it follows that (d_n) is a sequence in \mathcal{N} which converges in the weak operator topology to $\lim w_n - \lim c_n$. Since \mathcal{N} is sequentially closed in the weak operator topology and $\lim c_n$ is in W, it follows that $\lim w_n$ is in W. So W is σ -closed in A''. Also, by Lemma 5.4.3,

$$k(\lim w_n) = k(\lim c_n) = c = \bigvee_{n>1} c_n = \bigvee_{n>1} k(w_n).$$

We now come to the representation theorem for monotone σ -complete C^* -algebras.

Theorem 5.4.5 Let A be monotone σ -complete. There exists a σ -homomorphism q from A^{∞} onto A such that q(a) = a for each $a \in A$. Then $A^{\infty} \cap \mathcal{N}$ is a σ -ideal of A^{∞} and is the kernel of q. So A is isomorphic to $A^{\infty}/(A^{\infty} \cap \mathcal{N})$.

Proof The smallest σ -closed subspace of A''_{sa} which contains A_{sa} is A^{∞}_{sa} . So $A^{\infty} \subset W$. Let q be the restriction of k to A^{∞} . The result follows from Corollary 5.4.4. \square

We recall that the algebra \mathcal{N} is the complex linear span of \mathcal{M}^+ . We shall see from the results of Sect. 5.6, that in Theorem 5.6.5, we may replace $A^{\infty} \cap \mathcal{N}$ by $A^{\infty} \cap \mathcal{M}$.

When specialised to *commutative* algebras, Theorem 5.4.5 corresponds to the Loomis-Sikorski theorem for Boolean σ -algebras [153].

By applying results of Birkhoff-Ulam for complete Boolean algebras, see Theorem 4.1.3, every commutative monotone complete C^* -algebra can be represented as follows. Let S be the spectrum of a commutative monotone complete C^* -algebra then C(S) is isomorphic to the quotient of the algebra of bounded Borel measurable functions on S modulo the ideal of meagre Borel functions. This may be thought of as a special case, for commutative algebras, of the following representation theorem. See Theorem 4.2.9.

Theorem 5.4.6 Let A be monotone complete. There exists a σ -homomorphism q from A^b onto A such that q(a) = a for each $a \in A$. Then $A^b \cap \mathcal{N}$ is a σ -ideal of A^b and is the kernel of q. So A is isomorphic to $A^b/(A^b \cap \mathcal{N})$. Let (c_λ) be a norm bounded increasing net in A_{sa} with least upper bound c in A_{sa} . Let $\lim c_\lambda$ be its strong (and so weak) operator limit in A'' (and so is in A^b), then

$$q(\lim c_{\lambda}) = c.$$

Furthermore, given $f \in A^b_{sa}$, $q(f) \le 0$ if and only if $f \le 0$ a.e.. So q(f) = 0 if, and only if, f = 0 a.e.

Proof Let (c_{λ}) be a norm-bounded, upward directed net in A_{sa} . Then $\lim c_{\lambda}$ is in A^b . By Lemma 5.4.3 $\lim c_{\lambda}$ is also in W. By definition, A^b_{sa} is the smallest σ -closed subspace of A''_{sa} which contains all x that correspond to lower semicontinuous affine functions on K. So $A^b \subset W$. Let q be the restriction of k to A^b .