

$$\begin{split} V(\alpha,\beta,\gamma) &= U\left(\hat{x}_{3},\frac{\alpha}{2}\right)U\left(\hat{x}_{2},\frac{\beta}{2}\right)U\left(\hat{x}_{3},\frac{\gamma}{2}\right) \\ &= \begin{pmatrix} e^{-i\alpha/2} & 0\\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2)\\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix} \begin{pmatrix} e^{-i\gamma/2} & 0\\ 0 & e^{i\gamma/2} \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\alpha/2}\cos(\beta/2)e^{-i\gamma/2} & -e^{-i\alpha/2}\sin(\beta/2)e^{i\gamma/2}\\ e^{i\alpha/2}\sin(\beta/2)e^{-i\gamma/2} & e^{i\alpha/2}\cos(\beta/2)e^{i\gamma/2} \end{pmatrix} \\ &= \begin{pmatrix} \xi_{0} & -\xi_{1}^{*}\\ \xi_{1} & \xi_{0}^{*} \end{pmatrix} \end{split}$$
(5.1.10)

with

$$\xi_0 = e^{-i\alpha/2}\cos(\beta/2)e^{-i\gamma/2} \ \xi_1 = e^{i\alpha/2}\sin(\beta/2)e^{-i\gamma/2}$$
 (5.1.11)

The four matrix elements appearing in this relation are the so-called Cayley-Klein parameters. (See Equation 3.4.43 in Section 3.4.2.)

It is a general property of the matrices of the algebra  $A_2$ , that they can be represented either in terms of components or in terms of matrix elements. We have arrived at the conclusion that the representation of a unitary matrix in terms of elements is suitable for the parametrization of orientational configuration, while the rotation operator is represented in terms of components (axisangle variables).

There is one more step left to express this result most efficiently. We introduce the two-component complex vectors (spinors) of  $\mathcal{V}(2,C)$  already mentioned at the beginning of the chapter. In particular, we define two conjugate column vectors, or ket spinors:

$$|\xi\rangle = \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix}, \quad |\bar{\xi}\rangle = \begin{pmatrix} -\xi_1^* \\ \xi_0^* \end{pmatrix}$$
 (5.1.12)

and write the unitary V matrix symbolically as

$$V = (\langle \xi || \mid \bar{\xi} \rangle) \tag{5.1.13}$$

We define the corresponding bra vectors by splitting the Hermitian conjugate V horizontally into row vectors:

$$V^{\dagger} = \begin{pmatrix} \xi_0^* & \xi_1^* \\ -\xi_1 & \xi_0 \end{pmatrix} = \begin{pmatrix} \langle \xi | \\ \langle \bar{\xi} | \end{pmatrix}$$
 (5.1.14)

or

$$\langle \xi | = (\xi_0^*, \xi_1^*); \quad \langle \bar{\xi} | = (-\xi_1, \xi_0)$$
 (5.1.15)

The condition of unitarity of V can be expressed as

$$V^{\dagger}V = \begin{pmatrix} \langle \xi | \\ \langle \bar{\xi} | \end{pmatrix} (|\xi\rangle, |\bar{\xi}\rangle)$$
 (5.1.16)

$$= \begin{pmatrix} \langle \xi \mid \xi \rangle & \langle \xi \mid \bar{\xi} \rangle \\ \langle \bar{\xi} \mid \xi \rangle & \langle \bar{\xi} \mid \bar{\xi} \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
 (5.1.17)

yielding at once the conditions of orthonormality

$$\begin{aligned}
\langle \xi \mid \xi \rangle &= \langle \bar{\xi} \mid \bar{\xi} \rangle = 1 \\
\langle \xi \mid \bar{\xi} \rangle &= \langle \bar{\xi} \mid \xi \rangle = 0
\end{aligned} (5.1.18)$$

These can be, of course, verified by direct calculation. The orthogonal spinors are also called conjugate spinors. We see from these relations that our definition of spin conjugation is, indeed, a sensible one. However, the meaning of this concept is richer than the analogy with the ortho-normality relation in the real domain might suggest.

First of all we express spin conjugation in terms of a matrix operation. The relation is nonlinear, as it involves the operation of complex conjugation  $\mathcal{K}$ .