We need to show  $w_{\lambda} < 0$  in  $\Sigma_{\lambda}$  for any  $\lambda \in (0, a)$ . This implies in particular that  $w_{\lambda}$  assumes along  $\partial \Sigma_{\lambda} \cap \Omega$  its maximum in  $\Sigma_{\lambda}$ . By Theorem 2.5 (the Hopf lemma) we have for any such  $\lambda \in (0, a)$ 

$$D_{x_1} w_{\lambda} \big|_{x_1 = \lambda} = 2 D_{x_1} u \big|_{x_1 = \lambda} < 0.$$

For any  $\lambda$  close to a, we have  $w_{\lambda} < 0$  by Proposition 2.13 (the maximum principle for a narrow domain) or Theorem 2.32. Let  $(\lambda_0, a)$  be the largest interval of values of  $\lambda$  such that  $w_{\lambda} < 0$  in  $\Sigma_{\lambda}$ . We want to show  $\lambda_0 = 0$ . If  $\lambda_0 > 0$ , by continuity,  $w_{\lambda_0} \leq 0$  in  $\Sigma_{\lambda_0}$  and  $w_{\lambda_0} \not\equiv 0$  on  $\partial \Sigma_{\lambda_0}$ . Then Theorem 2.7 (the strong maximum principle) implies  $w_{\lambda_0} < 0$  in  $\Sigma_{\lambda_0}$ . We will show that for any small  $\varepsilon > 0$ 

$$w_{\lambda_0-\varepsilon}<0$$
 in  $\Sigma_{\lambda_0-\varepsilon}$ .

Fix  $\delta > 0$  (to be determined). Let K be a closed subset in  $\Sigma_{\lambda_0}$  such that  $|\Sigma_{\lambda_0} \setminus K| < \frac{\delta}{2}$ . The fact that  $w_{\lambda_0} < 0$  in  $\Sigma_{\lambda_0}$  implies

$$w_{\lambda_0}(x) \le -\eta < 0$$
 for any  $x \in K$ .

By continuity we have

$$w_{\lambda_0-\varepsilon}<0$$
 in  $K$ .

For  $\varepsilon > 0$  small,  $|\Sigma_{\lambda_0 - \varepsilon} \setminus K| < \delta$ . We choose  $\delta$  in such a way that we may apply Theorem 2.32 (the maximum principle for a domain with small volume) to  $w_{\lambda_0 - \varepsilon}$  in  $\Sigma_{\lambda_0 - \varepsilon} \setminus K$ . Hence we get

$$w_{\lambda_0-\varepsilon}(x) \leq 0 \quad \text{in } \Sigma_{\lambda_0-\varepsilon} \setminus K$$

and then by Theorem 2.10

$$w_{\lambda_0-\varepsilon}(x)<0$$
 in  $\Sigma_{\lambda_0-\varepsilon}\setminus K$ .

Therefore we obtain for any small  $\varepsilon > 0$ 

$$w_{\lambda_0-\varepsilon}(x)<0$$
 in  $\Sigma_{\lambda_0-\varepsilon}$ .

This contradicts the choice of  $\lambda_0$ .