In this section we will adopt Einstein summation notation except when we say otherwise. We define a new connection $\tilde{\nabla}$ on $\tilde{S}(M)$ according to

$$\tilde{\nabla}_i = \nabla_i + \frac{1}{2} k_{ij} e_j e_0.$$

(From a spacetime perspective, the connection $\tilde{\nabla}$ comes from the ambient Levi-Civita connection on $T\mathcal{M}$, while ∇ comes from the intrinsically defined Levi-Civita connection on TM.) We now define the *hypersurface Dirac operator* $\tilde{\mathcal{D}}$ on $\tilde{S}(M)$ by

$$\tilde{\mathcal{D}} = e_i \cdot \tilde{\nabla}_i$$

$$= \mathcal{D} + \frac{1}{2} k_{ij} e_i e_j e_0$$

$$= \mathcal{D} - \frac{1}{2} (\operatorname{tr} k) e_0,$$

where \mathcal{D} is the usual Dirac operator on $\tilde{S}(M)$ as defined in Chapter 5, and we used symmetry considerations in the last line. (The trace of k is computed with respect to g.) Next we obtain a version of the Schrödinger-Lichnerowicz formula (Theorem 5.10) for initial data sets.

Theorem 8.21 (Witten). Let (M, g, k) be a spin initial data set. For any $\psi \in C^{\infty}(\tilde{S}(M))$,

$$\tilde{\mathcal{D}}^2 \psi = \tilde{\nabla}_i^* \tilde{\nabla}_i \psi + \frac{1}{2} (\mu + J e_0) \cdot \psi,$$

where $\tilde{\nabla}^*$ is the formal adjoint of $\tilde{\nabla}$ on $\tilde{S}(M)$.

Proof. We will take advantage of the work we already did to prove Theorem 5.10 in Chapter 5. As usual, we choose an orthonormal basis e_1, \ldots, e_n that is parallel at the point where we are computing. For any $\psi \in C^{\infty}(\tilde{S}(M))$, we have

$$\tilde{\mathcal{D}}^{2}\psi = \mathcal{D}^{2}\psi - \frac{1}{2}e_{i} \cdot \nabla_{i}[(\operatorname{tr} k)e_{0} \cdot \psi] - \frac{1}{2}(\operatorname{tr} k)e_{0}e_{i} \cdot \nabla_{i}\psi - \frac{1}{4}(\operatorname{tr} k)^{2}\psi$$

$$= \left(\nabla^{*}\nabla\psi + \frac{1}{4}R\psi\right) - \frac{1}{2}\nabla_{i}(\operatorname{tr} k)e_{i}e_{0} \cdot \psi - \frac{1}{4}(\operatorname{tr} k)^{2}\psi$$

$$= \nabla^{*}\nabla\psi + \frac{1}{2}\left[\frac{1}{2}(R - (\operatorname{tr} k)^{2}) - \nabla(\operatorname{tr} k)e_{0}\right] \cdot \psi,$$

where we used Theorem 5.10 in the second line. On the other hand, since the formal adjoint of ∇ on $\tilde{S}(M)$ is

$$\tilde{\nabla}_i^* = -\nabla_i + \frac{1}{2}k_{ij}e_je_0,$$