Example 2. As another application of the triple scalar product, let's find the Jacobian we used in Chapter 5, Section 4 for changing variables in a multiple integral. As you know, in rectangular coordinates the volume element is a rectangular box of volume dx dy dz. In other coordinate systems, the volume element may be approximately a parallelepiped as in Figure 3.1. We want a formula for the volume element in this case. (See, for example, the cylindrical and spherical coordinate volume elements in Chapter 5, Figures 4.4 and 4.5.)

Suppose we are given formulas for x, y, z as functions of new variables u, v, w. Then we want to find the vectors along the edges of the volume element in the u, v, w system. Suppose vector \mathbf{A} in Figure 3.1 is along the direction in which u increases while v and w remain constant. Then if $d\mathbf{r} = \mathbf{i} dx + \mathbf{j} dy + \mathbf{k} dz$ is a vector in this direction, we have

$$\mathbf{A} = \frac{\partial \mathbf{r}}{\partial u} du = \left(\mathbf{i} \frac{\partial x}{\partial u} + \mathbf{j} \frac{\partial y}{\partial u} + \mathbf{k} \frac{\partial z}{\partial u} \right) du.$$

Similarly if **B** is along the increasing v edge of the volume element and **C** is along the increasing w edge, we have

$$\mathbf{B} = \frac{\partial \mathbf{r}}{\partial v} dv = \left(\mathbf{i} \frac{\partial x}{\partial v} + \mathbf{j} \frac{\partial y}{\partial v} + \mathbf{k} \frac{\partial z}{\partial v} \right) dv,$$

$$\mathbf{C} = \frac{\partial \mathbf{r}}{\partial w} dw = \left(\mathbf{i} \frac{\partial x}{\partial w} + \mathbf{j} \frac{\partial y}{\partial w} + \mathbf{k} \frac{\partial z}{\partial w} \right) dw.$$

Then by (3.2)

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} du \, dv \, dw = J \, du \, dv \, dw$$
$$\begin{vmatrix} \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \\ \frac{\partial z}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

where J is the Jacobian of the transformation from x, y, z to u, v, w. Recall from the discussion of (3.2) that the triple scalar product may turn out to be positive or negative. Since we want a volume element to be positive, we use the absolute value of J. Thus the u, v, w volume element is |J| du dv dw as stated in Chapter 5, Section 4.

Applications of the Triple Vector Product In Figure 3.8 (compare Figure 2.6), suppose the particle m is at rest on a rotating rigid body (for example, the earth). Then the angular momentum \mathbf{L} of m about point O is defined by the equation $\mathbf{L} = \mathbf{r} \times (m\mathbf{v}) = m\mathbf{r} \times \mathbf{v}$. In the discussion of Figure 2.6, we showed that $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. Thus, $\mathbf{L} = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$. See Problem 16 and also Chapter 10, Section 4.

As another example, it is shown in mechanics that the centripetal acceleration of m in Figure 3.8 is $\mathbf{a} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$. See Problem 17.

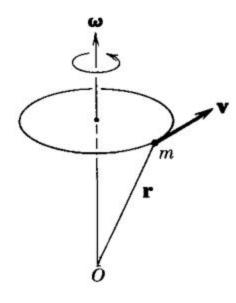


Figure 3.8