1 Problem 1

1.

$$
\begin{aligned}
min \quad & |x_1 - x_2| + |y_1 - y_2| + |x_1 - 300| + |y_1 - 1200| + |x_1 - 0| + |y_1 - 600| \\
& + |x_2 - 0| + |y_2 - 600| + |x_2 - 600| + |y_2 - 0| \\
& st.\ 0 \leq x_i \leq 900 \quad \forall_i = 1,2 \\
& 0 \leq y_i \leq 1500 \quad \forall_i = 1,2
\end{aligned}
$$

2.

$$
\begin{aligned}
min \quad & z_1 + z_2 + z_3 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8 + z_9 + z_{10} \\
& st.\ 0 \leq x_i \leq 900 \quad \forall_i = 1,2 \\
& 0 \leq y_i \leq 1500 \quad \forall_i = 1,2 \\
& -z_1 \leq x_1 - x_2 \leq z_1 \\
& -z_2 \leq y_1 - y_2 \leq z_2 \\
& -z_3 \leq x_1 - 300 \leq z_3 \\
& -z_4 \leq y_1 - 1200 \leq z_4 \\
& -z_5 \leq x_1 - 0 \leq z_5 \\
& -z_6 \leq y_1 - 600 \leq z_6 \\
& -z_7 \leq x_2 - 0 \leq z_7 \\
& -z_8 \leq y_2 - 600 \leq z_8 \\
& -z_9 \leq x_2 - 600 \leq z_9 \\
& -z_{10} \leq y_2 - 0 \leq z_{10}
\end{aligned}
$$

3. The optimal solution from Xpress is:
$(x_1, y_1) = (0, 600)$
$(x_2, y_2) = (0, 600)$

The two facilities are overlapping.

4. These constraints appear as the following modelled as absolute value functions:

$$
\begin{aligned}
min \quad & |x_1 - x_2| + |y_1 - y_2| + |x_1 - 300| + |y_1 - 1200| + |x_1 - 0| + |y_1 - 600| \\
& + |x_2 - 0| + |y_2 - 600| + |x_2 - 600| + |y_2 - 0| \\
& st.\ 0 \leq x_i \leq 900 \quad \forall_i = 1,2 \\
& 0 \leq y_i \leq 1500 \quad \forall_i = 1,2 \\
& |x_1 - 300| + |y_1 - 1200| \leq 500 \\
& |x_2 - 600| + |y_2 - 0| \leq 700
\end{aligned}
$$

Modelled as linear constraints they appear as the following:

$$\begin{aligned}
min \quad & z_1 + z_2 + z_3 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8 + z_9 + z_{10} \\
& st.\ 0 \le x_i \le 900 \quad \forall_i = 1,2 \\
& 0 \le y_i \le 1500 \quad \forall_i = 1,2 \\
& z_3 + z_4 \le 500 \\
& z_9 + z_{10} \le 700 \\
& -z_1 \le x_1 - x_2 \le z_1 \\
& -z_2 \le y_1 - y_2 \le z_2 \\
& -z_3 \le x_1 - 300 \le z_3 \\
& -z_4 \le y_1 - 1200 \le z_4 \\
& -z_5 \le x_1 - 0 \le z_5 \\
& -z_6 \le y_1 - 600 \le z_6 \\
& -z_7 \le x_2 - 0 \le z_7 \\
& -z_8 \le y_2 - 600 \le z_8 \\
& -z_9 \le x_2 - 600 \le z_9 \\
& -z_{10} \le y_2 - 0 \le z_{10}
\end{aligned}$$

The optimal solution from Xpress is:
$(x_1, y_1) = (300, 700)$
$(x_2, y_2) = (500, 600)$
Xpress Code Below:

```
model hw1prob1Fac !the name of the model
uses "mmxprs"; !gain access to the Xpress-Optimizer solver for solving
linear programs

declarations
  Objective:linctr !Objective is declared as a linear constraint,
  which is defined later in the code.
  aux = 1 .. 10 ! number of auxiliary variables. This is a fixed range.
  fac = 1 .. 2 ! number of new facilities. This is a fixed range.
  x : array(fac) of mpvar !mpvar declares a variable, and Xpress automatically
  assumes a variable to be nonnegative
  y : array(fac) of mpvar !array(fac) constructs an array of 2 elements
  z : array(aux) of mpvar
end-declarations

! define Objective function
Objective := (z(1) + z(2)) + (z(3) + z(4)) + (z(5) + z(6)) + (z(7)+z(8)) +
(z(9)+z(10))

! define all the constraints (they need to be linear):
```

```
x(1) <= 900
x(2) <= 900
y(1) <= 1500
y(2) <= 1500

x(1) - x(2) <= z(1)
x(1) - x(2) >= -z(1)
y(1) - y(2) <= z(2)
y(1) - y(2) >= -z(2)
x(1) - 300 <= z(3)
x(1) - 300 >= -z(3)
y(1) - 1200 <= z(4)
y(1) - 1200 >= -z(4)
x(1) <= z(5)
x(1) >= -z(5)
y(1) - 600 <= z(6)
y(1) - 600 >= -z(6)
x(2) <= z(7)
x(2) >= -z(7)
y(2) - 600 <= z(8)
y(2) - 600 >= -z(8)
x(2) - 600 <= z(9)
x(2) - 600 >= -z(9)
y(2) <= z(10)
y(2) >= -z(10)

z(3) + z(4) <= 500
z(9) + z(10) <= 700

! now we are ready to invoke the solver to minimize the objective function
minimize (Objective)

! if your model runs correctly, the following two lines of texts should be printed
in Output/Input tab on the right
writeln("Begin running model")
writeln("End running model")
! then the optimal solution of the problem should be printed in Output/Input
tab
forall (i in fac)
writeln("(x(",i,"),y(",i,"))=","(",getsol(x(i)),",",getsol(y(i)),")")
end-model
```

