ISyE6669 Deterministic Optimization Homework 1

Solutions

September 23, 2018

1 A warm-up for using XpressThe concrete poles example and Xpress(20 points)

1. (5 points)

$$
\begin{aligned}
\min \quad & |x_1 - x_2| + |y_1 - y_2| + |x_1 - 300| + |y_1 - 1200| + |x_1 - 0| + |y_1 - 600| \\
& + |x_2| + |y_2 - 600| + |x_2 - 600| + |y_2| \\
\text{s.t.} \quad & 0 \leq x_i \leq 900 \quad \forall i = 1, 2 \\
& 0 \leq y_i \leq 1500 \quad \forall i = 1, 2.
\end{aligned}
$$

2. (5 points)

$$
\begin{aligned}
\min \quad & z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8 + z_9 + z_{10} \\
\text{s.t.} \quad & 0 \leq x_i \leq 900 \quad \forall i = 1, 2 \\
& 0 \leq y_i \leq 1500 \quad \forall i = 1, 2 \\
& -z_1 \leq x_1 - x_2 \leq z_1 \\
& -z_2 \leq y_1 - y_2 \leq z_2 \\
& -z_3 \leq x_1 - 300 \leq z_3 \\
& -z_4 \leq y_1 - 1200 \leq z_4 \\
& -z_5 \leq x_1 - 0 \leq z_5 \\
& -z_6 \leq y_1 - 600 \leq z_6 \\
& -z_7 \leq x_2 - 0 \leq z_7 \\
& -z_8 \leq y_2 - 600 \leq z_8 \\
& -z_9 \leq x_2 - 600 \leq z_9 \\
& -z_{10} \leq y_2 \leq z_{10}
\end{aligned}
$$

3. (5 points for right answer) The optimal solution is $(x_1, y_1) = (x_2, y_2) = (0, 600)$. Yes, the two facilities overlap each other.

4. (5 points, -2 if contraints are wrong, -2 if reformulation is wrong, -1 if the final answer is wrong)First, the conditions can be modeled by constraints

$$
\begin{aligned}
&|x_1 - 300| + |y_1 - 1200| \leqslant 500, \\
&|x_2 - 600| + |y_2 - 0| \leqslant 700.
\end{aligned}
$$

Second, a linear reformulation of the above constraints can be written as

$$z_3 + z_4 \leqslant 500,$$
$$z_9 + z_{10} \leqslant 700.$$

Third, the optimal solution given by the modified code is $(x_1, y_1) = (300, 700)$, $(x_2, y_2) = (500, 600)$. The two new facilities do not overlap.

2 Quadratic programming and piecewise linear approximation(30 points)

1. (5 points)We define

$$z_1 := |x_1|, \quad z_2 := |x_2|$$

Then, problem (P) can be reformulated as

$$\begin{aligned}(P) \quad & \min\ f(\boldsymbol{x}) = (x_1 - 1.5)^2 + (x_2 - 0.8)^2 \\ & \text{s.t.} \quad z_1 + z_2 \leq 1, \\ & \qquad - z_1 \leq x_1 \leq z_1, \\ & \qquad - z_2 \leq x_2 \leq z_2, \\ & \qquad z_i \geq 0, i = 1, 2.\end{aligned}$$

2. (5 points)

$$\frac{\partial f}{\partial x_1} = 2(x_1 - 1.5), \quad \frac{\partial f}{\partial x_2} = 2(x_2 - 0.8)$$
$$H_i := 2(x_1^i - 1.5)(x_1 - x_1^i) + 2(x_2^i - 0.8)(x_2 - x_2^i) + (x_1^i - 1.5)^2 + (x_2^i - 0.8)^2$$
$$\hat{f}(\boldsymbol{x}) = \max\{H_1, \ldots, H_{10}\}$$

