PROOF: Let $\epsilon > 0$ be given. Choose N sufficiently large that $\sum_{k>N} a_k < \epsilon/(4B)$ and pick N' large enough so that for all $n \geq N'$

$$|b_n - b_{n-1}| \le \frac{\epsilon}{2aN}.$$

Then for $n \geq N + N'$, we have

$$\left| \sum_{k=0}^{n} a_{k}(b_{n} - b_{n-k}) \right| \leq \sum_{k=0}^{N} a_{k} |b_{n} - b_{n-k}| + \sum_{k=N+1}^{n} a_{k}(|b_{n}| + |b_{n-k}|)$$

$$\leq \sum_{k=0}^{N} a_{k} \sum_{j=0}^{k-1} |b_{n-j} - b_{n-j-1}| + 2B \sum_{k=N+1}^{n} a_{k}$$

$$\leq \sum_{k=0}^{N} a_{k} \sum_{j=0}^{k-1} \frac{\epsilon}{2aN} + \frac{\epsilon}{4B} \cdot 2B, \text{ since } n - j \geq N'$$

$$\leq \frac{\epsilon}{2a} \sum_{k=0}^{N} a_{k} + \frac{\epsilon}{2}$$

$$\leq \epsilon.$$

Theorem 9-7: Let i, j, and k be arbitrary states in a recurrent Markov chain which is either null or noncyclic ergodic. Then

$$\lim_{n\to\infty} \left[(N_{kk}^{(n)} - N_{ik}^{(n)}) \alpha_j / \alpha_k + N_{ij}^{(n)} - N_{kj}^{(n)} \right] = {}^k N_{ij}.$$

PROOF: We may assume that neither i nor j equals k, since otherwise both sides are clearly zero. We begin by establishing four equations:

(1)
$$N_{kk}^{(n)} = \sum_{\nu=0}^{\infty} F_{ik}^{(\nu)} N_{kk}^{(n)}$$

(2)
$$N_{ik}^{(n)} = \sum_{\nu=0}^{n} F_{ik}^{(\nu)} N_{kk}^{(n-\nu)}$$

(3)
$$N_{kj}^{(n)} = \sum_{v=0}^{\infty} F_{ik}^{(v)} N_{kj}^{(n)}$$

(4)
$$N_{ij}^{(n)} = \sum_{\nu=0}^{n} F_{ik}^{(\nu)} N_{kj}^{(n-\nu)} + {}^{k} N_{ij}^{(n)}.$$

Equations (1) and (3) follow from the fact that $\sum F_{ik}^{(v)} = H_{ik} = 1$. Equation (2) comes from Theorem 4-11 with the random time $\mathbf{t} = \min(\mathbf{t}_k, n)$, and equation (4) is a similar result, except that the sum has been broken into two parts representing what happens after and before state k is reached for the first time.