ISyE6669 Deterministic Optimization
Homework 5
Due on November 28, 2018

1 Two-Stage Stochastic Programming

Iteration 1:
The augmented (RMP) is

$$\begin{aligned} \min \quad & 100x_1 + 150x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 120, \\ & x_1 \ge 40, \\ & x_2 \ge 20. \end{aligned}$$

The optimal solution is $(x_1, x_2) = (40, 20)$ with $\theta^1 = -\infty$, $LB = -\infty$, $UB = -\infty$.
For $s = 1$,

$$\begin{aligned} \theta_1 = \min \; & -24y_1 - 28y_2 \\ \text{s.t.} \; & 6y_1 + 10y_2 \le 2400, \\ & 8y_1 + 5y_2 \le 1600, \\ & 0 \le y_1 \le 500, \\ & 0 \le y_2 \le 100. \end{aligned}$$

The solution is $\theta_1 = -6100, y = (137.5, 100), \pi_1 = (0, -3, 0, -13)$.
For $s = 2$,

$$\begin{aligned} \theta_1 = \min \; & -28y_1 - 32y_2 \\ \text{s.t.} \; & 6y_1 + 10y_2 \le 2400, \\ & 8y_1 + 5y_2 \le 1600, \\ & 0 \le y_1 \le 300, \\ & 0 \le y_2 \le 300. \end{aligned}$$

The solution is $\theta_2 = -8384, y = (80, 192), \pi_2 = (-2.32, -1.76, 0, 0)$.
$UB = 100x_1 + 150x_2 + p_1\theta_1 + p_2\theta_2 = -470.4$. Add the cut:

$$\begin{aligned} & \theta_1 \ge (60, 80, 500, 100)\pi_1 = -240x_2 - 1300 \\ & \theta_2 \ge (60, 80, 300, 300)\pi_2 = -139.2x_1 - 140.8x_2 \\ & \theta \ge p_1\theta_1 + p_2\theta_2 \ge -83.52x_1 - 180.48x_2 - 520 \end{aligned}$$

Iteration 2:
The augmented (RMP) is

$$\begin{aligned} \min \quad & 100x_1 + 150x_2 + \theta \\ \text{s.t.} \quad & x_1 + x_2 \le 120, \\ & x_1 \ge 40, \\ & x_2 \ge 20, \\ & 83.52x_1 + 180.48x_2 + \theta \ge -520. \end{aligned}$$

The optimal solution is $(x_1, x_2) = (40, 80)$ with $\theta^2 = -18299.2$, $LB = -2299.2$, $UB = -470.4$.
For $s = 1$,

$$\begin{aligned}
\theta_1 = \min\ & -24y_1 - 28y_2 \\
\text{s.t. } & 6y_1 + 10y_2 \le 2400, \\
& 8y_1 + 5y_2 \le 6400, \\
& 0 \le y_1 \le 500, \\
& 0 \le y_2 \le 100.
\end{aligned}$$

The solution is $\theta_1 = -9600, y = (400, 100), \pi_1 = (-4, 0, 0, 0)$.
For $s = 2$,

$$\begin{aligned}
\theta_1 = \min\ & -28y_1 - 32y_2 \\
\text{s.t. } & 6y_1 + 10y_2 \le 2400, \\
& 8y_1 + 5y_2 \le 6400, \\
& 0 \le y_1 \le 300, \\
& 0 \le y_2 \le 300.
\end{aligned}$$

The solution is $\theta_2 = -10320, y = (300, 60), \pi_2 = (-3.2, 0, -8.8, 0)$.
$UB = \min\{100x_1 + 150x_2 + p_1\theta_1 + p_2\theta_2, -470.4\} = -470.4$. Add the cut:

$$\begin{aligned}
&\theta_1 \ge (60, 80, 500, 100)\pi_1 = -240x_1 \\
&\theta_2 \ge (60, 80, 300, 300)\pi_2 = -192x_2 - 2640 \\
&\theta \ge p_1\theta_1 + p_2\theta_2 \ge -211.2x_1 - 1584
\end{aligned}$$

