## CHAPTER NINE

We have spent most of our time on parabolic equations; non-parabolic equations have made only token appearances, such as at the beginning of these notes when we took a brief glance at the wave equation, which is hyperbolic. It is fitting to end with a brief glance at a token elliptic, Laplace's equation.

We will give one existence and uniqueness theorem for bounded regions, and then see how such equations arise as the limits of parabolic equations. In particular, we will look at the limits of the Brownian density process as  $t \to \infty$ .

Let D be a bounded domain in R<sup>d</sup> with a smooth boundary. Consider

$$\begin{cases} \Delta u = f & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

If f is bounded and continuous, the solution to (9.1) is

(9.2) 
$$u(y) = \int K(x,y) f(x) dx = K(f,y),$$

where K is the Green's function for (9.1). Notice that in particular,  $\Delta K(f,y) = f(y).$ 

Let M be an L<sup>2</sup>-valued measure on R<sup>d</sup> (not a martingale measure, for there is no t in the problem!) Set  $Q(A,B) = E\{M(A)M(B)\}$  and suppose that there exists a positive definite measure  $\widetilde{Q}$  on R<sup>d</sup> × R<sup>d</sup> such that  $|Q(A,B)| \leq \widetilde{Q}(A\times B)$  for all Borel A, B  $\subseteq$  R<sup>d</sup>. This assures us of a good integration theory. We also assume for convenience that  $M(\partial D) = 0$ .

Let T be a kth order differential operator on  $R^d$  with smooth coefficients (0  $\leq$  k <  $^\infty$ ) and consider the SPDE

$$\begin{cases}
\Delta U = TM & \text{in } D \\
U = 0 & \text{in } \partial D
\end{cases}$$

Let us get the weak form of (9.3). Multiply by a test function  $\phi$  and integrate over  $\mathbf{R}^{\mathbf{d}}$ , pretending  $\mathbf{M}$  is smooth. Suppose  $\phi$  = 0 on  $\partial D$ . We can then do two integrations by parts to get

$$\int U(x)\Delta\phi(x)dx = \int T \dot{M}(x)\phi(x)dx.$$

Let T\* be the formal adjoint of T. If T is a zeroth or first order