$$\int_0^1 \psi \circ f \, dx = l\psi(a) + m\psi(\lambda a + \mu b) + n\psi(b)$$

$$= \psi(\lambda a + \mu b) - l/\lambda \left(\psi(\lambda a + \mu b) - \lambda\psi(a) - \mu\psi(b)\right)$$

$$< \psi(\lambda a + \mu b).$$

So $\int_0^1 \psi \circ f dx < \psi \left(\int_0^1 f dx \right)$, a contradiction.

11. (i) If $a_i \ge 0$, $b_i \ge 0$, i = 1, 2, ..., n and p > 1, 1/p + 1/q = 1, then

$$\sum_{i=1}^{n} a_{i}b_{i} \leq \left(\sum_{i=1}^{n} a_{i}^{p}\right)^{1/p} \left(\sum_{i=1}^{n} b_{i}^{q}\right)^{1/q}.$$

(ii) If
$$p \ge 1$$
, $\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}$.

(iii) If
$$a_i \ge 0$$
, $b_i \ge 0$, $i = 1, \ldots, n$, $\sum_{i=1}^n a_i b_i \le \left(\sum_{i=1}^n a_i\right) \max_{1 \le i \le n} b_i$. The proof of (i), for example, is obtained by taking $X = [1, \ldots, n]$, $a(i) = a_i$, $\mu([i]) = 1$ so that $\sum_{i=1}^n a_i = \int a \, d\mu$, and applying Holder's inequality.

- 12. They imply $\|\sin x \cos x\|_2 = \|(f \sin x) (f \cos x)\|_2 \le \|f \sin x\|_2 + \|f \cos x\|_2 \le 1$. But the first term is $\sqrt{\pi}$.
- 13. Apply the Schwarz inequality.
- 14. (i) is a special case of (ii). Write $|f|^p = F$, $|g|^q = G$, $\alpha = 1/p$, $\beta = 1/q$, then $F \in L^{\alpha}(\mu)$, $G \in L^{\beta}(\mu)$, so by Theorem 7, $FG \in L^{1}(\mu)$.
- 15. (i) Minkowski's inequality gives $||f||_2 ||f_n||_2 | \le ||f f_n||_2$.

(ii)
$$\left| \int_a^t f \, dx - \int_a^t f_n \, dx \right| = \left| \int_a^b \chi_{(a,t)} (f - f_n) \, dx \right| \le \sqrt{(t-a)} \, \|f - f_n\|_2$$
, by Hölders inequality.

- (iii) To verify (i), integrate explicitly and use $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$. To verify
- (ii), integrate and use the standard Fourier Series for t^2 .
- 16. By Minkowski's inequality $|\|f_n\|_p \|f\|_p| \le \|f_n f\|_p \to 0$.
- 17. By Example 20, p. 67, we can find ψ such that $\psi(t)t^{p-1}$ $f^p \in L^1$, $\psi \ge 1$ on [0,1], $\psi(0+) = \infty$. Then

$$F(x) = \int_{x}^{1} f \, dt = \int_{x}^{1} \frac{1}{\psi^{1/p} t^{(p-1)/p}} \psi^{1/p} t^{(p-1)/p} f \, dt$$

$$\leq \left(\int_{x}^{1} \psi^{-q/p} t^{-1} \, dt \right)^{1/p} \left(\int_{x}^{1} t^{p-1} \psi f^{p} \, dt \right)^{1/p},$$

by Hölder's inequality, p and q being conjugate indices. So