$$(\phi \circ \bar{\phi}^{-1})(a^1, a^2, \dots, a^n) = (-a^1, a^2, \dots, a^n),$$

the absolute value of its Jacobian determinant is

$$|J(\phi \circ \bar{\phi}^{-1})| = |-1| = 1. \tag{23.10}$$

Therefore,

$$\int_{\bar{\phi}(U)} (\bar{\phi}^{-1})^* \tau = -\int_{\bar{\phi}(U)} (f \circ \bar{\phi}^{-1}) dr^1 \cdots dr^n \quad (\text{by } (23.9))$$

$$= -\int_{\bar{\phi}(U)} (f \circ \phi^{-1}) \circ (\phi \circ \bar{\phi}^{-1}) |J(\phi \circ \bar{\phi}^{-1})| dr^1 \cdots dr^n \quad (\text{by } (23.10))$$

$$= -\int_{\phi(U)} (f \circ \phi^{-1}) dr^1 \cdots dr^n \quad (\text{by the change-of-variables formula})$$

$$= -\int_{\phi(U)} (\phi^{-1})^* \tau.$$

The treatment of integration above can be extended almost word for word to oriented manifolds with boundary. It has the virtue of simplicity and is of great utility in proving theorems. However, it is not practical for actual computation of integrals; an n-form multiplied by a partition of unity can rarely be integrated as a closed expression. To calculate explicitly integrals over an oriented n-manifold M, it is best to consider integrals over a parametrized set.

Definition 23.10. A parametrized set in an oriented n-manifold M is a subset A together with a C^{∞} map $F: D \to M$ from a compact domain of integration $D \subset \mathbb{R}^n$ to M such that A = F(D) and F restricts to an orientation-preserving diffeomorphism from $\operatorname{int}(D)$ to $F(\operatorname{int}(D))$. Note that by smooth invariance of domain for manifolds (Remark 22.5), $F(\operatorname{int}(D))$ is an open subset of M. The C^{∞} map $F: D \to A$ is called a parametrization of A.

If A is a parametrized set in M with parametrization $F: D \to A$ and ω is a C^{∞} n-form on M, not necessarily with compact support, then we define $\int_A \omega$ to be $\int_D F^* \omega$. It can be shown that the definition of $\int_A \omega$ is independent of the parametrization and that in case A is a manifold, it agrees with the earlier definition of integration over a manifold. Subdividing an oriented manifold into a union of parametrized sets can be an effective method of calculating an integral over the manifold. We will not delve into this theory of integration (see [31, Theorem 25.4, p. 213] or [25, Proposition 14.7, p. 356]), but will content ourselves with an example.

Example 23.11 (*Integral over a sphere*). In spherical coordinates, ρ is the distance $\sqrt{x^2 + y^2 + z^2}$ of the point $(x, y, z) \in \mathbb{R}^3$ to the origin, φ is the angle that the vector $\langle x, y, z \rangle$ makes with the positive z-axis, and θ is the angle that the vector $\langle x, y \rangle$ in the (x, y)-plane makes with the positive x-axis (Figure 23.3(a)). Let ω be the 2-form on the unit sphere S^2 in \mathbb{R}^3 given by