ISyE6669 Deterministic Optimization
Homework 5
Due November 28, 2018 (On Campus)
Due December 5, 2018 (DL)

1 Two-Stage Stochastic Programming

Consider the following two-stage stochastic linear program appeared in the lecture note.

$$
\begin{aligned}
Z^* = \min_{x_1,x_2,y_1,y_2} \quad & 100x_1 + 150x_2 + \sum_{s=1}^{2} p_s(q_1^s y_1^s + q_2^s y_2^s) \\
\text{s.t.} \quad & x_1 + x_2 \le 120, \\
& x_1 \ge 40, \\
& x_2 \ge 20, \\
& 6y_1^s + 10y_2^s \le 60x_1, \quad \forall s = 1,2, \\
& 8y_1^s + 5y_2^s \le 80x_2, \quad \forall s = 1,2, \\
& 0 \le y_1^s \le d_1^s, \quad \forall s = 1,2, \\
& 0 \le y_2^s \le d_2^s, \quad \forall s = 1,2,
\end{aligned}
$$

where the random data has two scenarios: $(d_1^1, d_2^1, q_1^1, q_2^1) = (500, 100, -24, -28)$ with probability 0.4 and $(d_1^2, d_2^2, q_1^2, q_2^2) = (300, 300, -28, -32)$ with probability 0.6, the first-stage decision is $\boldsymbol{x} = (x_1, x_2)$, and the second-stage decision is $\boldsymbol{y}^s = (y_1^s, y_2^s)$ for scenarios $s = 1, 2$.

The lecture note implemented two iterations of the Benders decomposition algorithm. Now we want to carry on the steps until an optimal solution of the two-stage stochastic program is found.

1. Write an Xpress code to solve the subproblem in each iteration. You will use this code to solve all the subproblems by entering the corresponding parameter values. You need to submit your code on t-square and turn in a hard-copy with your homework.

2. Write an Xpress code to solve the restricted master problem (RMP) in each iteration. You will update this code by adding a Benders cut in each iteration. You should incorporate the two Benders cuts found in the first two iterations of the Benders decomposition. You need to submit your code on t-square and turn in a hard-copy with your homework.

3. Now carry out the Benders decomposition from the third iteration. Follow the format given in the numerical example of the lecture, i.e. write down the RMP, write down scenario subproblems, solve them by the codes your build above, write down the optimal solutions, find the upper bound UB and the lower bound LB, and construct the optimality cut. You should only terminate when UB is equal to LB.

4. Write an Xpress code to solve the extensive formulation. Record the optimal solution and objective value. Compare it to the result of the Benders decomposition. Are they the same?

2 IP modeling

1. Let three binary variables x_1, x_2, x_3 represent "event i is chosen if $x_i = 1$, and event i is not chosen if $x_i = 0$" for $i = 1, 2, 3$. Write down all feasible binary solutions of the nonlinear constraint $(1 - x_1) \cdot (1 - x_2) \cdot x_3 = x_1$. Then, reformulate the nonlinear constraint using linear constraints to describe the same set of feasible binary solutions.

2. Write *one* linear constraint to model the **disjunction** "Either event 1 is chosen or event 2 is chosen or event 3 is not chosen". Note that the "or" is inclusive without specification. Use binary variable $x_i = 1$ if event i is chosen, and $x_i = 0$ if event i is not chosen.

3. Write linear constraints for the **disjunction**: "$2x_1 + x_2 \leq 2$ or $-x_1 + 2x_2 \geq 2$". Here x_1, x_2 are continuous variables in the box $[-3, 3] \times [-3, 3]$ (i.e. $-3 \leq x_1 \leq 3$ and $-3 \leq x_2 \leq 3$). If you need to use a big M in a constraint, then you need to find the smallest valid value for M for each constraint. Draw the feasible region by hand.

4. Write linear constraints for the **exclusive or (xor)** statement: "$2x_1 + x_2 \leq 2$ or $-x_1 + 2x_2 \geq 2$ but not both" and x_1, x_2 are in the box $[-2, 2] \times [-2, 2]$. Find the best big M's in your formulation. Then, draw the feasible region defined by the above **xor** statement in the plane of (x_1, x_2).

5. We have to place n facilities in n locations. The data are the amount f_{kl} of goods that has to be shipped from facility k to facility l, for $k = 1, \ldots, n$ and $l = 1, \ldots, n$, and the distance d_{ij} between locations i, j, for $i = 1, \ldots, n$ and $j = 1, \ldots, n$. The problem is to assign facilities to locations so as to minimize the total cumulative distance traveled by the goods. For example, if $f_{12} = 2, d_{34} = 3$ and facility 1 is placed at location 3 and facility 2 is placed at location 4, then the total distance traveled by the goods from facility 1 to facility 2 is $f_{12}d_{34} = 6$. Now define variable x_{ki} to be 1 if facility k is placed at location i, and 0 otherwise.

   (a) Write down an integer programming model for this problem in the x_{ki} variables defined above. The objective function should be a quadratic function. You need to explain each constraint and the objective of your model.

   (b) Write an equivalent *linear* integer programming model that linearizes the quadratic objective function. You need to introduce new variables for this purpose. Call these new variables y and you can name each y with appropriate sub-index.