2 Problem 2

1.

$$
\begin{aligned}
|x_1| + |x_2| &\leq 1 \\
z_1 + z_2 &\leq 1 \\
-z_1 \leq x_1 &\leq z_2 \\
-z_2 \leq x_2 &\leq z_2 \\
0 &\leq z_1 \\
0 &\leq z_2
\end{aligned}
$$

2.

$$
\begin{aligned}
\hat{f}(x) &= f(x_o) + f'(x_o)(x - x_o) \\
\hat{f}(x) &= f(x_1^i, x_2^i) + \frac{\partial(x_1, x_2)}{\partial f(x_1)}(x_1 - x_1^i) + \frac{\partial(x_1, x_2^i)}{\partial f(x_2)}(x_2 - x_2^i) \\
\hat{f}(x) &= (x_1^i - 1.5)^2 + (x_2^i - 0.8)^2 + 2(x_1 - 1.5)(x_1 - x_1^i) + 2(x_2^i - 0.8)(x_2 - x_2^i)
\end{aligned}
$$

3.

$$
\begin{aligned}
\hat{f}(x) &\leq z \\
(x_1^i - 1.5)^2 + (x_2^i - 0.8)^2 + 2(x_1 - 1.5)(x_1 - x_1^i) + 2(x_2^i - 0.8)(x_2 - x_2^i) &\leq z \\
min \quad &z
\end{aligned}
$$

4.

$$x_1 = [0.10, 0.15, 0.60, 0.75, 0.85, 0.14, 0.27, 0.30, 0.44, 0.37]$$

$$x_2 = [0.80, 0.45, 0.02, 0.05, 0.01, 0.49, 0.34, 0.09, 0.17, 0.12]$$

5.

```
model hw1prob2LP
uses "mmxprs"; !gain access to the Xpress-Optimizer solver
uses "mmsystem"


declarations
! you may need to define additional variables
 x : array(1..2) of mpvar
 obj : linctr
 z : array(1..3) of mpvar
 x1 : array(1..10) of real
```

```
 x2 : array(1..10) of real
end-declarations

! write the linear constraints here:
 x1 :: [0.1, 0.15, 0.60, 0.75, 0.85, 0.14, 0.27, 0.30, 0.44, 0.37]
 x2 :: [0.8, 0.45, 0.02, 0.05, 0.01, 0.49, 0.34, 0.09, 0.17, 0.12]
forall(i in 1..10)
(x1(i) - 1.5)^2 + (x2(i) - 0.8)^2 + 2*(x1(i) - 1.5)*(x(1) - x1(i)) +
2*(x2(i) - 0.8)*(x(2) - x2(i)) <= z(1)
z(2) + z(3) <= 1
x(1) >= -z(2)
x(1) <= z(2)
x(2) >= -z(3)
x(2) <= z(3)
z(2) >= 0
z(3) >= 0

! The objective function should be linear

obj := z(1)

! solve the problem and print solution
minimize(obj)
writeln("Solution: ", getobjval)

! you need to print out all the variables
forall(i in 1..2) writeln(getsol(x(i)))

end-model
```

Solution:

```
Fri Sep 07 2018 19:31:22 GMT-0400 (EDT)
FICO Xpress Mosel 64-bit v4.8.4
(c) Copyright Fair Isaac Corporation 2001-2018. All rights reserved
Compiling hw1prob2LP.mos with -g
Running model
Solution: 0.825
0.848571
0.151429

Process exited with code: 0
```

