Notice that

$$|\mu_{k_p}|\left(\bigcup_n E_{k_n}\right) + |\mu_{k_p}|\left(\bigcup_n E_n \setminus \bigcup_n E_{k_n}\right) = |\mu_{k_p}|\left(\bigcup_n E_n\right) \le 1,$$

which, since

$$\bigcup_{\substack{j \neq k_p \\ j \in N_p}} E_j \subseteq \bigcup_n E_n \setminus \bigcup_n E_{k_n},$$

gives us

$$|\mu_{k_p}| \left(\bigcup_n E_{k_n}\right) \leq 1 - \varepsilon$$

for all p.

Repeat the above argument starting this time with the sequences $\mu'_n = \mu_{k_n}$ and $E'_n = E_{k_n}$; our starting point now will be the inequality

$$|\mu'_n|\Big(\bigcup_n E'_n\Big) \leq 1-\varepsilon.$$

Proceeding as above, either we arrive immediately at a suitable subsequence or extract a subsequence (j_{k_n}) of (k_n) for which another ε can be shaved off the right side of the above inequality making

$$|\mu_{j_{k_p}}|\left(\bigcup_n E_{j_{k_n}}\right) \leq 1 - 2\varepsilon$$

hold for all p.

Whatever the first n is that makes $1 - n\varepsilon < 0$, the above procedure must end satisfactorily by n steps or face the possibility that $0 \le 1 - n\varepsilon < 0$.

From Rosenthal's lemma and the Nikodym-Grothendieck boundedness theorem we derive another classic convergence theorem pertaining to l_{∞}^* .

Phillips's Lemma. Let $\mu_n \in ba(2^N)$ satisfy $\lim_n \mu_n(\Delta) = 0$ for each $\Delta \subseteq N$. Then

$$\lim_{n}\sum_{j}|\mu_{n}(\{j\})|=0.$$

PROOF. The Nikodym-Grothendieck theorem tells us that $\sup_n ||\mu_n|| < \infty$, and so the possibility of applying Rosenthal's lemma arises.

Were the conclusion of Phillips's lemma not to hold, it would be because for some $\delta > 0$ and some subsequence [which we will still refer to as (μ_n)] of (μ_n) we have

$$\sum_{j} |\mu_n(\{j\})| \ge 6\delta$$

for all n.