(2.15) Example. Let $R \subseteq S$ be commutative rings, and let $g \in S$ be such that $g^2, g^3 \in R$. This implies (easily) that $g^n \in R$ for all $n \ge 2$, but g itself may or may not be in R. For $r \in R$, let $P_r = (1 + rg, g^2)$ be the R-submodule of S generated by the indicated elements.⁷ For $r, s \in R$:

$$P_r P_s = (1 + rg, g^2)(1 + sg, g^2) = (1 + (r + s)g + rsg^2, g^2 + rg^3, g^2 + sg^3, g^4)$$

contains $g^2(1+(r+s)g+rsg^2)-rsg^4=g^2+(r+s)g^3$, so it also contains rg^3 , sg^3 and g^2 , 1+(r+s)g. Therefore,

$$P_r P_s = (1 + (r + s)g, g^2, rg^3, sg^3).$$

From $g^3(1+(r+s)g)-(r+s)g^2g^2=g^3$, we see further that

(2.15A)
$$P_r P_s = (1 + (r + s)g, g^2) = P_{r+s}.$$

In particular, $P_r P_{-r} = P_0 = (1, g^2) = R$, so $\{P_r : r \in R\}$ is a family of invertible (hence projective) R-submodules of S, with $P_r^* = P_r^{-1} = P_{-r}$. The criterion for P_r to be free turns out to be the following:

(2.15B)
$$P_r$$
 is R -free iff $u(1+rg) \in R$ for some $u \in U(R[g])$.

In fact, if P_r is R-free, by (2.14)(4) we have $P_{-r} = uR$ for some $u \in U(S)$. Since

$$u \in P_{-r} = (1 - rg, g^2) \subseteq R[g]$$
 and $u^{-1} \in P_r = (1 + rg, g^2) \subseteq R[g]$,

we have $u \in U(R[g])$ and $u(1+rg) \in uP_r = R$. Conversely, suppose $u(1+rg) \in R$ for some $u \in U(R[g])$. Since $ug^2 \in R[g]g^2 \subseteq R$, we have $uP_r = (u(1+rg), ug^2) \subseteq R$ so $P_r \subseteq u^{-1}R$. We finish by showing that $P_r = u^{-1}R$. Let $J = \{c \in R : cg \in R\}$. This is an ideal of R[g] (called the "conductor" of the pair $R \subseteq R[g]$). Clearly, $Rg^2 \subseteq J$. Also, for any $c \in J$:

$$c = c(1 - r^2g^2 + r^2g^2) = [c(1 - rg)](1 + rg) + [cr^2]g^2 \in P_r$$

so $J \subseteq P_r$. Let $t = u(1 + rg) \in R$. Since $(1 + rg)(1 - rg) \in 1 + J$, we have $\overline{1 + rg} \in U(R[g]/J)$, and hence $\overline{t} \in U(R[g]/J)$. Using the fact that $R \subseteq R[g]$ is an integral extension, we see that $\overline{t} \in U(R/J)$. Therefore,

$$P_r/J = (1 + rg) \cdot (R/J) = u^{-1}t \cdot (R/J) = u^{-1} \cdot (R/J).$$

This implies that $u^{-1}R \subseteq P_r$, as desired.

To further simplify the criterion for P_r to be free, we can impose an additional hypothesis.

(2.15C) Proposition. In the preceding example, assume that

(2.15D) For
$$u \in U(R[g])$$
, $u(1+rg) \in R \implies u \in R$.

⁷The construction of $P_1 = (1 + g, g^2)$ was first given by S. Schanuel. It is, therefore, reasonable to call the P_r 's Schanuel modules. The direct calculation checking $P_r P_s = P_{r+s}$ in (2.15A) appears to be new.