Let $a_n = f^{(n)}(a)/n!$, so that for any a we have $|a_n| > n^n$. By Hadamard's formula, the radius of convergence is given by

$$\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{1/n} = \limsup_{n \to \infty} n = \infty$$

so that the power series defined in the right hand side is valid only at a. Yet, as an analytic function it should be analytic in some radius of convergence for R > 0. Of course, if g(n) is any increasing function (instead of n^n) such that $g(n) \to \infty$, then we arrive at the same conclusion.

Here's another way to go about the problem, which seems to be more of what Ahlfors intended by putting this problem here. Cauchy's estimate tells us that if R > 0,

$$|f^{(n)}(z)| \le \frac{n!M}{R^n}$$

where $M = \max_{|z|=R} |f(z)|$. Note that M is finite since f(z) is continuous and |z| = R is compact. If also $|f^{(n)}(z)| > n!n^n$, then $n^n \leq M/R^n$ for all $n \geq 0$. For n large enough this inequality fails, since we can rearrange it to $(nR)^n \leq M$, and the left hand side is unbounded for every R > 0.

4.2.3.6. A more general form of Lemma 3 reads as follows:

Let the function $\varphi(z,t)$ be continuous as a function of both variables when z lies in a region Ω and $a \le t \le b$. Suppose further that $\varphi(z,t)$ is analytic as a function of $z \in \Omega$ for any fixed t. Then

$$F(z) = \int_{\alpha}^{\beta} \varphi(z,t) \ dt$$

is analytic in z and

$$F'(z) = \int_{\alpha}^{\beta} \frac{\partial \varphi(z, t)}{\partial z} dt \qquad (\star).$$

To prove this represent $\varphi(z,t)$ as a Cauchy integral

$$\varphi(z,t) = \frac{1}{2\pi i} \int_C \frac{\varphi(\zeta,t)}{\zeta-z} d\zeta.$$

Fill in the necessary details to obtain

$$F(z) = \int_C \left(\frac{1}{2\pi i} \int_{\alpha}^{\beta} \varphi(\zeta, t) \ dt \right) \frac{d\zeta}{\zeta - z}.$$

and use Lemma 3 to prove (\star) .

Solution: Recall that Cauchy's integral formula tells us that

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta) \ d\zeta}{\zeta - z}$$

for z in the bounded region determined by C. Applying this with F(z) gives

$$F(z) = \frac{1}{2\pi i} \int_C \frac{F(\zeta) d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_C \left(\int_\alpha^\beta \varphi(\zeta, t) dt \right) \frac{d\zeta}{\zeta - z} = \int_C \left(\frac{1}{2\pi i} \int_\alpha^\beta \varphi(\zeta, t) dt \right) \frac{d\zeta}{\zeta - z}$$

(one may also view it as a consequence of Fubini's theorem). Defining $\Phi(\zeta) = 1/2\pi i \int_{\alpha}^{\beta} \varphi(\zeta, t) dt$, Lemma 3 tells us that

$$F'(z) = \int_C \frac{\Phi(\zeta) \ d\zeta}{(\zeta - z)^2} = \int_C \left(\frac{1}{2\pi i} \int_{\alpha}^{\beta} \varphi(\zeta, t) \ dt \right) \frac{d\zeta}{(\zeta - z)^2}.$$

Now apply Lemma 3 once more to $\varphi(z,t)$ directly:

$$\frac{\partial \varphi(z,t)}{\partial z} = \frac{1}{2\pi i} \int_C \frac{\varphi(\zeta,t) \ d\zeta}{(\zeta-z)^2}.$$

Integrating this over $\alpha \leq t \leq \beta$ and using Fubini's theorem to switch integrals gives the result.

4.3. Local Properties of Analytical Functions.