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ff 653 July 65

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Translation of "Optimal'nyye formy tel s prisoyedinennymi udarnymi volnami." Izvestiya Akademii Nauk SSSR. Mekhanika Zhidkosti i Gaza, No. 4, pp. 9-18, 1966.

602	N67-15765						
FORM 6	(ACCESSION NUMBER)	(THRU)					
HLITY	(PAGES)	(CODE)					
PA	(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)					

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION WASHINGTON D.C. JANUARY 1967

OPTIMUM FORMS OF BODIES WITH ASSOCIATED SHOCK WAVES

A. V. Shipilin (Moscow)

The problem is discussed of finding the form of two-dimensional and axisymmetric bodies guaranteeing the minimum wave drag in the supersonic flow of an ideal gas. The solution is sought among those bodies which — when flow passes around them — give rise to associated shock waves. Limitations on the contour of the body are arbitrary: One may pre-assign the dimensions of the body, its volume, area, etc. Similar problems with arbitrary isoperimentric conditions can be solved by a method proposed in (Refs. 1, 2). This method consists of using the exact equations of gas dynamics, describing the flow, as supplementary constraints. In the articles (Refs. 3-6) this method was further developed during the solution of a number of problems.

1. Let x and y be the coordinate axes (Figure 1), ab -- the contour of the desired body, ac -- the associated shock wave, bc -- the characteristic of the second family, cd -- the characteristic of the first family. It is assumed that inside the region abc the flow is supersonic, and there are no shock waves. The incoming stream is uniform. Let u and v be the projections of the velocity on the axes x and y, let p denote the pressure, ρ -- the density of the gas, ψ -- the stream function, where

$$d\psi = y^{\vee}_{0} (udy - vdx) \tag{1.0}$$

 ν = 0 and 1 correspondingly, respectively, to the two-dimensional and the axi-symmetric case. Here all the quantities are considered to be dimensionless, and ψ at the body will be taken as equal to zero.

The problem is analyzed in terms of the variables ψ , y. The stationary, non-isotropic flow of the gas inside the region abc is described by the equations

$$L_{1} \equiv \frac{\partial y^{\mathsf{v}} p}{\partial \psi} - \frac{\partial u}{\partial y} = 0, \qquad L_{2} \equiv \frac{\partial}{\partial \psi} \frac{u}{v} + \frac{\partial}{\partial y} \frac{1}{y^{\mathsf{v}} \rho v} = 0$$

$$\frac{w^{2}}{2} + \frac{\kappa p}{(\varkappa - 1)\rho} = \frac{1}{2} \frac{\varkappa + 1}{\varkappa - 1}, \qquad \frac{p}{\rho^{\varkappa}} = \varphi^{\varkappa - 1}(\psi)$$

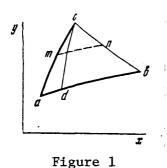
$$w = \sqrt{u^{2} + v^{2}} \qquad (1.1)$$

Here x is the adiabatic exponent; ϕ is the entropy function. The wave drag is expressed by the functional

$$\chi = \int_{y_{-}}^{y_{b}} p[x(y)] y^{\nu} dy$$
 (1.2)

The coordinate y_b may be either fixed or arbitrary, depending on the conditions imposed on the contour of the body. The quantity x_b should be

^{*} Numbers in the margin indicate pagination in the original foreign text.



pre-assigned. In the general case, we may pre-assign n isoperimetric conditions which are written in the following way:

$$r^{i} = \int_{v_{a}}^{v_{b}} f^{i} \left[x'_{i}(y), \ x'(y), \ y \right] dy$$
 (1.3)

Here x(y) is the function describing the contour of the body; x'(y) is its /10 derivative with respect to y. Along the contour ab, the following condition is satisfied:

$$vx'(y) - u = 0 {(1.4)}$$

The problem is formulated in the following way. Let us find a function x(y) realizing the extremum of the functional (1.2), satisfying the isoperimetric conditions (1.3), and at the same time satisfying the constraint (1.4) along ab and the equations (1.1) in the region abc. We shall employ S to denote the region enclosed inside abc. We shall form the Lagrangian functional

$$I = \int_{y_a}^{y_b} [py^{\nu} + \gamma_0(y)(vx' - u)] dy + \sum_{i=1}^{n} \gamma_i r^i +$$

$$+ \iint_{\mathbb{R}} [h_1(\psi, y) L_1 + h_2(\psi, y) L_2] dy d\psi$$
(1.5)

Here the γi 's are constants, and γ_0 (y), h_1 (ψ , y) and h_2 (ψ , y) are variable Lagrange multipliers. We shall determine the functions x(y), p, ρ , u, v, h_1 , h_2 γ_0 in such a way that the functional (1.5) assumes its stationary value. Then, according to (Ref. 7), the functions which are found provide the extremum of the original functional (1.2) for the pre-assigned restrictions.

2. We shall calculate the first-order variation of the functional I. We should note that in the plane ψy the location ab of the body and the location ac of the shock wave are fixed. Only the location of the points b and c may change. The form of the characteristic bc changes when the flow is varied in the region s. However, the variation of the double integral, associated with a change of the boundary of the region s is missing since the integrands are equal to zero.

