Further, the vertices in any component of $T \setminus \{k\}$ are either all positive, all negative or all zero. It follows that the subgraph of T induced by the zero vertices is connected. The proof of the second part is similar to the one given for (i). $\square$

8.4 Bounds for Algebraic Connectivity

The following representation for the second smallest eigenvalue of a symmetric matrix will be used. It is easily derived from the spectral theorem.

Lemma 8.15 *Let A be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n$. Let u be an eigenvector of A corresponding to λ_n. Then*

$$\lambda_{n-1} = \min \left\{ \frac{x'Ax}{x'x} \right\},$$

where the minimum is taken over all nonzero vectors x, orthogonal to u.

We introduce some notation. Let G be a connected graph with $V(G) = \{1, \ldots, n\}$. If $i, j \in V(G)$, then as usual the distance between i and j, denoted $d(i, j)$, is defined to be the length (that is, the number of edges) in the shortest (ij)-path. We set $d(i, i) = 0, i = 1, \ldots, n$. If $V_1, V_2 \subset V(G)$ are nonempty sets then define

$$d(V_1, V_2) = \min\{d(i, j) : i \in V_1, j \in V_2\}.$$

If $V_1 = \{i\}$ we write $d(V_1, V_2)$ as $d(i, V_2)$.

Theorem 8.16 *Let G be a connected graph with $V(G) = \{1, \ldots, n\}$. Let V_1 and V_2 be nonempty disjoint subsets of $V(G)$, and let G_1 and G_2 be the subgraphs induced by V_1 and V_2, respectively. Let L be the Laplacian of G and μ the algebraic connectivity. Then*

$$\mu \leq \frac{1}{d(V_1, V_2)^2} \left(\frac{1}{|V_1|} + \frac{1}{|V_2|} \right) (|E(G)| - |E(G_1)| - |E(G_2)|).$$

Proof Let

$$g(i) = \frac{1}{|V_1|} - \frac{1}{d(V_1, V_2)} \left(\frac{1}{|V_1|} + \frac{1}{|V_2|} \right) \min\{d(i, V_1), d(V_1, V_2)\},$$

$i = 1, \ldots, n$. Note that if $i \in V_1$ then $g(i) = \frac{1}{|V_1|}$, and if $i \in V_2$ then $g(i) = -\frac{1}{|V_2|}$. Also, if $i \sim j$ then $|d(i, V_1) - d(j, V_2)| \leq 1$ and hence

$$|g(i) - g(j)| \leq \frac{1}{d(V_1, V_2)} \left(\frac{1}{|V_1|} + \frac{1}{|V_2|} \right). \tag{8.12}$$