Iteration 3:
Master program has solution $(x_1, x_2) = (66.828, 53.172)$, $\theta^3 = -15697.994$. Add the cut

$$115.2x_1 + 96x_2 + \theta \ge -2104.$$

Iteration 4:
Master program has solution $(x_1, x_2) = (40, 33.75)$, $\theta^3 = -9952$. Add the cut

$$133.44x_1 + 130.56x_2 + \theta \ge 0.$$

Iteration 5:
Solve the first stage program

$$\begin{aligned}
\min\quad & 100x_1 + 150x_2 + \theta \\
\text{s.t.}\quad & x_1 + x_2 \le 120, \\
& x_1 \ge 55, \\
& x_2 \ge 25, \\
& 83.52x_1 + 180.48x_2 + \theta \ge -520, \\
& 211.2x_1 + \theta \ge -1584, \\
& 115.2x_1 + 96x_2 + \theta \ge -2104, \\
& 133.44x_1 + 130.56x_2 + \theta \ge 0.
\end{aligned}$$

The optimal solution is $(x_1, x_2) = (46.667, 36.25)$ with $\theta^5 = -10960$, $LB = -855.83$, $UB = -681.5$.
For $s = 1$,

$$\begin{aligned}
\theta_1 = \min\ & -24y_1 - 28y_2 \\
\text{s.t. } & 6y_1 + 10y_2 \leq 2800, \\
& 8y_1 + 5y_2 \leq 2900, \\
& 0 \leq y_1 \leq 500, \\
& 0 \leq y_2 \leq 100.
\end{aligned}$$

The solution is $\theta_1 = -10000, y = (300, 100), \pi_1 = (0, -3, 0, -13)$.
For $s = 2$,

$$\begin{aligned}
\theta_1 = \min\ & -28y_1 - 32y_2 \\
\text{s.t. } & 6y_1 + 10y_2 \leq 2800, \\
& 8y_1 + 5y_2 \leq 2900, \\
& 0 \leq y_1 \leq 300, \\
& 0 \leq y_2 \leq 300.
\end{aligned}$$

The solution is $\theta_2 = -11600, y = (300, 100), \pi_2 = (-2.32, -1.76, 0, 0)$.
$UB = \min\{100x_1 + 150x_2 + p_1\theta_1 + p_2\theta_2, -681.5\} = -855.33$. $UB = LB$, the algorithm should terminate. The optimal solution is $(46.6667, 36.25)$.

2 (45 pts) IP modeling

1. The above logic statement can be written as the following linear constraints:

$$\begin{aligned}
x_1 &= 0, \\
x_2 &\geq x_3, \\
x_1, x_2, x_3 &\in \{0, 1\}.
\end{aligned}$$

2. The disjunction logic statement can be written as the following linear constraint:

$$x_1 + (1 - x_3) + x_2 \geq 1$$

3. Here, for the disjunction logic statement, we will use one binary variable z and constants M_1 and M_2. The linear constraints are:

$$\begin{aligned}
2x_1 + x_2 &\leq 2 + M_1 z, \\
-x_1 + 2x_2 &\geq 2 - M_2(1 - z).
\end{aligned}$$

   Picking the smallest values for M_1 and M_2 considering the boundaries $[-3, 3] \times [-3, 3]$, we have:

$$\begin{aligned}
2x_1 + x_2 \leq 9 &= 2 + M_1, \\
2 - M_2 = -9 &\leq -x_1 + 2x_2, \\
M_1 &\geq 7, \\
M_2 &\geq 11.
\end{aligned}$$

Finally, the linear constraints can b written as:

$$\begin{aligned} 2x_1 + x_2 &\leq 2 + 7z, \\ -x_1 + 2x_2 &\geq 2 - 11(1-z), \\ y &\in \{0,1\}. \end{aligned}$$

4. The exclusive or logic statement (xor) implies that only one of the given conditions must hold. Consider z as a binary variable and four constants, M_1, M_2, M_3, and M_4. We can then express the xor statement as the following linear constraints:

$$\begin{aligned} 2x_1 + x_2 &\leq 2 + M_1(1-z), \\ -x_1 + 2x_2 &\leq 2 - M_2(1-z), \\ 2x_1 + x_2 &\geq 2 - M_3 z, \\ -x_1 + 2x_2 &> 2 - M_4 z. \end{aligned}$$

Using the given bounds, we have that:

$$\begin{aligned} 2 - M_3 = -6 \leq 2x_1 + x_2 \leq 6 = 2 + M_2, \\ 2 - M_4 < -6 \leq -x_1 + 2x_2 \leq 6 = 2 + M_2. \end{aligned}$$

Thus, we have that $M_1 = 4, M_2 = 4$, and $M_3 = M_4 = 8$.

