Theorem 4 (Compactness criterion in $L_1[0, 1]$). For a bounded subset $D \subset L_1[0, 1]$ to be precompact it is necessary and sufficient that it is equicontinuous in the mean.

Proof. Suppose D is precompact. Since, by the lemma, $L_{\tau} \to I$ pointwise, one has that $L_{\tau} \to I$ uniformly on D. This uniform convergence is just another formulation of the requisite equicontinuity in the mean.

Now the converse. Suppose the set D is equicontinuous in the mean. Let us show that in this case the averaging operators E_n introduced in Example 3 of Subsection 11.2.2 converge uniformly on D to the unit operator. Since the sequence (E_n) is an approximate identity in $L_1[0,1]$, this will establish the precompactness of D. Thus, given an arbitrary $\varepsilon > 0$, take a $\delta > 0$ as in the definition of the equicontinuity in the mean: $\int_0^1 |f(t+\tau) - f(t)| dt < \varepsilon$ for all $f \in D$ and all $\tau \in [-\delta, \delta]$. Then for any $n > 1/\delta$ and any $f \in D$ we have

$$||E_n(f) - f|| = \int_0^1 \left| \sum_{k=1}^n n \int_{\Delta_{n,k}} f(x) dx \mathbb{1}_{\Delta_{n,k}}(t) - f(t) \right| dt$$

$$= \int_{0}^{1} \left| \sum_{k=1}^{n} n \left(\int_{\Delta_{n,k}} [f(x) - f(t)] dx \right) \mathbb{1}_{\Delta_{n,k}}(t) \right| dt$$

$$\leq \int_0^1 \sum_{k=1}^n n \int_{\Delta_{n,k}} |f(x) - f(t)| dx \mathbb{1}_{\Delta_{n,k}}(t) dt = n \sum_{k=1}^n \int_{\Delta_{n,k}} \int_{\Delta_{n,k}} |f(x) - f(t)| dx dt.$$

Using the fact that all pairs $(x, t) \in \bigcup_{k=1}^n \Delta_{n,k} \times \Delta_{n,k}$ obey the conditions $0 \le t \le 1$ and $t - \frac{1}{n} \le x \le t + \frac{1}{n}$, and making the change of variables $x \to t + \tau$, we complete the estimate to

$$||E_n(f) - f|| \le \int_{[-1/n, 1/n]} \int_0^1 |f(t+\tau) - f(t)| dt d\tau < 2\varepsilon.$$

Exercises

- 1. In the definitions of continuity and equicontinuity in the mean, instead of $\tau \in [-\delta, \delta]$ one can write $\tau \in [0, \delta]$.
- **2.** In the space ℓ_{∞} , the operators P_n , acting as $P_n((x_j)_{j=1}^{\infty}) = (x_1, \dots, x_n, 0, 0, \dots)$, do not form an approximate identity.
- **3.** The space ℓ_{∞} admits no approximate identity (indeed, we note that in ℓ_{∞} no convenient compactness criterion is known).
- **4.** Give an example of a precompact set $D \subset \ell_p$ that admits no joint majorant $z \in \ell_p$.