The first-order variation of the functional (1.5) has the form

$$\delta I = \left(py^{\nu} + \sum_{i=1}^{n} \gamma_{i} f^{i}\right) \delta y_{b} + \int_{y_{a}}^{y_{b}} \left[\sum_{i=1}^{n} \gamma_{i} \frac{\partial f^{i}}{\partial x} \delta x + \left(\gamma_{0} v + \sum_{i=1}^{n} \gamma_{i} \frac{\partial f^{i}}{\partial x}\right) \delta x' + y^{\nu} \delta p + \gamma_{0} x' \delta v - \gamma_{0} \delta u\right] dy +$$

$$+ \iint_{S} \left\{h_{1}(\psi, y) \left[\frac{\partial \delta(y^{\nu} p)}{\partial \psi} - \frac{\partial \delta u}{\partial y}\right] + h_{2}(\psi, y) \left[\frac{\partial \delta(uv^{-1})}{\partial \psi} + \frac{\partial \delta(y^{\nu} \rho v)^{-1}}{\partial y}\right]\right\} d\psi dy$$

$$(2.1)$$

The derivatives of the variations of the functions are eliminated from the double integrals with the aid of Green's theorem. The variation δx ' in the interval along ab is eliminated by means of integration by parts, taking into consideration the fact that $x(y_b)$ is fixed. Making use of the terminal equations of system (1.1), we shall express the variations δp and $\delta \rho$ in terms of the variations δu , δv and $\delta \phi$ according to the formulas

$$\delta p = -\rho u \delta u - \rho v \delta v - \frac{p}{\varphi} \delta \varphi, \quad \delta \rho = -\frac{\rho u}{a^2} \delta u - \frac{\rho v}{a^2} \delta v - \frac{\rho}{\varphi} \delta \varphi$$
 (2.2)

where $a^2 = \kappa p \rho^{-1}$ is the speed of sound squared. In addition, let us note that the functions describing the flow, at points lying in the shock wave, depend solely on σ — the angle of inclination of the shock wave to the x-axis — and on the parameters of the incoming flow. The variation of any of these functions f is related to the variation $\delta\sigma$ by the relation

$$\delta f = \frac{df}{ds} \, \delta s \tag{2.3}$$

Taking into consideration all the above statements, and using (2.2) and (2.3), we may transform expression (2.1) to the form /11

$$\begin{split} \delta I &= W_1 \delta y_b + \int\limits_{y_a}^{y_b} (U_1 \delta u + V_1 \delta v + \Phi_1 \delta \phi + W_2 \delta x) \, dy + \\ &+ \int\limits_{\psi_b}^{\psi_c} (U_2 \delta u + V_2 \delta v + \Phi_2 \delta \phi) \, d\psi + \int\limits_{\psi_c}^{\phi_a} W_3 \delta \sigma d\psi + \\ &+ \iint\limits_{\mathbb{R}} (U_3 \delta u + V_3 \delta v + \Phi_3 \delta \phi) \, d\psi dy \end{split}$$

The coefficients in front of the variations of u, v, ϕ , σ , y_b , x are known functions of the flow parameters and the Lagrange multipliers. We shall determine h_1 (ψ , y) and h_2 (ψ , y) in the region S by equating to zero the expressions next to δu and δv in the double integral.

We shall obtain the following system of partial differential equations:

$$U_{3} \equiv y^{\nu} \rho u \frac{\partial h_{1}}{\partial \psi} + \frac{\partial h_{1}}{\partial y} - \frac{1}{v} \frac{\partial h_{2}}{\partial \psi} - \frac{u}{y^{\nu} \rho v a^{2}} \frac{\partial h_{2}}{\partial y} = 0$$

$$V_{3} \equiv y^{\nu} \rho v \frac{\partial h_{1}}{\partial \psi} + \frac{u}{v^{2}} \frac{\partial h_{2}}{\partial \psi} - \frac{v^{2} - a^{2}}{y^{\nu} \rho v^{2} a^{2}} \frac{\partial h_{2}}{\partial y} = 0$$

$$(2.4)$$

The system (2.4) is hyperbolic for $w>\alpha$ and has two families of characteristics coinciding with the characteristics of the system (1.1). Along the characteristics, the following relations are satisfied

Here the upper sign refers to the characteristics of the first family.

By equating to zero the expression next to δx in the integral along ab, we shall obtain

$$\frac{d\gamma_0 v}{dy} = \sum_{i=1}^n \gamma_i \left[\frac{\partial f^i}{\partial x} - \frac{d}{dy} \left(\frac{\partial f^i}{\partial x'} \right) \right]$$
 (2.6)

After equating to zero the expressions multiplying δu and δv in the integral along ab and eliminating γ_0 (y) from them after having integrated equations (2.6) -- we obtain the boundary conditions for h_1 and h_2 along ab

$$h_1 = 1, h_2 = \sum_{i=1}^{n} \gamma_i \int_{y}^{y_b} \left[\frac{\partial f^i}{\partial x} - \frac{d}{dy} \left(\frac{\partial f^i}{\partial x'} \right) \right] dy + \gamma^*$$
 (2.7)

Here γ^* is an integration constant. By virtue of the first condition in (2.7), the coefficient in front of $\delta\phi$ in the integral along ab is canceled. In problems where \mathbf{y}_b is fixed, the part of the variation δI lying outside the integral vanishes because $\delta \mathbf{y} = 0$ for $\mathbf{y} = \mathbf{y}_b$. If -- according to the conditions of the problem -- \mathbf{y}_b is free, then it is necessary that the following equality be satisfied at the point \mathbf{y}_b .

$$W_1 \equiv py^{\nu} + \sum_{i=1}^{n} \gamma_i f^i = 0 \tag{2.8}$$

After equating to zero the coefficients standing by δu and δv in the integral along characteristic bc, we obtain the following after simple transformations

$$h_2 - y^{\nu}_0 v^2 t g \alpha h_1 = 0 {(2.9)}$$

We should note that the variation $\delta \varphi$ in the entire flow retains its value along the streamlines, and is expressed in terms of $\delta \sigma$ by means of formula (2.3).

