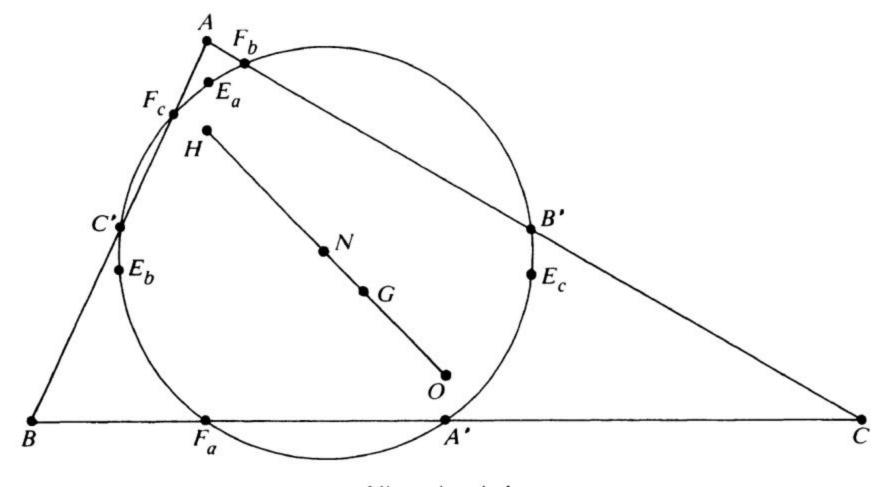
14 Classical Theorems

Charles-Julian Brianchon (1785–1864) and Jean-Victor Poncelet (1788–1867) showed that the circumcircle of the Euler triangle coincides with the other two circumcircles. Therefore, as you would expect and as we shall prove next, the nine-point circle does pass through nine noteworthy points. Poncelet was introduced briefly in the last section. He gave the main impetus to the revival of projective geometry, having laid the foundations of modern projective geometry while a prisoner of the Russians in the Napoleonic Wars. Brianchon is most famous for the theorem of projective geometry he proved as a student. The theorem is known as *Brianchon's Theorem: The three diagonals of a hexagon circumscribed about a conic are concurrent*.

We have defined the ninepoint circle of $\triangle ABC$ to be the circumcircle of the medial triangle $\triangle A'B'C'$. We first wish to show this circle, which has center N, also passes through the feet of the altitudes. Let F_a be the foot of the altitude through A. So F_a and A' are both on \overrightarrow{BC} . We wish to show $NF_a = NA'$. We suppose $F_a \neq A'$. See Figure 14.17. Now, points H and F_a are on the perpendicular to \overrightarrow{BC} at F_a , and points O and A' are on the perpendicular to \overrightarrow{BC} at A'. Then, since N is the midpoint of H and O, point N must be on the perpendicular bisector of segment $\overrightarrow{F_aA'}$. So $NF_a = NA'$, as desired. Likewise, $NF_b = NB'$ and $NF_c = NC'$, where F_b and F_c are the feet of the altitudes from B and C, respectively. Therefore, the vertices of the orthic triangle are on the ninepoint circle.

Now let E_a , E_b , E_c be the midpoints between H and A, B, C, respectively. See Figure 14.17. We wish to show these Euler points are also on the nine-point circle of $\triangle ABC$. Since the product $\delta_{G,-1/2}\delta_{H,2}$ has dilation ratio -1 and fixes point N, then the product is σ_N . So $\sigma_N(E_a) = A'$, $\sigma_N(E_b) = B'$, and $\sigma_N(E_c) = C'$. Hence, $NE_a = NA'$, $NE_b = NB'$, and $NE_c = NC'$, as desired. We have proved the *Ninepoint Circle Theorem* of Brianchon and Poncelet.



Ninepoint circle

Figure 14.17