let P be a unit mass at x (that is, P(A) is 1 or 0 according as x lies in A or not) and let P_n be a unit mass at x_n . Now if 2/n is less than the smallest nonzero t_i , then x and x_n either both lie in $\pi_{t_1 \cdots t_k}^{-1} H$ or else neither one does, so that $P\pi_{t_1 \cdots t_k}^{-1}(H) = P_n\pi_{t_1 \cdots t_k}^{-1}(H)$. Therefore there is weak convergence $P_n\pi_{t_1 \cdots t_k}^{-1} \Rightarrow P\pi_{t_1 \cdots t_k}^{-1}$ in R^k for each t_1, \dots, t_k . On the other hand, the set $\{y: |y(t)| \le 1/2, 0 \le t \le 1\}$, the sphere of radius 1/2 about x, is a P-continuity set and $P_n(A) = 1$ does not converge to P(A) = 0. Thus P_n does not converge weakly to P.

This example shows that if there is convergence of the finite-dimensional distributions, that is, if

$$(5.1) P_n \pi_{t_1 \cdots t_k}^{-1} \Rightarrow P \pi_{t_1 \cdots t_k}^{-1}$$

for all k and t_1, \dots, t_k , it does not follow that there is weak convergence of P_n to P:

$$(5.2) P_n \Rightarrow P.$$

(The converse proposition of course does hold because of Corollary 2 to Theorem 3.3.) Thus weak convergence in C involves considerations going beyond finite-dimensional ones, which is why it is useful (see the introduction).

On the other hand, (5.1) does imply (5.2) in the presence of relative compactness. THEOREM 5.1. If (5.1) holds for all k and t_1, \dots, t_k , and if $\{P_n\}$ is relatively compact, then (5.2) holds.

Proof. Since $\{P_n\}$ is relatively compact, each subsequence $\{P_{n_i}\}$ contains a further subsequence $\{P_{n_{i_m}}\}$ such that $P_{n_{i_m}} \Rightarrow Q$ as $m \to \infty$ for some probability measure Q on C. But then $P_{n_{i_m}} \pi_{t_1 \cdots t_k}^{-1} \Rightarrow Q \pi_{t_1 \cdots t_k}^{-1}$, so that, because of (5.1), $Q \pi_{t_1 \cdots t_k}^{-1} = P \pi_{t_1 \cdots t_k}^{-1}$. Thus P and Q have the same finite-dimensional distributions and, as observed above, this implies P = Q. Thus each subsequence of $\{P_n\}$ contains a further subsequence converging weakly to P, and (5.2) follows by Theorem 2.2.

Theorem 4.1 characterizes relative compactness by tightness. In order to apply Theorem 5.1 in concrete cases, we shall in turn characterize tightness by means of the Arzelà-Ascoli theorem.

For $x \in C$ and $\delta > 0$, the modulus of continuity is defined by

$$w_x(\delta) = \sup\{|x(s) - x(t)| : 0 \le s, t \le 1, |s - t| < \delta\}.$$

According to the Arzelà-Ascoli theorem, a set A in C has compact closure if and only if

$$\sup_{x \in C} |x(0)| < \infty$$

and

(5.4)
$$\lim_{\delta \to 0} \sup_{x \in C} w_x(\delta) = 0.$$

THEOREM 5.2. A family Π of probability measures on C is tight (hence relatively compact) if and only if for each η there exists an a such that

(5.5)
$$P\{x:|x(0)| > a\} < \eta, \quad P \in \Pi,$$