Proof. As we know, every abelian function can be written in the form

$$\frac{f}{g}, \quad f, g \in [\tilde{Q}, \tilde{l}, \tilde{E}],$$

for a suitable triple $(\tilde{Q}, \tilde{l}, \tilde{E})$.

We choose the natural number r as in Lemma 9.1. Then we choose an arbitrary theta function

$$h \in [rQ - \tilde{Q}, rl - \tilde{l}, rE - \tilde{E}], \quad h \neq 0.$$

The existence of h is ensured, since $rQ-\tilde{Q}$ leads to a nondegenerate Riemannian form. We have

$$\frac{f}{g} = \frac{fh}{gh}$$
 and $fh, gh \in [rQ, rl, rE]$.

Exercises for Sect. VI.9

1. The polynomial ring $A=\mathbb{C}[X_1,\ldots,X_m]$ admits the grading

$$A_r:=\{P\in A;\quad P\text{ homogeneous of degree }r\}.$$

Can there be an isomorphism from A(Q, l, A) onto A (for suitable m) which is compatible with this grading?

2. Can there be an isomorphism from A(Q, l, A) onto the graded ring of elliptic modular forms ([FB], Sect. V.3) which respects the gradings?

10. A Nondegenerateness Theorem

In principle, it could be possible that every abelian function for a lattice L is periodic with respect to a bigger lattice \tilde{L} , even if L admits a nondegenerate Riemannian form. Our next goal is to prove that such a pathological behavior is not possible at least for theta functions. We start with some notation.

Let A be an alternating nondegenerate bilinear form on $\mathbb{C}^n \times \mathbb{C}^n$ which takes only integral values on $L \times L$. We can define the dual lattice with respect to A (compare Remark 1.3):

$$L_* := \{ z \in \mathbb{C}^n; \quad A(z, a) \in \mathbb{Z} \text{ for all } a \in L \}.$$

It is easy to show that L_* is a lattice and that $L \subset L_*$.