1. Convex Sets

To prove (b) \Rightarrow (a) it suffices to show that

$$C \subset \operatorname{conv}(\operatorname{ext} C)$$
. (4)

(In fact, suppose that (4) holds. Since the opposite inclusion of (4) is obvious, it then follows that C = conv(ext C). But then we also have C = conv M for any subset M of C containing ext C.) We shall prove (4) by induction on the dimension of C. For dim C = -1, 0 there is nothing to prove. For dim C = 1the statement is clearly valid. Suppose that the statement is valid for all compact convex sets of dimension $\langle e \rangle$, where $e \geq 2$, and let C be a compact convex set of dimension e. Let x be any point in C; we shall prove that x is a convex combination of extreme points of C, cf. Theorem 2.2. If x itself is an extreme point, there is nothing to prove. If x is not an extreme point, then there is a segment in C having x in its relative interior. Extending the segment, if necessary, we see that there are in fact points $y_0, y_1 \in rb$ C such that $x \in]y_0, y_1[$. Let F_0 and F_1 be the smallest faces of C containing y_0 and y_1 , respectively. Then F_0 and F_1 are proper faces of C, cf. Corollary 5.7. They are, in particular, compact convex sets, cf. Theorem 5.1, and they both have dimension $\langle e, cf. Corollary 5.5$. Then, by the induction hypothesis, there are points $x_{01}, \ldots, x_{0p} \in \text{ext } F_0 \text{ and } x_{11}, \ldots, x_{1q} \in \text{ext } F_1 \text{ such that } y_0 \text{ is a convex}$ combination of the x_{0i} 's and y_1 is a convex combination of the x_{1j} 's. Since x is a convex combination of y_0 and y_1 , it follows that x is a convex combination of the x_{0i} 's and x_{1j} 's. To complete the proof, we note that the x_{0i} 's and x_{1j} 's are in fact extreme points of C; this follows from Theorem 5.2.

Corollary 5.11. Let C be a compact convex set in \mathbb{R}^d with dim C = n. Then each point of C is a convex combination of at most n + 1 extreme points of C.

PROOF. Combine Theorem 5.10(c) and Corollary 2.4.

EXERCISES

- 5.1. Show that ext C is closed when C is a 2-dimensional compact convex set.
- 5.2. Let C be the convex hull of the set of points $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ such that

$$\alpha_1 = \alpha_2 = 0, \qquad \alpha_3 \in [-1, 1],$$

or

$$\alpha_3 = 0, \quad (\alpha_1 - 1)^2 + \alpha_2^2 = 1.$$

Show that ext C is non-closed.

- 5.3. Let C be a closed convex set in \mathbb{R}^d . Show that if a convex subset F of C is a face of C, then $C \setminus F$ is convex. Show that the converse does not hold in general.
- 5.4. Let C be a non-empty closed convex set in \mathbb{R}^d . An affine subspace A of \mathbb{R}^d is said to support C if $A \cap C \neq \emptyset$ and $C \setminus A$ is convex. Show that the supporting hyperplanes of C in the sense of Section 4 are the hyperplanes that support C in the sense just