ISyE6669 Deterministic Optimization
Homework 4

Question 1: Cutting Stock Problem 1 (50 points)

In this problem, we want to walk you through **one** iteration of the column generation in solving the cutting stock problem. Consider the following formulation

$$\begin{aligned}
\min \quad & \sum_{j=1}^{n} x_i \\
\text{s.t.} \quad & \sum_{j=1}^{N} \boldsymbol{A}_j x_j = \boldsymbol{b} \\
& x_j \geq 0, \qquad \forall j = 1, \dots, N.
\end{aligned}$$

The problem has the following data. Customers need three types of smaller widths: $w_1 = 20, w_2 = 40, w_3 = 50$ with quantities $b_1 = 250, b_2 = 120, b_3 = 100$. The width of a big roll is $W = 170$.

1. Assume the column generation algorithm starts from the following initial patterns:

$$\boldsymbol{A}_1 = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{A}_2 = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}, \boldsymbol{A}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

   Write down the restricted master problem (RMP) using these patterns.

   Solution: The restricted master problem (RMP) can be written as follows,

$$\begin{aligned}
\min \quad & x_1 + x_2 + x_3 \\
\text{s.t.} \quad & \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} x_3 = \begin{bmatrix} 250 \\ 120 \\ 100 \end{bmatrix}, \\
& \mathbf{x} = (x_1, x_2, x_3) \geq 0.
\end{aligned}$$

2. Solve RMP in Xpress. Write down the optimal solution, the optimal basis $\boldsymbol{B}$, and its inverse $\boldsymbol{B}^{-1}$. Find the optimal dual solution $\hat{\boldsymbol{y}}^\top = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}$. To take the inverse of $\boldsymbol{B}$, you can use your favorite calculator or computer program. In this iteration, you could solve this LP by hand. But we ask you to set up the code in Xpress and solve it using Xpress. This code will be used in later iterations.

   Solution: The optimal solution is

$$\mathbf{x} = (62.5, 30, 50).$$

   The optimal basis is $\{x_1, x_2, x_3\}$ and thus the corresponding matrix

$$\boldsymbol{B} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Its inverse is

$$\boldsymbol{B}^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

The optimal dual solution is

$$\hat{\boldsymbol{y}}^\top = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1} = (1/4, 1/4, 1/2).$$

3. Write down the pricing problem, i.e. the knapsack problem using the above data and the optimal dual solution you found.

   Solution: The pricing problem can be written as

$$\begin{aligned} \hat{Z} = \max_{a_1,a_2,a_3} \ & \frac{a_1}{4} + \frac{a_2}{4} + \frac{a_3}{2} \\ \text{s.t. } & 20a_1 + 40a_2 + 50a_3 \leq 170, \\ & a_j \geq 0,\ a_j \text{ is integer},\ j = 1, 2, 3. \end{aligned}$$

   The minimum reduced cost is $1 - \hat{Z}$.

4. Solve the pricing problem in Xpress. Should we terminate the column generation algorithm at this point? Explain. If the column generation should continue, what is the new pattern generated by the pricing problem?

   Solution: Using Xpress, the solution to the pricing problem is

$$(a_1, a_2, a_3) = (8, 0, 0),$$

   and the minimum reduced cost is $1 - (\frac{1}{4}8 + 0 + 0) = -1 < 0$. Therefore, we should continue the column generation algorithm with the new column $A_4 = \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix}$.

5. If the column generation should continue, then augment (RMP) with the new column and solve it in Xpress again. You can easily modify your Xpress code by incorporating the new column. Write down the optimal solution, the optimal basis $\boldsymbol{B}$, and the inverse $\boldsymbol{B}^{-1}$. Compute the dual variable. Then solve the pricing problem again by modifying the date in your code. Should you terminate the column generation at this iteration? Explain. If the column generation should continue, do the same for all the following iterations until the column generation terminates.

