Exercice 4 :

Soit $(X_1, X_2, --, X_n)$ un n-echantillon de loi de Bernouille

$$S_n = \sum_{K=1}^{n} X_K \quad , \quad \overline{X}_n = \frac{S_n}{n}$$

a) Montrons que $\overline{X}_n$ est un estimateur sans biais

$$\mathbb{E}(\overline{X}_n) = \mathbb{E}\left(\frac{S_n}{n}\right) = \mathbb{E}\left(\frac{1}{n}\sum_{K=1}^{n} X_K\right) \quad \text{or } X_K \sim \mathbb{E}(\mathcal{B})$$

$$E(X_K) = P$$

$$E(\overline{X}_n) = \frac{1}{n}\sum_{K=1}^{n} \mathbb{E}(X_K)$$

$$= \frac{1}{n}\sum_{K=1}^{n} P$$

$$= \frac{1}{n} \times nP = P$$

$$E(\overline{X}_n) = P$$

Donc $\overline{X}_n$ est un estimateur sans biais

Montrons que $\overline{X}_n$ est un estimateur de constant P

__Rappel__

On dit qu'un estimateur $\hat{\theta}_n$ est constant, si

$$\lim_{n \to +\infty} Var(\hat{\theta}_n) = 0$$

$$Var(\overline{X}_n) = Var\left(\frac{S_n}{n}\right) = Var\left(\frac{1}{n}\sum_{K=1}^{n} X_K\right)$$

$$= \frac{1}{n^2} Var(X_K) \quad \text{avec } Var(X_K) = P(1-P)$$

$$= \frac{1}{n^2}\sum_{K=1}^{n} P(1-P)$$

$$= \frac{1}{n^2} \times n(P(1-P))$$

$$= \frac{1}{n} P(1-P)$$

$$\lim_{n \to +\infty} \frac{1}{n} P(1-P) = 0$$

Donc l'estimateur est consistant

b) $\sigma^2 = p(1-p)$ et $U_n = \bar{X}_n(1-\bar{X}_n)$ (2)

1) calculons l'espérance et la variance de S_n

$$\mathbb{E}(S_n) = \mathbb{E}(n\bar{X}_n)$$
$$= n\,\mathbb{E}(\bar{X}_n)$$
$$\mathbb{E}(S_n) = np$$

$$Var(S_n) = Var(n\bar{X}_n)$$
$$= n^2 Var(\bar{X}_n)$$
$$= n^2 \times \frac{1}{n} p(1-p)$$
$$Var(S_n) = np(1-p)$$

2) Montrons que U_n est un estimateur biaisé de σ^2

$$U_n = \bar{X}_n - \bar{X}_n^2$$
$$\mathbb{E}(U_n) = \mathbb{E}(\bar{X}_n - \bar{X}_n^2)$$
$$= \mathbb{E}(\bar{X}_n) - \mathbb{E}(\bar{X}_n^2)$$
$$= p - \mathbb{E}(\bar{X}_n^2)$$

cherchons $\mathbb{E}(\bar{X}_n^2)$?

$$\mathbb{E}(\bar{X}_n^2) = Var(\bar{X}_n) + [\mathbb{E}(X_n)]^2$$
$$= \frac{p(1-p)}{n} + p^2$$
$$= \frac{p-p^2}{n} + p^2$$
$$= \frac{p-p^2+np^2}{n}$$

$$\mathbb{E}(U_n) = p - \frac{(p-p^2+np^2)}{n}$$
$$= \frac{np - p + p^2 - np^2}{n}$$
$$= \frac{\cancel{np(1-p)}\,(n-1)p - (n-1)p^2}{n}$$
$$= \frac{n-1}{n}(p-p^2)$$

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$$\mathbb{E}(U_n) = \frac{n-1}{n} p(1-p)$$

(3)

$$\boxed{\mathbb{E}(U_n) = \frac{n-1}{n} \sigma^2}$$

d'où U_n est un estimateur biaisé de σ^2

3) Donnons l'estimateur de V_n sans biaisé de σ^2 en fonct° de U_n

$$\mathbb{E}(V_n) = \sigma^2$$

$$\mathbb{E}(U_n) = \frac{n-1}{n} \sigma^2 \Rightarrow \sigma^2 = \frac{n}{n-1} \mathbb{E}(U_n)$$

$$\Rightarrow \sigma^2 = \mathbb{E}\left(\frac{n}{n-1} U_n\right)$$

Par identification:

$$\boxed{V_n = \frac{n}{n-1} U_n}$$

Exercice 5;

1) Donnons un intervalle de confiance de m et de σ

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\overline{X} = \frac{1}{10} (490 + 492 + 502 + 505 + 490 + 495 + 492 + 490 + 497)$$

$$\boxed{\overline{X} = 495}$$

$$Var(\overline{X}) = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \overline{x}^2$$

$$= \frac{1}{10} \left((490)^2 + \cdots + (497)^2\right) - (495)^2$$

$$Var(\overline{X}) = 25 \Rightarrow S^2 = \frac{n}{n-1} Var(\overline{X}) = \frac{250}{9}$$

