Observe that $e: L \times L \times \mathbb{Z}$ defines by reduction modulo m a symplectic pairing $e_m: L/mL \times L/mL \to \mathbb{Z}/m$, and for m dividing n, we have a commutative diagram

$$L/nL \times L/nL \xrightarrow{e_n} \mathbb{Z}/n$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$L/mL \times L/mL \xrightarrow{e_m} \mathbb{Z}/m.$$

(2.2) **Remark.** The symplectic pairings e and e_m are nondegenerate. This means for example that for any linear map $u: L \to \mathbb{Z}$ there exists a unique y in L with u(x) = e(x, y) for all x in L.

Using the definitions going into the formula for the dual isogeny in (1.2), we can obtain a useful formula for e(x, y). First, let sgn(x, y) equal +1, 0, -1 when Im(x/y) is > 0, = 0, and < 0, respectively. Then it is easy to check that

$$e(x, y) = \operatorname{sgn}(x, y) \cdot \frac{a(\mathbb{Z}x + \mathbb{Z}y)}{a(L)}.$$

(2.3) **Remark.** With this formula for e, we show that for any isogeny $\lambda : E = \mathbb{C}/L \to E' = \mathbb{C}/L'$ with $\lambda L \subset L'$ and dual isogeny $\hat{\lambda} : E' \to E$ that the relation $e_{L'}(\lambda x, x') = e_L(x, \hat{\lambda} x')$ holds for x in L and x' in L'. For we have the following inclusions between lattices

$$\lambda L \subset L'$$

$$\cup \qquad \qquad \cup$$

$$\mathbb{Z}\lambda x + \mathbb{Z}nx' \subset \mathbb{Z}\lambda x + \mathbb{Z}x'.$$

Now calculate

$$e_{L'}(x, \hat{\lambda}x') = \operatorname{sgn}(x, \hat{\lambda}x') \cdot \frac{a(\mathbb{Z}x + \mathbb{Z}(n/\lambda)x')}{a(L)}$$

$$= \operatorname{sgn}(\lambda x, x') \cdot \frac{a(\mathbb{Z}\lambda x + \mathbb{Z}nx')}{a(\lambda L)}$$

$$= \operatorname{sgn}(\lambda x, x')[\lambda L : \mathbb{Z}\lambda x + \mathbb{Z}nx']$$

$$= \operatorname{sgn}(\lambda x, x')[L' : \mathbb{Z}\lambda x + \mathbb{Z}x']$$

$$= e_{L'}(\lambda x, x').$$

This verifies the formula.

The above discussion takes place on latices. Now we reinterpret the pairing on the division points ${}_N E(\mathbb{C}) = (I/N)L/L \subset E(\mathbb{C})$ contained in the complex points of the curve. This will lead later to the algebraic definition of the symplectic pairing which is due to A. Weil. The results over the complex numbers are summarized in the following proposition.