6. A company sets an auction for n objects numbered $N = \{1, 2, \ldots, n\}$. Bidders submit their bids for some subsets of the n objects that they like. The auction house has received m bids, namely bids b_j dollars for subset $S_j \subseteq N$ of objects, for $j = 1, \ldots, m$. We can use a matrix A to store the S_j's in the following way: each row A_i of A corresponds to bidder i's bid and the component $A_{ij} = 1$ if bidder i chooses object j and 0 otherwise. So A is a given 0-1 matrix of m rows and n columns.

   The auction house is faced with the problem of choosing the winning bidders so that profit is maximized and each of the 10 objects is given to at most one bidder (multiple winning bidders can be chosen). Formulate a linear integer program for the auction house to solve this problem using the matrix A defined above. You need to define all variables carefully and explain each constraint in your IP.

7. The demand for a product is known to be d_t units in periods $t = 1, \ldots, n$. If we produce the product in period t, we incur a machine setup cost f_t which does not depend on the number of units produced plus a production cost p_t per unit produced. We may produce any number of units in any period. Any inventory carried over from period t to period $t+1$ incurs an inventory cost r_t per unit carried over. Initial inventory is s_0. Formulate a mixed integer linear program in order to meet the demand over the n periods while minimizing overall costs.

8. Sharpen your pencil and try if you can crack the following Sudoku by hand. You can take as long as you need. Now write down a binary program to solve the Sudoku. Then code it in Xpress, solve it, and fill in your answer in the blanks. For your Xpress code, you can define a list, e.g. $\{1, 3, 5\}$, inside `declarations` by $L = [1, 3, 5]$.

8								
		3	6					
	7			9		2		
	5				7			
				4	5	7		
			1				3	
		1					6	8
		8	5				1	
	9					4		

P.S. Either you cracked this Sudoku by hand or by your binary program, congratulations! You just solved the world's hardest Sudoku.

3 Convex hull of mixed-integer program

1. Consider the following integer program.

$$
\begin{aligned}
(P) \quad \max \quad & -x_1 + 2x_2 \\
\text{s.t.} \quad & -x_1 + x_2 \leq 1.2 \\
& -2x_1 + x_2 \leq 1 \\
& 6x_1 + x_2 \leq 12 \\
& x_2 \geq 0 \\
& x_1 \in \mathbb{Z},\ x_2 \in \mathbb{Z}.
\end{aligned}
$$

   (a) Draw the feasible region of (P).
   (b) Find the optimal solution and optimal objective value of the linear relaxation of (P) by inspection.
   (c) Draw the convex hull of the feasible region of (P), denoted as $\mathrm{conv}(P)$.
   (d) Write down the minimal set of linear constraints for $\mathrm{conv}(P)$.
   (e) Find the optimal solution and optimal objective value of (P) by inspection.

2. Consider the following mixed-integer set.

$$S = \left\{ (x_1, x_2, y) : y + x_1 \leq 1.5,\ y + x_2 \leq 2,\ x_1 \in \mathbb{Z}_+,\ x_2 \in \mathbb{Z}_+,\ y \geq 0 \right\},$$

where x_1, x_2 are nonnegative integer variables and y is a continuous nonnegative variable.

(a) Draw the set S in of (x_1, x_2, y), with y as the axis vertical to the plane (x_1, x_2).

(b) On the same plot as in the above question, draw the convex hull of S, denoted as $\text{conv}(S)$. You need to be careful about the facets of the $\text{conv}(S)$.

(c) $\text{Conv}(S)$ should be a polyhedron. Write down a minimal set of linear constraints for $\text{conv}(S)$.

(d) On a different plot, draw the feasible region of the linear programming relaxation of S, denoted as S'. Is S' the same as $\text{conv}(S)$?

4 Integer Hull, Branch-and-Bound, and Cutting-Plane

Consider the following integer programming problem:

$$
\begin{aligned}
(P) \quad \min \quad & -x_1 - 2x_2 \\
\text{s.t.} \quad & 14x_1 + 10x_2 \leq 39 && (1) \\
& -5x_1 + 4x_2 \leq 10 && (2) \\
& 2x_1 - 3x_2 \leq 3 && (3) \\
& x_1, x_2 \geq 0, \ \text{integer}.
\end{aligned}
$$

1. Solve the LP relaxation of (P) graphically. What are the optimal cost and optimal solution of the linear programming relaxation?

2. Solve the IP graphically. What are the optimal cost and optimal solution of the integer programming problem?

3. What is the integer hull of the set of feasible integer solutions? Find the minimal set of defining linear inequalities for the integer hull.

4. Solve the problem by branch-and-bound. At the root node, branch on x_1 variable. Solve the LP relaxations in the bounding process graphically.

5. Now we want to derive a Chvatal-Gomory cut following the lecture notes. First write (P) in standard form. After you solve the LP relaxation of (P) graphically, you will find that x_1, x_2 are both basic variables. Use the standard form of Eq. (1) and Eq. (2) to derive a new linear equality which has no x_1 term and has x_2 term with coefficient 1. Write down this linear equality. Now do rounding down on both sides of it and write down the resulting Chvatal-Gomory cut. Does it cut off the LP relaxation solution?