## **Probability and Statistics**

## Team (5 problems)

**Problem 1.** One hundred passengers board a plane with exactly 100 seats. The first passenger takes a seat at random. The second passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. The third passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. This process continues until all the 100 passengers have boarded the plane. What is the probability that the last passenger takes his own seat?

**Problem 2.** Assume a sequence of random variables  $X_n$  converges in distribution to a random variable X. Let  $\{N_t, t \geq 0\}$  be a set of positive integer-valued random variables, which is independent of  $(X_n)$  and converges in probability to  $\infty$  as  $t \to \infty$ . Prove that  $X_{N_t}$  converges in distribution to X as  $t \to \infty$ .

**Problem 3.** Suppose  $T_1, T_2, \ldots, T_n$  is a sequence of independent, identically distributed random variables with the exponential distribution of the density function

$$p(x) = \begin{cases} e^{-x}, & x \ge 0; \\ 0, & x < 0. \end{cases}$$

Let  $S_n = T_1 + T_2 + \cdots + T_n$ . Find the distribution of the random vector

$$V_n = \left\{ \frac{T_1}{S_n}, \frac{T_2}{S_n}, \cdots, \frac{T_n}{S_n} \right\}.$$

**Problem 4.** Suppose that X and Z are jointly normal with mean zero and standard deviation 1. For a strictly monotonic function  $f(\cdot)$ ,  $\operatorname{cov}(X,Z) = 0$  if and only if  $\operatorname{cov}(X,f(Z)) = 0$ , provided the latter covariance exists. **Hint:** Z can be expressed as  $Z = \rho X + \varepsilon$  where X and  $\varepsilon$  are independent and  $\varepsilon \sim N(0,\sqrt{1-\rho^2})$ .

**Problem 5.** Consider the following penalized least-squares problem (Lasso):

$$\frac{1}{2}\|\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}\|^2 + \lambda\|\boldsymbol{\beta}\|_1$$

Let  $\widehat{\beta}$  be a minimizer and  $\Delta = \widehat{\beta} - \beta^*$  for any given  $\beta^*$ . If  $\lambda > 2 \|\mathbf{X}^T(\mathbf{Y} - \mathbf{X}\beta^*)\|_{\infty}$ , show that

1. 
$$\|\mathbf{Y} - \mathbf{X}^T \widehat{\boldsymbol{\beta}}\|^2 - \|\mathbf{Y} - \mathbf{X}^T \boldsymbol{\beta}^*\|^2 > -\lambda \|\boldsymbol{\Delta}\|_1$$
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