ISyE6669 Deterministic Optimization Homework 3

Solutions

October 20, 2018

1 (20 points) Form the Dual in two different ways and form the dual of the dual

1. Assign dual variables y_1,y_2,y_3 to the three primal constraints, respectively. The Lagrangian relaxation problem is formulated as:

$$\begin{aligned} Z(\boldsymbol{y}) = \min_{x_1,x_2,x_3,x_4} \quad & x_1 - x_2 + x_3 - y_1(x_1 + x_2 - x_3 + 2x_4) - y_2(3x_1 + x_2 + 4x_3 - 2x_4 - 4) - y_3(-x_1 - x_2 + 2x_3 + x_4 - 5) \\ \text{s.t.} \quad & x_1 \geq 0,\ x_2 \leq 0,\ x_3, x_4 \text{ free,} \end{aligned}$$

where $y_1 \geq 0$, y_2 free, $y_3 \leq 0$.

2. After grouping primal variables, the Lagrangian relaxation problem can be written as

$$\begin{aligned} Z(\boldsymbol{y}) = \min_{x_1,x_2,x_3,x_4} \quad & (1 - (y_1 + 3y_2 - y_3))x_1 + (-1 - (y_1 + y_2 - y_3))x_2 + (1 - (-y_1 + 4y_2 + 2y_3))x_3 + (-(2y_1 - y_2 + y_3))x_4 + \\ \text{s.t.} \quad & x_1 \geq 0,\ x_2 \leq 0,\ x_3, x_4 \text{ free.} \end{aligned}$$

This Lagrangian relaxation problem has a decomposable structure. We can solve it by minimizing over each variable of $\boldsymbol{x}$ separately as follows.

$$\min_{x_1 \geq 0} (1 - (y_1 + 3y_2 - y_3))x_1 = \begin{cases} 0 & \text{if } (1 - (y_1 + 3y_2 - y_3)) \geq 0 \\ -\infty & \text{o.w.} \end{cases}$$

$$\min_{x_2 \leq 0} (-1 - (y_1 + y_2 - y_3))x_2 = \begin{cases} 0 & \text{if } (-1 - (y_1 + y_2 - y_3)) \leq 0 \\ -\infty & \text{o.w.} \end{cases}$$

$$\min_{x_3 \text{ free}} (1 - (-y_1 + 4y_2 + 2y_3))x_3 = \begin{cases} 0 & \text{if } (1 - (-y_1 + 4y_2 + 2y_3)) = 0 \\ -\infty & \text{o.w.} \end{cases}$$

$$\min_{x_4 \text{ free}} (-(2y_1 - y_2 + y_3))x_4 = \begin{cases} 0 & \text{if } (-(2y_1 - y_2 + y_3)) = 0 \\ -\infty & \text{o.w.} \end{cases}$$

Combing the above four subproblems, we have

$$L(\boldsymbol{y}) = \begin{cases} 4y_2 + 5y_3 & \text{if } (1 - (y_1 + 3y_2 - y_3)) \geq 0,\ (-1 - (y_1 + y_2 - y_3)) \leq 0, \\ & \quad (1 - (-y_1 + 4y_2 + 2y_3)) = 0,\ (-(2y_1 - y_2 + y_3)) = 0 \\ -\infty & \text{o.w.} \end{cases}$$

3.

$$\begin{aligned} \max \quad & 4y_2 + 5y_3 \\ \text{s.t.} \quad & 1 - (y_1 + 3y_2 - y_3) \geq 0 && (1) \\ & -1 - (y_1 + y_2 - y_3) \leq 0 && (2) \\ & 1 - (-y_1 + 4y_2 + 2y_3) = 0 && (3) \\ & -(2y_1 - y_2 + y_3) = 0 && (4) \\ & y_1 \geq 0,\ y_2 \text{ free},\ y_3 \leq 0. \end{aligned}$$

4. To derive the dual of the dual, assign dual variables x_1, x_2, x_3, x_4 to the four dual constraints (5)-(8), respectively. Form the Lagrangian relaxation problem as follows.