5. (a)

$$\begin{aligned} \min \ & \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} f_{kl} d_{ij} x_{ki} x_{lj} \\ \text{s.t. } & \sum_{k=1}^{n} x_{ki} = 1 \quad \forall i = 1, \ldots, n, \\ & \sum_{i=1}^{n} x_{ki} = 1 \quad \forall k = 1, \ldots, n, \\ & x_{ki} \in \{0,1\} \quad \forall i, k = 1, \ldots, n, \end{aligned}$$

(b)

$$
\begin{aligned}
\min \; & \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} f_{kl} d_{ij} y_{ijkl} \\
\text{s.t. } & \sum_{k=1}^{n} x_{ki} = 1 \quad \forall i = 1, \ldots, n, \\
& \sum_{i=1}^{n} x_{ki} = 1 \quad \forall k = 1, \ldots, n, \\
& y_{ijkl} \geq x_{ki} + x_{lj} - 1, \\
& y_{ijkl} \leq x_{ki}, \\
& y_{ijkl} \leq x_{lj}, \\
& x_{ki} \in \{0, 1\} \quad \forall i, k = 1, \ldots, n, \\
& y_{ijkl} \in \{0, 1\} \quad \forall i, j, k, l = 1, \ldots, n,
\end{aligned}
$$

6.

$$
\begin{aligned}
P = \min \; & \sum_{i=1}^{m} b_i y_i \\
\text{s.t. } & x_{ij} \leq A_{ij} \quad \forall i = 1, \ldots, m, j = 1, \ldots, n \\
& \sum_{i=1}^{m} x_{ij} \leq 1 \quad \forall j = 1, \ldots, n, \\
& \sum_{j=1}^{n} x_{ij} = y_i \sum_{j=1}^{n} A_{ij} \quad i = 1, \ldots, m \\
& y_i, x_{ij} \in \{0, 1\} \quad \forall i = 1, \ldots, m, j = 1, \ldots, n.
\end{aligned}
$$

7.

$$
\begin{aligned}
P = \min \; & \sum_{t=1}^{n} f_t y_t + p_t x_t + s_t r_t \\
\text{s.t. } & x_t + s_{t-1} \geq d_t \quad \forall t = 1, \ldots, n \\
& s_t = s_{t-1} + x_t - d_t \quad \forall t = 1, \ldots, n, \\
& x_t \leq \sum_{i=1}^{n} d_i y_t \quad \forall t = 1, \ldots, n, \\
& x_t \geq 0, y_t \in \{0, 1\} \quad \forall t = 1, \ldots, n.
\end{aligned}
$$

8.

8	1	2	7	5	3	6	4	9
9	4	3	6	8	2	1	7	5
6	7	5	4	9	1	2	8	3
1	5	4	2	3	7	8	9	6
3	6	9	8	4	5	7	2	1
2	8	7	1	6	9	5	3	4
5	2	1	9	7	4	3	6	8
4	3	8	5	2	6	9	1	7
7	9	6	3	1	8	4	5	2

3 Convex hull of mixed-integer program

1. (a)
   (b) $X^* = (1.54, 2.74),\ f(X^*) = 3.94.$
   (c)
   (d)
   $$\text{conv}(P) = \{(x_1, x_2) :\ x_1 - x_2 \geq -1, 2x_1 + x_2 \leq 4, x_1, x_2 \geq 0\}.$$
   (e) $X^* = (1, 2),\ f(X^*) = 3.$

2. (12 pts)
   (a)
   (b)
   (c) The convex hull is defined by the following points $(1, 1, 0)$, $(1, 2, 0)$, $(1, 1, 0.5)$.
   $$\text{conv}(S) = \{(x_1, x_2) :\ 2y + x_2 \leq 2, x_1 = 1, x_2 \geq 1, y \geq 0\}.$$
   (d) S'' is not the same as $conv(S)$.

