and (12.30) show that

$$L(f) = \lim_{p \to \infty} L\left(\sum_{n=1}^{p} f \xi_{F_n}\right) = \lim_{p \to \infty} \sum_{n=1}^{p} L(f \xi_{F_n})$$

$$= \lim_{p \to \infty} \sum_{n=1}^{p} \int_{F_n} f \bar{g} d\mu = \lim_{p \to \infty} \int_{X} \sum_{n=1}^{p} f \xi_{F_n} \bar{g} d\mu$$

$$= \int_{X} f \bar{g} d\mu. \quad \Box$$

(20.20) **Theorem.** Let  $(X, \mathcal{A}, \mu)$  be a decomposable measure space (19.25). Then the mapping T defined by

$$T(g) = L_{\tilde{g}}$$

[see (20.16)] is a norm-preserving linear mapping of  $\mathfrak{L}_{\infty}$  onto the conjugate space  $\mathfrak{L}_{1}^{*}$ . Thus, as Banach spaces,  $\mathfrak{L}_{\infty}$  and  $\mathfrak{L}_{1}^{*}$  are isomorphic.

**Proof.** The fact that T is a norm-preserving mapping from  $\mathfrak{L}_{\infty}$  into  $\mathfrak{L}_{1}^{*}$  is (20.16). It follows from (20.19) that T is onto  $\mathfrak{L}_{1}^{*}$ . It is trivial that T is linear. Since T is both linear and norm-preserving, it is one-to-one.  $\square$ 

- (20.21) Note. As we have shown in (20.17), the conclusion in (20.20) fails for some nondecomposable measure spaces. However J. Schwartz has found a representation of  $\mathfrak{L}_1^*(X, \mathscr{A}, \mu)$  for arbitrary  $(X, \mathscr{A}, \mu)$  [Proc. Amer. Math. Soc. 2 (1951), 270–275], to which the interested reader is referred.
- (20.22) Exercise. Let X be a locally compact Hausdorff space and let  $\iota$  be an outer measure on  $\mathscr{P}(X)$  as in § 9. Prove that the definitions of local  $\iota$ -nullity given in (9.29) and in (20.11) are equivalent.
- (20.23) Exercise. Let  $(X, \mathcal{A}, \mu)$  be a degenerate measure space such that  $\mu(X) = \infty$  [see (10.3) for the definition]. Is this measure space decomposable? Find  $\mathfrak{L}_1$ ,  $\mathfrak{L}_1^*$ , and  $\mathfrak{L}_{\infty}$  explicitly for this measure space.
- (20.24) Exercise. Let  $(X, \mathscr{A}, \mu)$  be any measure space and let  $f \in \mathfrak{L}_1(X, \mathscr{A}, \mu)$ . Define L on  $\mathfrak{L}_{\infty}(X, \mathscr{A}, \mu)$  by

$$L(g) = \int_X g \, \bar{f} \, d\mu .$$

Prove that  $L \in \mathfrak{L}_{\infty}^*$  and that  $||L|| = ||f||_1$ .

(20.25) Exercise. Prove that  $\mathfrak{L}_1([0,1])$  [with Lebesgue measure] is not reflexive by showing that not every  $L \in \mathfrak{L}_{\infty}^*([0,1])$  has the form described in (20.24). [Hint. Use the Hahn-Banach theorem to produce an  $L \neq 0$  such that L(g) = 0 for all  $g \in \mathfrak{L}_{\infty}$  for which g is essentially continuous, i.e.,  $||g - h||_{\infty} = 0$  for some  $h \in \mathfrak{C}([0,1])$ .]

(20.26) Exercise. (a) Prove that  $\mathfrak{L}_{\infty}([0, 1])$  is not separable.

(b) Find necessary and sufficient conditions on a measure space that its  $\mathfrak{L}_{\infty}$  space be separable. [Do not forget (20.23).]

Having found the conjugate space of  $\mathfrak{L}_p(X, \mathscr{A}, \mu)$  for  $1 and any measure space <math>(X, \mathscr{A}, \mu)$ , and of  $\mathfrak{L}_1(X, \mathscr{A}, \mu)$  for a large class of