$$y_{1,t}(t,x) = \alpha_1\, y_{1,xx}(t,x), \quad 0 < t < T,\ 0 < x < u(t), \tag{5.1.21}$$

$$y_{2,t}(t,x) = \alpha_2\, y_{2,xx}(t,x), \quad 0 < t < T,\ u(t) < x < b, \tag{5.1.22}$$

$$y_1(0,x) = \varphi_1(x),\ x \in [0,u_0], \quad y_2(0,x) = \varphi_2(x),\ x \in [u_0,b], \tag{5.1.23}$$

$$y_1(t,0) = 0, \quad y_2(t,b) = 0, \tag{5.1.24}$$

$$y_1(t,u(t)) = y_2(t,u(t)) = 0, \quad 0 \le t \le T, \tag{5.1.25}$$

$$u_t(t) = \beta_2\, y_{2,x}(t,u(t)) - \beta_1\, y_{1,x}(t,u(t)), \quad 0 < t \le T, \tag{5.1.26}$$

$$u(0) = u_0 \in \,]0,b[\,. \tag{5.1.27}$$

Here, y_1, y_2 represent the "temperatures" in the two phases (solid and liquid, say), which coexist in the one-dimensional "container" $[0,b]$, separated by an interface located at time t at the space point $x = u(t)$, $0 \le t \le T$. The positive constants α_i, β_i, $i = 1,2$, are related to the physics of the problem (thermal conductivity, *Stefan condition* on the interface), and the initial temperature distributions φ_i, $i = 1,2$, in the phases and the initial location $u(0)$ of the interface are known.

Now let $v \in L^2(0,T)$ be a "control" function. We replace the Stefan condition (5.1.26) and (5.1.27) by

$$u'(t) = v(t) \quad \text{in } [0,T], \tag{5.1.28}$$

$$u(0) = u_0, \tag{5.1.29}$$

where v belongs to a prescribed set U_{ad} that usually is chosen compact in $C[0,T]$. In this way, a class of "admissible free boundaries" is defined. The original problem may then be reformulated as the optimal shape design problem

$$\operatorname*{Min}_{v \in U_{ad}} \left\{ \int_0^T [\beta_2\, y_{2,x}(t,u(t)) - \beta_1\, y_{1,x}(t,u(t)) - v(t)]^2\, dt \right\}, \tag{5.1.30}$$

subject to the system (5.1.21)–(5.1.25) and (5.1.28), (5.1.29).

We remark that here the optimal shape design problem could be stated directly as an optimal control problem, since we considered the one-dimensional case. This is not possible in higher dimensions of space.

The above relationship between free boundary problems and shape optimization problems is reflected in the scientific literature by the use of similar methods. A survey along these lines may be found in the work of Hoffmann and Tiba [1995]. Several important techniques will be discussed below in this chapter.