corresponds to the character det : $P \to \mathbb{C}^{\times}$ which extends to a character of G (check this!). That's why (10.5) is the pull-back on G/P of a G-linearized bundle on G/G, and is hence trivial as a line bundle. The corresponding G-linearized bundle on G/G is not trivial as a G-bundle (since det : $G \to \mathbb{C}^{\times}$ does not equal the trivial homomorphism), and (10.5) is also not trivial as a G-bundle on G/P.

Now consider in more detail the case when m=n-1 and P=B is the Borel subgroup of upper-triangular matrices in GL(V) for $V=\mathbb{C}^n$. Here the correspondence from Proposition 10.2 has the following explicit form. Order the basis of \mathbb{C}^n as e_1, \ldots, e_n . Then the group of characters $Hom(B, \mathbb{C}^{\times})$ is identified with \mathbb{Z}^n . The G-linearized line bundle corresponding to $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ is simply

$$S_1^{\lambda_1} \otimes (S_2/S_1)^{\lambda_2} \otimes \cdots \otimes (\tilde{\mathbb{C}}^n/S_{n-1})^{\lambda_n}.$$

Verifying this is a non-difficult but essential computation. The simplest case is n = 2: here one needs to check that the *B*-character in the fiber of the bundle $S_1^{\lambda_1} \otimes (\tilde{\mathbb{C}}^2/S_1)^{\lambda_2}$ at the point $B \in G/B$ is precisely the character $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$. We hope that the reader will verify this explicitly.

For a general connected reductive group G and a Borel subgroup $B \subset G$, we denote by $O(\lambda)$ the G-linearized line bundle on G/B corresponding to a character $\lambda: B \to \mathbb{C}^{\times}$, i.e., such that B acts via λ in the fiber of $O(\lambda)$ at the point $B \in G/B$. If we choose G to be SL(V), then every line bundle on G/B admits a unique G-linearization and the bijection of Proposition 10.2 induces an equality $\operatorname{Pic} G/B = \operatorname{Hom}(B, \mathbb{C}^{\times})$, where B is now a Borel subgroup of SL(V).

The isomorphisms (10.3) enable us to compute explicitly the cohomology of any GL(2)-linearized line bundle on $GL(2)/P = \mathbb{P}(V)$ for $V = \mathbb{C}^2$. Indeed, notice that in this case

$$O(\lambda) = S_1^{\lambda_1} \otimes (\tilde{V}/S_1)^{\lambda_2} = S_1^{\lambda_1 - \lambda_2} \otimes (\Lambda^2(\tilde{V}))^{\otimes \lambda_2},$$

where $(\Lambda^2(\tilde{V}))^{\otimes \lambda_2}$ is a trivial bundle on $\mathbb{P}(V)$ with a nontrivial *G*-linearization. Hence (10.3) implies for $\lambda_2 - \lambda_1 \geq 0$,

$$H^{0}(\mathbb{P}(V), S_{1}^{\lambda_{1}} \otimes (\tilde{V}/S_{1})^{\lambda_{2}}) = S^{\lambda_{2} - \lambda_{1}}(V^{*}) \otimes (\Lambda^{2}(V))^{\otimes \lambda_{2}},$$
(10.6)
$$H^{1}(\mathbb{P}(V), S_{1}^{\lambda_{1}} \otimes (\tilde{V}/S_{1})^{\lambda_{2}}) = 0;$$

for $\lambda_2 - \lambda_1 \leq -2$,

$$H^{0}(\mathbb{P}(V), S_{1}^{\lambda_{1}} \otimes (\tilde{V}/S_{1})^{\lambda_{2}}) = 0,$$

$$H^{1}(\mathbb{P}(V), S_{1}^{\lambda_{1}} \otimes (\tilde{V}/S_{1})^{\lambda_{2}}) = S^{\lambda_{1} - \lambda_{2} - 2}(V) \otimes (\Lambda^{2}(V))^{\otimes \lambda_{2} + 1};$$
(10.7)