two linear forms that generate the ideal of the line, together with any form of degree 7 vanishing on the points but not on the line. But Theorem 4.2(c) tells us that since the 7 points of X are in linearly general position the Castelnuovo–Mumford regularity of S_X (defined in Chapter 4) is 2, or equivalently, that the Betti diagram of S_X fits into 3 rows. Moreover, the ring S_X is reduced and of dimension 1 so it has depth 1. The Auslander–Buchsbaum Formula A2.15 shows that the resolution will have length 3. Putting this together, and using Corollary 1.9 we see that the minimal free resolution of S_X must have Betti diagram of the form

where the $\beta_{i,j}$ that are not shown are zero. In particular, the ideal of X is generated by quadrics and cubics.

Using Corollary 1.10 we compute successively $\beta_{1,2} = 3$, $\beta_{1,3} - \beta_{2,3} = 1$, $\beta_{2,4} - \beta_{3,4} = 6$, $\beta_{3,5} = 3$, and the Betti diagram has the form

(This is the same diagram as at the end of the previous section. Here is the connection: Extending the ground field if necessary to make it infinite, we could use Lemma A2.3 and choose a linear form $x \in S$ that is a nonzerodivisor on S_X . By Lemma 3.15 the graded Betti numbers of S_X/xS_X as an S/xS-module are the same as those of S_X as an S-module. Using our knowledge of the Hilbert function of S_X and the exactness of the sequence

$$0 \longrightarrow S_X(-1) \stackrel{x}{\longrightarrow} S_X \longrightarrow S_X/xS_x \longrightarrow 0,$$

we see that the cyclic (S/xS)-module S_X/xS_x has Hilbert function with values 1, 3, 3. This is what we used in Section 2B.)

... and Other Information in the Resolution

We see that even in this simple case the Hilbert function does not determine the $\beta_{i,j}$, and indeed they can take different values. It turns out that the difference reflects a fundamental geometric distinction between different sets X of 7 points in linearly general position in \mathbb{P}^3 : whether or not X lies on a curve of degree 3.

Up to linear automorphisms of \mathbb{P}^3 there is only one irreducible curve of degree 3 not contained in a plane. This *twisted cubic* is one of the *rational normal curves* studied in Chapter 6. Any 6 points in linearly general position in \mathbb{P}^3 lie