**Proposition 5.1** (Correspondence Theorem for Rings). If I is a proper ideal in a commutative ring R, then there is an inclusion-preserving bijection  $\varphi$  from the set of all ideals J in R containing I to the set of all ideals in R/I, given by

$$\varphi \colon J \mapsto J/I = \{a + I \colon a \in J\}.$$

**Proof.** If we forget its multiplication, the commutative ring R is merely an additive abelian group and its ideal I is a (normal) subgroup. The Correspondence Theorem for Groups, Theorem 1.82, now applies to the natural map  $\pi: R \to R/I$ , and it gives an inclusion-preserving bijection

 $\Phi: \{ \text{all subgroups of } R \text{ containing } I \} \to \{ \text{all subgroups of } R/I \},$  where  $\Phi(J) = \pi(J) = J/I.$ 

If J is an ideal, then  $\Phi(J)$  is also an ideal, for if  $r \in R$  and  $a \in J$ , then  $ra \in J$ , and

$$(r+I)(a+I) = ra + I \in J/I.$$

Let  $\varphi$  be the restriction of  $\Phi$  to the set of intermediate ideals;  $\varphi$  is an injection because  $\Phi$  is an injection. To see that  $\varphi$  is surjective, let  $J^*$  be an ideal in R/I. Now  $\pi^{-1}(J^*)$  is an intermediate ideal in R, for it contains  $I = \pi^{-1}((0))$ , and  $\varphi(\pi^{-1}(J^*)) = \pi(\pi^{-1}(J^*)) = J^*.$ 

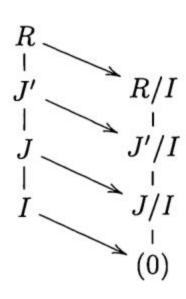


Figure 5.1. Correspondence Theorem.

Usually, the Correspondence Theorem for Rings is invoked, tacitly, by saying that every ideal in the quotient ring R/I has the form J/I for some unique ideal J with  $I \subseteq J \subseteq R$ .

**Example 5.2.** Let I = (m) be a nonzero ideal in  $\mathbb{Z}$ . If J is an ideal in  $\mathbb{Z}$  containing I, then J = (a) for some  $a \in \mathbb{Z}$  (because  $\mathbb{Z}$  is a PID). Since  $(m) \subseteq (a)$  if and only if  $a \mid m$ , the Correspondence Theorem for Rings shows that every ideal in the ring  $\mathbb{Z}/I = \mathbb{I}_m$  has the form J/I = ([a]) for some divisor a of m.

**Definition.** An ideal I in a commutative ring R is called a **prime ideal** if it is a proper ideal, that is,  $I \neq R$ , and  $ab \in I$  implies that  $a \in I$  or  $b \in I$ .

## Example 5.3.

(i) The ideal (0) is a prime ideal in a ring R if and only if R is a domain.

<sup>&</sup>lt;sup>1</sup>If X and Y are sets,  $f: X \to Y$  is a function, and S is a subset of Y, then  $ff^{-1}(S) \subseteq S$ ; if f is surjective, then  $ff^{-1}(S) = S$ .