6.

```
model hw1prob2QP
uses "mmxprs"; !gain access to the Xpress-Optimizer solver
uses "mmsystem"
uses "mmquad" !notice you need to call this module to solve QP

declarations
 x : array(1..2) of mpvar
 obj : qexp ! notice qexp is needed to define quadratic expression
 z : array(1..2) of mpvar
end-declarations

! write the linear constraints here:
z(1) + z(2) <= 1
x(1) >= -z(1)
x(1) <= z(1)
x(2) >= -z(2)
x(2) <= z(2)
z(1) >= 0
z(2) >= 0



! The objective function is a quadratic expression
! for example, obj := x(1)^2 + x(2)^2 + 2*x(1)*x(2)
obj := (x(1) - 1.5)^2 +  (x(2) - 0.8)^2

! solve the problem and print solution
minimize(obj)
writeln("Solution: ", getobjval)
forall(i in 1..2) writeln(getsol(x(i)))

end-model
```

Solution

```
Fri Sep 07 2018 19:36:03 GMT-0400 (EDT)
FICO Xpress Mosel 64-bit v4.8.4
(c) Copyright Fair Isaac Corporation 2001-2018. All rights reserved
Compiling hw1prob2QP.mos with -g
Running model
Solution: 0.845
0.849998
0.150002
```

```
Process exited with code: 0
```

The two solutions are similar showing that our linear program was a good approximation of the quadratic program.

3 Problem 3

(a)

$$
\begin{aligned}
max \quad 60,000x_1 + 40,000x_2 + 30,000x_3 + 30,000x_4 + 15,000x_5 \\
s.t. \quad x_1 + x_2 + x_3 + x_4 + x_5 \leq 7 \\
4x_1 + 2x_2 + 2x_3 + 2x_4 + x_5 \leq 8 \\
x_1 \leq 1.8 \\
x_3 \leq 0.3 \\
x_1 + +x_2 + x_3 \leq 3.8 \\
x_4 + x_5 \leq 3.2 \\
0.5 \leq x_2 \\
0.5 \leq x_4 \\
0.4 \leq x_5 \\
0 \leq x_1 \\
0 \leq x_2 \\
0 \leq x_3 \\
0 \leq x_4 \\
0 \leq x_5
\end{aligned}
$$

(b)

```
model hw1prob3LP
uses "mmxprs"; !gain access to the Xpress-Optimizer solver
uses "mmsystem"


declarations
! you may need to define additional variables
 x : array(1..5) of mpvar
 obj : linctr
end-declarations

! write the linear constraints here:
x(1) + x(2) + x(3) + x(4) + x(5) <= 7
```

```
4*x(1) + 2*x(2) + 2*x(3) + 2*x(4) + x(5) <= 8
x(1) <= 1.8
x(3) <= 0.3
x(1) + x(2) + x(3) <= 3.8
x(4) + x(5) <= 3.2
x(2) >= 0.5
x(4) >= 0.5
x(5) >= 0.4
x(1) >= 0
x(2) >= 0
x(3) >= 0
x(4) >= 0
x(5) >= 0
! The objective function should be linear

obj := 60000*x(1) + 40000*x(2) + 30000*x(3) + 30000*x(4) + 15000*x(5)


! solve the problem and print solution
maximize(obj)
writeln("Solution: ", getobjval)

! you need to print out all the variables
forall(i in 1..5) writeln(getsol(x(i)))

end-model
```

Solution:

```
Fri Sep 07 2018 22:03:22 GMT-0400 (EDT)
FICO Xpress Mosel 64-bit v4.8.4
(c) Copyright Fair Isaac Corporation 2001-2018. All rights reserved
Compiling hw1prob3.mos with -g
Running model
Solution: 153000
0
3.3
0
0.5
0.4

Process exited with code: 0
```

4 Problem 4

1.

$$f(x) = max \quad \{a_1^T x + b_1 ... a_m^T x + b_m\}$$
$$\forall x, y \in \mathbb{R}^n \qquad \lambda \in [0,1]$$

We want to show:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$
$$f(\lambda x + (1-\lambda)y) \leq max \quad \{a_1^T x(\lambda x + (1-\lambda y)) + b_1 ... a_m^T(\lambda x + (1-\lambda y)) + b_m\}$$

We know that:

$$(1-\lambda)f(y) = (1-\lambda)max \quad \{a_1^T y + b_1 ... a_m^T y + b_m\}$$
$$(\lambda)f(x) = (\lambda)max \quad \{a_1^T x + b_1 ... a_m^T x + b_m\}$$

is convex.
thus:

