

Figure 3.1. Fusion simplifies over.

Fusion

We've now established that for every stopper G, there is a K = G with no dominated or reversible options, and this K is unique up to similarity. But similarity is not quite isomorphism, as shown in Figure 2.1 on page 292. Different forms of **over** (for example) might have different numbers of subpositions, even though no subpositions of either form have any dominated or reversible options. (The theory of Chapter II shows that this situation isn't possible for short games.)

Similarity can be strengthened to isomorphism with the help of a new type of simplification: fusion of equivalent subpositions. Figure 3.1 shows how **over** might be condensed in two stages. First, the two terminal positions (which are obviously equal) are fused into a single vertex; then the two positions of value **over** are fused in a separate step. Fusion can be seen as a loopy analogue of the Replacement Lemma (Lemma II.2.2 on page 64).

Theorem 3.5 (Fusion Lemma). Let G be a stopper, and assume that no subposition of G has any dominated or reversible options. Suppose there are distinct subpositions H and J of G with H = J. Let G' be obtained from G by replacing all moves to J (throughout all subpositions of G) with moves to H. Then G' is also a stopper and G' = G.

Proof. Step 1: We first show that G' is a stopper. Suppose (for contradiction) that there is an infinite alternating run

$$X'_0, X'_1, X'_2, \dots$$

starting from some subposition X'_0 of G'. Let

$$X_0, X_1, X_2, \dots$$

be the corresponding sequence of subpositions of G. To obtain a contradiction, we'll construct an alternating run

$$Y_0, Y_1, Y_2, \ldots$$