Intuitively, this holds since Theorem 4.4.4 implies

$$H(t) = \int_0^t h(t-s)\,dU(s)$$

and Theorem 4.4.3 implies $dU(s) \to ds/\mu$ as $s \to \infty$. We will define directly Riemann integrable in a minute. We will start doing the proof and then figure out what we need to assume.

Proof. Suppose

$$h(s) = \sum_{k=0}^{\infty} a_k 1_{[k\delta,(k+1)\delta)}(s)$$

where $\sum_{k=0}^{\infty} |a_k| < \infty$. Since $U([t, t+\delta]) \le U([0,\delta]) < \infty$, it follows easily from Theorem 4.4.3 that

$$\int_0^t h(t-s)dU(s) = \sum_{k=0}^{\infty} a_k U((t-(k+1)\delta, t-k\delta]) \to \frac{1}{\mu}\sum_{k=0}^{\infty} a_k\delta$$

(Pick K so that $\sum_{k\ge K} |a_k| \le \epsilon/2U([0,\delta])$ and then T so that

$$|a_k| \cdot |U((t-(k+1)\delta, t-k\delta]) - \delta/\mu| \le \frac{\epsilon}{2K}$$

for $t \ge T$ and $0 \le k < K$.) If h is an arbitrary function on $[0,\infty)$, we let

$$I^\delta = \sum_{k=0}^{\infty} \delta \sup\{h(x) : x \in [k\delta, (k+1)\delta)\}$$

$$I_\delta = \sum_{k=0}^{\infty} \delta \inf\{h(x) : x \in [k\delta, (k+1)\delta)\}$$

be upper and lower Riemann sums approximating the integral of h over $[0,\infty)$. Comparing h with the obvious upper and lower bounds that are constant on $[k\delta, (k+1)\delta)$ and using the result for the special case,

$$\frac{I_\delta}{\mu} \le \liminf_{t\to\infty} \int_0^t h(t-s)\,dU(s) \le \limsup_{t\to\infty} \int_0^t h(t-s)\,dU(s) \le \frac{I^\delta}{\mu}$$

If I^δ and I_δ both approach the same finite limit I as $\delta \to 0$, then h is said to be **directly Riemann integrable**, and it follows that

$$\int_0^t h(t-s)\,dU(y) \to I/\mu \qquad \blacksquare$$

Remark. The word "direct" in the name refers to the fact that although the Riemann integral over $[0,\infty)$ is usually defined as the limit of integrals over $[0,a]$, we are approximating the integral over $[0,\infty)$ directly.

In checking the new hypothesis in Theorem 4.4.5, the following result is useful.

Lemma 4.4.6. *If $h(x) \ge 0$ is decreasing with $h(0) < \infty$ and $\int_0^\infty h(x)\,dx < \infty$, then h is directly Riemann integrable.*