the inequality 3.4.21 that $x \in N_{\phi}$ if and only if $\phi(y^*x) = 0 \ \forall y \in A$; this implies that N_{ϕ} is a vector subspace of A which is in fact a left-ideal (i.e., $x \in N_{\phi} \Rightarrow zx \in N_{\phi} \ \forall \ z \in A$).

Deduce now that the equation

$$\langle x + N_{\phi}, y + N_{\phi} \rangle = \phi(y^*x)$$

defines a genuine inner product on the quotient space $V = A/N_{\phi}$. For notational convenience, let us write $\eta(x) = x + N_{\phi}$ so that $\eta: A \to V$; since N_{ϕ} is a left-ideal in A, it follows that each $x \in A$ unambiguously defines a linear map $L_x: V \to V$ by the prescription: $L_x \eta(y) = \eta(xy)$.

We claim now that each L_x is a bounded operator on the inner product space V and that $||L_x||_{\mathcal{L}(V)} \leq ||x||_A$. This amounts to the assertion that

$$\phi(y^*x^*xy) = ||L_x\eta(y)||^2 \le ||x||^2||\eta(y)||^2 = ||x||^2\phi(y^*y)$$

for all $x, y \in A$. Notice now that, for each fixed $y \in A$, if we consider the functional $\psi(z) = \phi(y^*zy)$, then ψ is a positive linear functional; consequently, we find from Proposition 3.4.11 that $||\psi|| = \psi(1) = \phi(y^*y)$; in particular, we find that for arbitrary $x, y \in A$, we must have $\phi(y^*x^*xy) = \psi(x^*x) \leq ||\psi|| \cdot ||x^*x||$; in other words, $\phi(y^*x^*xy) \leq ||x||^2\phi(y^*y)$, as asserted.

Since V is a genuine inner product space, we may form its completion – call it \mathcal{H}_{ϕ} – where we think of V as a dense subspace of \mathcal{H}_{ϕ} . We may deduce from the previous paragraph that each L_x extends uniquely to a bounded operator on \mathcal{H}_{ϕ} , which we will denote by $\pi_{\phi}(x)$; the operator $\pi_{\phi}(x)$ is defined by the requirement that $\pi_{\phi}(x)\eta(y) = \eta(xy)$; this immediately implies that π_{ϕ} is an unital algebra homomorphism of A into $\mathcal{L}(\mathcal{H}_{\phi})$. To see that π_{ϕ} preserves adjoints, note that if $x, y, z \in A$ are arbitrary, then

$$\langle \pi_{\phi}(x)\eta(y), \eta(z) \rangle = \phi(z^*(xy))$$

$$= \phi((x^*z)^*y)$$

$$= \langle \eta(y), \pi_{\phi}(x^*)\eta(z) \rangle ,$$

which implies, in view of the density of $\eta(A)$ in \mathcal{H}_{ϕ} , that $\pi_{\phi}(x)^* = \pi_{\phi}(x^*)$, so that π_{ϕ} is indeed a representation of A on \mathcal{H}_{ϕ} . Finally, it should be obvious that $\xi_{\phi} = \eta(1)$ is a cyclic vector for this representation.

Conversely, if (\mathcal{H}, π, ξ) is another triple which also "works" for ϕ as asserted in the statement of the second half of Theorem 3.4.13, observe that for arbitrary $x, y \in A$, we have

$$\langle \pi(x)\xi, \pi(y)\xi \rangle_{\mathcal{H}} = \phi(y^*x) = \langle \pi_{\phi}(x)\xi_{\phi}, \pi_{\phi}(y)\xi_{\phi} \rangle_{\mathcal{H}_{\phi}}$$

for all $x, y \in A$; the assumptions that ξ and ξ_{ϕ} are cyclic vectors for the representations π and π_{ϕ} respectively imply, via Exercise 3.4.12, that there exists a unique unitary operator $U: \mathcal{H} \to \mathcal{H}_{\phi}$ with the property that $U(\pi(x)\xi) = \pi_{\phi}(x)\xi_{\phi}$ for all $x \in A$; it is clear that U has the properties asserted in Theorem 3.4.13.