The expression Φ_3 , standing in front of $\delta\phi$ in the double integral, contains the partial derivatives $\partial h_1/\partial\psi$ and $\partial h_2/\partial y$. Using (2.4), we shall express $\partial h_1/\partial\psi$ in terms of $\partial h_1/\partial y$ and $\partial h_2/\partial y$. Let mn be any streamline (Figure 1), with ψ being the corresponding value. We may transform the double integral to the form

$$\iint\limits_{\mathcal{S}} \Phi_3 \delta \phi d\psi dy = - \int\limits_{\psi_a}^{\psi_c} \left\{ \frac{1}{\phi} \frac{d\phi}{d\sigma} \, \delta \sigma \left[\int\limits_{h_1(\psi, y_m)}^{h_1(\psi, y_m)} \frac{u \sin^2 \alpha}{\varkappa} \, dh_1 + \int\limits_{h_2(\psi, y_m)}^{h_2(\psi, y_m)} \frac{\varkappa - \cos^2 \alpha}{\varkappa y \, \mathrm{'p} v} \, dh_2 \right] \right\} d\psi$$

Making use of the relations connecting the parameters of the flow in front of the shock wave to those behind it, one can simplify the expression W_3 standing in front of $\delta\sigma$ in the integral along the shock wave αc . Combining all the expressions standing by $\delta\sigma$, which enter the variation δI , and equating to zero the sum obtained, we find that for each ψ the following equality should be satisfied

$$E \equiv Nh_2 + A = 0, \quad \frac{1}{N} = \frac{1}{\varphi} \left(y^{\nu} \rho_0 w_0 \sin^2 \sigma \frac{d\varphi}{d\sigma} \right) \quad \left(\vartheta = \operatorname{arctg} \frac{v}{u} \right)$$
 (2.10)

$$A = \int_{h_1(\psi_1, y_m)}^{h_1(\psi_1, y_m)} \frac{u \sin^2 \alpha}{\varkappa} dh_1 + \int_{h_2(\psi_1, y_m)}^{h_2(\psi_1, y_m)} \frac{\varkappa - \cos^2 \alpha}{\varkappa y^{\mathsf{v}} \rho v} dh_2 + h_1 \frac{p \sin (\vartheta - \alpha)}{\rho w \sin \alpha} - \frac{h_2}{y^{\mathsf{v}} \rho v}$$
(2.10)

Here ρ_0 is the density of the gas in the incoming flow. The first term in equation (2.10) is calculated for the shock wave, and the terms lying outside the integral, which are contained in A, are calculated along the characteristic bc for the corresponding ψ . By virtue of the conditions (2.4), (2.6) through (2.10), the first-order variation of the functional (1.5) vanishes.

The equations obtained constitute a system of necessary conditions determining the optimum contour. Given a certain contour ab, one can compute -for a given incoming flow -- the flow in the region ${\mathcal S}$ which is a region of influence for ab. On the basis of the flow found, one can solve the Cauchy boundary problem for the multipliers \boldsymbol{h}_1 and \boldsymbol{h}_2 satisfying the system of equations (2.4) and the boundary conditions (2.7). The contour ab should be chosen in such a way that the flow parameters and the functions h_1 , h_2 satisfy the conditions (2.9) and (2.10). In fact, the part ad (Figure 1) of the contour determines the flow in the region adc and the shock wave ac -- i.e., the angle σ as a function of ψ . The part db for data pertaining to the characteristic dcdetermines the flow in the region bcd and the characteristic bc. Thus, two functions are not pre-assigned, and by controlling them one can satisfy the conditions (2.9) and (2.10). Moreover, we have n arbitrary constants γ_i , which are chosen in such a way that they should satisfy n isoperimetric conditions (1.3). The constant of integration γ^* is found from the condition that the equalities (2.7) and (2.9) should be simultaneously satisfied at the point b. If the value of \mathbf{y}_h is not pre-assigned, then to find it we have the condition (2.8).

3. We shall now study the relations (2.9) and (2.10) at the point c. The integral terms, entering in (2.10), vanish at this point. The conditions (2.9) and (2.10) with respect to h_1 and h_2 represent a homogeneous system of two linear algebraic equations. If h_1 and h_2 are not equal to zero, then the determinant of this system should vanish. Since all the flow parameters at the point in question are determined in terms of σ , we obtain the same condition for σ in the case of $\psi = \psi_c$

$$\varepsilon \equiv \kappa \rho w \sin^2 \theta - \frac{1}{\varphi} \frac{d\varphi}{d\sigma} \rho_0 w_0 \sin^2 \sigma \left[\kappa \sin \theta - \cos \alpha \sin (\theta - \alpha) \right] = 0$$
 (3.1)

This means that the solution of the problem formulated in this way exists only for certain special relations between the parameters of the incoming flow and the conditions (1.3) imposed on the body. In the general case, if h_1 and h_2 are continuous at c, one cannot satisfy the conditions (2.9) and (2.10) simultaneously. Therefore, it is necessary to introduce a discontinuity at c in the Lagrange multipliers.

In connection with the introduction of discontinuous multipliers, we /13 should mention the articles (Refs. 3, 7, 8). In (Ref. 3), the discontinuous Lagrange multipliers were first discussed in connection with the application to variational problems of gas dynamics. It was shown that it is only the characteristics which can serve as the lines of discontinuity, and the derivation of the conditions for the discontinuities was given. The problems of gas dynamics in connection with the utilization of discontinuous Lagrange multipliers were also investigated in (Ref. 5).

