EE-559 – Deep learning

3.2. Probabilistic view of a linear classifier

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Consider the following class populations

$\forall y \in \{0, 1\}, x \in \mathbb{R}^D$,

$$\mu_{X|Y=y}(x) = \frac{1}{\sqrt{(2\pi)^D |\Sigma|}} \exp\left(-\frac{1}{2}(x - m_y)\Sigma^{-1}(x - m_y)^T\right).$$

That is, they are Gaussian with **the same covariance matrix $\Sigma$.** This is the **homoscedasticity** assumption.

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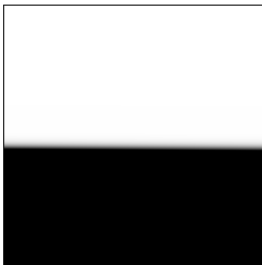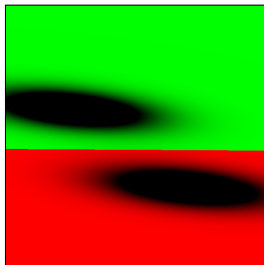$$

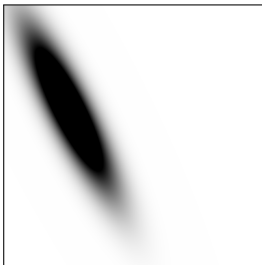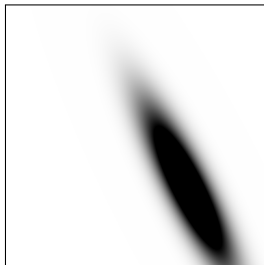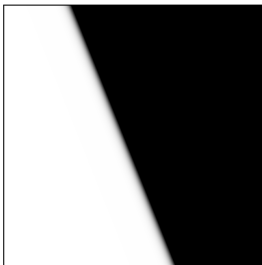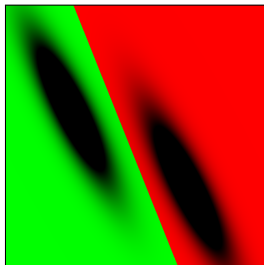**The homoscedasticity makes the second-order terms vanish.**

$\mu_{X|Y=0}$          $\mu_{X|Y=1}$
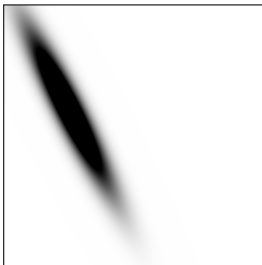
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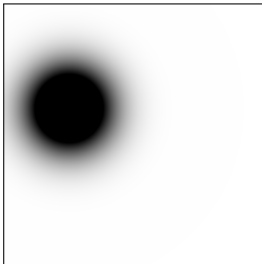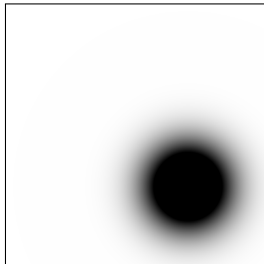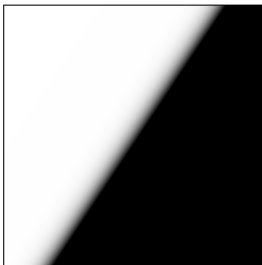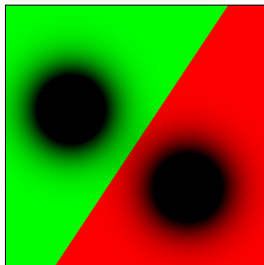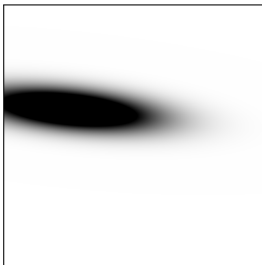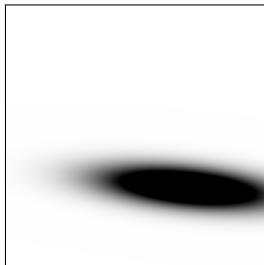
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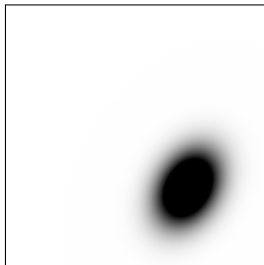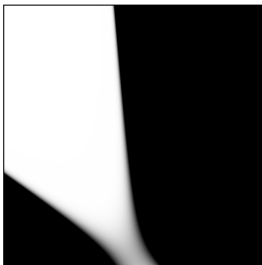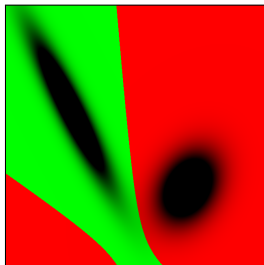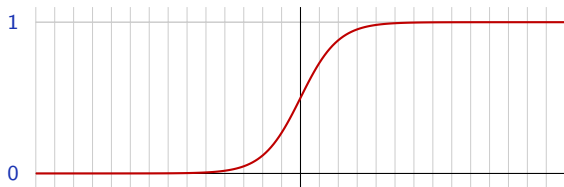$P(Y = 1 \mid X = x)$

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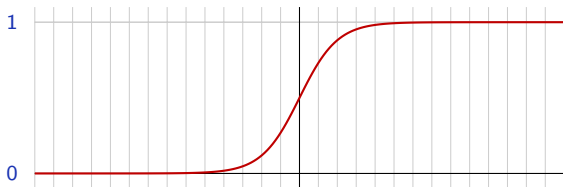$$\sigma(x) = \frac{1}{1 + e^{-x}},$$

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So the overall model

$$f(x; w, b) = \sigma(w \cdot x + b)$$

looks very similar to the perceptron.

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First, to simplify the next slide, note that we have

$$1 - \sigma(x) = 1 - \frac{1}{1 + e^{-x}} = \sigma(-x),$$

hence if $Y$ takes value in $\{-1, 1\}$ then

$$\forall y \in \{-1, 1\}, \quad P(Y = y \mid X = x) = \sigma(y(w \cdot x + b)).$$

We have

$$\log \mu_{W,B}(w, b \mid \mathcal{D} = \mathbf{d})$$
$$= \log \frac{\mu_{\mathcal{D}}(\mathbf{d} \mid W = w, B = b) \, \mu_{W,B}(w, b)}{\mu_{\mathcal{D}}(\mathbf{d})}$$
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This is the **logistic regression**, whose loss aims at minimizing

$$- \log \sigma(y_n f(x_n)).$$

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We will come back sometime to a probabilistic interpretation, but most of the methods will be envisioned from the signal-processing and optimization angles.

The end