   Solution: Iteration 1:
   The augmented (RMP) is

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 + x_4 \\ \text{s.t.} \quad & \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} x_3 + \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix} x_4 = \begin{bmatrix} 250 \\ 120 \\ 100 \end{bmatrix}, \\ & \mathbf{x} = (x_1, x_2, x_3, x_4) \geq 0. \end{aligned}$$

The optimal solution is $(x_1, x_2, x_3, x_4) = (0, 30, 50, 31.25)$. The optimal basis is $\{x_2, x_3, x_4\}$, and the corresponding matrices are

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 8 \\ 4 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \quad \boldsymbol{B}^{-1} = \begin{bmatrix} 0 & 1/4 & 0 \\ 0 & 0 & 1/2 \\ 1/8 & 0 & 0 \end{bmatrix}.$$

The dual variables are

$$\hat{\boldsymbol{y}}^\top = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1} = (1/8, 1/4, 1/2).$$

The pricing problem can be written as

$$\begin{aligned} \hat{Z} = \max_{a_1,a_2,a_3} \ & (1/8, 1/4, 1/2) \cdot (a_1, a_2, a_3)^\top \\ \text{s.t. } & 20a_1 + 40a_2 + 50a_3 \leq 170, \\ & a_j \geq 0,\ a_j \text{ is integer},\ j = 1, 2, 3. \end{aligned}$$

The minimum reduced cost is $1 - \hat{Z}$. The solution to the pricing problem is

$$(a_1, a_2, a_3) = (1, 0, 3),$$

and the minimum reduced cost is $1 - (1/8 + 3/2) < 0$. The algorithm should continue with the added column $A_5 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

Iteration 2:
The augmented (RMP) is

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 + x_4 + x_5 \\ \text{s.t.} \quad & \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} x_3 + \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} x_5 = \begin{bmatrix} 250 \\ 120 \\ 100 \end{bmatrix}, \\ & \mathbf{x} = (x_1, x_2, x_3, x_4, x_5) \geq 0. \end{aligned}$$

The optimal solution is $(x_1, x_2, x_3, x_4, x_5) = (0, 30, 0, 27.833, 33.33)$. The optimal basis is $\{x_2, x_4, x_5\}$, and the corresponding matrices are

$$\boldsymbol{B} = \begin{bmatrix} 0 & 8 & 1 \\ 4 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \boldsymbol{B}^{-1} = \begin{bmatrix} 0 & 1/4 & 0 \\ 1/8 & 0 & -1/24 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

The dual variables are

$$\hat{\boldsymbol{y}}^\top = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1} = (1/8, 1/4, 7/24).$$

The pricing problem can be written as

$$\begin{aligned} \hat{Z} = \max_{a_1,a_2,a_3} \ & (1/8, 1/4, 7/24) \cdot (a_1, a_2, a_3)^\top \\ \text{s.t. } & 20a_1 + 40a_2 + 50a_3 \leq 170, \\ & a_j \geq 0,\ a_j \text{ is integer},\ j = 1, 2, 3. \end{aligned}$$

The minimum reduced cost is $1 - \hat{Z}$. The solution to the pricing problem is

$$(a_1, a_2, a_3) = (0, 3, 1),$$

and the minimum reduced cost is $1 - Z = -0.04167 < 0$. The algorithm should continue with the added column $A_6 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$.

Iteration 3:

The augmented (RMP) is

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\ \text{s.t.} \quad & \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} x_3 + \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} x_5 + \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} x_6 = \begin{bmatrix} 250 \\ 120 \\ 100 \end{bmatrix}, \\ & \mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \geq 0. \end{aligned}$$

The optimal solution is $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 28.75, 20, 40)$. The optimal basis is $\{x_4, x_5, x_6\}$, and the corresponding matrices are

$$\boldsymbol{B} = \begin{bmatrix} 8 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix}, \quad \boldsymbol{B}^{-1} = \begin{bmatrix} -1/8 & 1/72 & -1/24 \\ 0 & -1/9 & 1/3 \\ 0 & 1/3 & 0 \end{bmatrix}.$$

The dual variables are

$$\hat{\boldsymbol{y}}^\top = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1} = (1/8, 17/72, 7/24).$$

The pricing problem can be written as

$$\begin{aligned} \hat{Z} = \max_{a_1, a_2, a_3} \; & (1/8, 17/72, 7/24) \cdot (a_1, a_2, a_3)^\top \\ \text{s.t. } & 20a_1 + 40a_2 + 50a_3 \leq 170, \\ & a_j \geq 0,\ a_j \text{ is integer, } j = 1, 2, 3. \end{aligned}$$

The minimum reduced cost is $1 - \hat{Z}$. The solution to the pricing problem is

$$(a_1, a_2, a_3) = (6, 0, 1),$$

and the minimum reduced cost is $1 - Z < 0$. The algorithm should continue with the added column $A_6 = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix}$.