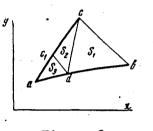


Figure 2

We shall divide the region S into three subregions, as it is shown in Figure 2. Here dc_1 is the characteristic of the second family. Let us consider the equations (1.1) in the subregions S_{2} with the help of the multipliers $h_1^{(i)}$ and $h_2^{(i)}$. Inside each of the subregions, the multipliers are determined with the aid of the system of

equations (2.4), and the multipliers satisfy the conditions (2.7) in the parts ad and db. In calculating the first-order variation, we must deal with integrals along the characteristics dc_1 and cd, containing under the integral sign the variations $\delta u,\ \delta v,$ and $\delta \sigma.$ Considering that the flow parameters and their variations are continuous in dc_1 and cd, we find that the coefficients of $\delta \mathbf{u}$, $\delta \mathbf{v}$, and $\delta \sigma$ will contain Δh_1 and Δh_2 where Δh_1 , Δh_2 are the discontinuities of the multipliers. Equating to zero the expressions multiplying δu and δv , we shall obtain two conditions for the discontinuities. Along the characteristics, these conditions are linearly dependent. We have (Ref. 3)

$$y_0^* v^2 \operatorname{tg} \Delta h_1 \pm \Delta h_2 = 0 \tag{3.2}$$

where the upper sign refers to the characteristics of the first family. for example, if Δh_1 is given at one point of the characteristic, one can determine from (3.2) and (2.5) the discontinuity of the multipliers along the entire characteristic. The condition (2.9) will not change with the introduction of discontinuous multipliers, and the condition (2.10) will assume the form

$$E \equiv \begin{cases} Nh_{2}^{(2)} - \Delta h_{1}^{(1)} B + A = 0 & (\psi_{c_{1}} \leqslant \psi \leqslant \psi_{c}) \\ Nh_{2}^{(3)} - \Delta h_{1}^{(1)} B + \Delta h_{1}^{(2)} C + A = 0 & (\psi_{a} \leqslant \psi \leqslant \psi_{c_{1}}) \\ \Delta h_{1}^{(1)} = h_{1}^{(2)} - h_{1}^{(1)}, & \Delta h_{1}^{(2)} = h_{1}^{(3)} - h_{1}^{(2)} \end{cases}$$

$$(3.3)$$

$$B = \frac{p \sin (\theta + \alpha)}{\rho w \sin \alpha} - v \operatorname{tg} \alpha, \qquad C = \frac{p \sin (\theta - \alpha)}{\rho w \sin \alpha} - v \operatorname{tg} \alpha$$

 $B = \frac{p\sin(\theta + \alpha)}{\rho w \sin \alpha} - v \operatorname{tg} \alpha, \qquad C = \frac{p\sin(\theta - \alpha)}{\rho w \sin \alpha} - v \operatorname{tg} \alpha$ Here $\Delta h_1^{(1)}$ and B are taken along the characteristic cd, and $\Delta h_1^{(2)}$ \mathcal{C} -- along dc_1 . By integrating (3.2) and (2.5) we find

$$\Delta h_1^{(1)} = \frac{\mu_1}{\sqrt{G}}, \quad \Delta h_2^{(1)} = -\mu_1 \sqrt{G}$$
 (3.4)

$$\Delta h_1^{(2)} = \frac{\mu_2}{\sqrt{G}}, \quad \Delta h_2^{(2)} = \mu_2 \sqrt{G}$$

$$(G = y^{\nu} \rho v^2 \operatorname{tg} \alpha)$$
(3.4)

Here the constant μ_1 is determined by the condition that the equalities (2.9) and (2.10) should simultaneously be satisfied at c, and $\mu_2 = {}^{-\mu}1^{\bullet}$. In this case, $h_1 = 1$ at the point d when we approach it either from the left or from the right. It may be readily seen that the second condition (2.7) is not satisfied on the left of the discontinuity. Therefore, at d the flow parameters should break down, and this is provided by the introduction of a break in the contour ab at that point. The discontinuities $\Delta h_1(2)$ and $\Delta h_2(2)$, propagating along dc_1 , will reach the point c_1 located on the shock wave. At this point, there is only one arbitrary choice possible — the choice of the angle σ . The condition (3.3) should be continuous for $\psi = \psi_{c_1}$. In the general case, this does not take $\frac{14}{2}$ place. Therefore it is necessary to introduce at point c_1 a discontinuity of multipliers, which will be propagated (Figure 3) along the characteristic of the first family c_1d_1 . At point d_1 , the contour should again suffer a break, and the reflected discontinuity of the multipliers is propagated along the characteristic of the second family d_1c_2 . In other words, the same considerations apply to the region adc_1 , as to the region abc. We may deal in an analogous way with the region adc_1 , as to the region abc. We may deal in an analogous way with the region adc_1 , etc. Regions appear $c_1d_1c_2$, $d_1c_2d_2$, $c_2d_2c_3$, $d_2c_3d_3$,, $d_{k-1}c_kd_k$, $c_kd_kc_{k+1}$, ..., in which the equations (1.1) are taken into consideration with the help of the Lagrange multipliers $h^{(4)}$, $h^{(5)}$, $h^{(5)}$, ..., $h^{(5)}$, ..., $h^{(2k+1)}$, $h^{(2k+2)}$, $h^{(2k+2)}$, $h^{(2k+2)}$, $h^{(2k+2)}$, $h^{(2k+2)}$,