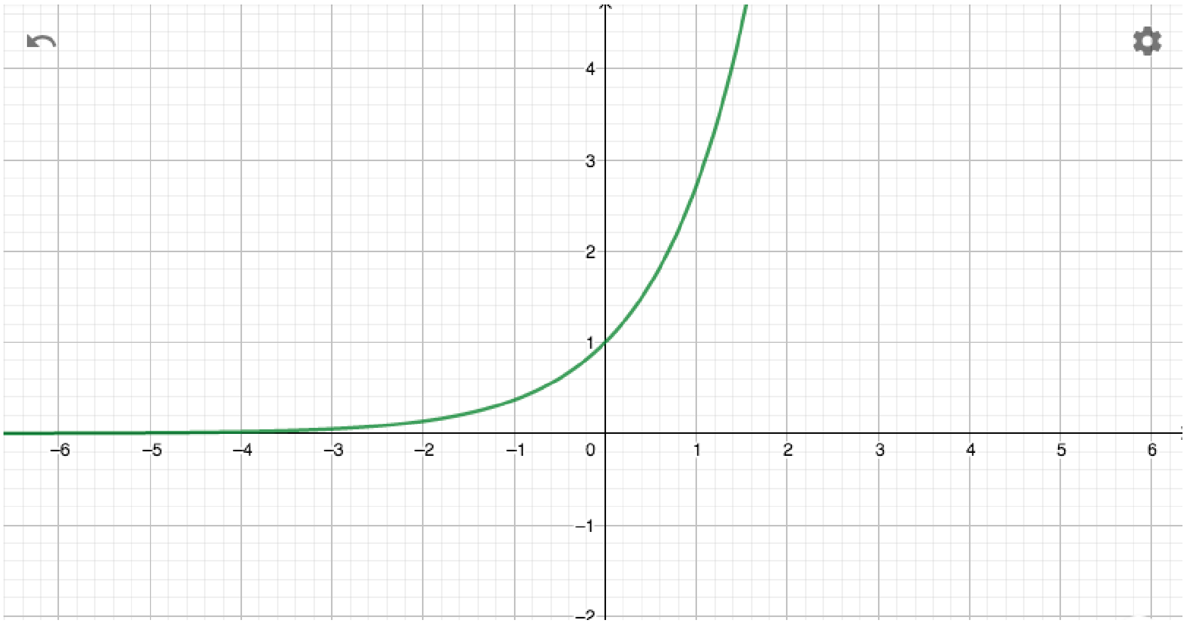
$$f(\lambda x + (1-\lambda)y) \leq max \quad \{a_1^T x(\lambda x + (1-\lambda y)) + b_1 ... a_m^T(\lambda x + (1-\lambda y)) + b_m\}$$
$$= max \quad \{\lambda a_1^T x \ + (1-\lambda)a_1^T y + \lambda b_1 + (1-\lambda)b_1 ..\}$$
$$max \quad \{\lambda(a_1^T x + b_1) + (1-\lambda)(a_1^T y + b_1) .... \lambda(a_m^T x + b_m) + (1-\lambda)(a_m^T y + b_m)\} \leq$$
$$max\{\lambda(a_1^T x + b_1), ... \lambda(a_m^T x + b_m)\} + max\{(1-\lambda)(a_1^T y + b_1), ...(1-\lambda)(a_m^T y + b_m)\}$$

Which is the same as:

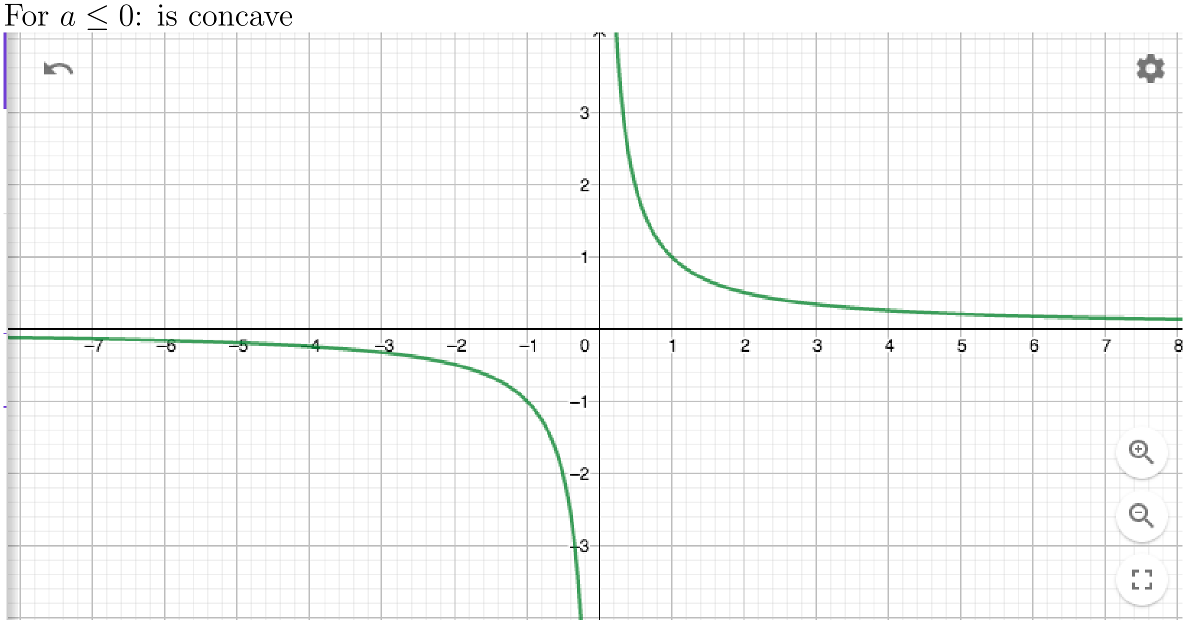
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

Thus proved by Jensen's inequality.

