

18.3: Algebraic Constraints

In general we would like to design feedback controllers to attenuate both noise and disturbances at the output. We have examined SISO and MIMO conditions that guarantee rejection of low frequency disturbances as well as similar conditions for the rejection of high frequency noise. However, one might wonder if we can

- 1. minimize the influence of either noise or disturbances over all frequencies, and/or
- 2. minimize the influence of both noise and disturbances at the same frequency.

Let us begin this discussion by recalling the following:

- ullet $S=(I+PK)^{-1}$ is the transfer function mapping disturbances to the output;
- $T = PK(I + PK)^{-1}$ is the transfer function mapping noise to the output.

As we mentioned earlier, in a control design it is usually desirable to make both S and T small. However, because of algebraic constraints, both goals are not simultaneously achievable at the same frequency. These constraints are as follows.

General Limitations

S+T=I for all complex (Laplace domain) frequencies s. This is easily verified, since

$$S+T = (I+PK)^{-1} + PK(I+PK)^{-1}$$

= $(I+PK)(I+PK)^{-1}$
= I

This result implies that if $\sigma_{\max}[S(j\omega)]$ is small in some frequency range, $\sigma_{\max}[T(j\omega)] \sim 1$. The converse is also true.

Fortunately, we rarely need to make both of these functions small in the same frequency region.

Limitations Due to RHP Zeros and Poles

Before we discuss these limitations, we quote the following fact from complex analysis:

Let H(s) be a stable, causal, linear time-invariant continuous-time system. The *maximum modulus* principle implies that

$$\sigma_{\max}[H(s)] \leq \sup_{\omega} \sigma \max[H(j\omega)] = \|H\|_{\infty} \quad orall s \in ext{RHP}$$

In other words, a stable function, which is analytic in the RHP, achieves its maximum value over the RHP when evaluated on the imaginary axis.

Using this result, we can arrive at relationships between poles and zeros of the plant P located in the RHP and limitations on performance (e.g., disturbance and noise rejection).

SISO Systems: Disturbance Rejection

Consider the stable sensitivity function $S = (1 + PK)^{-1}$ for any stabilizing controller, K; then,

$$egin{aligned} S\left(z_i
ight) &= \left(1 + P\left(z_i
ight)K\left(z_i
ight)
ight)^{-1} = 1 & ext{ for all RHP zeros } z_i ext{ of } P \ S\left(p_i
ight) &= \left(1 + P\left(p_i
ight)K\left(p_i
ight)
ight)^{-1} = 0 & ext{ for all RHP poles } p_i ext{ of } P \end{aligned}$$

Since the \mathcal{H}_{∞} norm bounds the gain of a system over all frequencies,

$$1 = |S\left(z_i\right)| \le \|S\|_{\infty}$$

This means that we cannot uniformly attenuate disturbances over the entire frequency range if there are zeros in the RHP.

SISO Systems: Noise Rejection

Since the transfer function relating a noise input to the output is $T = PK(1 + PK)^{-1}$, an argument for T similar to S can be made, but with the roles of poles and zeros interchanged. In this case, RHP poles of the plant restrict us from uniformly attenuating noise over the entire frequency range.