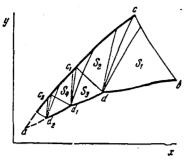
The condition (3.3) will assume the form

$$E \equiv \begin{cases} h_{2}^{(2)}N - \Delta h_{1}^{(1)}B + A = 0 & (\psi_{c_{1}} \leqslant \psi \leqslant \psi_{c}) \\ h_{2}^{(4)}N - \Delta h_{1}^{(1)}B + \Delta h_{1}^{(2)}C - \Delta h_{1}^{(3)}B_{1} + A = 0 & (\psi_{c_{2}} \leqslant \psi \leqslant \psi_{c_{1}}) \\ h_{2}^{(6)}N - \Delta h_{1}^{(1)}B + \Delta h_{1}^{(2)}C - \Delta h_{1}^{(3)}B_{1} + \Delta h_{1}^{(4)}C_{1} - \Delta h_{1}^{(6)}B_{2} + A = 0 & (\psi_{c_{2}} \leqslant \psi \leqslant \psi_{c_{2}}) \\ \vdots & \vdots & \vdots & \vdots \\ h_{2}^{(2k)}N - \Delta h_{1}^{(1)}B + \Delta h_{1}^{(2)}C - \dots + \Delta h_{1}^{(2k-2)}C_{k-2} - \Delta h_{1}^{(2k-1)}B_{k-1} + A = 0 \\ \vdots & \vdots & \vdots & \vdots \\ h_{2}^{(2k)}N - \Delta h_{1}^{(1)}B + \Delta h_{1}^{(2)}C - \dots + \Delta h_{1}^{(2k-2)}C_{k-2} - \Delta h_{1}^{(2k-1)}B_{k-1} + A = 0 \\ \vdots & \vdots & \vdots & \vdots \\ h_{2}^{(2k)}N - \Delta h_{1}^{(2k)}B + \Delta h_{1}^{(2)}C - \dots + \Delta h_{1}^{(2k-2)}C_{k-2} - \Delta h_{1}^{(2k-1)}B_{k-1} + A = 0 \\ \vdots & \vdots & \vdots & \vdots \\ h_{2}^{(2k)}N - \Delta h_{1}^{(2k)}B + \Delta h_{1}^{(2)}C - \dots + \Delta h_{1}^{(2k-2)}C_{k-2} - \Delta h_{1}^{(2k-1)}B_{k-1} + A = 0 \\ \vdots & \vdots & \vdots & \vdots \\ h_{2}^{(2k)}N - \Delta h_{1}^{(2k)}B + \Delta h_{1}^{(2)}C - \dots + \Delta h_{1}^{(2k-2)}C_{k-2} - \Delta h_{1}^{(2k-2)}B_{k-1} + A = 0 \\ \vdots & \vdots & \vdots \\ h_{2}^{(2k)}N - \Delta h_{1}^{(2k)}B + \Delta h_{1}^{(2)}C - \dots + \Delta h_{1}^{(2k-2)}C_{k-2} - \Delta h_{1}^{(2k-2)}B_{k-1} + A = 0 \\ \vdots & \vdots & \vdots & \vdots \\ h_{2}^{(2k)}N - \Delta h_{1}^{(2k)}B + \Delta h_{1}^{(2)}C - \dots + \Delta h_{1}^{(2k-2)}C_{k-2} - \Delta h_{1}^{(2k-2)}B_{k-1} + A = 0 \\ \vdots & \vdots & \vdots & \vdots \\ h_{2}^{(2k)}N - \Delta h_{1}^{(2k)}B + \Delta h_{1}^{(2)}C - \dots + \Delta h_{1}^{(2k-2)}C_{k-2} - \Delta h_{1}^{(2k-2)}B_{k-1} + A = 0 \\ \vdots & \vdots & \vdots & \vdots \\ h_{2}^{(2k)}N - \Delta h_{1}^{(2k)}B + \Delta h_{1}^{(2k)}C - \dots + \Delta h_{1}^{(2k-2)}C_{k-2} - \Delta h_{1}^{(2k-2)}B_{k-1} + A = 0 \\ \vdots & \vdots & \vdots & \vdots \\ h_{2}^{(2k)}N - \Delta h_{1}^{(2k)}B + \Delta h_{1}^{(2k)}C - \dots + \Delta h_{1}^{(2k-2)}C_{k-2} - \Delta h_{1}^{(2k-2)}B_{k-1} + A = 0 \\ \vdots & \vdots & \vdots & \vdots \\ h_{2}^{(2k)}N - \Delta h_{1}^{(2k)}B + \Delta h_{1}^{(2k)}C - \dots + \Delta h_{1}^{(2k-2)}C_{k-2} - \Delta h_{1}^{(2k-2)}B_{k-1} + A = 0 \\ \vdots & \vdots & \vdots & \vdots \\ h_{2}^{(2k)}N - \Delta h_{1}^{(2k)}B + \Delta h_{1}^{(2k)}C - \dots + \Delta h_{1}^{(2k)}B + \Delta h_{1}^{(2k)}C + \dots + \Delta h_{1}^{(2k)}C + \dots + \Delta h_{1}^{(2k)}B + \Delta h_{1}^{(2k)}C + \dots + \Delta h_{1}^{(2k)}B + \Delta h_{1}^{(2k)}C + \dots + \Delta h_{1}^{(2k)}C + \dots + \Delta h_{1}^{(2k)}C + \dots + \Delta h_{1}^{(2k)}C +$$

Here B_i , C_i have the same meaning as B and C, but are taken along the characteristics $d_i c_i$, $d_i c_{i+1}$, correspondingly.