4 Integer hull, Branch-and-bound, and Cutting-plane

1. The feasible region for (P) is highlighted in Figure 1.

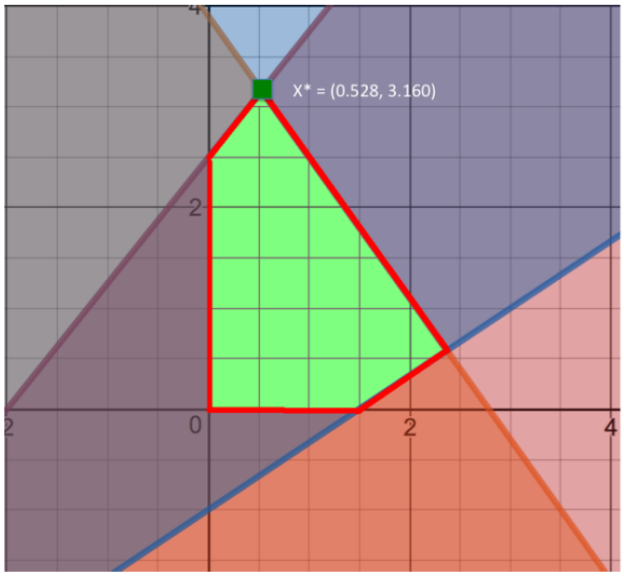

Figure 1: The feasible region for (P) with optimal solution X^* highlighted in green

The optimal solution is located in the intersection between constraints (1) and (2), which yields

$$\begin{aligned} 2(14x_1 + 10x_2) - 5(-5x_1 + 4x_2) = 2 \cdot 39 - 5 \cdot 10, \\ x_1 = \frac{28}{53},\ x_2 = \frac{335}{106}. \end{aligned}$$

Therefore, we have that $X^* = (0.52, 3.16)$ and $f(X^*) = -6.85$.

2. The integer optimal solution of (P) is highlighted in Figure 2.

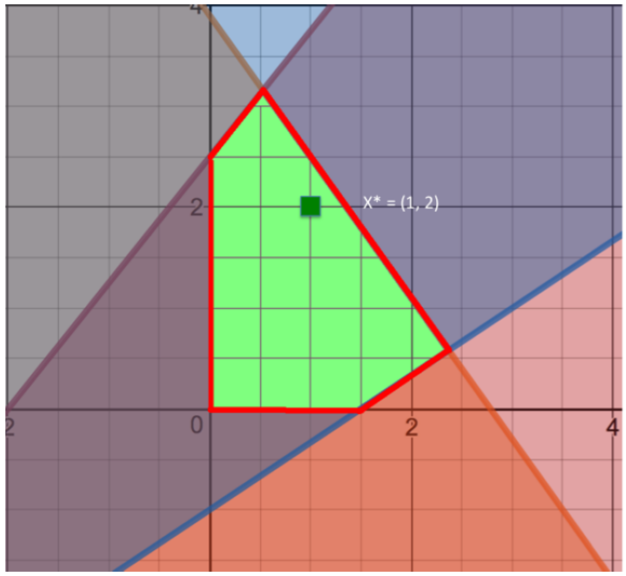

Figure 2: The feasible region for (P) with optimal solution X^*_{int} highlighted in green

The optimal integer solution is located in $(1, 2)$, with $f(X^*_{\text{int}}) = -5$.