$$\begin{aligned} \tilde{Z}(\boldsymbol{x}) = \max_{y_1,y_2,y_3} \ & 4y_2 + 5y_3 + x_1(1 - (y_1 + 3y_2 - y_3)) + x_2(-1 - (y_1 + y_2 - y_3)) \\ & + x_3(1 - (-y_1 + 4y_2 + 2y_3)) + x_4(-(2y_1 - y_2 + y_3)) \\ \text{s.t. } & y_1 \geq 0,\ y_2 \text{ free},\ y_3 \leq 0, \end{aligned}$$

where $x_1 \geq 0$, $x_2 \leq 0$, x_3, x_4 are free. Notice that the signs of x_i's should match the way you multiple them with the constraints. In other words, if you want to add $-x_1(1-(y_1+3y_2-y_3))$ in the Lagrangian relaxation, then x_1 should be ≤ 0 instead of ≥ 0.

Regroup the $\boldsymbol{y}$ variables, the Lagrangian relaxation becomes

$$\begin{aligned} \tilde{Z}(\boldsymbol{x}) = \max_{y_1,y_2,y_3} \ & x_1 - x_2 + x_3 + (-x_1 - x_2 + x_3 - 2x_4)y_1 + (4 - 3x_1 - x_2 - 4x_3 + x_4)y_2 + (5 + x_1 + x_2 - 2x_3 - x_4)y_3 \\ \text{s.t. } & y_1 \geq 0,\ y_2 \text{ free},\ y_3 \leq 0. \end{aligned}$$

Solving the above Lagrangian relaxation problem by maximizing over each y_i variable separately, we get

$$\tilde{Z}(\boldsymbol{x}) = \begin{cases} x_1 - x_2 + x_3 & \text{if } -x_1 - x_2 + x_3 - 2x_4 \leq 0, 4 - 3x_1 - x_2 - 4x_3 + x_4 = 0, 5 + x_1 + x_2 - 2x_3 - x_4 \geq 0 \\ \infty & \text{o.w.} \end{cases}$$

The Lagrangian dual problem is

$$\begin{aligned} \min \quad & x_1 - x_2 + x_3 \\ \text{s.t.} \quad & -x_1 - x_2 + x_3 - 2x_4 \leq 0 \\ & 4 - 3x_1 - x_2 - 4x_3 + x_4 = 0 \\ & 5 + x_1 + x_2 - 2x_3 - x_4 \geq 0 \\ & x_1 \geq 0, x_2 \leq 0, x_3, x_4 \text{ free.} \end{aligned}$$

Yes, the dual of the dual and the primal are equivalent.

5. The relaxation problem is

$$Z(\boldsymbol{y}) = \max_{x_1,x_2,x_3,x_4} x_1 - x_2 + x_3 + (-x_1 - x_2 + x_3 - 2x_4)y_1 + (4 - 3x_1 - x_2 - 4x_3 + x_4)y_2 + (5 + x_1 + x_2 - 2x_3 - x_4)y_3 + ($$

$$\text{s.t. } x_1\text{free}, x_2 \le 0, x_3, x_4 \text{ free.}$$

$$\begin{aligned} Z(\boldsymbol{x}) = \max_{y_1,y_2,y_3} 4y_2 + 5y_3 + &\min_{x_1:x_1\text{free}} x_1(1 - (y_1 + 3y_2 - y_3 - y_4)) + \min_{x_2:x_2\le 0} x_2(-1 - (y_1 + y_2 - y_3)) \\ &+ \min_{x_3:x_3\text{free}} x_3(1 - (-y_1 + 4y_2 + 2y_3)) + \min_{x_4:x_4\text{free}} x_4(-(2y_1 - y_2 + y_3)) \end{aligned}$$

We can solve it by minimizing over each variable of $\boldsymbol{x}$ separately as follows.

