that  $\delta_n^{c_n} = \prod_{1 \leq j < k \leq n} |z_k - z_j|$ . Show that  $z_1, \ldots, z_n$  belong to  $\partial \hat{K}$  (the outer boundary of K).

## §12 The Refinement of a Subharmonic Function

Because of the intimate connection of subharmonic functions with so many of the properties of capacity and the solution of the Dirichlet problem, it is no surprise that the ability to manufacture these functions will extend the power of the theory and increase the depth of the results. In this section a new technique is developed that increases our proficiency at constructing subharmonic functions. In the next two sections this augmented skill will be put to good use as we prove Wiener's criterion for regularity.

Let G be a hyperbolic open subset of  $\mathbb C$  and suppose E is an arbitrary subset of G. For a negative subharmonic function u on G define

$$I_E^u(z) \equiv \sup \{\phi(z) : \phi \text{ is a negative subharmonic function on } G$$
  
that is not identically  $-\infty$  on any component of  $G$   
and  $\phi \leq u$  on  $E\}$ .

The function  $I_E^u$  is called the *increased function* of u relative to E. Be aware that in the notation for the increased function the role of G is suppressed. Also note that  $I_E^u$  may fail to be use and thus may not be subharmonic. To correct for this, define

$$\hat{I}_E^u(z) = \limsup_{w \to z} I_E^u(w)$$

for all z in G. The function  $\hat{I}_E^u$  is called the *refinement* of u relative to E. The term for this function most often seen in the literature is the French word "balayage." This word means "sweeping" or "brushing." The term and concept go back to Poincaré and the idea is that the subharmonic function u is modified and polished (or brushed) to produce a better behaved function that still resembles u on the set E. That the function  $\hat{I}_E^u$  accomplishes this will be seen shortly. We are avoiding the word "sweep" in this context as it was used in §2 in a different way.

Here are some of the properties of the increased function and the refinement. Let  $\Phi_E^u$  be the set of negative subharmonic functions used to define  $I_E^u$ .

**12.1 Proposition.** If G is a hyperbolic open set, E and F are subsets of G, and u and v are negative subharmonic functions on G, then:

- (a)  $\hat{I}_E^u$  is subharmonic on G;
- (b)  $u \leq I_E^u \leq \hat{I}_E^u \leq 0$  on G;