We shall determine the discontinuity at a point c_k . For the continuity of the condition (3.5) at points c_1 , c_2 , ..., c_k ..., we must require that

$$\Delta h_{2}^{(2k+1)} N \, + \, \Delta h_{2}^{(2k)} N \, + \, \Delta h_{1}^{(2k)} C_{k-1} - \Delta h_{1}^{(2k+1)} B_{k} \, = \, 0$$





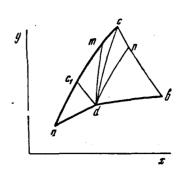


Figure 4

Taking into account (3.2), after certain transformations we obtain the following at \boldsymbol{c}_{k}

$$\Delta h_1^{(2k+1)} = \Delta h_1^{(2k)} \frac{\varepsilon}{\varepsilon + 2w \sin^2 \alpha \cos \theta \kappa^{-1}}$$
 (3.6)

where ε is the left-hand side of equation (3.1). According to this formula, we can also determine the initial discontinuity at c for $\Delta h_1^{(0)} = 1$. The discontinuity $\Delta h_1^{(2k)}$ on the body at d_{k-1} is determined from the condition $h_1^{(2k+1)} = 1$. Formula (3.6) shows that $\Delta h_1^{(2k+1)} = 0$, if at c_k the angle σ satisfies equation (3.1). Moreover, $\Delta h_1^{(2k+1)} \leq \Delta h_1^{(2k)}$ if $\cos \theta > 0$ and σ at c_k is such that $\varepsilon > 0$. Thus, a discontinuity of the Lagrange multipliers, when there is a reflection from the shock wave, is damped for a certain region of angles σ . The question of damping a discontinuity for the reflection from the contour of the body can be solved numerically.

Thus, a contour ab, containing infinitely many breaks, can be a solution of our problem. In the special case, when the condition (3.1) is satisfied at c, the contour ab is smooth.

- 4. We shall investigate the problem of determining a contour, which guarantees the minimum wave drag and which passes through the given points a and b.
- G.G. Chernyy, when studying the flow around bodies similar to a wedge, showed that, for bodies with a preassigned elongation, the wedge is not the optimum form (Ref. 9). Only in the special case, when the preassigned aspect ratio is such that the condition (3.1) is satisfied, does the wedge guarantee minimum wave drag. For the axisymmetric case, in the case of an associated shock wave the corresponding class of smooth bodies was found in the work (Ref. 10). An analogous result for a non-uniform, incoming flow was obtained in (Ref. 11). In the article (Ref. 12) with the same formulation of the problem as in (Ref. 9), but for larger Mach numbers, it was shown that the wave drag for a fixed lifting power and bodies similar to a wedge decreases with the introduction of an infinite number of breaks of positive and negative signs.

For purposes of simplicity, we shall consider first the flow structure/15 (figure 4) which is determined by one break of the contour at the point d. In

 this case, the isoperimetric condition over the length, with (1.4) being taken into consideration, is expressed in the following way:

$$x_b - x_a = \int_{y_a}^{y_d} \frac{u}{v} \, dy + \int_{y_d}^{y_b} \frac{u}{v} \, dy$$

The Lagrangian functional has the form

$$I = \int_{y_{a}}^{y_{d}} \left(py^{\nu} + \lambda \frac{u}{v} \right) dy + \int_{y_{d}}^{y_{b}} \left(py^{\nu} + \lambda \frac{u}{v} \right) dy + \\ + \sum_{i=1}^{3} \int_{S_{i}}^{y} \left[h_{1}^{(i)}(\psi, y) L_{1} + h_{2}^{(i)}(\psi, y) L_{2} \right] dy d\psi$$

$$(4.1)$$

Here λ is a constant Lagrange multiplier; S_1 is the region bcd of flow; S_2 is the region cdc_1 ; S_3 denotes the region adc_1 . The region dmcnd corresponds to the flow of rarefaction, caused by the break at point d. The location of the point d is unrestricted. The necessary conditions for an extremum in the regions amd and bnd may be written just as in Section 2. We shall compute the variation of the part of the functional I, referring to the region dmcnd, which will be denoted by τ . The partial derivatives with respect to y, which are contained in the differential equations (1.1), become infinite at point d. Therefore, the variation of the double integral — when the point d is varied — may not be equal to zero.

The calculations were performed in accordance with the work by A. N. Krayko (Ref. 6). For this purpose, the double integral over the region τ will be written in terms of the new independent variables ω, ψ^* which are related to y, ψ by the formulas

$$\psi^* = \psi, \ \omega = \operatorname{arctg}(\psi/y - y_d) \tag{4.2}$$

In terms of these variables, the derivatives contained in the integrand become finite, and the process of calculating the variation is analogous to the process described above. The expressions multiplying the variations δu , δv , and $\delta \sigma$ in the double integral — after returning to the variables ψ , y — coincide with the corresponding expressions obtained in Section 2. Moreover, the function y, which enters in equations (1.1) and is not an independent variable, is also varied. From the conversion formulas (4.2), it follows that at any point of the region τ the equality $\delta y = \delta y_d$, is satisfied, where δy_d is the displacement of the point y_d — a quantity which can be taken in front of the integral sign.

Let f be one of the functions describing the flow in the regions dc_1^{md} and dnbd, and f^* — the same function in the region τ . The variations of this function at the boundary of the region τ can be expressed in the following way:

$$\delta f = \Delta f - \frac{\partial f}{\partial y} \Delta y$$
 $\delta f^* = \Delta f - \frac{\partial f^*}{\partial \omega} \Delta \omega$

