230 Special Limits

8. Iterated Ends and Limits

We now describe when the "double integral" can be obtained as an "iterated" integral (Fubini!).

Proposition. Let $S: P^{op} \times P \times C^{op} \times C \to X$ be a functor such that the end $\int_{c} S(p,q,c,c)$ exists for all pairs $\langle p,q \rangle$ of objects of P; by the parameter theorems, regard these ends as a bifunctor $P^{op} \times P \to X$, and regard S as a bifunctor $(P \times C)^{op} \times (P \times C) \to X$. Then there is an isomorphism

$$\theta: \int_{\langle p,c\rangle} S(p,c,p,c) \cong \int_{p} \left[\int_{c} S(p,p,c,c)\right].$$

Indeed, the "double end" on the left exists if and only if the end \int_{p}^{p} on the right exists, and then there is a unique arrow θ in X such that the diagram

$$\int_{\langle p,c\rangle} S(p,p,c,c) \xrightarrow{\xi_{\langle p,c\rangle}} S(p,p,c,c) \\
\downarrow^{\theta \mid \downarrow} \qquad \qquad \parallel \\
\int_{p} \left[\int_{c} S(p,p,c,c) \right] \xrightarrow{\varrho_{p}} \int_{c} S(p,p,c,c) \xrightarrow{\omega_{p,p,c}} S(p,p,c,c)$$

commutes, where the horizontal arrows ξ , ϱ , and ω are the universal wedges belonging to the corresponding ends; moreover, the arrow θ is an isomorphism.

Proof. For each $\langle p, q \rangle \in P \times P$ we are given the end

$$\omega_{p,q}: \int_{c} S(p,q,c,c) \xrightarrow{\cdot \cdot \cdot} S(p,q,-,-).$$

For any $x \in X$ each *P*-indexed family $\rho_p : x \to \int_c S(p, p, c, c)$ of arrows of *X* determines a $(P \times C)$ -indexed family $\xi_{p,c}$ as the composites

$$\xi_{p,c}: x \xrightarrow{\varrho_p} \int_c S(p,p,c,c) \xrightarrow{\omega_{p,p,c}} S(p,p,c,c);$$

for p fixed, $\xi_{\langle p, - \rangle}$ is trivially a wedge in c. Conversely, since $\omega_{p,p}$ is universal, every $(P \times C)$ -indexed family which is natural in c for each p is such a composite, for a unique family ϱ . Now ϱ or ξ is extranatural in p (the latter for some c) if and only if the corresponding square below

$$\begin{array}{c|c}
x & \xrightarrow{\varrho_p} & \int_c S(p, p, c, c) \\
\downarrow \varrho_q & \downarrow & \downarrow \zeta S(p, s, c, c) \\
\int_c S(q, q, c, c) & \xrightarrow{\xi_{p,c}} & S(p, p, c, c) \\
\int_c S(q, q, c, c) & \downarrow \zeta S(p, q, c, c), \\
\downarrow S(q, q, c, c) & \downarrow S(p, q, c, c)
\end{array}$$

$$\begin{array}{c}
X & \xrightarrow{\xi_{p,c}} & S(p, p, c, c) \\
\downarrow \xi_{q,c} & \downarrow S(p, p, c, c)
\end{array}$$

$$\begin{array}{c}
S(p, p, c, c) \\
\downarrow S(p, s, c, c)
\end{array}$$

$$\begin{array}{c}
S(q, q, c, c) & \xrightarrow{S(s, q, c, c)} & S(p, q, c, c)
\end{array}$$