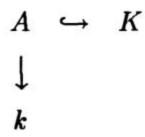
The principal ideal πA generated by π can be described in terms of v as $\{x \in K : v(x) > 0\}$. It is a maximal ideal, since every element of A not in πA is a unit. The quotient ring $k = A/\pi A$ is therefore a field, called the residue field associated to the valuation v.

Example. Let K be the field \mathbf{Q} of rational numbers, and let p be a prime number. The p-adic valuation on \mathbf{Q} is defined by setting v(x) equal to the exponent of p in the prime factorization of x. More precisely, given $x \in \mathbf{Q}^*$, write $x = p^n u$, where n is a (possibly negative) integer and u is a rational number whose numerator and denominator are not divisible by p; then v(x) = n. The valuation ring A is the ring of fractions a/b with $a, b \in \mathbf{Z}$ and b not divisible by p. [The ring A happens to be the localization of \mathbf{Z} at p, but we will not make any use of this.] The residue field k is the field \mathbf{F}_p of integers mod p; one sees this by using the homomorphism $A \twoheadrightarrow \mathbf{F}_p$ given by $a/b \mapsto (a \mod p)(b \mod p)^{-1}$, where a and b are as above.

The valuation ring A in this example can be described informally as the largest subring of \mathbf{Q} on which reduction mod p makes sense. It is thus the natural ring to introduce if one wants to relate the field \mathbf{Q} to the field \mathbf{F}_p . This illustrates our point of view toward valuations: We will be interested in studying things (namely, matrix groups) defined over a field K, and we wish to "reduce" to a simpler field k as an aid in this study; a discrete valuation makes this possible by providing us with a nice ring k to serve as intermediary between k and k:



Returning now to the general theory, we note that the study of the arithmetic of A (e.g., ideals and prime factorization) is fairly trivial:

Proposition 1. A discrete valuation ring A is a principal ideal domain, and every non-zero ideal is generated by π^n for some $n \geq 0$. In particular, πA is the unique non-zero prime ideal of A.

PROOF: Let I be a non-zero ideal and let $n = \min\{v(a) : a \in I\}$. Then I contains π^n , and every element of I is divisible by π^n ; hence $I = \pi^n A$. \square

One consequence of this is that we can apply the basic facts about modules over a principal ideal domain (e.g., a submodule of a free module is free). Let's recall some of these facts, in the form in which we'll need them later. Let V be the vector space K^n . By a lattice (or A-lattice) in V we will mean an A-submodule $L \subset V$ of the form $L = Ae_1 \oplus \cdots \oplus Ae_n$ for some basis e_1, \ldots, e_n of V. In particular, L is a free A-module of rank n. If we take e_1, \ldots, e_n to be the standard basis of V, then the resulting lattice is A^n , which we call the standard lattice.