Soit $Z = \sqrt{n} \times \frac{\overline{X}_n - m}{\sqrt{S^2}} \rightsquigarrow T(n-1)$

$$\rightsquigarrow T(9)$$

$$\mathbb{P}(|Z| \leq t_\alpha) \ ?$$

$$\mathbb{P}(-t_{1-\frac{\alpha}{2}} \leq Z \leq t_{1-\frac{\alpha}{2}})$$

$$\mathbb{P}\left(-t_{1-\frac{\alpha}{2}} \leq \sqrt{n} \times \frac{\bar{X}_n - m}{\sqrt{S^2}} \leq t_{1-\frac{\alpha}{2}}\right) = \mathbb{P}\left(-\frac{t_{1-\frac{\alpha}{2}}}{\sqrt{n}} \leq \frac{\bar{X}_n - m}{\sqrt{S^2}} \leq \frac{t_{1-\frac{\alpha}{2}}}{\sqrt{n}}\right)$$

$$= \mathbb{P}\left(-\frac{t_{1-\frac{\alpha}{2}}}{\sqrt{n}} \times \sqrt{S^2} \leq \bar{X}_n - m \leq \frac{t_{1-\frac{\alpha}{2}}}{\sqrt{n}} \times \sqrt{S^2}\right)$$

$$= \mathbb{P}\left(-\frac{t_{1-\frac{\alpha}{2}}}{\sqrt{n}} \times \sqrt{S^2} - \bar{X}_n \leq -m \leq \frac{t_{1-\frac{\alpha}{2}}}{\sqrt{n}} \times \sqrt{S^2} - \bar{X}_n\right)$$

$$= \mathbb{P}\left(\bar{X}_n - \frac{t_{1-\frac{\alpha}{2}}}{\sqrt{n}} \times \sqrt{S^2} \leq m \leq \bar{X}_n + \frac{t_{1-\frac{\alpha}{2}}}{\sqrt{n}} \times \sqrt{S^2}\right)$$

$\alpha = 0,1$

on $t_{1-\frac{\alpha}{2}} = t_{1-\frac{0,1}{2}} = t_{0,95} = 1,833$

$$IC_{90\%}(m) = \left[495 - \frac{1,833}{\sqrt{10}} \times \sqrt{\frac{250}{9}}, 495 + \frac{1,833}{\sqrt{10}} \times \sqrt{\frac{250}{9}}\right)$$

$$IC_{90\%}(m) = [491,9 \ , \ 498,05]$$

Soit $Z' = \frac{n-1}{\sigma^2} S^2 \rightsquigarrow \chi^2(n-1)$

~~$\mathbb{P}(|Z'| \leq x_\alpha) = ?$~~ $\mathbb{P}(x_{\frac{\alpha}{2}} \leq Z' \leq x_{1-\frac{\alpha}{2}})$

$$\mathbb{P}(+x_{\frac{\alpha}{2}} \leq Z' \leq x_{\frac{\alpha}{2}}) = \mathbb{P}\left(+x_{\frac{\alpha}{2}} \leq \frac{n-1}{\sigma^2} S^2 \leq x_{\frac{\alpha}{2}}\right)$$

$$= \mathbb{P}\left(+\frac{x_{\alpha/2}}{(n-1)S^2} \leq \frac{1}{\sigma^2} \leq \frac{x_{1-\frac{\alpha}{2}}}{(n-1)S^2}\right)$$

$$= \mathbb{P}\left(\frac{S^2(n-1)}{x_{1-\frac{\alpha}{2}}} \leq \sigma^2 \leq \frac{S^2(n-1)}{x_{\frac{\alpha}{2}}}\right)$$

$x_{1-\frac{\alpha}{2}} = x_{0,95} = 16,919$

$x_{\frac{\alpha}{2}} = x_{0,05} = 3,325$

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$$IC_{90\%}(\sigma) = \left[\sqrt{\frac{\frac{250}{9}(9)}{16,919}} , \sqrt{\frac{\frac{250}{9}(9)}{3,325}} \right]$$

$$IC_{90\%}(\sigma) = [3,84 , 8,67]$$

b) $m = 500$

Soit $Z = \frac{n-1}{\sigma^2} s^2 \sim \chi^2(n-1)$

$$P(x_{\frac{\alpha}{2}} \leq Z \leq x_{1-\frac{\alpha}{2}}) = P\left(x_{\frac{\alpha}{2}} \leq \frac{n-1}{\sigma^2} s^2 \leq x_{1-\frac{\alpha}{2}}\right)$$

$$S^2 = \frac{n}{n-1} \sum_{i=1}^{n} (x_i - m)^2 = \frac{10}{9} \left[(490-500)^2 + (492-500)^2 + \cdots + (487-500)^2 \right]$$

s^2

$$P\left(\frac{s^2(n-1)}{x_{1-\frac{\alpha}{2}}} \leq \sigma^2 \leq \frac{s^2(n-1)}{x_{\frac{\alpha}{2}}} \right)$$

$$IC_{90\%}(\sigma) = \left[\sqrt{\frac{9 \times \frac{500}{9}}{16,918}} , \sqrt{\frac{500}{3,325}} \right]$$

$$IC_{90\%}(\sigma) = [17,19 , 38,77]$$

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