a finite result – whence $Mf \in \mathcal{S}$ – and implying the estimate

$$(32.6) \qquad N_r(Mf) \le c_r N_{r+1}(f)$$

with a constant c_r whose exact value is of little importance, because (6) is enough to establish the continuity of M.

As a map of $\mathcal{S}$ into $\mathcal{S}$, the Fourier transform F is also *continuous* in each sense. To see this without much calculating, first remark that

$$N_{p,q}(f) = N_0 \left(M^p D^q f \right) = \| M^p D^q f \|_{\mathbb{R}}$$

and then $N_{p,q}(\hat{f}) = N_0 \left(M^p D^q F f \right) = N_0 \left(M^p F M^q f \right) = N_0 \left(F D^p M^q f \right)$ by the "commutation formulae" (31.2") and (31.6). Now, in general,

$$N_0(Ff) = \sup \left| \int f(x) \overline{\mathbf{e}(xy)} dx \right| \le \int |f(x)| dx = \|f\|_1;$$

since the function $(x^2 + 1) f(x)$ is bounded by $N_2(f)$, by (3), one finds

$$N_0(Ff) \le N_2(f) \int \left(x^2 + 1 \right)^{-1} dx,$$

with a convergent integral whose exact value, $c = \pi$, is not important. It follows that
$$N_{p,q}(\hat{f}) = N_0 \left(F D^p M^q f \right) \le c N_2 \left(D^p M^q f \right);$$

on applying (5) p times to the function $M^q f$ one finds a result $\le N_{p+2} \left(M^q f \right)$ up to a constant factor, and by applying (6) q times to f one obtains a relation of the form $N_{p,q}(\hat{f}) \le N_{p+q+2}(f)$ up to a constant factor. Remembering the definition (3) of N_r, we finally have

$$(32.7) \qquad N_r(\hat{f}) \le c_r' N_{r+2}(f)$$

where c_r' is a new constant. This proves the continuity of the Fourier transform. Since it is bijective and quasi identical to its inverse map by virtue of the relation $\widehat{\widehat{f}}(x) = f(-x)$, we conclude that the Fourier transform is a bijective and bicontinuous map of $\mathcal{S}$ onto $\mathcal{S}$, in other words what in topology one calls a *homeomorphism* (linear too) of $\mathcal{S}$ onto $\mathcal{S}$.

With their systematic recourse to the operators D, M and F, these calculations can appear a little abstract. But to write explicitly the integrals and derivatives which they mask would be even less enticing.

We can now return to distribution theory. Following Schwartz, we will call any *continuous* linear form $T : \mathcal{S} \to \mathbb{C}$ a *tempered distribution* on $\mathbb{R}$. The inequality $|T(f)| < \varepsilon$ has to hold for every $f \in \mathcal{S}$ "close enough" to 0; this means that there exists an $r \in \mathbb{N}$ and a $\delta > 0$ such that