$$\min_{x_1\text{free}} (1 - (y_1 + 3y_2 - y_3 - y_4))x_1 = \begin{cases} 0 & \text{if } (1 - (y_1 + 3y_2 - y_3 - y_4)) = 0 \\ -\infty & \text{o.w.} \end{cases}$$

$$\min_{x_2\le 0} (-1 - (y_1 + y_2 - y_3))x_2 = \begin{cases} 0 & \text{if } (-1 - (y_1 + y_2 - y_3)) \le 0 \\ -\infty & \text{o.w.} \end{cases}$$

$$\min_{x_3 \text{ free}} (1 - (-y_1 + 4y_2 + 2y_3))x_3 = \begin{cases} 0 & \text{if } (1 - (-y_1 + 4y_2 + 2y_3)) = 0 \\ -\infty & \text{o.w.} \end{cases}$$

$$\min_{x_4 \text{ free}} (-(2y_1 - y_2 + y_3))x_4 = \begin{cases} 0 & \text{if } (-(2y_1 - y_2 + y_3)) = 0 \\ -\infty & \text{o.w.} \end{cases}$$

The dual maximization problem is

$$\begin{aligned} (D') \max \quad & 4y_2 + 5y_3 \\ \text{s.t.} \quad & 1 - (y_1 + 3y_2 - y_3 - y_4) = 0 && (5) \\ & -1 - (y_1 + y_2 - y_3) \le 0 && (6) \\ & 1 - (-y_1 + 4y_2 + 2y_3) = 0 && (7) \\ & -(2y_1 - y_2 + y_3) = 0 && (8) \\ & y_1 \ge 0,\ y_2 \text{ free},\ y_3 \le 0, y_4 \le 0. \end{aligned}$$

6. (D) and (D') are equivalent. We have $1 - (y_1 + 3y_2 - y_3) = -y_4$ from (D'). Since $y_4 \le 0$, we have $1 - (y_1 + 3y_2 - y_3) \ge 0$, which is exactly the first constraint in (D).

2 (10 points) More Duals

1. The dual LP is:

$$
\begin{aligned}
(D) \quad \max \quad & -4y_1 - y_2 \\
\text{s.t.} \quad & -2y_1 - y_2 \leq -1 \\
& -3y_1 + y_2 \leq 1 \\
& y_1, y_2 \leq 0
\end{aligned}
$$

2. The dual LP is:

$$
\begin{aligned}
(D) \quad \max \quad & 0 \\
\text{s.t.} \quad & -y_1 + 3y_2 \leq 1 \\
& 3y_1 + y_3 \leq -2 \\
& -y_2 + 2y_3 = -1 \\
& y_1 \leq 0, y_2 \leq 0, y_3 \leq 0.
\end{aligned}
$$

3 (20 points) Use Weak and Strong Duality Theorems

1. The dual LP is:

$$
\begin{aligned}
(D) \quad \min \quad & \sum_{i=1}^{m} 0y_i \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij}y_i \geq c_j \quad \forall j = 1, ..., n \\
& y_i \geq 0 \quad \forall i = 1, ..., m
\end{aligned}
$$

2. No, it can't. Since the primal is a maximization problem, the dual is minimization and by the weak duality theorem we know that $\boldsymbol{y}^T\boldsymbol{b} \geq \boldsymbol{c}^T\boldsymbol{x}$ for any pair of a primal feasible solution $\boldsymbol{x}$ and a dual feasible solution $\boldsymbol{y}$. Note that the solution $x_j = 0, \forall j = 1, ..., n$ is always feasible, regardless of the parameters, with $\boldsymbol{c}^T\boldsymbol{x} = 0$, then the dual optimum is bounded by zero for any possible parameters.

3. Yes, it can. Consider for example a set of parameters where $c_1 > 0$, and $a_{i1} \leq 0$. Start with the solution from part 1, note that you can fix $x_j = 0, \forall i = 2, ..., n$ and let x_1 go to infinity, since all the constraints are of the form $\sum_{j=1}^{n} a_{ij}x_j \leq 0$ and the objective function is maximization, as x_1 increases the solution continues to be feasible, and the objective function improves, hence the primal has an unbounded optimum, and from the weak duality theorem it follows that, with these parameters the dual is infeasible.

