necessarily nondegenerate quadratic form on V would suffice, but here we have no need to investigate the most general possible construction. On the contrary, for our purposes it suffices to take \mathbb{R}^n with its standard Euclidean scalar product. An orthonormal basis will be denoted by e_1, \ldots, e_n .

Definition 1.8.1 The Clifford algebra Cl(V), also denoted Cl(n), is the quotient of the tensor algebra $\bigoplus_{k>0} V \otimes \ldots \otimes V$ generated by V by the two sided

ideal generated by all elements of the form $v \otimes v + ||v||^2$ for $v \in V$.

Thus, the multiplication rule for the Clifford algebra Cl(V) is

$$vw + wv = -2\langle v, w \rangle \tag{1.8.1}$$

In particular, in terms of our orthonormal basis e_1, \ldots, e_n , we have

$$e_i^2 = -1 \text{ and } e_i e_j = -e_i e_j \text{ for } i \neq j.$$
 (1.8.2)

From this, one easily sees that a basis of Cl(V) as a real vector space is given by

$$e_0 := 1, \quad e_{\alpha} := e_{\alpha_1} e_{\alpha_2} \dots e_{\alpha_k}$$

with $\alpha = \{\alpha_1, \dots, \alpha_k\} \subset \{1, \dots, n\}$ and $\alpha_1 < \alpha_2 \dots < \alpha_k$. For such an α , we shall put $|\alpha| := k$ in the sequel. Thus, as a vector space, Cl(V) is isomorphic to $\Lambda^*(V)$ (as algebras, these two spaces are of course different). In particular, the dimension of Cl(V) as a vector space is 2^n . Also, declaring this basis as being orthonormal, we obtain a scalar product on Cl(V) extending the one on V.

We define the degree of e_{α} as being $|\alpha|$. The e_{α} of degree k generate the subset $\operatorname{Cl}^k(V)$ of elements of degree k. We have

$$Cl^0 = \mathbb{R}$$
$$Cl^1 = V.$$

Finally, we let $Cl^{ev}(V)$ and $Cl^{odd}(V)$ be the subspaces of elements of even, resp. odd degree. The former is a subalgebra of Cl(V), but not the latter.

Lemma 1.8.1 The center of Cl(V) consists of those elements that commute with all $v \in Cl^1(V) = V$. For n even, the center is $Cl^0(V)$, while for n odd, it is $Cl^0(V) \oplus Cl^n(V)$.

Proof. It suffices to consider basis vectors $e_{\alpha} = e_{\alpha_1} \dots e_{\alpha_k}$ as above. For $j \notin \alpha$, we have

$$e_{\alpha}e_j = (-1)^{|\alpha|}e_j e_{\alpha},$$

and thus $|\alpha|$ has to be even for e_{α} to commute with e_{j} , while

$$e_{\alpha}e_{\alpha_j} = (-1)^{|\alpha|-1}e_{\alpha_j}e_{\alpha},$$