10. *Proof.* We omit (a) since is standard. For (b), if u attains an interior maximum, then the conclusion follows from strong maximum principle.

If not, then for some  $x^0 \in \partial U, u(x^0) > u(x) \ \forall x \in U$ . Then Hopf's lemma implies  $\frac{\partial u}{\partial \nu}(x^0) > 0$ , which is a contradiction.

**Remark** 2. A generalization of this problem to mixed boundary conditions is recorded in Gilbarg-Trudinger, Elliptic PDEs of second order, Problem 3.1.

11. Proof. Define

$$B[u, v] = \int_{U} \sum_{i,j} a^{ij} u_{x_i} v_{x_j} dx \text{ for } u \in H^1(U), v \in H^1_0(U).$$

By Exercise 5.17,  $\phi(u) \in H^1(U)$ . Then, for all  $v \in C_c^{\infty}(U)$ ,  $v \geq 0$ ,

$$B[\phi(u), v] = \int_{U} \sum_{i,j} a^{ij} (\phi(u))_{x_i} v_{x_j} dx$$

$$= \int_{U} \sum_{i,j} a^{ij} \phi'(u) u_{x_i} v_{x_j} dx, \quad (\phi'(u) \text{ is bounded since u is bounded})$$

$$= \int_{U} \sum_{i,j} a^{ij} u_{x_i} (\phi'(u)v)_{x_j} - \sum_{i,j} a_{ij} \phi''(u) u_{x_i} u_{x_j} v dx$$

$$\leq 0 - \int_{U} \phi''(u) v |Du|^2 dx \leq 0, \text{ by convexity of } \phi.$$

(We don't know whether the product of two  $H^1$  functions is weakly differentiable. This is why we do not take  $v \in H_0^1$ .) Now we complete the proof with the standard density argument.  $\square$ 

12. Proof. Given  $u \in C^2(U) \cap C(\bar{U})$  with  $Lu \leq 0$  in U and  $u \leq 0$  on  $\partial U$ . Since  $\bar{U}$  is compact and  $v \in C(\bar{U}), \ v \geq c > 0$ . So  $w := \frac{u}{v} \in C^2(U) \cap C(\bar{U})$ . Brutal computation gives us

$$\begin{split} -a^{ij}w_{x_ix_j} &= \frac{-a^{ij}u_{x_ix_j}v + a^{ij}v_{x_ix_j}u}{v^2} + \frac{a^{ij}v_{x_i}u_{x_j} - a^{ij}u_{x_i}v_{x_j}}{v^2} - a^{ij}\frac{2}{v}v_{x_j}\frac{v_{x_i}u - vu_{x_i}}{v^2} \\ &= \frac{(Lu - b^iu_{x_i} - cu)v + (-Lv + b^iv_{x_i} + cv)u}{v^2} + 0 + a^{ij}\frac{2}{v}v_{x_j}w_{x_i}, \text{ since } a^{ij} = a^{ji}. \\ &= \frac{Lu}{v} - \frac{uLv}{v^2} - b^iw_{x_i} + a^{ij}\frac{2}{v}v_{x_j}w_{x_i} \end{split}$$

Therefore,

$$Mw := -a^{ij}w_{x_ix_j} + w_{x_i} \left[ b^i - a^{ij} \frac{2}{v} v_{x_j} \right] = \frac{Lu}{v} - \frac{uLv}{v^2} \le 0 \text{ on } \{ x \in \bar{U} : u > 0 \} \subseteq U$$

If  $\{x \in \overline{U} : u > 0\}$  is not empty, Weak maximum principle to the operator M with bounded coefficients (since  $v \in C^1(\overline{U})$ ) will lead a contradiction that

$$0 < \max_{\{\overline{u} > 0\}} w = \max_{\partial \{u > 0\}} w = \frac{0}{v} = 0$$

Hence  $u \leq 0$  in U.