**Proposition 1.122.** Let R be a commutative Noetherian ring without nontrivial idempotents and let

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0$$

$$\downarrow^{\varphi_1} \qquad \downarrow^{\varphi_2} \qquad \downarrow^{\varphi_3}$$

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0$$

be an exact commutative diagram of modules of finite projective dimension. Then

$$P_{\varphi_2}(t) = P_{\varphi_1}(t) \cdot P_{\varphi_3}(t).$$

**Proof.** This follows directly from the preceding comments.

Remark 1.123. If E is a graded module and  $\varphi$  is homogeneous, then  $P_{\varphi}(t)$  is a homogeneous polynomial and  $\deg E = \deg P_{\varphi}(t)$ .

## **Minimal Polynomial**

Under special conditions one can obtain the minimal polynomial of certain endomorphisms instead of the characteristic polynomial. This may occur in the setting of an affine domain A and one of its Noether normalizations  $R = k[z_1, \ldots, z_d]$ . For  $u \in A$ , the kernel of the homomorphism

$$R[t] \longrightarrow A, \quad t \mapsto u,$$

is an irreducible polynomial  $h_u(t)$  (appropriately homogeneous if A is a graded algebra and u is homogeneous). It is related to the characteristic polynomial  $f_u(t)$  of u by an equality of the form

$$f_u(t) = h_u(t)^r.$$

## The Determinant of an Endomorphism

Let R be a commutative ring and A a finite R-module. An element  $a \in A$  defines an endomorphism

$$f_a: A \to A, f_a(x) = ax,$$

of R-modules. We seek ways to define the *determinant* of  $f_a$  relative to R. For example, if A is a free R-module, we may use the standard definition. More generally, if A has a finite free R-resolution  $\mathbb{F}$ ,

$$0 \to F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$
,

as above we lift  $f_a$  to an endomorphism of  $\mathbb{F}$ , and 'define'  $\det(f_a)$  as the alternating product

$$\det(f_a) = \prod_{i=0}^{n} \det(f_i)^{(-1)^i},$$