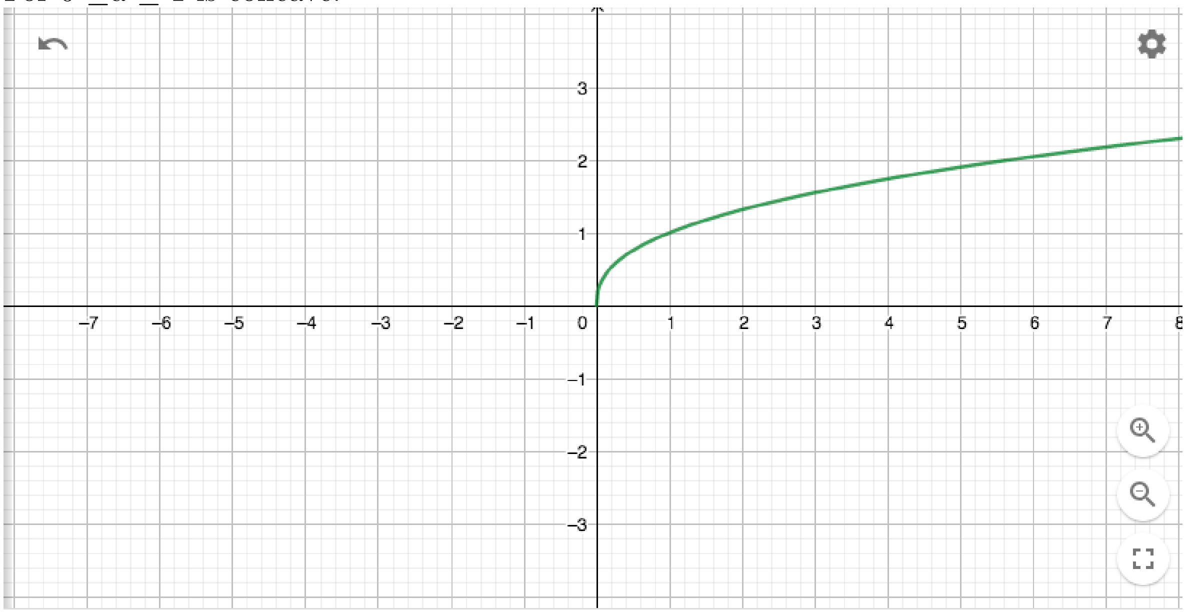
2.
(a)
$f'(x) = a\mathrm{e}^{ax}$
$f''(x) = a^2\mathrm{e}^{ax}$
thus: $f''(x) \geq 0 \quad \forall x \in \mathbb{R} \quad and \quad a \in \mathbb{R}$ proved by second order differential