3. (5 points) The problem (P) can be reformulated as

$$\begin{aligned}(P) \quad & \min\ z \\ & \text{s.t.}\ 2(x_1^i - 1.5)(x_1 - x_1^i) + 2(x_2^i - 0.8)(x_2 - x_2^i) + (x_1^i - 1.5)^2 + (x_2^i - 0.8)^2 \leq z \quad i = 1, \ldots, 10 \\ & \qquad z_1 + z_2 \leq 1, \\ & \qquad - z_1 \leq x_1 \leq z_1, \\ & \qquad - z_2 \leq x_2 \leq z_2, \\ & \qquad z_i \geq 0 \quad i = 1, 2.\end{aligned}$$

4. (5 points, -1 for each wrong pair)$(0.5, 0.5) : 1.09$, $(-0.5, -0.5) : 5.69$, $(0.5, -0.5) : 2.69$, $(-0.5, 0.5) : 4.09$, $(1, 0) : 0.89$, $(0, 1) : 2.29$, $(-1, 0) : 6.89$, $(0, -1) : 5.49$, $(0, 0) : 2.89$, $(0.1, 0.9) : 1.97$.

5. (5 points, -2 for each wrong answer) An optimal solution to the quadratic program given by Xpress is

$$\text{obj} = 0.74, \quad (x_1, x_2) = (0.75, 0.25).$$

6. (5 points, -2 for each wrong answer) An optimal solution to the quadratic program given by Xpress is

$$\text{obj} = 0.845002, \quad (x_1, x_2) = (0.849986, 0.150012).$$

The LP approximation solutions are close to the QP problem.

3 Production planning by a computer manufacturer (20 points; 10 points for the formulation, -1 for each wrong constraint and obj function; 10 points for code, -1 for each error) The full model formulation is

$$\begin{aligned}
\max \quad & 60x_1 + 40x_2 + 30x_3 + 30x_4 + 15x_5 \quad \text{(total revenue)} \\
\text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 + x_5 \leq 7 \quad \text{(CPU availability)} \\
& 4x_1 + 2x_2 + 2x_3 + 2x_4 + x_5 \leq 8 \quad \text{(256K availability)} \\
& x_1 \leq 1.8 \quad \text{(max demand for GP-1)} \\
& x_3 \leq 0.3 \quad \text{(max demand for GP-3)} \\
& x_1 + x_2 + x_3 \leq 3.8 \text{(max demand for GP family)} \\
& x_4 + x_5 \leq 3.2 \text{(max demand for WS family)} \\
& x_2 \geq 0.5 \quad \text{(min demand for GP-2)} \\
& x_4 \geq 0.5 \quad \text{(min demand for WS-1)} \\
& x_5 \geq 0.4 \text{(min demand for WS-2)} \\
& x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{aligned}$$

Check out the attached DEC.mos and DEC.dat files.

4 Convex functions(30 points)

1. (5 points)

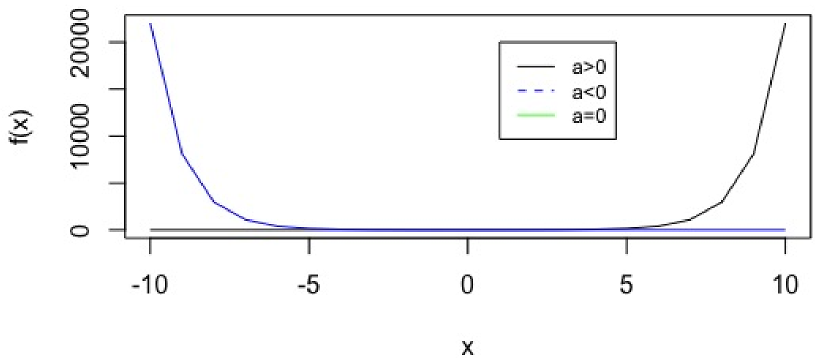
Proof. For $\forall 0 < \lambda < 1$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we have