4. If the dual is feasible the optimal cost of the primal is 0 (zero). This follows because the objective function of the dual for ANY dual feasible solution is $\boldsymbol{b}^T\boldsymbol{y} = \boldsymbol{0}^T\boldsymbol{y} = 0$, we know if both problems have a feasible solution, then they both have a finite optimum, then by strong duality the objective value of ANY dual optimal solution is equal to the objective value of ANY primal optimal solution. Since the objective of any dual feasible solution is zero, then, its optimal value is zero and the optimal value of the primal must also be zero.

4 (20 points) Treasure Island and Complementary Slackness

1. The dual problem is

$$
\begin{aligned}
\min \quad & 700y_1 + 700y_2 \\
\text{s.t.} \quad & 4y_1 + 4y_2 \geq 2 \\
& y_1 + 2y_2 \geq 1 \\
& 2y_1 + y_2 \geq 4 \\
& 3y_1 + 5y_2 \geq 15 \\
& y_1, y_2 \geq 0
\end{aligned}
$$

2. The graph is presented in Figure1. We include the cutoffs with the coordinate axis for clarity.

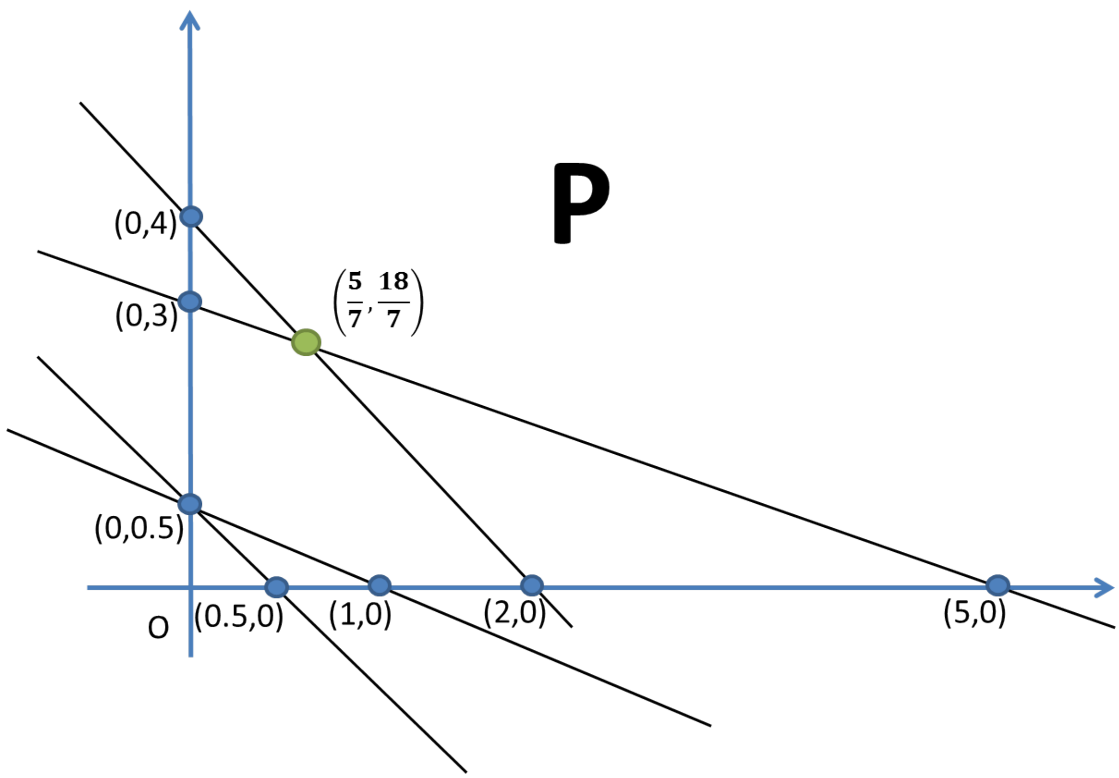


Figure 1: Graphic solution to the dual

3. The solution is highlighted by the green circle in the graph. The numerical values are obtained by solving the system of equations:

$$
\begin{aligned}
2y_1 + y_2 &= 4 \\
3y_1 + 5y_2 &= 15
\end{aligned}
$$

as these 2 intersect at the optimal point. The solution is $\boldsymbol{y} = \left(\frac{5}{7}, \frac{18}{7}\right)$.

