Definition 3.28.

A parametrized surface is a smooth function $\sigma: U \to \mathbb{R}^3$ (where U is an open set in \mathbb{R}^2) such for all $q \in U$, $d\sigma_q$ has rank 2.

Notice that a regular surface is a set, while a parametrized surface is a function. Every surface patch for a regular surface is a parametrized surface. A parametrized surface need not be one-to-one. For example, the function $\sigma: \mathbb{R}^2 \to \mathbb{R}^3$ defined as $\sigma(u,v) = (\cos u, \sin u, v)$ is a parametrized surface that wraps the plane infinitely many times around the cylinder; the restriction of this function to a smaller domain was used in Example 3.23 as a surface patch for the cylinder. Exercise 3.27 in this section (illustrated in Fig. 3.17) shows a parametrized surface that fails to be one-to-one because of more complicated kinds of self-intersections. In studying only local properties, it doesn't matter whether one works with regular surfaces or parametrized surfaces, because of the following proposition:

Proposition 3.29.

If $\sigma: U \to \mathbb{R}^3$ is a parametrized surface, then for each $q_0 \in U$, there exists an open set $\tilde{U} \subset U$ containing q_0 such that the image $S = \sigma(\tilde{U})$ is a regular surface covered by a single surface patch (namely, the restriction of σ to \tilde{U}).

PROOF. Let $q_0 \in U$. Choose any $N \in \mathbb{R}^3$ with $N \notin \text{span}\{\sigma_u(q_0), \sigma_v(q_0)\}$. Consider the smooth function $f: U \times \mathbb{R} \to \mathbb{R}^3$ defined as follows:

$$f(q,t) = \sigma(q) + tN.$$

Set $p_0 = \sigma(q_0) = f(q_0, 0)$. The derivative $df_{(q_0,0)} : \mathbb{R}^3 \to \mathbb{R}^3$ is invertible, because it sends the basis $\{e_1, e_2, e_3\}$ to the basis $\{\sigma_u(q_0), \sigma_v(q_0), N\}$. By the inverse function theorem (on page 120), f restricts to a diffeomorphism from a neighborhood, \mathcal{A} , of $(q_0, 0)$ in $U \times \mathbb{R} \subset \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ to a neighborhood, \mathcal{B} , of p_0 in \mathbb{R}^3 . We can assume (after possibly shrinking the neighborhoods) that \mathcal{A} has the form $\mathcal{A} = \tilde{U} \times (-\epsilon, \epsilon)$, where $\epsilon > 0$ and $\tilde{U} \subset U$ is a neighborhood of q_0 .

Define $S = \sigma(\tilde{U})$. Since f is injective on \mathcal{A} , it follows that σ is injective on \tilde{U} , so it has an inverse $\sigma^{-1}: S \to \tilde{U}$. It remains to prove that σ^{-1} is smooth. For this, let $P: \mathcal{A} \to U$ denote the natural projection that maps $(q,t) \mapsto q$. The function $P \circ f^{-1}: \mathcal{B} \to \tilde{U}$ is a smooth function that agrees with σ^{-1} on $\mathcal{B} \cap S$, which verifies that σ^{-1} is smooth.

In light of Proposition 3.29, you might guess that the image of every injective parametrized surface is a regular surface, but counterexamples to this guess will be given in Sect. 10.