3. The integer hull of the set of feasible integer solutions is highlighted in Figure 3. It contains the following points: $(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)$, and $(2, 1)$. The minimal set of defining linear inequalities is given by:

$$\text{IntConv}(P) = \{(x_1, x_2) : \; x_1 - x_2 \leq 1, x_1 + x_2 \leq 3, x_2 \leq 2, x_1, x_2 \geq 0\}.$$

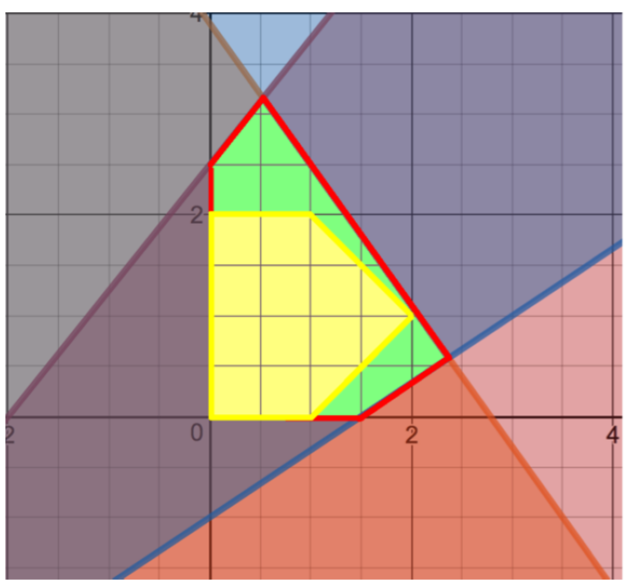

Figure 3: The integer hull of the feasible integer points of (P) highlighted in yellow

4. The branch-and-bound tree is showed in Figure 4. The optimal solutions at each node are highlighted in Figure 5. The optimal integer solution is found at node 6. See that since we are always adding a new constraint at every node, the solution found in node 3 cannot be improved further in its branch. Thus, the algorithm stops and the optimal integer solution $X^*_{\text{int}} = (1, 2)$ is reported.

