Proof. For fixed $m \in \mathbb{N}$ the balls $B(x, \frac{1}{m})$, $x \in K$, cover K, so there is a finite set $F_m \subseteq K$ such that

$$K\subseteq \bigcup_{x\in F_m}B(x,\frac{1}{m}).$$

Then the set $F := \bigcup_{m \in \mathbb{N}} F_m$ is countable and dense in K.

Theorem 4.7. A compact topological space K is metrizable if and only if C(K) is separable.

Proof. Suppose that C(K) is separable, and let $(f_n)_{n \in \mathbb{N}}$ be a sequence in C(K) such that $\{f_n : n \in \mathbb{N}\}$ is dense in C(K). Define

$$\Phi: K \to \Omega := \prod_{n \in \mathbb{N}} \mathbb{C}, \qquad \Phi(x) := (f_n(x))_{n \in \mathbb{N}},$$

where Ω carries the usual product topology. Then Φ is continuous and injective, by Urysohn's lemma and the density assumption. The topology on Ω is metrizable (see Appendix A.5). Since K is compact, Φ is a homeomorphism from K onto $\Phi(K)$. Consequently, K is metrizable.

For the converse suppose that $d: K \times K \to \mathbb{R}_+$ is a metric that induces the topology of K. By Lemma 4.6 there is a countable set $A \subseteq K$ with $\overline{A} = K$. Consider the countable(!) set

$$D := \{ f \in C(K) : f \text{ is a finite product of functions } d(\cdot, y), y \in A \} \cup \{1\}.$$

Then lin(D) is a conjugation invariant subalgebra of C(K) containing the constants and separating the points of K. By the Stone–Weierstraß theorem, $\overline{lin}(D) = C(K)$ and hence C(K) is separable.

4.2 The Space C(K) as a Commutative C^* -Algebra

In this section we show how the compact space K can be recovered if only the space C(K) is known (see Theorem 4.11 below).

The main idea is readily formulated. Let K be any compact topological space. To $x \in K$ we associate the functional

$$\delta_x : C(K) \to \mathbb{C}, \qquad \langle f, \delta_x \rangle := f(x) \qquad (f \in C(K))$$

called the **Dirac** or **evaluation functional** at $x \in K$. Then $\delta_x \in C(K)'$ with $\|\delta_x\| = 1$. By Urysohn's lemma, C(K) separates the points of K, which means that the map

$$\delta: K \to C(K)', \qquad x \mapsto \delta_x$$