that $q \equiv 1$ (m). We have to investigate the number N_s of elements in $\overline{H}_f(F_s)$, where $[F_s:F] = s$. Theorem 2 of Chapter 10 shows that N_s is given by

$$q^{s(n-1)} + q^{s(n-2)} + \cdots + q^{s} + 1$$

$$+ \frac{1}{q^{s}} \sum_{\chi_{0}^{(s)}, \dots, \chi_{n}^{(s)}} \chi_{0}^{(s)}(a_{0}^{-1}) \cdots \chi_{n}^{(s)}(a_{n}^{-1}) g(\chi_{0}^{(s)}) \cdots g(\chi_{n}^{(s)}), \quad (4)$$

where q^s is the number of elements in F_s , and the $\chi_i^{(s)}$ are multiplicative characters of F_s such that $\chi_i^{(s)m} = \varepsilon$, $\chi_i^{(s)} \neq \varepsilon$, and $\chi_0^{(s)} \chi_1^{(s)} \cdots \chi_n^{(s)} = \varepsilon$.

We must analyze the terms $\chi_i^{(s)}(a_i^{-1})$ and $g(\chi_i^{(s)})$. To do this we first relate characters of F_s to characters of F.

Let χ be a character of F and set $\chi' = \chi \circ N_{F_s/F}$; i.e., for $\alpha \in F_s$, $\chi'(\alpha) = \chi(N_{F_s/F}(\alpha))$. Then one sees, using Proposition 11.2.2, that χ' is a character of F_s , and moreover that

- (a) $\chi \neq \rho$ implies that $\chi' \neq \rho'$.
- (b) $\chi^m = \varepsilon$ implies that $\chi'^m = \varepsilon$.
- (c) $\chi'(a) = \chi(a)^s$ for all $a \in F$.