Iteration 4:

The augmented (RMP) is

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\ \text{s.t.} \quad & \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} x_3 + \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} x_5 + \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} x_6 + \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix} x_7 = \begin{bmatrix} 250 \\ 120 \\ 100 \end{bmatrix}, \\ & \mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \geq 0. \end{aligned}$$

The optimal solution is $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (0, 0, 0, 0, 6.4706, 40, 40.5882)$. The optimal basis is $\{x_5, x_6, x_7\}$, and the corresponding matrices are

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 3 & 0 \\ 3 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{B}^{-1} = \begin{bmatrix} -1/17 & -2/17 & 6/17 \\ 0 & 1/3 & 0 \\ 3/17 & 1/51 & -1/17 \end{bmatrix}.$$

The dual variables are

$$\hat{\boldsymbol{y}}^\top = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1} = (2/17, 4/17, 5/17).$$

The pricing problem can be written as

$$\begin{aligned} \hat{Z} = \max_{a_1, a_2, a_3} \ & (2/17, 4/17, 5/17) \cdot (a_1, a_2, a_3)^\top \\ \text{s.t. } & 20a_1 + 40a_2 + 50a_3 \leq 170, \\ & a_j \geq 0, \ a_j \text{ is integer}, \ j = 1, 2, 3. \end{aligned}$$

The minimum reduced cost is $1 - \hat{Z}$. The solution to the pricing problem is

$$(a_1, a_2, a_3) = (0, 3, 1),$$

and the minimum reduced cost is $1 - Z = 0$. The algorithm should terminate.

6. Write down the final optimal solution, the optimal basis, and the optimal objective value.

   Solution: The finial optimal solution is $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (0, 0, 0, 0, 6.4706, 40, 40.5882)$. The optimal basis is $\{x_5, x_6, x_7\}$ with the matrix

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 3 & 0 \\ 3 & 1 & 1 \end{bmatrix}.$$

   The optimal objective value is 87.0588. Since the problem require that the solution should be integer, we round them and get $(x_5, x_6, x_7) = (7, 40, 41)$ and 88 is the objective value,

7. (5 pt) For this problem, you need to submit on t-square all your codes for all the steps separately. Name them as question1_RMP_step1.mos, question1_Pricing_step1.mos, question1_RMP_step2.mos, and so on.

   Solution: See the codes on T-square.

Question 2: Dantzig-Wolfe decomposition (50 points)

Consider the following linear program:

$$\begin{aligned} \min \quad & x_1 - 2x_2 + 3x_3 - 4x_4 && (1) \\ \text{s.t.} \quad & x_1 - x_2 + x_3 - x_4 \leq 1 && (2) \\ & x_1 + x_2 \leq 1 && (3) \\ & -x_1 + x_2 \leq 1 && (4) \\ & x_2 \geq 0 && (5) \\ & 0 \leq x_3 \leq 1 && (6) \\ & 0 \leq x_4 \leq 1 && (7) \end{aligned}$$

Consider constraint (2) as the complicating constraint and the rest (3)-(7) as easy constraints. Notice that the easy constraints are *separable*, i.e. constraints (3)-(5) only involve x_1 and x_2, while constraints (6)-(7) only involve x_3 and x_4. Therefore, define the polyhedron for variables x_1 and x_2 defined by (3)-(5) as P_1, and P_2 as the polyhedron for x_3 and x_4 defined by (6)-(7).

1. First, argue that both P_1 and P_2 are bounded.

   Solution: Sum the inequalities (3) and (4), we know that $x_2 \leq 1$. Together with (5), this means $0 \leq x_2 \leq 1$. We substitute the range of x_2 into (3) and (4), and obtain that $-1 \leq x_1 \leq 1$. Since P_1 is bounded in each component, it is bounded.

