which can be achieved if M'(1) > 0 (since $\varepsilon \rho \to 1$ and $\frac{1}{\varepsilon} - \rho \sim |\log \varepsilon|$).

For the Dirichlet case (3.30) and (3.32) we define similarly the functionals $\widetilde{J}_{\varepsilon,D}$ and $\Psi_{\varepsilon,D}$ and the expansion corresponding to (3.34) now reads as follows:

$$\varepsilon^{n-1}\Psi_{\varepsilon,D}(\rho) = M(\varepsilon\rho) \left(\alpha + \beta e^{-2\lambda(\frac{1}{\varepsilon}-\rho)}\right) + \text{higher-order terms.}$$
 (3.35)

Comparing (3.35) to (3.34), we see that only the sign for the term $\beta e^{-2\lambda(\frac{1}{\varepsilon}-\rho)}$ is different, which reflects the different, or opposite, effects of Dirichlet and Neumann boundary conditions. Heuristically, when $V \equiv 1$, the first term in (3.34) and (3.35) is due to the volume energy which always has a tendency to "shrink" (in order to minimize), while the second term $\pm \beta e^{-2\lambda(\frac{1}{\varepsilon}-\rho)}$ in (3.35) and (3.34), respectively, indicates that in the Dirichlet case the boundary "pushes" the mass of the solution away from the boundary (therefore only single-peak solutions are possible), but in the Neumann case the boundary "pulls" the mass of the solution and thereby reaches a balance at $\varepsilon \rho = r_{\varepsilon} \sim 1 - \varepsilon |\log \varepsilon|$ creating an extra solution.

We remark that the method described above also applies to the annulus case and yields the following interesting results for $V \equiv 1$, which illustrate the opposite effects between Dirichlet and Neumann boundary conditions most vividly.

Theorem 3.11 (see [AMN3]).

- (i) For every p > 1 and ε small, the Neumann problem (3.1) with $\Omega = \{x \in \mathbb{R}^n \mid 0 < a < |x| < b\}$ possesses a solution concentrating at $|x| = r_{\varepsilon}$, where $b r_{\varepsilon} \sim \varepsilon |\log \varepsilon|$, near the outer boundary |x| = b.
- (ii) For every p > 1 and ε small, the Dirichlet problem (3.2) with $\Omega = \{x \in \mathbb{R}^n \mid 0 < a < |x| < b\}$ possesses a solution concentrating at $|x| = r_{\varepsilon}$, where $r_{\varepsilon} a \sim \varepsilon |\log \varepsilon|$, near the inner boundary |x| = a.

Observe that from the "moving plane" method [GNN1] it follows easily that the Dirichlet problem (3.2) does not have a solution concentrating on a sphere near the outer boundary |x| = b.

In conclusion, we mention that the method of Theorem 3.10 can be extended to produce solutions with k-dimensional concentration sets, but again, some symmetry assumptions are needed. Other interesting progress in this direction includes a one-dimensional concentration set in the interior of a two-dimensional domain due to [WY]. The conjecture stated at the beginning of this section remains largely a major open problem.

3.1.4 Remarks

In Section 3.1, we have considered the various concentration phenomena for essentially just one equation, namely,

$$\varepsilon^2 \Delta u - u + u^p = 0 \tag{3.36}$$

in a bounded domain Ω under either Dirichlet or Neumann boundary conditions in (3.2) or (3.1), respectively. However, since (3.36) is quite basic, similar phenomena could be