and so

$$\int_{B_R} |u - u_{B_R}|^{1*} dx = \left(1 - \frac{\mathcal{L}^N(B_R \cap E)}{\mathcal{L}^N(B_R)}\right)^{1*} \mathcal{L}^N(B_R \cap E) + \left(\frac{\mathcal{L}^N(B_R \cap E)}{\mathcal{L}^N(B_R)}\right)^{1*} \mathcal{L}^N(B_R \setminus E).$$

If  $\mathcal{L}^N(B_R \cap E) \leq \mathcal{L}^N(B_R \setminus E)$ , then

$$\left(\int_{B_R} |u - u_{B_R}|^{1*} dx\right)^{1/1*} \ge \frac{\mathcal{L}^N(B_R \setminus E)}{\mathcal{L}^N(B_R)} (\mathcal{L}^N(B_R \cap E))^{1/1*}$$

$$\ge \frac{1}{2} (\mathcal{L}^N(B_R \cap E))^{1/1*}$$

$$= \frac{1}{2} \min\{\mathcal{L}^N(B_R \cap E), \mathcal{L}^N(B_R \setminus E)\}^{1/1*}.$$

The other case is analogous.

By applying Poincaré's inequality for balls (see the previous exercise), we get that the left-hand side of the previous inequality is bounded from above by  $c||D(\chi_E)||(B_R)$ , and so

$$\frac{1}{2}\min\{\mathcal{L}^N(B_R\cap E), \mathcal{L}^N(B_R\setminus E)\}^{1/1^*} \le c\|D(\chi_E)\|(B_R)$$
$$\le c\|D(\chi_E)\|(\mathbb{R}^N).$$

Hence, the claim is proved.

By letting  $R \to \infty$  in the previous inequality and using Proposition B.9, it follows that either E or  $\mathbb{R}^N \setminus E$  has finite Lebesgue measure.  $\square$ 

Thus, we have shown that the Sobolev–Gagliardo–Nirenberg embedding theorem in BV implies the isoperimetric inequality. Next we show that the opposite is also true.

**Theorem 14.45.** Assume that the isoperimetric inequality (14.45) holds for all sets with finite perimeter. Then there exists a constant c = c(N) > 0 such that

$$||u||_{L^{1^*}(\mathbb{R}^N)} \le c||Du||(\mathbb{R}^N)$$

for all  $u \in BV(\mathbb{R}^N)$ .

**Proof.** Assume first that  $u \geq 0$  and that  $u \in C^{\infty}(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N)$ . For  $t \in \mathbb{R}$ , define  $A_t := \{x \in \mathbb{R}^N : u(x) > t\}$ . Then by the coarea formula (14.42) and the isoperimetric inequality (14.45),

(14.46) 
$$\int_{\mathbb{R}^N} \|\nabla u\| \, dx = \int_0^\infty P(A_t) \, dt \ge \frac{1}{c} \int_0^\infty (\mathcal{L}^N(A_t))^{1/1^*} \, dt.$$