1.4 Risk-Neutral Pricing

for any  $0 \le s \le t \le T$ . The process  $(\xi_t^{\theta})_{0 \le t \le T}$  is called the Radon-Nikodym process.

The main result of this section asserts that the process  $(W_i^*)$  given by (1.45) is a standard Brownian motion under the probability  $IP^*$ . This result in its full generality (when  $\theta$  is an adapted stochastic process) is known as Girsanov's theorem. In our simple case ( $\theta$  constant), it is easily derived by using the characterization (1.3) and formula (1.49) as follows:

$$\mathbb{E}^{*}\{e^{iu(W_{t}^{*}-W_{s}^{*})} \mid \mathcal{F}_{s}\} = \frac{1}{\xi_{s}^{\theta}} \mathbb{E}\{\xi_{t}^{\theta}e^{iu(W_{t}^{*}-W_{s}^{*})} \mid \mathcal{F}_{s}\} 
= e^{\theta W_{s}+\theta^{2}s/2} \mathbb{E}\{e^{-\theta W_{t}-\theta^{2}t/2}e^{iu(W_{t}-W_{s}+\theta(t-s))} \mid \mathcal{F}_{s}\} 
= e^{(-\theta^{2}/2+iu\theta)(t-s)} \mathbb{E}\{e^{i(u+i\theta)(W_{t}-W_{s})} \mid \mathcal{F}_{s}\} 
= e^{(-\theta^{2}/2+iu\theta)(t-s)}e^{-(u+i\theta)^{2}(t-s)/2} 
= e^{-u^{2}(t-s)/2}.$$

## 1.4.2 Self-Financing Portfolios

As in Section 1.3.1, a portfolio comprises  $a_t$  units of stock and  $b_t$  in bonds; we denote by  $V_t$  its value at time t:

$$V_t = a_t X_t + b_t e^{rt}.$$

The self-financing property (1.28), namely  $dV_t = a_t dX_t + rb_t e^{rt} dt$ , implies that the discounted value of the portfolio,  $\widetilde{V}_t = e^{-rt}V_t$ , is a martingale under the risk-neutral probability  $IP^*$ . This important property of self-financing portfolios is obtained as follows:

$$d\widetilde{V}_{t} = -re^{-rt}V_{t} dt + e^{-rt} dV_{t}$$

$$= -re^{-rt}(a_{t}X_{t} + b_{t}e^{rt}) dt + e^{-rt}(a_{t} dX_{t} + rb_{t}e^{rt} dt)$$

$$= -re^{-rt}a_{t}X_{t} dt + e^{-rt}a_{t} dX_{t}$$

$$= a_{t} d(e^{-rt}X_{t})$$

$$= a_{t} d\widetilde{X}_{t}$$

$$= \sigma a_{t}\widetilde{X}_{t} dW_{t}^{*} \text{ (by (1.46)),} \tag{1.50}$$

which shows that  $(\widetilde{V}_t)$  is a martingale under  $IP^*$  as a stochastic integral with respect to the Brownian motion  $(W_t^*)$ . Indeed, the same computation shows that if a portfolio satisfies  $d\widetilde{V}_t = a_t d\widetilde{X}_t$  then it is self-financing.

A simple calculation demonstrates the connection between martingales and no arbitrage. Suppose that  $(a_t, b_t)_{0 \le t \le T}$  is a self-financing arbitrage strategy; that is,

$$V_T \ge e^{rT} V_0 \quad (IP - a.s.), \tag{1.51}$$

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with

$$IP\{V_T > e^{rT}V_0\} > 0, (1.52)$$

so that the strategy never makes less than money in the bank and there is some chance of making more. But

$$I\!\!E^{\star}\{V_T\} = e^{rT}V_0$$

by the martingale property, so (1.51) and (1.52) cannot hold. This is because IP and  $IP^*$  are equivalent and so (1.51) and (1.52) also hold with IP replaced by  $IP^*$ .

## 1.4.3 Risk-Neutral Valuation

Assume that  $(a_t, b_t)$  is a self-financing portfolio satisfying the same integrability conditions of Section 1.3.1 and replicating the European-style derivative with nonnegative payoff H:

$$a_T X_T + b_T e^{rT} = H, (1.53)$$

where we assume that H is a square integrable  $\mathcal{F}_T$ -adapted random variable. This includes European calls and puts or more general standard European derivatives for which  $H = h(X_T)$ , as well as other European-style exotic derivatives presented in Section 1.2.3.

On one hand, a no-arbitrage argument shows that the price at time t of this derivative should be the value  $V_t$  of this portfolio. On the other hand, as shown in Section 1.4.2, the discounted values  $(\widetilde{V}_t)$  of this portfolio form a martingale under the risk-neutral probability  $\mathbb{P}^*$  and consequently

$$\widetilde{V}_t = I\!\!E^* \{ \widetilde{V}_T \mid \mathcal{F}_t \},$$

which gives

$$V_t = I\!\!E^* \{ e^{-r(T-t)} H \mid \mathcal{F}_t \}$$
 (1.54)

after reintroducing the discounting factor and using the replicating property (1.53).

Alternatively, given the risk-neutral valuation formula (1.54), we can find a self-financing replicating portfolio for the payoff H. The existence of such a portfolio is guaranteed by an application of the martingale representation theorem: for  $0 \le t \le T$ ,

$$M_t = IE^* \{e^{-rT}H \mid \mathcal{F}_t\}$$