PROOF. The quotient space E/F is Hausdorff and finite-dimensional and the natural projection  $\pi\colon E\to E/F$  is continuous. Its restriction  $\pi|_G$  to G is an algebraic, hence by the proven theorem also a topological isomorphism between G and E/F. The projection  $p_G\colon E\to G$  has the form  $p_G=(\pi|_G)^{-1}\circ\pi$ , hence is continuous.

**1.5.4. Definition.** A subset A of a topological vector space E is called precompact or totally bounded if, for every neighborhood of zero V in E, one can find a finite set  $\{a_1, \ldots, a_n\}$  in E such that  $A \subset \bigcup_{k=1}^n (a_k + V)$ .

The set  $\{a_1, \ldots, a_n\}$  is called a finite V-net (or an  $\varepsilon$ -net if V is a ball of radius  $\varepsilon$  in a metric space). It is easily seen that every compact set in a topological vector space is precompact. We observe (this fact is not needed now, so it will be proven in § 1.8 after we discuss completions of topological vector spaces) that a subset of a topological vector space is precompact precisely when its closure in the completion of this topological vector space is compact. However, the closure of a precompact set in an incomplete space may fail to be compact.

**1.5.5. Lemma.** Every precompact subset of a topological vector space is bounded.

PROOF. Let A be a precompact subset of a topological vector space E and let V be a neighborhood of zero in E; we have to prove that there exists  $\nu > 0$  such that  $A \subset tV$  if  $|t| > \nu$ . Let W be a circled neighborhood of zero such that  $W + W \subset V$  and let  $a_1, \ldots, a_n$  be elements of E for which  $A \subset \bigcup_{k=1}^n (a_k + W)$ . Let  $\nu > 1$  be such that  $\{a_1, \ldots, a_n\} \subset tW$  if  $|t| > \nu$ . Then for such numbers t we have

$$A \subset \bigcup_{k=1}^{n} (a_k + W) \subset tW + W \subset tW + tW = t(W + W) \subset tV,$$

which shows that A is bounded.

**1.5.6. Theorem.** A Hausdorff topological vector space E over the field  $\mathbb{R}$  or  $\mathbb{C}$  is finite-dimensional if and only if it possess a compact neighborhood of zero. Moreover, it is sufficient that it possess a precompact neighborhood of zero.

PROOF. The necessity is clear from Theorem 1.5.1, since any Hausdorff topological vector space (over a nondiscrete complete normed field  $\mathbb{K}$ ) of finite dimension n is isomorphic (as a topological vector space) to the space  $\mathbb{K}^n$ , and if S is a compact neighborhood of zero in  $\mathbb{K}$ , then the product of n copies of S is a compact neighborhood of zero in  $\mathbb{K}^n$ .

Let us prove the sufficiency (the given proof is due to Gleason). Let V be a precompact neighborhood of zero in E and let  $a_1, \ldots, a_n \in E$  be elements such that

$$V \subset \bigcup_{k=1}^{n} \left( a_k + \frac{1}{2}V \right). \tag{1.5.3}$$

We show that the linear span of the set  $A = \{a_1, \ldots, a_n\}$  coincides with the whole space, i.e., that every element in E is a linear combination of elements of