Here Δy is a change in the ordinate of the point on the boundary for fixed ψ , and $\Delta \omega$ is the corresponding change of ω . The quantity Δf represents the difference between the values of f at the initial limit and the varied limit for fixed ψ . It follows from the formulas (4.2) that $\Delta \omega = \psi^{-1}(\delta y_d - \Delta y) \sin^2 \omega$, We shall combine the terms, entering in the contour integral along dmend and the double integral over the region τ , containing δy_d . As was shown in (Ref. 6), the sum of all these terms, with the utilization of equations (1.1) and (2.4), may be reduced to

$$\delta y_d \int_{\omega_+}^{\omega_-} \left[h_1 y^{\nu} \frac{dp}{d\omega} + h_2 \frac{d}{d\omega} \left(\frac{u}{v} \right) \right] d\omega$$

where ω_{\perp} is the angle of inclination of the characteristic dm at point d, and ω_{-} is the angle of inclination of the characteristic dn at d. In the remaining terms of the contour integral, we shall change to the independent variables ψ ,y. On the sections md and nd, we shall collect the terms next to Δf to the right and to the left of the boundary. These expressions will vanish if $h_{\mathbf{1}}$ and $h_{
m p}$ are continuous. If they are discontinuous, however, then we shall obtain the condition (3.2) for the discontinuities. In the expressions, referr- $\frac{16}{16}$ ing to Δ y, along md and nd all the partial derivatives can be converted to total derivatives by using (1.1). By again collecting the terms standing in front of the total derivatives to the left and to the right of the boundary, we shall obtain the expressions which vanish identically. If h_1 , h_2 are continuous, this is obvious. If they are discontinuous, then, making use of (3.2), one can show that these expressions are the compatability conditions along the characteristics of the system (1.1). On the section mc of the shock wave, $\Delta y = 0$ and Δf are expressed in terms of $\delta\sigma$. On the part cn of the characteristic, the value $\Delta f \sim (\partial f/\partial y)$ Δy , which is equal to δf , is left from the variation δf^* . By examining the expression multiplying δf , we see that we obtain the same extremum conditions, as for the part nb.

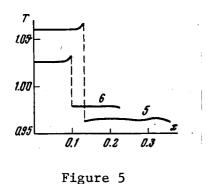
Finally, the extremum conditions will be the following. In the regions $S_{\vec{i}}$, the equations (2.4) serve for the determination of $h_1(i)$ and $h_2(i)$. On the parts ad and db of the contour, the formulas (2.7) hold. For the problem in question, they yield

$$h_1^{(1)} = h_1^{(3)} = 1, h_2^{(1)} = h_2^{(3)} = \lambda$$
 (4.3)

At the point d, the following condition should be satisfied

$$\left(y^{\nu}p + \lambda \frac{u}{v}\right)_{+} - \left(y^{\nu}p + \lambda \frac{u}{v}\right)_{-} + \int_{\omega_{+}}^{\omega_{-}} \left[h_{1}y^{\nu} \frac{dp}{d\omega} + h_{2} \frac{d}{d\omega} \left(\frac{u}{v}\right)\right] d\omega = 0 \qquad (4.4)$$

Here the plus sign refers to the values at the point d being approached from the left, and the minus sign — to the corresponding values when the point d is approached from the right. When the problem is studied as a whole (Figure 3), analogous equalities apply for the points d_1 , d_2 , ...



abc, the condition (3.3) should be satisfied.

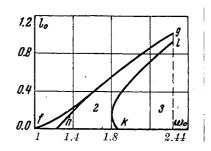


Figure 6

On the characteristic bc, it is necessary to satisfy the conditions (2.9), and on the characteristics dc and dc_1 — the conditions (3.2). For all ψ in

It may be readily seen that the equalities h_1 = 1, h_2 = λ give the solution of the system (2.4) for the regions ac_1d and dnb. We shall substitute these expressions in the condition (2.9).

We find that the following equality is satisfied along the characteristic nb

$$\lambda = y^{\nu} \rho v^2 \operatorname{tg} \alpha \tag{4.5}$$

if the contour bd has a minimum wave drag for the fixed characteristic dn. This relation was obtained in (Ref. 13) when the author examined the variational problem of constructing a generatrix with the fixed points d and b and a fixed characteristic dn. Hence it follows that it is more convenient to solve the opposite problem -- i.e., to find a shock wave ac and a characteristic bc, satisfying the extremum conditions. After solving equations (1.1) numberically, these may then be used to construct the required contour ab.

For the numerical solution of the problem, the following iteration process is proposed. A point m is given on the y-axis in the xy plane. The segment $y_m - y_a$ is divided into N equal parts with the ordinates y_k (k = 0,1,...,N). σ_k is given at each point y_k . In the xy plane, we construct the shock wave am which is the initial approximation to the original wave (Figure 4). From the data pertaining to the former, we may compute the flow in the region and, and may select the contour ad of the body. Furthermore, we choose a certain break angle $\Delta\theta_1$ from which we may calculate the flow of rarefaction in the region and and may compile a part mc of the shock wave. From a certain chosen value $\Delta\theta_2$ of the break angle, we may calculate the region dend. At the point n, we compute λ according to formula (4.5). The characteristic nb up to $\psi = 0$ is calculated from formula (4.5) and the relations for the characteristic of the second family.

Along the characteristics nd and nb, $h_1^{(1)}$ and $h_2^{(1)}$ are given, since they are defined over the entire region dnb. We shall calculate the term entering

in Equation (3.3) which refer to the characteristic bn. By the method of characteristics, and using (2.9), we determine $h_1^{(1)}$ and $h_2^{(1)}$ in the region dcnd. From formulas (3.4) we compute $h_1^{(2)}$ and $h_2^{(2)}$ along the characteristic cd. Here μ_1 is determined by the condition that (2.9) and (3.3) be simultaneously satisfied at the point c. In the region dmcd, the functions $h_1^{(2)}$ and $h_2^{(2)}$ are calculated by the method of characteristics from the data on the characteristic cd, and by drawing upon the condition (3.3) along the part cm of the shock wave. From formulas (3.4) and the known values of $h_1^{(3)}$ and $h_2^{(3)}$, we may determine $h_1^{(2)}$ and $h_2^{(2)}$ along the characteristic c_1d . The constant dc_2 is found from the condition of continuity of $h_1^{(3)}$ and $h_1^{(1)}$ at the point d. The quantities dc_1 and dc_2 are chosen in such a way that the condition (4.4) and $dc_2^{(2)}$ and $dc_2^{(2)}$ are chosen in such a way that the condition (4.4) and $dc_2^{(2)}$ and $dc_2^{(2)}$ and $dc_2^{(2)}$ and $dc_2^{(2)}$ and $dc_2^{(2)}$ and $dc_2^{(2)}$ in the region dc_1md .