$$\begin{aligned}
f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) =& \max\{\boldsymbol{a}_1^\top(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) + b_1, \ldots, \boldsymbol{a}_m^\top(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) + b_m\} \\
=& \max\{\lambda(\boldsymbol{a}_1^\top \boldsymbol{x} + b_1) + (1-\lambda)(\boldsymbol{a}_1^\top \boldsymbol{y} + b_1), \ldots, \lambda(\boldsymbol{a}_m^\top \boldsymbol{x} + b_m) + (1-\lambda)(\boldsymbol{a}_m^\top \boldsymbol{y} + b_m)\} \\
\leq& \lambda \max\{\boldsymbol{a}_1^\top \boldsymbol{x} + b_1, \ldots, \boldsymbol{a}_m^\top \boldsymbol{x} + b_m\} \\
& + (1-\lambda)\max\{\boldsymbol{a}_1^\top \boldsymbol{y} + b_1, \ldots, \boldsymbol{a}_m^\top \boldsymbol{y} + b_m\} \\
=& \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y})
\end{aligned}$$

□

2. (25 points)

   (a) (4 points, 2 points for proof, 2 points for plot) $f''(x) = a^2 e^{ax} \geq 0$, $f(x)$ is convex.



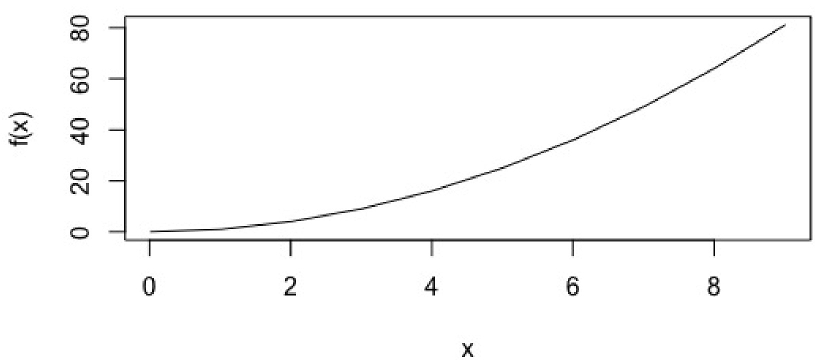


Figure 1: (b)$a \geq 1$

(b) (4 points, 2 points for proof, 2 points for plot) When $a \geq 1$, $f^{''}(x) = a(a-1)x^{a-2} \geq 0$, convex. When $a \leq 0$, $f^{''}(x) = a(a-1)x^{a-2} \geq 0$, convex. When $0 \leq a \leq 1$, $f(x)$ is concave.

(c) (4 points, 2 points for proof, 2 points for plot)$f^{''}(x) = \frac{1}{x^2} \geq 0$, $f(x)$ is convex.

(d) (4 points, 2 points for proof, 2 points for plot)$f^{''}(x) = \frac{1}{x} \geq 0$, $f(x)$ is convex.

(e) (4 points, 2 points for proof, 2 points for plot)The Hessian matrix H is

$$\begin{pmatrix} \frac{2}{y} & -\frac{2x}{y^2} \\ -\frac{2x}{y^2} & \frac{2x^2}{y^3} \end{pmatrix}.$$