4. Substituting on the objective function we get,

$$\boldsymbol{b}^T\boldsymbol{y} = 700\frac{5}{7} + 700\frac{18}{7} = 2300$$

so, we could have made \$ 200 more than Long John.

5. We see both dual variables are positive, so it follows from Primal Complementary Slackness ($y_i(\boldsymbol{a_i}^T\boldsymbol{x} - b_i) = 0$) that both primal constraints must be active.

6. We see that the third and fourth dual constraints are active, while the first and second are slack. From Dual Complementary Slackness ($x_j(c_j - \boldsymbol{y}^T\boldsymbol{A}_j) = 0$) it follows that $x_1 = x_2 = 0$ in the optimal solution.

7. Since we know some variables are zero, and both constraints are active, we can find the solution by solving the system of equations:

$$\begin{aligned} 2x_3 + 3x_4 &= 700 \\ x_3 + 5x_4 &= 700 \end{aligned}$$

The solution is $\boldsymbol{x} = (0, 0, 200, 100)$.

5 (20 points) Lagrangian relaxation for solving a hard combinatorial optimization problem

1.

$$\begin{aligned} (D) \quad Z_{LR}(y) = \quad & \max \quad \sum_{i=1}^{n} p_i x_i + y(W - \sum_{i=1}^{n} w_i x_i) \\ & \text{s.t.} \quad x_i \in \{0,1\} \quad \forall i = 1, ..., n \end{aligned}$$

$y \geq 0$, because $y(W - \sum_{i=1}^{n} w_i x_i) \geq 0$ and $W - \sum_{i=1}^{n} w_i x_i \geq 0$.

2.

$$\begin{aligned} Z_{LR}(y) = \min_{x_i} & \sum_{i=1}^{n} p_i x_i + y(W - \sum_{i=1}^{n} w_i x_i) \\ =& Wy + \max_{x_1}\{(p_1 - w_1 y)x_1\} + \ldots + \max_{x_n}\{(p_n - w_n y)x_n\} \\ =& Wy + Z_1(y) + \ldots + Z_n(y) \end{aligned}$$

where $Z_i(y) = \max_{x_i}\{(p_i - w_i y)x_i\}$. The closed form of each subproblem is

$$Z_i(y) = \max\{0, p_i - w_i y\}.$$

3. The Lagrangian dual (nonlinear form) is

$$\begin{aligned} \min \quad & yW + \sum_{i=1}^{n} Z_i \\ \text{s.t.} \quad & y \geq 0 \end{aligned}$$

   The linear form is

$$\begin{aligned} \min \quad & yW + \sum_{i=1}^{n} Z_i \\ \text{s.t.} \quad & p_i - w_i y \leq Z_i \quad \forall i = 1, ..., n \\ & Z_i \geq 0 \quad \forall i = 1, ..., n \end{aligned}$$

4. (a) Primal solution $Z^* = 34$, $x^\top = [0, 1, 0, 0, 1, 0, 0, 1]$
   Dual solution $Z_D = 34.6667$, $y = 2.66667$
   Percentage Duality gap 0.0196

   (b) Primal solution $Z^* = 67$, $x^\top = [1, 1, 0, 1, 1, 0, 1, 1]$
   Dual solution $Z_D = 69.1$, $y = 2.1$
   Percentage Duality gap 0.0313

   (c) Primal solution $Z^* = 88$, $x^\top = [1, 1, 0, 1, 1, 1, 1, 1]$
   Dual solution $Z_D = 89.8571$, $y = 1.85714$
   Percentage Duality gap 0.0211

6 (10 points) Lagrangian relaxation for solving a hard combinatorial optimization problem

