where $M_s = \{t \in (a, d) : P_w(t) > s > 0\}$. Now

$$\int_{a}^{d} \sqrt{P_{w}} w = \int_{a}^{d} \left(\int_{0}^{\sqrt{P_{w}}} ds \right) w = \int_{0}^{\infty} \int_{M_{s^{2}}} w \, ds = \left(\int_{0}^{1} + \int_{1}^{\infty} \int_{M_{s^{2}}} w \, ds \right)$$

$$\leq \int_{a}^{d} w + \int_{a}^{d} \frac{1}{|p|} w \int_{1}^{\infty} \frac{3}{s^{2}} \, ds = \int_{a}^{d} w + 3 \int_{a}^{d} \frac{1}{|p|} < \infty.$$

It follows that $\sqrt{|\lambda|} |m(t,\lambda)|$ is bounded on (a,x) by a function integrable with weight w. The theorem now follows, under the restriction of the size of gaps of supp w assumed, by dominated convergence from the pointwise result of Theorem 6.3.1.

To complete the proof we must remove this restriction. We have just proved that the theorem is true for x up to and including the left endpoint d in the first gap violating the restriction. For x in this gap we have

$$\psi(x,\lambda) = \psi(d,\lambda)u(x) + p\psi'(d,\lambda)v(x) ,$$

where u and v are solutions of -(pu')' + qu = 0 with appropriate λ -independent initial data in d. Since $m(d, \cdot)$ is a non-trivial Nevanlinna function it follows that $p\psi'(x,\lambda)/p\psi'(d,\lambda) = \exp\{O(\log |\lambda|)\}$ so the theorem holds up to and including the right endpoint of the gap and therefore up to and including the first point in the next large gap (if any). Since the support of w can only have a finite number of such large gaps in any compact interval this completes the proof.

Exercises

Exercise 6.4.1 Give a result analogous to Theorem 6.4.4 but for the kernel of $u \mapsto p(R_{\lambda}u)'$. Also prove corresponding results for the left-definite case.

Exercise 6.4.2 Prove that the radius of the circle defined by $\|\theta(\cdot, \lambda) + m\varphi(\cdot, \lambda)\|_x^2 \le \text{Im } m/\text{Im } \lambda$ in both the right- and left-definite cases is given by $(2 |\text{Im } \lambda| \|\varphi\|_x^2)^{-1}$.

Hint: The inequality $\|\psi\|_x^2 \le \text{Im } m/\text{Im } \lambda$ may be rewritten as $|m-c|^2 \le r^2$, where $r^2 = A(2 \text{Im } \lambda \|\varphi\|_x^2)^{-2}$ and

$$A = |2i \operatorname{Im} \lambda \langle \theta, \varphi \rangle_{x} + 1|^{2} + 2i \operatorname{Im} \lambda ||\theta||_{x}^{2} 2i \operatorname{Im} \lambda ||\varphi||_{x}^{2}.$$

Use the differential equation and integration by parts to write $\lambda \langle \theta, \varphi \rangle_x$ as an integrated term plus $\overline{\lambda} \langle \theta, \varphi \rangle_x$. Similarly for $\lambda \|\theta\|_x^2$ and $\lambda \|\varphi\|_x^2$. Now rearrange the resulting terms to show that $A = \left| \mathcal{W}_p(\varphi, \theta) \right|^2 = 1$.

Exercise 6.4.3 Prove Theorem 6.4.1 in the left-definite case.