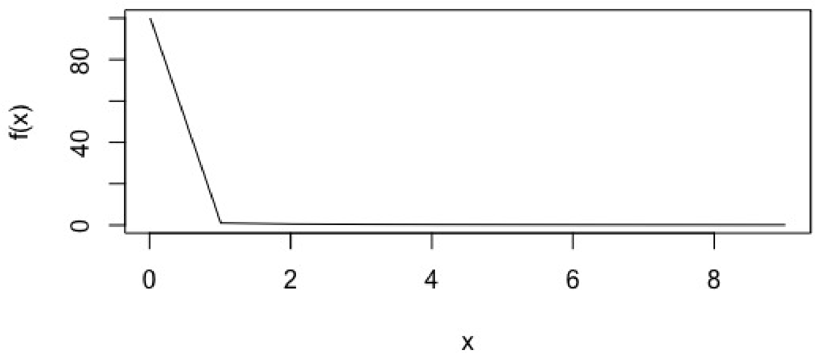


Figure 2: (b)$a \leq 0$

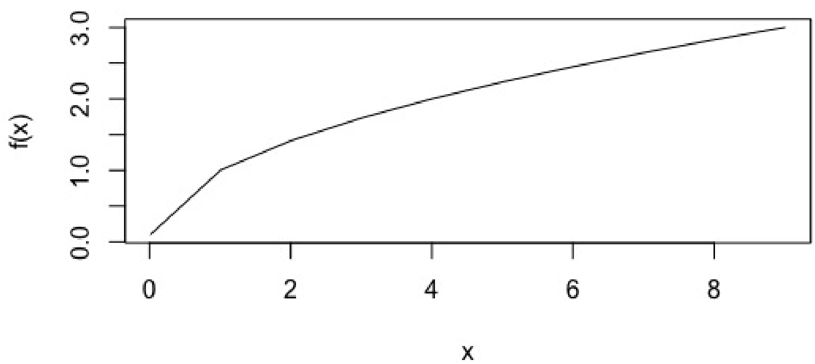


Figure 3: (b)$0 \leq a \leq 1$

For $\forall a, b \in \mathbb{R}$,

$$
\begin{aligned}
&\begin{pmatrix} a & b \end{pmatrix} H \begin{pmatrix} a \\ b \end{pmatrix} \\
=&\frac{2}{y}\left(a^2 - \frac{2x}{y}ab + \frac{x^2}{y^2}b^2\right) \\
=&\frac{2}{y}\left(a - \frac{x}{y}b\right)^2 \geq 0
\end{aligned}
$$

Thus, H is postive semidefinite, and f is convex.

(f) (5 points) Let $f(\boldsymbol{x}) = (|x_1|^p + |x_2|^p)^{1/p}$. Let $\boldsymbol{x}$, $\boldsymbol{y}$ be vectors such that $f(\boldsymbol{x}) = f(\boldsymbol{y}) = 1$. We call such vectors unit vectors. Since the function $|x|^p$ $(p \geq 1)$ is

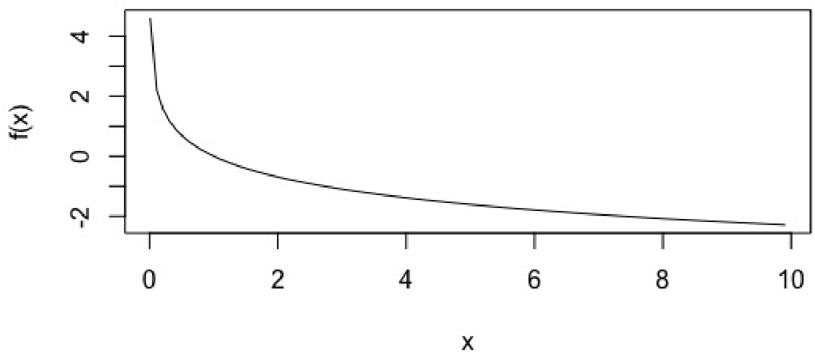


Figure 4: (c)

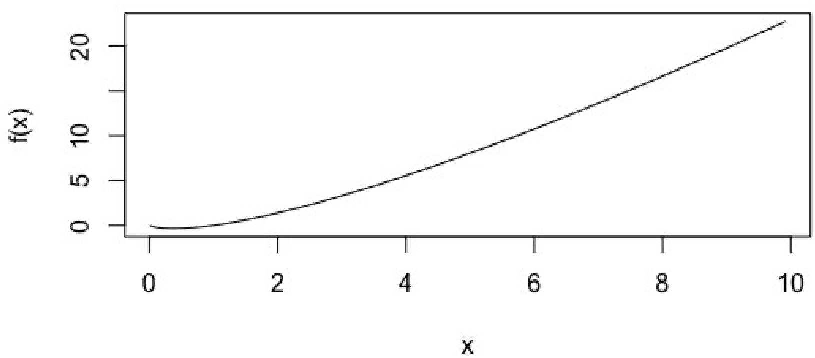


Figure 5: (d)

