

Figure 24.2

$$\tau_1 \tau_2^{-1}(\mathbf{x}) = \tau_1((-\mathbf{v}) + \mathbf{x})$$

$$= \mathbf{u} + ((-\mathbf{v}) + \mathbf{x})$$

$$= (\mathbf{u} - \mathbf{v}) + \mathbf{x}.$$

So $\tau_1 \tau_2^{-1}$ is translation by $\mathbf{u} - \mathbf{v}$ and therefore belongs to T. Let O denote the subgroup of E_2 which consists of the *orthogonal transformations*. In other words the elements of O are rotations about the origin and reflections in lines through the origin. The discussion in the previous paragraph shows that $E_2 = TO$.

The intersection of T and O is just the identity transformation because every non-trivial translation moves the origin, whereas every element of O keeps the origin fixed. The usual argument now shows that each isometry can be written in *only one* way as an orthogonal transformation followed by a translation. For if $g = \tau f = \tau' f'$ where τ , $\tau' \in T$ and f, $f' \in O$, then $(\tau')^{-1}\tau = f' f^{-1}$ lies in $T \cap O$ and hence $\tau = \tau'$, f = f'. If $g = \tau f$ and if f is a rotation, then g is called a *direct isometry*. In the other case, when f is a reflection, g is said to be an *opposite isometry*.

Suppose $f \in O$, $\tau \in T$ and $\tau(\mathbf{0}) = \mathbf{v}$. Then for each $\mathbf{x} \in \mathbb{R}^2$ we have

$$f\tau f^{-1}(\mathbf{x}) = f(\mathbf{v} + f^{-1}(\mathbf{x}))$$

$$= f(\mathbf{v}) + f(f^{-1}(\mathbf{x})) \quad \text{because } f \text{ is linear}$$

$$= f(\mathbf{v}) + \mathbf{x}.$$

Therefore the conjugate $f\tau f^{-1}$ is translation by the vector $f(\mathbf{v})$. Since the elements of T and O together generate E_2 , we see (using (15.2)) that T is a normal subgroup of E_2 .

We can now understand the product structure of our group in terms of the

decomposition $E_2 = TO$. If $g = \tau f$, $h = \tau_1 f_1$ where τ , $\tau_1 \in T$ and f, $f_1 \in O$, then $gh = \tau f \tau_1 f_1 = (\tau f \tau_1 f^{-1})(f f_1)$

expresses gh as an orthogonal transformation followed by a translation. Put another way the correspondence

$$g \to (\tau, f)$$

is an isomorphism between E_2 and the semidirect product $T \times_{\varphi} O$ where $\varphi \colon O \to \operatorname{Aut}(T)$ is given by conjugation.

Specific calculations are best carried out using rather different notation. Suppose $g = \tau f$ where $\tau \in T$ and $f \in O$. If $\mathbf{v} = \tau(\mathbf{0})$, and if M is the orthogonal matrix which represents f in the standard basis for \mathbb{R}^2 , then

$$g(\mathbf{x}) = \mathbf{v} + f_{\mathbf{M}}(\mathbf{x}) = \mathbf{v} + \mathbf{x}M^{t} \tag{*}$$

for all $\mathbf{x} \in \mathbb{R}^2$. Conversely, given $\mathbf{v} \in \mathbb{R}^2$ and $M \in O_2$, the formula (*) determines an isometry of the plane. We may therefore think of each isometry as an ordered pair (\mathbf{v}, M) in which $\mathbf{v} \in \mathbb{R}^2$ and $M \in O_2$, with multiplication given by

$$(\mathbf{v}, M)(\mathbf{v}_1, M_1) = (\mathbf{v} + f_M(\mathbf{v}_1), MM_1).$$

If we are pressed to be very precise we explain that we have identified E_2 with the semidirect product $\mathbb{R}^2 \times_{\psi} O_2$, the homomorphism $\psi \colon O_2 \to \operatorname{Aut}(\mathbb{R}^2)$ being the usual action of O_2 on \mathbb{R}^2 . Notice that (\mathbf{v}, M) is a direct isometry when $\det M = +1$ and an opposite isometry when $\det M = -1$.

The "simplest" isometries are easily described as ordered pairs. Let

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \qquad B = \begin{bmatrix} \cos \varphi & \sin \varphi \\ \sin \varphi & -\cos \varphi \end{bmatrix}$$

and let *l*, *m* be the lines shown in Figure 24.3.

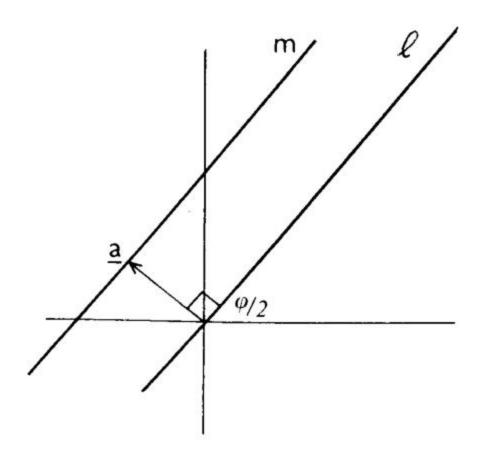


Figure 24.3