by (8.15), $|\mathbb{E}[X|\mathcal{G}](\omega_n)| \leq \int_{G_n} |X|dP/P(G_n)$ and

$$\int_{\Omega} |\mathbb{E}[X|\mathcal{G}]|dP = \sum_{n=1}^{\infty} \int_{G_n} |\mathbb{E}[X|\mathcal{G}]|dP = \sum_{n=1}^{\infty} |\mathbb{E}[X|\mathcal{G}](\omega_n)| \cdot P(G_n)$$

$$\leq \sum_{n=1}^{\infty} \int_{G_n} |X|dP = \int_{\Omega} |X|dP = \mathbb{E}[|X|].$$

Hence $\mathbb{E}[X|\mathcal{G}]$ is integrable and $\mathbb{E}[|\mathbb{E}[X|\mathcal{G}]|] \leq \mathbb{E}[|X|]$.

The trivial σ -field \mathcal{F}_{\emptyset} is countably generated and, since $P(\Omega) = 1$, it follows that $\mathbb{E}[X|\mathcal{F}_{\emptyset}](\omega) = \mathbb{E}[X]$ for all $\omega \in \Omega$. Hence we may *identify* $\mathbb{E}[X]$ and the constant random variable $\mathbb{E}[X|\mathcal{F}_{\emptyset}]$ and regard the expectation defined in the previous chapter as a special case of conditional expectation.

Our next result characterizes conditional expectations in the countably generated case and extends, although we do not prove it, with a slightly weaker form of uniqueness to arbitrary conditional expectations.

Proposition 8.4. Let (Ω, \mathcal{F}, P) denote a probability space and let \mathcal{G} denote a σ -field on Ω generated by a countable partition $(G_n)_{n=1}^{\infty}$ of Ω . We suppose $\mathcal{G} \subset \mathcal{F}$ and $P(G_n) > 0$ for all n. If X is an integrable random variable on (Ω, \mathcal{F}, P) , then $\mathbb{E}[X|\mathcal{G}]$ is the unique \mathcal{G} measurable integrable random variable on (Ω, \mathcal{F}, P) satisfying

(8.17)
$$\int_{A} \mathbb{E}[X|\mathcal{G}]dP = \int_{A} XdP$$

for all $A \in \mathcal{G}$.

Proof. Let n be arbitrary and let $\omega \in G_n$. Since $\mathbb{E}[X|\mathcal{G}]$ is constant on each G_n , it is \mathcal{G} measurable and

$$\int_{G_n} \mathbb{E}[X|\mathcal{G}]dP = \mathbb{E}[X|\mathcal{G}](\omega) \cdot \int_{G_n} dP = \left(\frac{1}{P(G_n)} \int_{G_n} XdP\right) \cdot P(G_n)$$
$$= \int_{G_n} XdP.$$

If $A \in \mathcal{G}$, then $A = \bigcup_{n \in M} G_n$ for some $M \subset \mathbb{N}$. Hence

$$\int_{A} \mathbb{E}[X|\mathcal{G}]dP = \int_{\bigcup_{n \in M} G_{n}} \mathbb{E}[X|\mathcal{G}]dP = \sum_{n \in M} \int_{G_{n}} \mathbb{E}[X|\mathcal{G}]dP$$

$$= \sum_{n \in M} \int_{G_{n}} XdP = \int_{\bigcup_{n \in M} G_{n}} XdP$$

$$= \int_{A} XdP.$$