This can also be written as

$$X_{t} = S_{\Delta t} X_{t_{0}} + \left(\int_{0}^{\Delta t} S_{s} ds \right) F(X_{t_{0}}) + \left(O_{t} - S_{\Delta t} O_{t_{0}} \right)$$

$$+ \int_{t_{0}}^{t} S_{(t-s)} F'(X_{t_{0}}) \Delta O_{s} ds + O\left((\Delta t)^{1 + \min(1, 2\theta)} \right)$$
(7.35)

and is a Taylor approximation of order $1 + \min(1, 2\theta)$. It will be used in Section 7.7 to derive the numerical scheme (7.68).

Taylor expansion of order $1 + \min(1, 3\theta)$

The remainder term (7.34) consists of three parts, namely,

$$I_*^1[I_*^0](t) = O\left((\Delta t)^2\right), \qquad I_*^2[I_*^0, \Delta O](t) = O\left((\Delta t)^{2+\theta}\right),$$
$$I_*^2[\Delta O, \Delta O](t) = O\left((\Delta t)^{1+2\theta}\right).$$

Since $\theta < \frac{1}{2}$ and hence $\min(2, 2+\theta, 1+2\theta) = 1+2\theta$ in the examples in Section 7.6, the stochastic process $I_*^2[\Delta O, \Delta O]$ will be expanded here. Applying Proposition 5 to this term yields

$$I_*^2[\Delta O, \Delta O] = I^2[\Delta O, \Delta O] + I_*^3[I_*^0, \Delta O, \Delta O] + I_*^3[\Delta O, \Delta O, \Delta O],$$

and inserting this into (7.33) then gives

$$\Delta X = I^{0} + \Delta O + I^{1}[\Delta O] + I^{2}[\Delta O, \Delta O] + R, \tag{7.36}$$

where the remainder $R \in \mathcal{C}$ reads

$$R = I_*^1[I_*^0] + I_*^2[I_*^0, \Delta O] + I_*^3[I_*^0, \Delta O, \Delta O] + I_*^3[\Delta O, \Delta O, \Delta O].$$
 (7.37)

By Proposition 6

$$I_*^3[I_*^0, \Delta O, \Delta O](t) = O\left((\Delta t)^{2+2\theta}\right), \qquad I_*^3[\Delta O, \Delta O, \Delta O](t) = O\left((\Delta t)^{1+3\theta}\right),$$

so $R = O\left((\Delta t)^{1+\min(1,3\theta)}\right)$ since $\min(2,2+\theta,2+2\theta,1+3\theta) = 1+\min(1,3\theta)$. Finally, the Taylor expansion written out fully is

$$X_{t} = S_{\Delta t} X_{t_{0}} + \left(\int_{0}^{\Delta t} S_{s} ds \right) F(X_{t_{0}}) + \left(O_{t} - S_{\Delta t} O_{t_{0}} \right) + \int_{t_{0}}^{t} S_{(t-s)} F'(X_{t_{0}}) \Delta O_{s} ds$$

$$+ \frac{1}{2} \int_{t_{0}}^{t} S_{(t-s)} F''(X_{t_{0}}) (\Delta O_{s}, \Delta O_{s}) ds + O\left((\Delta t)^{1 + \min(1, 3\theta)} \right). \tag{7.38}$$