We want to examine the third term in (8.7.1) more closely:

$$\begin{split} &\int\limits_{M} \langle \nabla_{\frac{\partial}{\partial x^{\alpha}}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial s} dx^{\alpha}, \frac{\partial f}{\partial x^{\beta}} dx^{\beta} \rangle_{T^{*}M \otimes f^{-1}TN} \\ &= -\int\limits_{M} \langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial s} dx^{\alpha}, \nabla_{\frac{\partial}{\partial x^{\alpha}}} \frac{\partial f}{\partial x^{\beta}} dx^{\beta} \rangle_{T^{*}M \otimes f^{-1}TN} \end{split}$$

since  $\nabla$  is metric and integrating by parts

$$= -\int_{M} \langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial s}, \operatorname{trace}_{M} \nabla df \rangle_{f^{-1}TN}.$$
 (8.7.2)

**Theorem 8.7.1** For a smooth family  $f_{st}: M \to N$  of finite energy maps between Riemannian manifolds, with  $f_{st}(x) = f_{00}(x)$  for all  $x \in \partial M$  (in case  $\partial M \neq \emptyset$ ) and all s, t, we have for the second variation of energy, with  $V = \frac{\partial f}{\partial s}|_{s=0}, W = \frac{\partial f}{\partial t}|_{t=0}$ 

$$\frac{\partial^{2} E(f_{st})}{\partial s \partial t}\Big|_{s=t=0}$$

$$= \int_{M} \langle \nabla V, \nabla W \rangle_{f^{-1}TN} - \int_{M} \operatorname{trace}_{M} \langle R^{N}(df, V)W, df \rangle_{f^{-1}TN}$$

$$+ \int_{M} \langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial s}, \operatorname{trace}_{M} \nabla df \rangle_{f^{-1}TN}.$$
(8.7.3)

If  $f_{00}$  is harmonic, or if  $\nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial s} \equiv 0$  for s = t = 0, then the second variation depends only on V and W, but not on higher derivatives of f w.r.t. s,t, and

$$I_{f}(V,W) := \frac{\partial^{2} E(f_{st})}{\partial s \partial t} = \int_{M} \langle \nabla V, \nabla W \rangle_{f^{-1}TN}$$
$$- \int_{M} \operatorname{trace}_{M} \langle R^{N}(df, V)W, df \rangle_{f^{-1}TN}. \tag{8.7.4}$$

*Proof.* (8.7.3) follows from (8.7.1), (8.7.2). (8.7.4) holds if either  $\nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial s} \equiv 0$ or trace<sub>M</sub> $\nabla df \equiv 0$ , and the latter is the harmonic map equation (cf. (8.1.14)).

We look at the special case where we only have one parameter:

$$f(x,t) = f_t(x), f : M \times (-\varepsilon, \varepsilon) \to N,$$

$$W := \frac{\partial f}{\partial t}_{|t=0}$$