$$\beta \in \mathscr{B}_{\Theta} \cap (\Theta + 1) \land \alpha \in \mathscr{B}_{b^{St}} \Rightarrow \alpha^{V} \in \mathscr{B}_{\beta^{V}}^{V}$$
 (9.41)

$$\beta \in \mathscr{B}_{\Theta+1} \wedge \alpha \in Cr(\beta) \Rightarrow \alpha^{V} \in Cr(\beta^{V}).$$
 (9.42)

Property (9.42) holds true since  $\alpha \in Cr(\beta)$  implies that there is a  $\beta_0 \geq \beta$  such that  $\alpha \in Cr(\beta_0) \setminus Cr(\beta_0 + 1)$ . Then  $\alpha = \bar{\varphi}_{\beta_0}(\eta)$  for some  $\eta$  and thus  $\alpha^V = \lceil \alpha \rceil^V = \bar{\varphi}_{\beta_0} \rceil_V(\lceil \eta \rceil^V) \in Cr(\beta_0^V) \subseteq Cr(\beta^V)$  since  $\beta^V \leq \beta_0^V$  by (9.40).

For  $\beta \in \mathcal{B}_{\Theta} \cap (\Theta + 1)$  the interpretation V is thus an embedding from  $\mathcal{B}_{\beta}$  into  $\mathcal{B}_{\beta^V}^V$  which preserves principality, strong criticality and  $\beta$ -criticality. We will now prove that this embedding is also onto. The proof will need a relativized version of Lemma 9.6.2 saying that if  $\psi_V(\alpha) \in \mathcal{B}_{\beta}^{V,n}$  there is an  $\alpha_0 \in \mathcal{B}_{\beta}^{V,n}$  such that  $\alpha_0 \in \mathcal{B}_{\alpha_0}^V$  and  $\psi_V(\alpha) = \psi_V(\alpha_0)$ , where  $\mathcal{B}_{\beta}^{V,n}$  is defined analogously to  $\mathcal{B}_{\beta}^n$ . Since the proof of Lemma 9.6.2 only needs  $\psi(\alpha) < \Omega$  it relativizes easily to interpretations which are good relative to some  $\Theta$ .

**9.7.8 Lemma** Let V be a good interpretation relative to  $\Theta$  and  $\beta \in \mathscr{B}_{\Theta} \cap (\Theta + 1)$ . Then for every  $\alpha \in \mathscr{B}^{V}_{\beta^{V}}$  there is a  $\gamma \in \mathscr{B}_{\beta}$  such that  $\alpha = \gamma^{V}$ . Moreover we have  $\alpha \in SC$  iff  $\gamma \in SC$  and  $\alpha \in \mathbb{H}$  iff  $\gamma \in \mathbb{H}$ .

*Proof* Let  $\alpha \in \mathscr{B}_{\beta}^{V,n}$ . We prove the lemma by induction on  $\beta^V$  with side induction on n.

If  $\alpha=0$  we put  $\gamma:=0$  and if  $\alpha=V(\Omega)$  we put  $\gamma:=\omega_1$ . Now assume  $\alpha=_{NF}$   $\alpha_1+\cdots+\alpha_n$ . Then  $\mathbb{H}\ni\alpha_i<\alpha$  for  $i=1,\ldots,n$ . By the main induction hypothesis there are ordinals  $\gamma_i\in\mathscr{B}_\beta$  such that  $\alpha_i=\gamma_i^V$  and  $\gamma_i\in\mathbb{H}$ . By equation (9.40) we obtain  $\gamma_1\geq\cdots\geq\gamma_n$  and put  $\gamma:=\gamma_1\cdots+\gamma_n$ . Then  $\gamma=_{NF}\gamma_1\cdots+\gamma_n$  and  $\gamma^V=\gamma_1^V\cdots+\gamma_n^V$ ,  $\gamma\in\mathscr{B}_\beta$  and  $\gamma\notin\mathbb{H}$ .

Next assume  $\alpha = \bar{\varphi}_{\alpha_1}(\alpha_2)$ . Then  $\alpha \in \mathbb{H} \setminus SC$  and  $\alpha_i < \alpha$ . By the main induction hypothesis there are ordinals  $\gamma_1, \gamma_2 \in \mathscr{B}_{\beta}$  such that  $\gamma_i^V = \alpha_i$  for i = 1, 2. Let  $\gamma = \bar{\varphi}_{\gamma_1}(\gamma_2)$ . Then  $\gamma \in \mathscr{B}_{\beta}$  and  $\gamma_i < \gamma$  for i = 1, 2.

Let  $\alpha = \psi_V(\eta)$  such that  $\eta \in \mathcal{B}_{\beta^V}^{V,n-1} \cap \beta^V$ . Then  $\alpha \in SC$ . By Lemma 9.6.2 there is an  $\eta_0 \in \mathcal{B}_{\beta^V}^{V,n-1} \cap \beta^V$  such that  $\eta_0 \in \mathcal{B}_{\eta_0}^V$  and  $\alpha = \psi_V(\eta_0)$ . By induction hypothesis there is an  $\alpha_0$  such that  $\eta_0 = \alpha_0^V$ , hence  $\alpha_0^V \in \mathcal{B}_{\alpha_0^V}^V$ , which implies  $\alpha_0 \in \mathcal{B}_{\alpha_0}$ . So  $\gamma := \psi(\alpha_0)$  implies  $\gamma \in SC$  and  $\gamma =_{NF} \psi(\alpha_0)$  and we obtain  $\alpha = \psi_V(\eta_0) = \psi_V(\alpha_0^V) = \psi(\alpha_0)^V = \gamma^V$ .

**9.7.9 Theorem** Let V be an interpretation which is good relative to  $\Theta$  and  $\beta \in \mathscr{B}_{\Theta} \cap (\Theta + 1)$ . Then  $(\mathscr{B}_{\beta})^V = \mathscr{B}_{\beta^V}^V$ .

*Proof* From  $\alpha \in \mathscr{B}_{\beta}$  we obtain  $\alpha^{V} \in \mathscr{B}_{\beta^{V}}^{V}$  by (9.41). Hence  $(\mathscr{B}_{\beta})^{V} := \{\alpha^{V} \mid \alpha \in \mathscr{B}_{\beta}\} \subseteq \mathscr{B}_{\beta^{V}}^{V}$ . Conversely we obtain for  $\alpha \in \mathscr{B}_{\beta^{V}}^{V}$  a  $\gamma \in \mathscr{B}_{\beta}$  such that  $\alpha = \gamma^{V} \in (\mathscr{B}_{\beta})^{V}$  by Lemma 9.7.8. Hence  $\mathscr{B}_{\beta^{V}}^{V} \subseteq (\mathscr{B}_{\beta})^{V}$ .