It may happen that the necessary values of the discontinuities of the multipliers h_1 and h_2 at the point c_1 are equal to zero, for the chosen computational accuracy. In this case, we can disregard the breaks at the points d_1, d_2, \ldots . Then the function E at each of the points y_k on the shock wave is calculated from the known flow and the known Lagrange multipliers. The inclination angles for the new shock wave am are found from the formula

$$\sigma_{j+1}(y_k) = \sigma_j(y_k) + \eta E_j(y_k)$$

where j is the number of the successive iteration, η is a certain number satisfying the condition $|\eta| < 1; k = 0,1,2,\ldots,N-1$. At the point m, the quantity $\sigma(y_N)$ remains fixed, since E = 0 by virtue of the choice of h_1 and h_2 along cm, when calculating the Lagrange multipliers inside the region dmcd. The calculations are continued until the magnitude of E is equal to zero at all points y_k . Then from the obtained shock wave cm, two break angles cm and cm and cm and the characteristic cm of the second family-subordinated to the condition (4.5)-we may construct the contour cm. This contour gives the solution of the problem, if the obtained point b coincides with the given point. Two arbitrary parameters cm and cm allow us to solve the direct problem. The iteration process in the case of breaks at points cm cm is analogous to the one described before.

5. The calculations were performed in the two-dimensional case for different values of the velocity of the incoming uniform flow \mathbf{w}_0 , referred to the critical flow velocity. The gas was considered to be ideal with the exponent of the adiabatic curve \times equal to 1.4. The table gives the most

characteristic results of the calculations,

No	1	2	3	4	5	6	7
w_0	1.4000	1.6000	2.1000	2.3000	2.3000	2.3000	2.4000
l_0	0.2128	0.3358	0.4370	0.2069	0.4801	0.7237	0.8489
c_{xm}	1.0926	1.1815	0.6090	0.1717	0.5380	0.9519	0.9178
c_{xk}	1.0927	1.1818	Q.6096	0.1721	0.5413	0.9533	0.9215
x_d	0.7138	1.0378	1.1226	1.9555	1.3248	1.0112	1.0422
y_d	0.1533	0.4021	0.5056	0.4289	0.6744	0.7507	0.8133
x_b	1.8354	1.6785	3.4001	7.5288	3.5811	2.2167	2.2688
$oldsymbol{y}_b$	0.3906	0.6475	1.4858	1.5578	1.7195	1.6043	1.6979
Θ	-0.0031	-0.0047	-0.0209	-0.0172	-0.0499	-0.0299	-0.0526
$\Delta \stackrel{\bigcirc}{h_{1c}}$ (1)	0.0280	0.0367	0.0685	0.1190	0.1536	0.0788	0.1303
$\Delta h_{1d}^{(1)}$	0.0280	0.0361	0.0686	0.1192	0.1552	0.0792	0.1318
$\Delta h_{1d}(2)$	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
$\Delta h_{1e_1}^{(2)}$	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
$\Delta h_{2c}^{(1)}$	-0.0050	-0.0370	-0.0518	-0.0150	-0.1759	-0.2089	-0.4056
$\Delta h_{2d}^{-1}(1)$	-0.0050	-0.0376	-0.0516	-0.0150	-0.1741	-0.2079	-0.4010
Δh_{2d}^{-2}	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
$\Delta h_{2c_1}^{2c_2}(2)$	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000

The following notation has been adopted: 1 is the inverse of the aspect ratio; c_{xm} and c_{xk} are the coefficients of the wave drag for the optimum contour and a wedge with the equivalent wedge aspect ratio; Θ is the combined break angle at the point d, equal to $\Delta\theta_1$ + $\Delta\theta_2$. Also the values of the discontinuities of the Lagrange multipliers at the points c,d, and c_1 are given. Here $\Delta h_{ic}^{(1)}$ and $\Delta h_{id}^{(1)}$ are the quantities equal to $h_i^{(2)} - h_i^{(1)}$, which may be determined at the points c and d, respectively. The values of $\Delta h_{id}^{(2)}$ and $\Delta h_{ic_1}^{(2)}$ are defined (i =1,2) as $h_i^{(3)} - h_i^{(2)}$, which may be calculated at the points d and c_1 . The point a in all the examples has the coordinates (0,0). From the table given it can be seen that the values of the reflected discontinuities at the point d are equal to zero, within the accuracy of the calculations in question. With this accuracy, in all the examples the optimum body profiles obtained have only one break at the point d. $\,$ For examples 5 and 6 from the table, Figure 5 gives the dependence of T = tan θ /tan θ _k on x for the optimum profile, where θ _k is the angle of the equivalentwedge. All the examples are located in the regions fghand 3, shown in Figure 6. The curve fg for each value of w determines the wedge aspect ratio for which acoustic streaming occurs when the flow passes around the wedge. The curves hg and kl correspond to the values of the aspect ratio and of the incoming flow (References 9,10), $\,$ for which the wedge (arepsilon=0) will be the optimum form. In the regions fgh and 3, the value of ϵ is positive, and in region 2 it is negative.

The author is grateful to V.M. Borisov, A.N. Krayko, and Yu.D. Shmyglevskiy for the attention shown to the work.

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