**Table 3.3** *P*-values for three samples,  $n_1 = n_2 = n_3 = 5$ ,  $\sigma^2 = 14$ 

$\overline{\overline{Y}}_1$	$\overline{Y}_2$	$\overline{Y}_3$	ANOVA	ISO	Reg
2	6	6	0.149	0.050	0.045
2	6	7	0.082	0.026	0.017
2	6	8	0.036	0.011	0.006

relative ease of using regression is probably why isotonic regression is not used more.

Finally, although the *p*-values reported in Table 3.3 are for the known-variance case, qualitatively similar results are obtained for the more complicated case of  $\sigma$  unknown.

There is a large literature on order-restricted inference. For testing for ordered alternatives, there has been more emphasis on  $T_{\rm LR}$  than on  $T_{\rm W}$  and  $T_{\rm S}$ . The classic references are Barlow et al. (1972) and Robertson et al. (1988), whereas a more recent account is Silvapulle and Sen (2005).

## 3.6.2 Null Hypotheses on the Boundary of the Parameter Space

When a null hypothesis value, say  $\theta_0$  lies on the boundary of the parameter space, then maximum likelihood estimators are often truncated at that boundary because by definition  $\widehat{\theta}_{MLE}$  must lie in the parameter space of  $\theta$ . Thus  $\widehat{\theta}_{MLE}$  is equal to the boundary value  $\theta_0$  with positive probability and correspondingly  $T_{LR}$  is zero for those cases. The result is that the limiting distribution of  $T_{LR}$  is a mixture of a point mass at zero and a chi-squared distribution. We illustrate first with an artificial example and then consider the one-way random effects model.

## 3.6.2a Normal Mean with Restricted Parameter Space

Suppose that  $Y_1,\ldots,Y_n\sim N(\mu,1)$ . Usually,  $\widehat{\mu}_{MLE}=\overline{Y}$ , but suppose that we restrict the parameter space for  $\mu$  to be  $[\mu_0,\infty)$  where  $\mu_0$  is some given constant, instead of  $(-\infty,\infty)$ . Then  $\widehat{\mu}_{MLE}=\overline{Y}$  if  $\overline{Y}\geq \mu_0$  and  $\widehat{\mu}_{MLE}=\mu_0$  if  $\overline{Y}<\mu_0$ . Now suppose that the null hypothesis is  $H_0:\mu=\mu_0$ . We first consider the three likelihood-based test statistics, showing that only the score statistic has a limiting  $\chi_1^2$  distribution. Then we provide a simple solution to this testing problem.

Under  $H_0$ , the Wald statistic is  $T_{\rm W} = n(\widehat{\mu}_{\rm MLE} - \mu_0)^2$ , which is thus  $T_{\rm W} = 0$  if  $\widehat{\mu}_{\rm MLE} = \mu_0$  and  $T_{\rm W} = n(\overline{Y} - \mu_0)^2$  if  $\overline{Y} \ge \mu_0$ . The score statistic is  $T_{\rm S} = n(\overline{Y} - \mu_0)^2$ , and the likelihood ratio statistic is the same as the Wald statistic. Thus, only the score statistic converges to a  $\chi_1^2$  distribution under  $H_0$ . The Wald and the likelihood ratio statistics converge to a distribution that is an equal mixture of a point mass at 0 and a  $\chi_1^2$  distribution, the same distribution as in (3.23) for k=2. In fact the