(i)
$$\int_{\mathbb{R}^d} (f(x) + g(x)) dx = \int_{\mathbb{R}^d} f(x) dx + \int_{\mathbb{R}^d} g(x) dx,$$

$$\int_{\mathbb{R}^d} \alpha f(x) dx = \alpha \int_{\mathbb{R}^d} f(x) dx \text{ (linearity)}.$$

(ii) If
$$f \leq g$$
, then also $\int_{\mathbb{R}^d} f(x) dx \leq \int_{\mathbb{R}^d} g(x) dx$.

(iii) If $b \in \mathbb{R}^d$, A an orthogonal $d \times d$ -matrix, then

$$\int_{\mathbb{R}^d} f(Ax+b)dx = \int_{\mathbb{R}^d} f(x)dx.$$

Proof. Parts (i) follows directly from the corresponding rules for the integral of elementary functions. The rule (iii) holds for elementary functions, because for an orthogonal matrix A and an elementary function t, the function $t \circ A$ is again elementary as A maps a cube onto a cube, and similarly for translation by a vector b. (ii) follows, as in case $f \leq g$, we can approximate f and g by sequences (t_n) and (s_n) , respectively, of elementary functions with $t_n \leq s_n$.

Furthermore, we have

Lemma 13.5 Let $f \in C_c(\mathbb{R}^d)$ with supp(f) in a cube of side length ℓ . Then

$$\left| \int_{\mathbb{R}^d} f(x)dx \right| \le \sup |f(x)| \cdot \ell^d \,. \tag{2}$$

The proof again follows directly from the corresponding property for elementary functions.

Lemmas 13.4 and 13.5 mean that the correspondence

$$f \mapsto \int_{\mathbb{R}^d} f(x) dx$$

is a linear, bounded (therefore continuous) real valued functional on each Banach space $C_c(W)$ (W a cube in \mathbb{R}^d) that is invariant under isometries. One can show that a functional with these properties and the normalisation

$$\int_{I^d} 1 dx = 1$$