(ii) Otherwise φ_{γ} has the form $\exists x \theta(x)$. In this case let $T_{\gamma+1}$ be $T_{\gamma} \cup \{\varphi_{\gamma}\} \cup \{\theta(c)\}$ where c is a constant in C that does not occur in $T_{\gamma} \cup \{\varphi_{\gamma}\}$. Since T_{γ} contains fewer than κ constants of C, such a c exists.

So if $\beta = \gamma + 1$, then $T_{\beta} = T_{\gamma+1}$ is obtained by adding at most a sentence or two to T_{γ} . Since T_{γ} contains at most $|\gamma| + \aleph_0$ of the constants in C, so does T_{β} . Moreover, T_{β} can be shown to be consistent in the same manner that T_{m+1} was shown to be consistent in the first claim in the proof of Theorem 4.2.

Now suppose that β is not a successor ordinal. Then it is a limit ordinal. In this case, define T_{β} as the set of all \mathcal{V}^+ -sentences that occur in T_{γ} for some $\gamma < \beta$. Again, we claim that T_{β} is consistent and contains at most $|\beta| + \aleph_0$ of the constants in C.

Claim 1 T_{β} is consistent.

Proof Suppose T_{β} is not consistent. Then $T_{\beta} \vdash \bot$ for some contradiction \bot . Since formal proofs are finite, $\Delta \vdash \bot$ for some finite subset Δ of T_{β} . Since it is finite, $\Delta \subset T_{\gamma}$ for some $\gamma < \beta$. But this contradicts our assumption that any such T_{γ} is consistent. We conclude that T_{β} must be consistent as was claimed.

Claim 2 T_{β} contains at most $|\beta|$ of the constants in C.

Proof For each $\gamma < \beta$, let C_{γ} be the set of constants in C that occur in T_{γ} . Then the constants occurring in T_{β} are $\bigcup_{\gamma < \beta} C_{\gamma}$. By assumption, $|C_{\gamma}| \leq |\gamma| + \aleph_0 \leq |\beta| + \aleph_0$. Since we are assuming that β is a limit ordinal, β is infinite. In particular, $|\beta| + \aleph_0 = |\beta|$. So each $|C_{\gamma}| \leq |\beta|$. It follows that the number of constants from C occurring in T_{β} is

$$\left| \bigcup_{\gamma < \beta} C_{\gamma} \right| \le \sum_{\gamma < \beta} |C_{\gamma}| \le \sum_{\gamma < \beta} |\beta| = |\beta| \cdot |\beta| = |\beta|.$$

This completes the proof of the claim.

So for each $\beta < \alpha$ we have successfully defined a \mathcal{V}^+ -theory T_{β} . These have been defined in such a way that $T_{\beta_1} \subset T_{\beta_2}$ for $\beta_1 < \beta_2 < \alpha$.

We now define T_{α} as the set of all \mathcal{V}^+ -sentences that occur in T_{β} for some $\beta < \alpha$. Like each T_{β} , T_{α} is a theory. This can be proved in the same manner as Claim 1 above. Unlike T_{β} for $\beta < \alpha$, T_{α} is a complete theory. This is because each \mathcal{V}^+ -sentence is enumerated as φ_{ι} for some $\iota < \alpha$. Either φ_{ι} or $\neg \varphi_{\iota}$ is in $T_{\iota+1}$ and, hence, in T_{α} as well.

Since $\Gamma = T_0 \subset T_\alpha$, T_α has Property 1. Moreover, part (b)ii of the definition of $T_{\gamma+1} \subset T_\alpha$ guarantees that T_α has Property 2. It was shown in the proof of Theorem 4.2 that any complete theory with Property 2 has a model. Therefore T_α has a model and Γ is satisfiable. \square