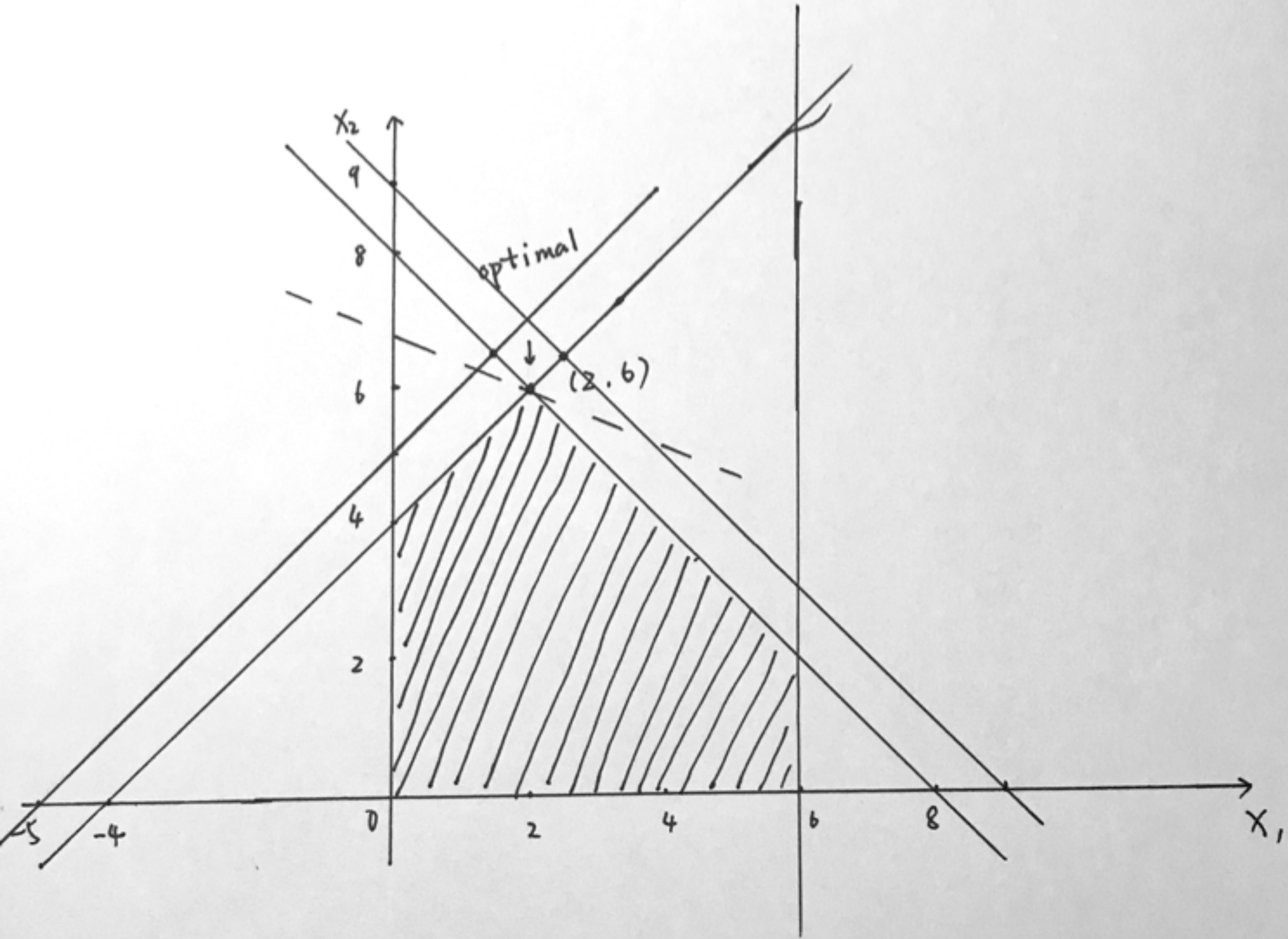
1. see attached picture.

