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which of course is no surprise. Next, we have

(3.106)
$$c_1(\Lambda) = \operatorname{ch}_1(\Lambda) = \pi_*\left(\frac{\gamma}{2}\right) = \frac{\kappa}{12}$$
,

where $\kappa = \kappa_1$ is the first tautological class. Similarly, to find $ch_2(\Lambda)$ we write

(3.107)
$$c_2(\Lambda) = \frac{\cosh_1(\Lambda)^2}{2} - \cosh_2(\Lambda) = \frac{\kappa^2}{288}$$

since $\operatorname{ch}_2(\Lambda) = 0$. In general, it's clear that the Grothendieck-Riemann-Roch in this case expresses each of the Chern classes of the Hodge bundle as a polynomial (with rational coefficients) in the tautological classes κ_i , and that the polynomial may be worked out explicitly in any given case. Note in particular that, while the λ_i are polynomials in the κ_i , the above examples already show that the converse is not true.

Next, we consider how this computation — at least in the case of the codimension 1 classes in $\overline{\mathcal{M}}_g$ — may be extended over all of the stable compactification $\overline{\mathcal{M}}_g$. Here we'll see the discussion of Section D used in practice. First of all, to define our terms, we will denote by ω the relative dualizing sheaf of $\overline{\mathcal{C}}_g$ over $\overline{\mathcal{M}}_g$, and call the direct image $\pi_*\omega$ on $\overline{\mathcal{M}}_g$ the Hodge bundle Λ . Note that the problem we were able to gloss over above has now become more serious: the universal curve now fails to be universal over a codimension 1 locus (all the points $[C] \in \Delta_1 \subset \overline{\mathcal{M}}_g$ correspond to curves with automorphisms). But now we have an alternative: by Proposition (3.93), in order to derive or prove any relation among divisor classes on the moduli space we simply have to verify the corresponding relation among the associated divisor classes on the base B of any family $X \to B$ of stable curves with smooth, one-dimensional base and smooth general fiber.

To do this, let $\rho: \mathcal{X} \longrightarrow B$ be any such one-parameter family of stable curves. We will use t to denote a local coordinate on the base B of the family. We make one modification: we let $\mu: \mathcal{Y} \longrightarrow \mathcal{X}$ be a minimal resolution of the singularities of the total space \mathcal{X} , and let $v = \rho \circ \mu: \mathcal{Y} \longrightarrow B$ be the composition. This has the effect, for each node p of a fiber of $\mathcal{X} \longrightarrow B$ with local coordinates x, y, t satisfying $xy = t^k$, of replacing the point p by a chain of k-1 rational curves. In this way we arrive at a family $v: \mathcal{Y} \longrightarrow B$ of semistable curves, with smooth total space and having k nodes lying over each node of a fiber of \mathcal{X} with local equation $xy - t^k$. To relate the invariants of the new family $v: \mathcal{Y} \longrightarrow B$ to those of the original, we have the:

EXERCISE (3.108) 1) Show that the relative dualizing sheaf of the new family is trivial on the exceptional divisor of the map μ , and hence that it's simply the pullback of the relative dualizing sheaf of $\rho: \mathcal{X} \longrightarrow B$, i.e.,

$$\omega_{Y/B} = \mu^* \omega_{X/B}$$
.