- 3. Let G be a compact Lie group and $M \times G \to M$ a differentiable right G-manifold. Assume further that the maps $\alpha_p \colon G \to M$, $g \mapsto pg$ are injections for every $p \in M$. Show that the G-manifold has the structure of a differentiable G-principal bundle.
- 4. Show that a bijective immersion of differentiable manifolds is a diffeomorphism, Hint: An immersion $f: M \to N$ is locally an embedding. If we had dim $N > \dim M$, then f(M) would have Lebesgue measure zero (locally) in N (see Bröcker and Jänich [1], §6).
- 5. Show that any two fibers of the Hopf fibration (4.10) are linked in S³; see Hopf [1]:

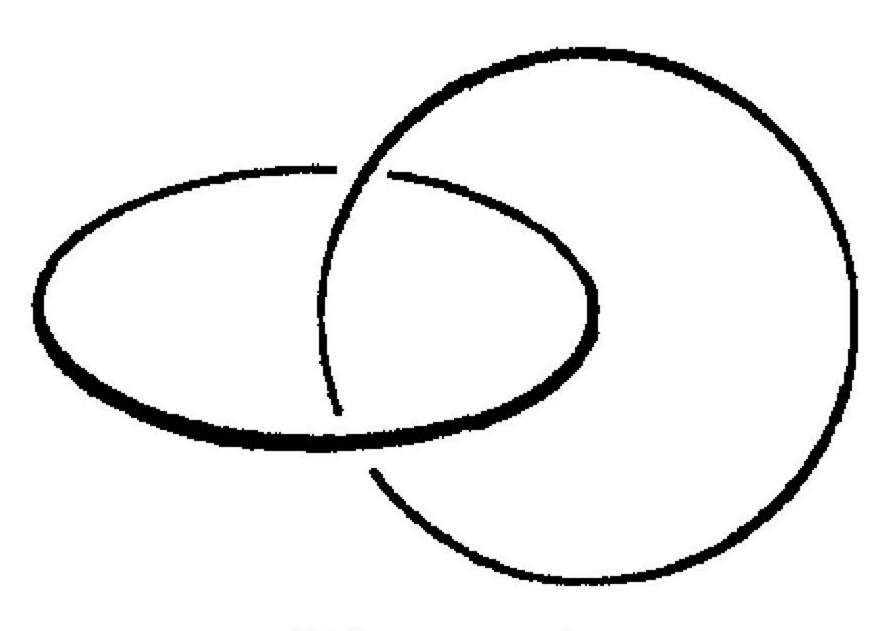


Figure 12

- 6. Give a diffeomorphism $SO(3) \cong \mathbb{R}P^3$, and, if you know enough about fundamental groups, show that $\pi_1(SO(n)) \cong \mathbb{Z}/2$ for n > 2 with a generator represented by the mapping $S^1 = SO(1) \stackrel{r}{\to} SO(n)$.
- 7. Show that Sp(n) is simply connected. Show that SU(n) is simply connected.
- 8. Show that there is a G-equivariant diffeomorphism of homogeneous spaces $G/H \stackrel{\cong}{\to} G/K$ if and only if H and K are conjugate in G.
- 9. Show that a homomorphism of tori $T^n \to T^k$ is induced by a linear map $\mathbb{R}^n \to \mathbb{R}^k$ whose associated matrix has integer coefficients.
- 10. Show that the only noncompact topologically cyclic Lie group is Z.
- 11. Let G be a Lie group and let $X, Y \in L(G)$. Show that [X, Y] = 0 if and only if $\exp(sX) \cdot \exp(tY) = \exp(tY) \cdot \exp(sX)$ for all $s, t \in \mathbb{R}$. If G is connected, then the Lie algebra of the center of G is $\{X \in LG | [X, Y] = 0 \text{ for all } Y \in LG \}$.

5. Invariant Integration

Let X be a locally compact space and $C_c^0(X)$ be the vector space of continuous real-valued functions on X with compact support. An *integral* on X is a monotone linear map

$$\int: C_c^0(X) \to \mathbb{R}, \quad f \mapsto \int f.$$

"Monotone" means that if $f(x) \le g(x)$ for all $x \in X$, then $\int f \le \int g$. The integral $\int f$ is often denoted $\int_X f(x) dx$.