If $M_i = 0$ for $i < n_0$, then

$$0 = F_{n_0-1}(M) \subseteq F_{n_0}(M) \subseteq \cdots \subseteq F_{+\infty}(M) = M,$$

and the filtration must terminate at some finite stage, i.e., $F_{n_1}(M) = M$ for some n_1 , since M is finitely generated and Noetherian. Note that there is a map of graded S-modules from $S \otimes_R M_i$ to $F_i(M)$, which induces by passage to the quotient a **surjective** map of graded S-modules

$$S \otimes_R Q_i \to F_i(M)/F_{i-1}(M)$$
.

Here we are viewing M_i and Q_i as graded modules concentrated in the single degree i. We will show this map is an isomorphism for each i.

For $i < n_0$ or $i > n_1$, this is obvious since both sides are zero. Suppose we know that $\operatorname{Tor}_1^S(R, F_i(M)) = 0$, which is at least the case for $i = n_1$ since $M \in \operatorname{Obj} \mathcal{F}$. From the short exact sequence of graded modules

$$0 \to F_{i-1}(M) \to F_i(M) \to F_i(M)/F_{i-1}(M) \to 0$$

and the fact that the natural map

$$R \otimes_S F_{i-1}(M) \to R \otimes_S F_i(M)$$

is injective with cokernel Q_i , we see first that $\operatorname{Tor}_1^S(R,\,F_i(M))=0$ implies also $\operatorname{Tor}_1^S(R,\,F_i(M)/F_{i-1}(M))=0$ and

$$\operatorname{Tor}_{1}^{S}(R, F_{i-1}(M)) \cong \operatorname{Tor}_{2}^{S}(R, F_{i}(M)/F_{i-1}(M)).$$

Then if K_i denotes the kernel of

$$S \otimes_R Q_i \to F_i(M)/F_{i-1}(M),$$

tensoring with R gives the exact sequence

$$0 = \operatorname{Tor}_1^S(R, F_i(M)/F_{i-1}(M)) \to R \otimes_S K_i$$

$$\to R \otimes_S (S \otimes_R Q_i) \to R \otimes_S (F_i(M)/F_{i-1}(M))$$

$$\parallel \qquad \qquad \parallel$$

$$Q_i = \qquad \qquad Q_i.$$