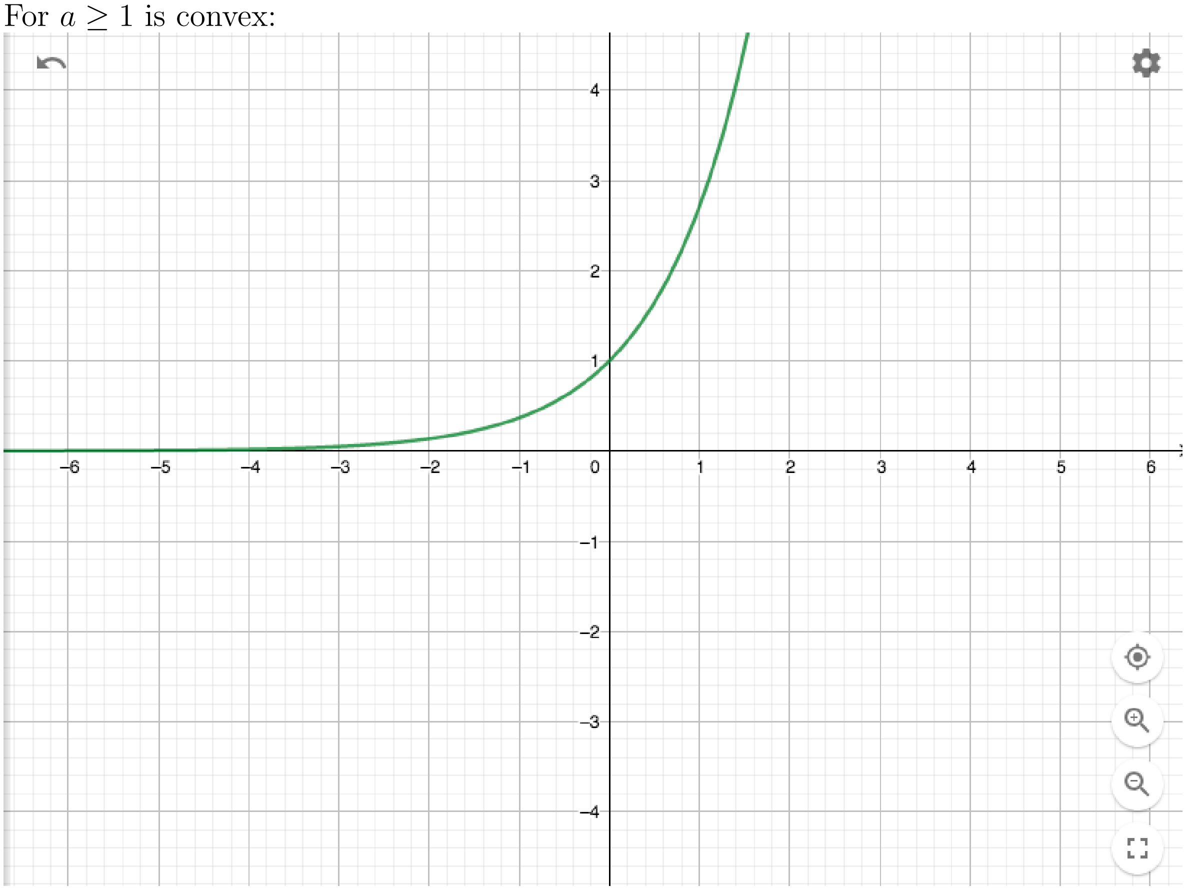
(b)
$f'(x) = ax^{a-1}$
$f''(x) = (a-1)\,ax^{a-2}$
For $a \leq 0$: is concave



For 0 $\leq$a $\leq$ 1 is concave:

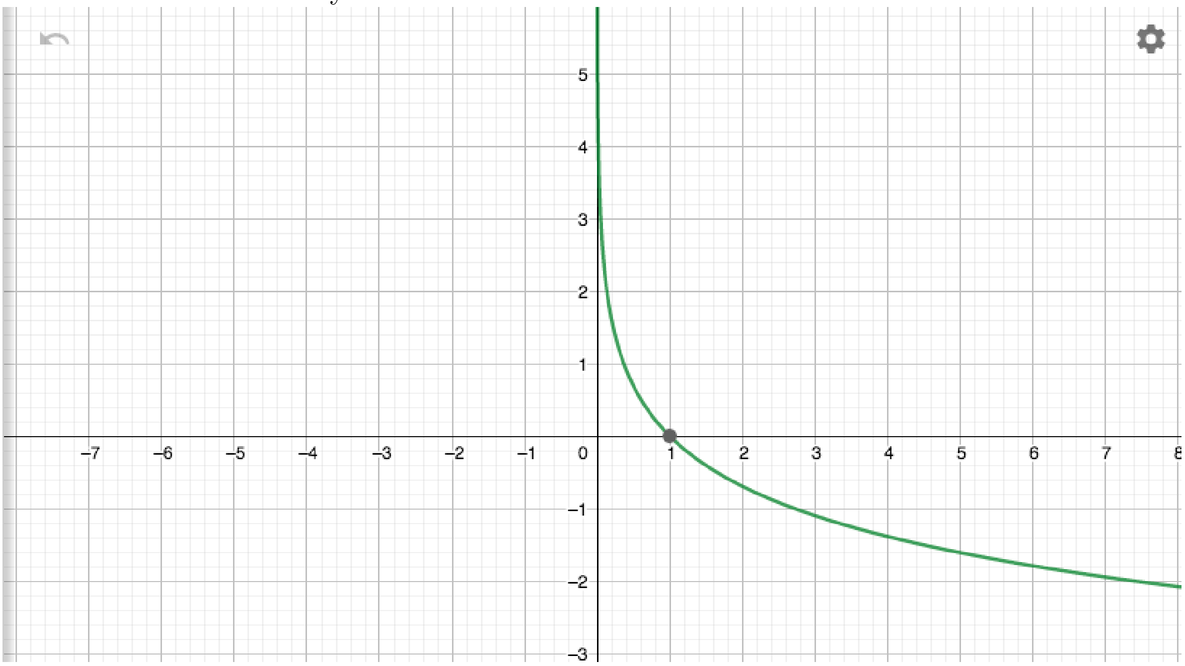


For $a \geq 1$ is convex:

