6 Radial SLE

6.1 Definitions

Motivated by the example of LERW (among others) given in the introductory chapter, we now want to find a nice way to encode growing families of compact subsets $(K_t, t \ge 0)$ of the closed unit disk that are growing from the boundary point 1 towards 0. As in the chordal case, we are in fact going to focus on the conformal geometry of the complement H_t of K_t in the unit disc \mathbb{U} . One first has to find a natural time-parametrization. It turns out to be convenient to define the conformal map g_t from H_t onto \mathbb{U} that is normalised by

$$g_t(0) = 0$$
 and $g'_t(0) > 0$.

Note that $g'_t(0) \geq 1$. This can be for instance derived using the fact that $\log g'_t(0)$ is the limit when $\varepsilon \to 0$ of $\log(1/\varepsilon)$ times the probability that a planar Brownian motion started from ε hits the circle of radius ε^2 before exiting H_t (an analyst would find this justification very strange, for sure).

Then (and this is simply because with obvious notation, $(\tilde{g}_s \circ g_t)(0) = \tilde{g}_s(0) \circ g'_t(0)$), one measures the "size" $a(K_t)$ of K_t via the derivative of g_t at the origin:

$$g_t'(0) = \exp(a(t)).$$

Hence, we will consider growing families of compact sets such that $a(K_t) = t$. Suppose now that $(\zeta_t, t \ge 0)$ is a continuous function on the unit circle $\partial \mathbb{U}$. Define for all $z \in \overline{\mathbb{U}}$, the solution $g_t(z)$ to the ODE

$$\partial_t g_t(z) = -g_t(z) \frac{g_t(z) + \zeta_t}{g_t(z) - \zeta_t} \tag{6.1}$$

such that $g_0(z) = z$. This solution is well-defined up to the (possibly infinite) time T(z) defined by

$$T(z) = \sup\{t > 0 : \min_{s \in [0,t)} |g_s(z) - \zeta_s| > 0\}.$$

We then define

$$K_t := \{ z \in \overline{\mathbb{U}} : T(z) \le t \}$$

and

$$U_t := \mathbb{U} \setminus K_t$$
.

The family $(K_t, t \ge 0)$ is called the (radial) Loewner chain associated to the driving function ζ .

The general statements that we described in the chordal case are also valid in this radial case. One can add one feature that has no analog in the chordal case: It is possible to estimate the Euclidean distance d_t from 0 to K_t in terms of a(t) = t. Indeed, since U_t contains the disc $d_t \times \mathbb{U}$, it is clear that $g'_t(0) \leq 1/d_t$. On the other hand, a classical result of the theory of