(ii) If $u \leq 0$ in Ω , then (2.7) holds trivially. Otherwise assume that $\max_{\bar{\Omega}} u = u(x_0) > 0$ and $x_0 \neq 0, 1$. Let (α, β) be the largest subinterval of Ω containing x_0 in which u > 0. We now have $\tilde{A}u := Au - cu \leq 0$ in (α, β) . Part (i), applied with the operator \tilde{A} in the interval (α, β) , therefore implies $u(x_0) = \max\{u(\alpha), u(\beta)\}$. But then α and β could not both be interior points of Ω , for then either $u(\alpha)$ or $u(\beta)$ would be positive, and the interval (α, β) would not be as large as possible with u > 0. This implies $u(x_0) = \max\{u(0), u(1)\}$ and hence (2.7).

As a consequence of this theorem we have the following stability estimate with respect to the maximum-norm, where we use the notation of Sect. 1.2.

Theorem 2.2. Let A be as in (2.1) and (2.2). If $u \in C^2$, then

$$||u||_{\mathcal{C}} \leq \max\{|u(0)|,|u(1)|\} + C||\mathcal{A}u||_{\mathcal{C}}.$$

The constant C depends on the coefficients of A but not on u.

Proof. We shall bound the maxima of $\pm u$. We set $\phi(x) = e^{\lambda} - e^{\lambda x}$ and define the two functions

$$v_{\pm}(x) = \pm u(x) - ||Au||_{\mathcal{C}} \phi(x).$$

Since $\phi \geq 0$ in Ω and $A\phi = ce^{\lambda} + (a\lambda^2 + (a'-b)\lambda - c)e^{\lambda x} \geq 1$ in $\bar{\Omega}$, if $\lambda > 0$ is chosen sufficiently large, we have, with such a choice of λ ,

$$Av_{\pm} = \pm Au - \|Au\|_{\mathcal{C}}A\phi \le \pm Au - \|Au\|_{\mathcal{C}} \le 0$$
 in Ω .

Theorem 2.1(ii) therefore yields

$$\begin{aligned} \max_{\bar{\Omega}}(v_{\pm}) &\leq \max \left\{ v_{\pm}(0), v_{\pm}(1), 0 \right\} \\ &\leq \max \left\{ \pm u(0), \pm u(1), 0 \right\} \leq \max \left\{ |u(0)|, |u(1)| \right\}, \end{aligned}$$

because $v_{\pm}(x) \leq \pm u(x)$ for all x. Hence,

$$\max_{\bar{\Omega}}(\pm u) = \max_{\bar{\Omega}} (v_{\pm} + \|\mathcal{A}u\|_{\mathcal{C}} \phi) \leq \max_{\bar{\Omega}}(v_{\pm}) + \|\mathcal{A}u\|_{\mathcal{C}} \|\phi\|_{\mathcal{C}}
\leq \max\{|u(0)|, |u(1)|\} + C\|\mathcal{A}u\|_{\mathcal{C}}, \text{ with } C = \|\phi\|_{\mathcal{C}},$$

which completes the proof.

From Theorem 2.2 we immediately conclude the uniqueness of a solution of (2.1). In fact, if u and v were two solutions, then their difference w = u - v would satisfy Aw = 0, w(0) = w(1) = 0, and hence $||w||_{\mathcal{C}} = 0$, so that u = v.

More generally, if u and v are two solutions of (2.1) with right hand sides f and g and boundary values u_0, u_1 and v_0, v_1 , respectively, then

$$||u-v||_{\mathcal{C}} \leq \max\{|u_0-v_0|, |u_1-v_1|\} + C||f-g||_{\mathcal{C}}.$$