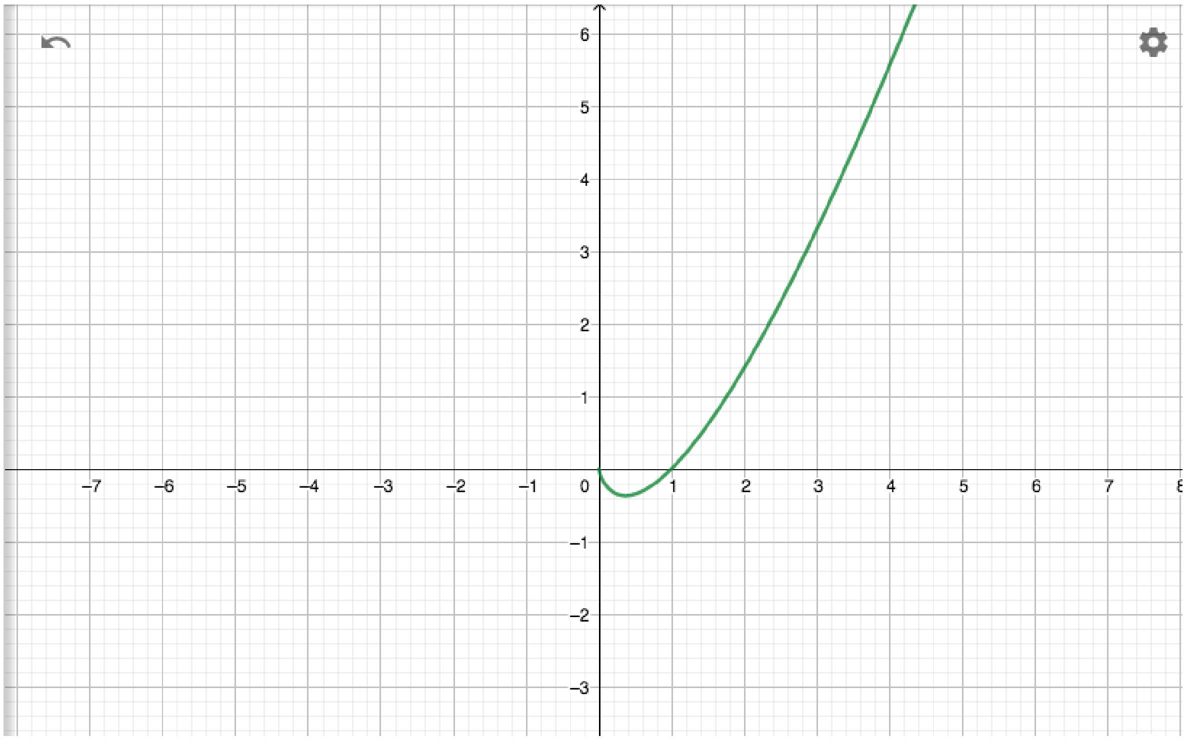


(c)
$f(x) = -log(x)$
$f'(x) = -\dfrac{1}{x}$

$f''(x) = \dfrac{1}{x^2}$
is convex for all $x > 0$ by second order differential



(d)
$f(x) = xlogx$
for $x > 0$
$f'(x) = \ln(x) + 1$
$f''(x) = \dfrac{1}{x}$
Thus $f''(x) > 0$ for all $x > 0$ by the second order differential

