Elementary symmetric polynomials are thus intimately connected with the roots of polynomials. Another reason for their importance is that they generate the algebra of all symmetric polynomials, but we will not need this fact. Our goal is to connect the elementary symmetric polynomials with the symmetric polynomials

$$p_j = p_j(\omega_1, \ldots, \omega_n) := \sum_{i=1}^{n} \omega_i^j, \quad j \geq 1.$$

One can verify that

$$e_1 = p_1, \quad 2e_2 = e_1 p_1 - p_2, \quad 3e_3 = e_2 p_1 - e_1 p_2 + p_3, \tag{6.7}$$

which indicates the general rule.

Lemma 6.35 (Newton's formulas) *For $k = 1, \ldots, n$, we have*

$$ke_k = \sum_{j=1}^{k} (-1)^{j-1} e_{k-j} p_j. \tag{6.8}$$

Proof From (6.6) it follows that

$$\sum_{j=0}^{n} (-1)^j e_{n-j} \omega_i^j = 0, \quad i = 1, \ldots, n.$$

Summing up over all i we obtain (6.8) for $k = n$. The case where $k < n$ follows easily from this one, basically we are facing only a problem in notation. The one that we are using suggests that n is fixed and k varies. While this setting is natural in view of applications, in the proof that follows it is inconvenient to regard n as fixed. Let us therefore write, just for the purpose of this proof, $e_{k,n}$ for e_k and $p_{j,n}$ for p_j. Our goal is to show that the polynomial

$$P := ke_{k,n} - \sum_{j=1}^{k} (-1)^{j-1} e_{k-j,n} p_{j,n}$$

is 0. Note that for any $i, j \leq k$ we have

$$e_{i,n}(\omega_1, \ldots, \omega_k, 0, \ldots, 0) = e_{i,k}(\omega_1, \ldots, \omega_k),$$

$$p_{j,n}(\omega_1, \ldots, \omega_k, 0, \ldots, 0) = p_{j,k}(\omega_1, \ldots, \omega_k).$$

Since (6.8) holds for $n = k$, it follows that $P(\omega_1, \ldots, \omega_k, 0, \ldots, 0) = 0$. This means that P does not contain nonzero monomials in $\omega_1, \ldots, \omega_k$. Similarly, by setting zeros at other places, we see that P does not contain nonzero monomials in any set of k indeterminates. However, from the definition of P it is obvious that its nonzero