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Lemma 0.1. *If* $x_1, \cdots, x_m, c_1, \cdots, c_m, C \geq 0$ *and* $x_1 + \cdots + x_m = C$ *then* $\prod x_i^{c_i}$ *is maximized when* $x_i = \frac{Cc_i}{D}$ *for* $1 \leq i \leq m$ *where* $D = c_1 + \cdots + c_m$.

Proof. Let $f(x_1, \cdots, x_m) = \prod x_i^{c_i}$, and $g(x_1, \cdots, x_m) = x_1, \cdots, x_m - 1$ representing the constraint of the problem. Then Lagrangian is:

$$\mathcal{L}(x_1, x_2, \ldots, x_m, \lambda) = f(x_1, \cdots, x_m) + \lambda g(x_1, \cdots, x_m) = \prod_{i=1}^{m} x_i^{c_i} + \lambda \left(\sum_{i=1}^{m} x_i - c \right)$$

Taking the partial derivative of $\mathcal{L}$ with respect to each x_i and λ, and setting them to zero, we get:

$$c_i x_i^{c_i - 1} \prod_{j \neq i} x_j^{c_j} + \lambda = 0 \quad \text{for } i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_i = c$$

Simplifying the first series of equations, we have the following steps:

$$\begin{aligned}
c_i x_i^{c_i - 1} \prod_{j \neq i} x_j^{c_j} + \lambda &= 0 \quad \text{for } i = 1, 2, \ldots, m \\
c_i x_i^{c_i} \prod_{j \neq i} x_j^{c_j} &= -\lambda x_i \quad \text{for } i = 1, 2, \ldots, m \\
c_i \prod_{j} x_j^{c_j} &= -\lambda x_i \quad \text{for } i = 1, 2, \ldots, m \\
c_i \prod_{j} x_j^{c_j} &= -\lambda x_i \quad \text{for } i = 1, 2, \ldots, m \\
\prod_{j} x_j^{c_j} &= -\lambda \frac{x_i}{c_i} \quad \text{for } i = 1, 2, \ldots, m.
\end{aligned}$$

Therefore, since in the last set of equations, $\prod_j x_j^{c_j}$ is a common term for all $1 \leq i \leq m$, we have:

$$-\lambda \frac{x_i}{c_i} = -\lambda \frac{x_j}{c_j} \quad \text{for } 1 \leq i, j \leq m. \tag{0.1}$$

Dividing both sides of the last equation by λ, we obtain:

$$\frac{x_i}{c_i} = \frac{x_j}{c_j} \quad \text{for } 1 \leq i, j \leq m. \tag{0.2}$$

By fixing $j = 1$, then we have:

$$\frac{x_i}{c_i} = \frac{x_1}{c_1} \quad \text{for } 2 \leq i \leq m, \tag{0.3}$$

therefore, for $2 \leq j \leq m$, we have:

$$x_i = \frac{c_i x_1}{c_1}. \tag{0.4}$$

Last equation gives us that $2 \leq i \leq m$, $x_i = \frac{c_i x_1}{c_1}$ for . Substituting, x_is in the constraint $x_1 + \cdots, x_m = C$ and factoring x_1, we obtain:

$$x_1 + \sum_{i=2}^{m} \frac{c_i x_1}{c_1} = x_1(1 + \sum_{i=2}^{m} \frac{c_i}{c_1}) = C. \tag{0.5}$$

Last equation will give us that $x_1 = \frac{C}{(1+\sum_{i=2}^{m} \frac{c_i}{c_1})}$, and also from Eq. 0.4, we obtain that for $2 \leq j \leq m$, $x_j = \frac{Cc_j}{(1+\sum_{i=2}^{m} \frac{c_i}{c_1})c_1}$. Notice that Lagrange Multiplier method gave us that $(\frac{Cc_1}{(1+\sum_{i=2}^{m} \frac{c_i}{c_1})c_1}, \cdots, \frac{Cc_m}{(1+\sum_{i=2}^{m} \frac{c_i}{c_1})c_1})$ is a critical point. Having $(1 + \sum_{i=2}^{m} \frac{c_m}{c_1})c_1 = D$, our critical point simplifies to $(\frac{Cc_1}{D}, \cdots, \frac{Cc_m}{D})$

Question: Why $(\frac{Cc_1}{D}, \cdots, \frac{Cc_m}{D})$ maximizes f? $\square$