2. Optimal solution: (2,6)

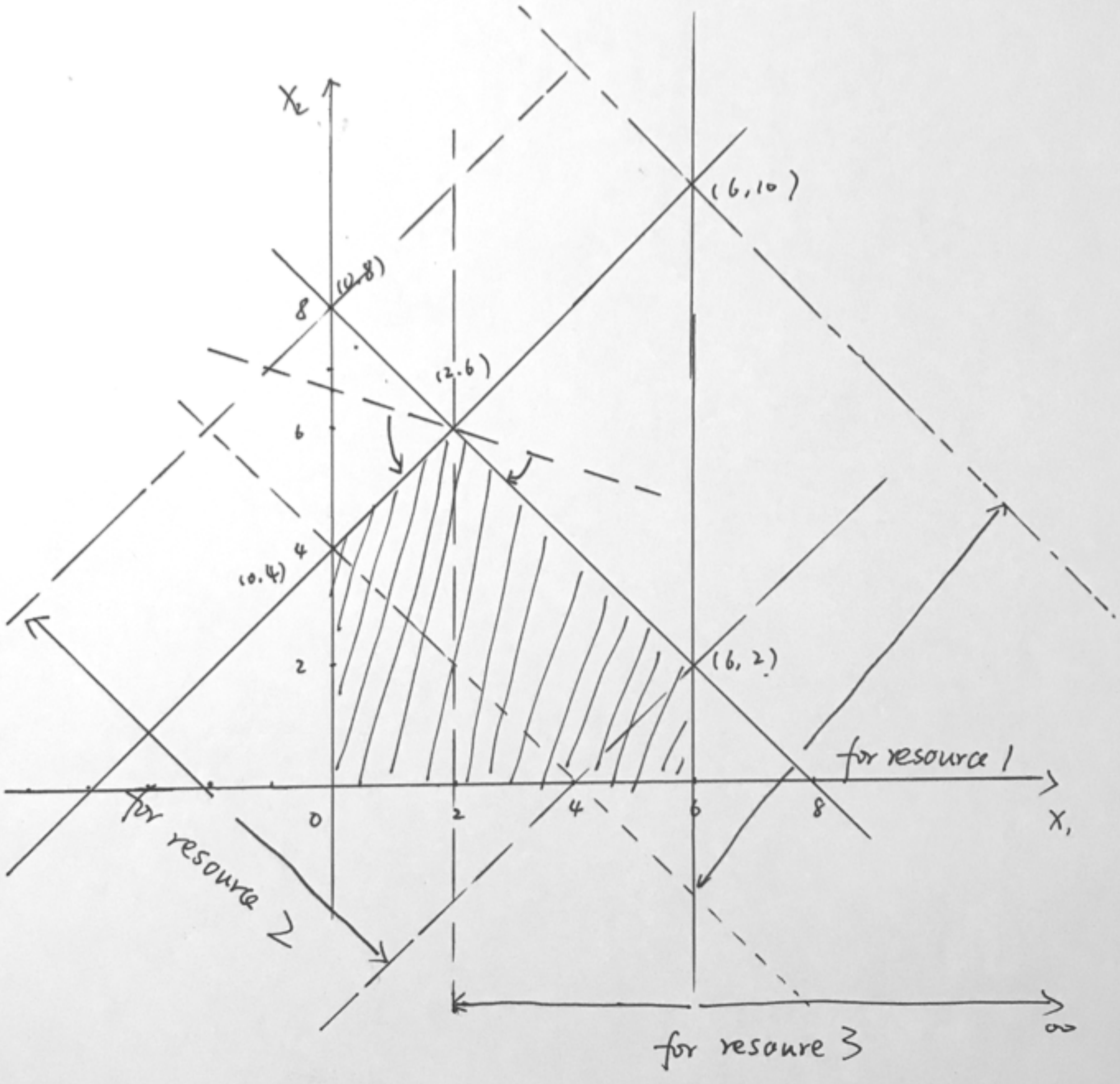
3. The shadow price for resource 1 is 2, for resource 2 is 1, and for resource 3 is 0.

4. range of objective coefficient of x_1 is $[-3, 3]$, and that of x_2 is $[-\infty, -1]$ or $[1, \infty]$.

5. The constant range for resource 1 is $[6, 16]$, for resource 2 is $[-4, 8]$, and for resource 3 is $[2, \infty]$.



P 1



X_2
X_1
(6,10)
(0,8)
(2,6)
(0,4)
(6,2)
0
2
4
6
8
for resource 1
for resoure 2
for resoure 3
∞