convex, we have

$$
\begin{aligned}
&f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y})^p \\
=&|\lambda x_1 + (1-\lambda)y_1|^p + |\lambda x_2 + (1-\lambda)y_2|^p \\
\leq&\lambda|x_1|^p + (1-\lambda)|y_1|^p + \lambda|x_2|^p + (1-\lambda)|y_2|^p \\
=&\lambda(|x_1|^p + |x_2|^p) + (1-\lambda)(|y_1|^p + y_2|^p) \\
=&\lambda f(\boldsymbol{x})^p + (1-\lambda)f(\boldsymbol{y})^p \\
=&1
\end{aligned}
$$

Therefore, for unit vectors, we have $f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leq 1$. Then we prove the statement for general $\boldsymbol{x}$ and $\boldsymbol{y}$ by normalizing them to unit vectors. Note that

$f(\boldsymbol{x})$ has the property that $af(\boldsymbol{x}) = f(a\boldsymbol{x})$ for any constant $a \geq 0$, because

$$f(a\boldsymbol{x}) = (|ax_1|^p + |ax_2|^p)^{1/p} = (a^p(|x_1|^p + |x_2|^p))^{1/p} = a(|x_1|^p + |x_2|^p)^{1/p} = af(\boldsymbol{x}).$$

For any vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ and $\lambda \in [0,1]$, let $a = \lambda f(\boldsymbol{x}) + (1-\lambda)f(\boldsymbol{y})$. Then we have

$$\begin{aligned}
&\frac{f(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y})}{\lambda f(\boldsymbol{x}) + (1-\lambda)f(\boldsymbol{y})} = \frac{1}{a}f(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \\
=&f\left(\frac{\lambda\boldsymbol{x}}{a} + \frac{(1-\lambda)\boldsymbol{y}}{a}\right) \\
=&f\left(\frac{f(\lambda\boldsymbol{x})}{a}\frac{\lambda\boldsymbol{x}}{f(\lambda\boldsymbol{x})} + \frac{f((1-\lambda)\boldsymbol{y})}{a}\frac{(1-\lambda)\boldsymbol{y}}{f((1-\lambda)\boldsymbol{y})}\right) \\
=&f\left(\frac{f(\lambda\boldsymbol{x})}{\lambda f(\boldsymbol{x}) + (1-\lambda)f(\boldsymbol{y})}\frac{\lambda\boldsymbol{x}}{f(\lambda\boldsymbol{x})} + \frac{f((1-\lambda)\boldsymbol{y})}{\lambda f(\boldsymbol{x}) + (1-\lambda)f(\boldsymbol{y})}\frac{(1-\lambda)\boldsymbol{y}}{f(1-\lambda)\boldsymbol{y}}\right) \\
\leq&1,
\end{aligned}$$

where the last inequality follows directly from the unit vector case, since

$$\frac{f(\lambda\boldsymbol{x})}{\lambda f(\boldsymbol{x}) + (1-\lambda)f(\boldsymbol{y})} + \frac{f((1-\lambda)\boldsymbol{y})}{\lambda f(\boldsymbol{x}) + (1-\lambda)f(\boldsymbol{y})} = 1,$$

and

$$f\left(\frac{\lambda\boldsymbol{x}}{f(\lambda\boldsymbol{x})}\right) = \frac{f(\lambda\boldsymbol{x})}{f(\lambda\boldsymbol{x})} = 1, \text{ and } f\left(\frac{(1-\lambda)\boldsymbol{y}}{f((1-\lambda)\boldsymbol{y})}\right) = \frac{f((1-\lambda)\boldsymbol{y})}{f((1-\lambda)\boldsymbol{y})} = 1.$$

Thus, $f(\lambda\boldsymbol{x} + (1-\lambda)y) \leq \lambda f(\boldsymbol{x}) + (1-\lambda)f(\boldsymbol{y})$.