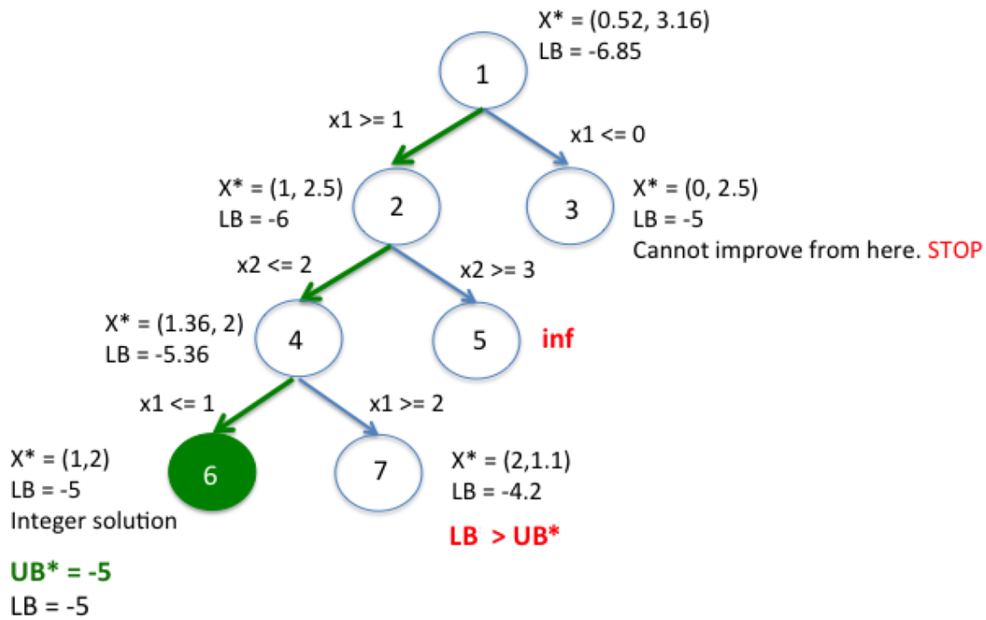

Figure 4: The branch-and-bound tree

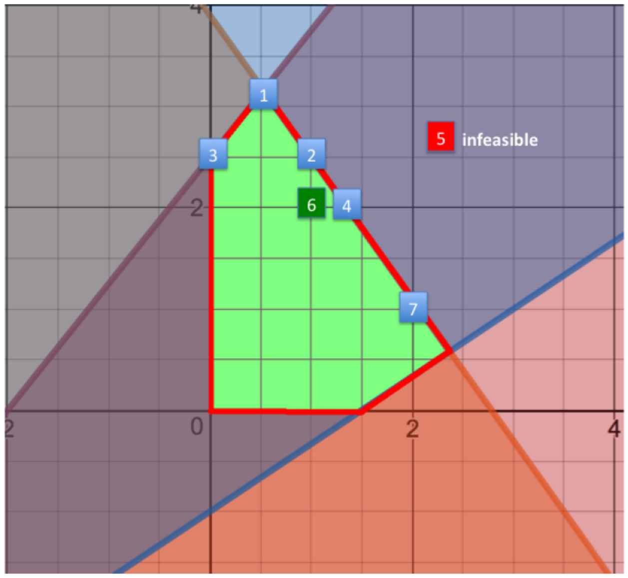

Figure 5: The branch-and-bound solutions found at each node. The optimal integer solution is highlighted in green

5. First, let's write (P) in standard form:

$$
\begin{aligned}
\text{(P)}\ \ \min \quad & -x_1 - 2x_2 \\
\text{s.t.} \quad & 14x_1 + 10x_2 + x_3 = 39, & (4)\\
& -5x_1 + 4x_2 + x_4 = 10, & (5)\\
& 2x_1 - 3x_2 + x_5 = 3, & (6)\\
& x_1, x_2, x_3, x_4 \geq 0, \text{integer}.
\end{aligned}
$$

Now, since we were able to find the optimal solution for the relaxed problem for x_1 and x_2, we can select x_3 and x_4 as non-basic variables. A basic feasible solution is then given by:

$$
\boldsymbol{B} = [\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_5] = \begin{bmatrix} 14 & 10 & 0 \\ -5 & 4 & 0 \\ 2 & -3 & 1 \end{bmatrix}, \ \boldsymbol{x}_B = \boldsymbol{B}^{-1} \begin{bmatrix} 39 \\ 10 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.52 \\ 3.16 \\ 11.42 \end{bmatrix}.
$$

Since the solution is not integer, we can derive the following new linear equality for the Chvatal-Gomory cut computing 5×(4)+14×(5), thus eliminating x_1:

$$
x_2 + \frac{5}{106}x_3 + \frac{14}{106}x_4 = \frac{335}{106}.
$$

Taking the floor function on both sides of the equation, we finally get:

$$
\begin{aligned}
x_2 + 0x_3 + 0x_4 &\leq 3, \\
\implies x_2 &\leq 3.
\end{aligned}
$$

Thus, the new Chvatal-Gomory cut found is $x_2 \leq 3$. As showed in Figure 6, the new linear constraint cuts the optimal solution for the relaxed problem.

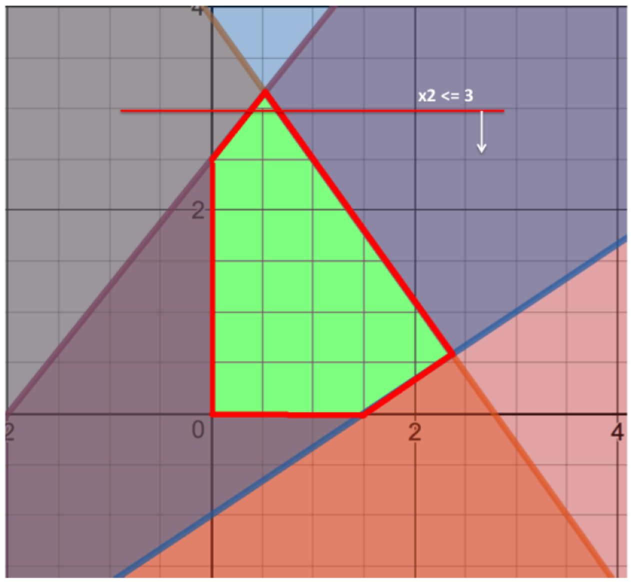

Figure 6: The feasible region for (P) and the new Chvatal-Gomory cut found