   From (6) and (7), it is also clear that P_2 is bounded in each component, and thus it is bounded.

2. Write down the master problem of the Dantzig-Wolfe decomposition. Since P_1 and P_2 are separable, we will use extreme point representation for P_1 and P_2 separately. That is, for every $(x_1, x_2) \in P_1$, we write $(x_1, x_2) = \sum_{i=1}^{N_1} \lambda_i (x_1^i, x_2^i)$ and λ_i's sum to one and are nonnegative, where (x_1^i, x_2^i) is an extreme point of P_1 and N_1 is the number of extreme points of P_1. Similarly, for every $(x_3, x_4) \in P_2$, we write $(x_3, x_4) = \sum_{j=1}^{N_2} \beta_j (x_3^j, x_4^j)$ and β_j's are convex weights. Use this representation to write down the Dantzig-Wolfe decomposition's master problem. How many columns are there in the master problem?

   Solution: The Dantzig-Wolfe decomposition's master problem can be written as

$$\begin{aligned}
\min_{\lambda,\beta} \quad & \sum_{i=1}^{N_1} \lambda_i (x_1^i - 2x_2^i) + \sum_{j=1}^{N_2} \beta_j (x_3^j - 4x_4^j) \\
\text{s.t.} \quad & \sum_{i=1}^{N_1} \lambda_i (x_1^i - x_2^i) + \sum_{j=1}^{N_2} \beta_j (x_3^j - x_4^j) \leq 1, \\
& \sum_{i=1}^{N_1} \lambda_i = 1, \\
& \sum_{j=1}^{N_2} \beta_j = 1, \\
& \lambda_i, \beta_j \geq 0, \forall i, j.
\end{aligned}$$

   There are 3 extreme points for P_1 and 4 extreme points for P_2. There are altogether $N_1 + N_2 = 7$ columns in the master problem.

3. Write down a reduced master problem (RMP) by choosing the smallest number of columns (i.e. just enough to form a basis) to start the Dantzig-Wolfe decomposition. Here, you need to write explicitly the numerical values of the objective coefficients and the matrix of the RMP. You do not need to solve this RMP.

   Solution: Pick $(1, 0)$ as an extreme point in P_1 and $(0, 0)$ as an extreme point in P_2. A reduced master problem can be thus written as

$$\begin{aligned} \min \quad & \lambda_1 \\ \text{s.t.} \quad & \lambda_1 \le 1, \\ & \lambda_1 = 1, \\ & \beta_1 = 1, \\ & \lambda_1, \beta_1 \ge 0. \end{aligned}$$

It is easy to see that the optimal dual variables are $(\hat{y}, \hat{u}, \hat{v}) = (0, 1, 0)$.

4. Write down the pricing problem. Note that since the easy constraints are separable into P_1 and P_2, your pricing problem should also be separable into subproblems only involving P_1 or P_2. You can denote the dual variables of RMP using some letters like $\boldsymbol{y}$ or $\boldsymbol{z}$, i.e. since you are not asked to solve the RMP, you can assume you know the dual variables and denote them as some vectors.

   Solution: The pricing problem can be written as

$$\begin{aligned} Z &= \min_{i,j} \left\{ x_1^i - 2x_2^i - \hat{y}(x_1^i - x_2^i) - \hat{u}, 3x_3^j - 4x_4^j - \hat{y}(x_3^j - x_4^j) - \hat{v} \right\} \\ &= \min \left\{ \min_{(x_1,x_2)\in P_1} \{x_1 - 2x_2 - \hat{y}(x_1 - x_2) - \hat{u}\}, \min_{(x_3,x_4)\in P_2} \{3x_3 - 4x_4 - \hat{y}(x_3 - x_4) - \hat{v}\} \right\}. \end{aligned}$$

   For this (RMP), the pricing problem is

$$Z = \min \left\{ \min_{(x_1,x_2)\in P_1} \{x_1 - 2x_2 - 1\}, \min_{(x_3,x_4)\in P_2} \{3x_3 - 4x_4\} \right\}.$$

5. Write down the condition under which the Dantzig-Wolfe decomposition should continue.

   Solution: The Dantzig-Wolfe decomposition should continue if $Z < 0$.