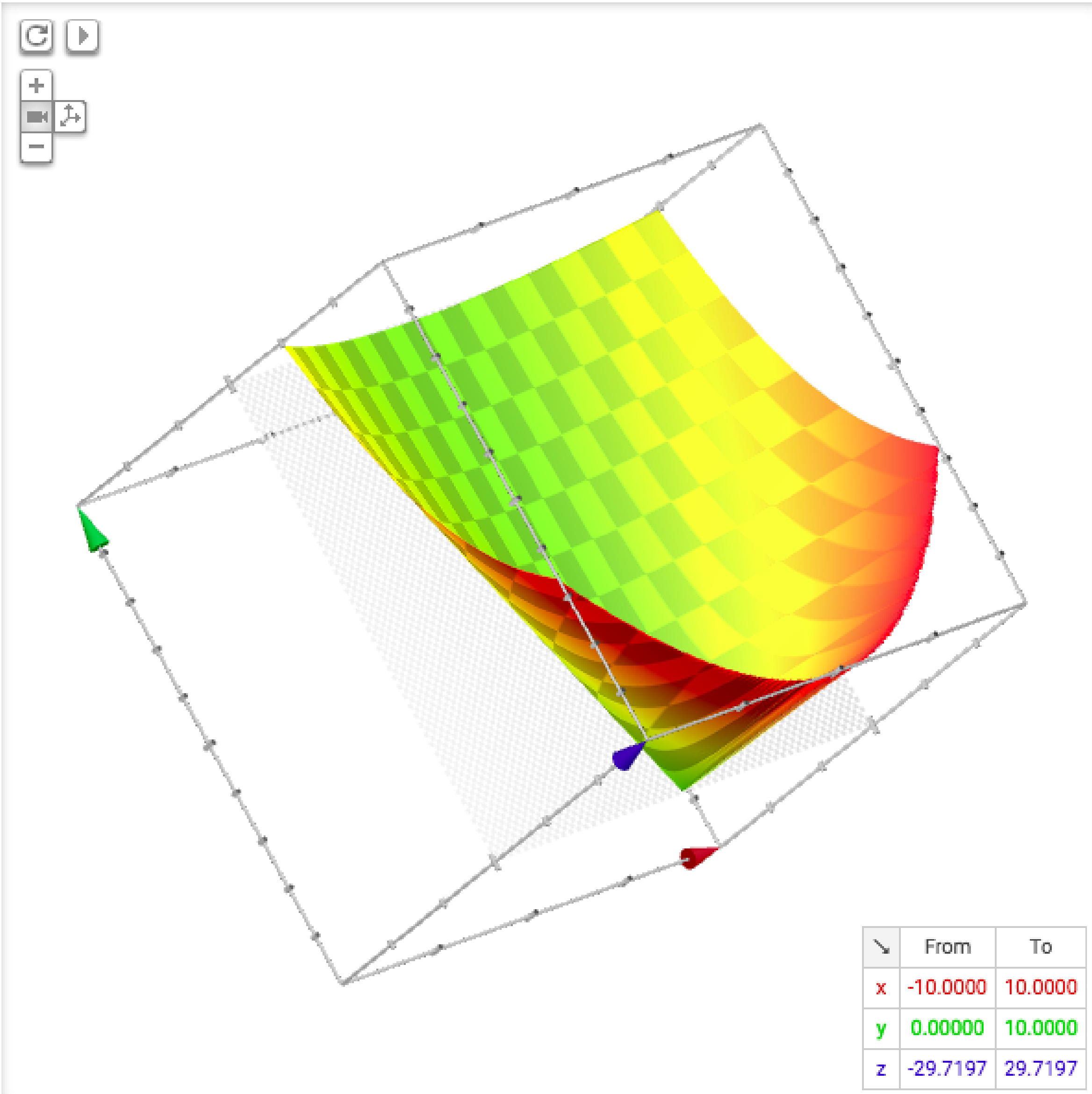


(e)
$f(x) = \frac{x^2}{y}$ for $y > 0$
$\frac{\partial}{\partial x} = \frac{2x}{y}$
$\frac{\partial}{\partial y} = \frac{x^2}{-y}$
$\frac{\partial}{\partial x \partial y} = \frac{-2x}{y^2}$
$\frac{\partial}{\partial y \partial x} = \frac{-2x}{y}$
$\frac{\partial_2}{\partial x_2} = \frac{2}{y}$
$\frac{\partial_2}{\partial y_2} = \frac{2x^3}{y^2}$
This creates the following Hessian matrix:

$$\nabla H = \begin{bmatrix} \frac{\partial_2}{\partial x_2} & \frac{\partial}{\partial x \partial y} \\ \frac{\partial}{\partial y \partial x} & \frac{\partial_2}{\partial y_2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{2}{y} & \frac{-2x}{y^2} \\ \frac{-2x}{y} & \frac{2x^3}{y^2} \end{bmatrix}$$

Since the determinant is non negative and the diagonals are positive this s a positive semi-definite matrix thus $f(x) = \frac{x^2}{y}$ is convex
Also $f''(x) > 0$ proven by second order differential.



(f)

$$f(x_1, x_2) = (|x_1|^p + |x_2|^p)^{\frac{1}{p}} \quad for(x_1, x_2) \quad \in \mathbb{R}_{\neq}$$

Using the norms theorem we know that:

$$||x_1 + x_2||_p \leq ||x_1||_p + ||x_2||_p$$

$$f(\lambda x_1 + (1-\lambda)x_1, \lambda x_2 + (1-\lambda)x_2) \leq \lambda f(x_1, x_2) + (1-\lambda)f(x_1, x_2)$$

$$= (|\lambda x_1 + (1-\lambda)x_1|^p + |\lambda x_2 + (1-\lambda)x_2|^p)^{\frac{1}{p}}$$

$$= ||\lambda x_1 + (1-\lambda)x_1 + \lambda x_2 + (1-\lambda)x_2||_p$$

$$= ||\lambda(x_1 + x_2) + (1-\lambda)(x_1 + x_2)||_p$$

$$\leq \lambda f(x_1, x_2) + (1-\lambda)(f(x_1, x_2)$$

proved by Jensen's Inequality.