

EE-559 – Deep learning

2.2. Over and under fitting

François Fleuret

<https://fleuret.org/ee559/>

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What about interviewing ten candidates and picking the best? What about interviewing one thousand?

With

$$Q_k^n \sim \mathcal{B}(0.5), \quad n = 1, \dots, 1000, \quad k = 1, \dots, 10,$$

independent standing for “candidate n answers question k correctly”, we have

$$\forall n, \quad P(\forall k, Q_k^n = 1) = \frac{1}{1024}$$

and

$$P(\exists n, \forall k, Q_k^n = 1) \simeq 0.62.$$

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So there is 62% chance that among 1,000 candidates answering completely at random, one will score perfectly.

Selecting a candidate based on a statistical estimator biases the said estimator for that candidate. And you need a greater number of “competence checks” if you have a larger pool of candidates.

Over and under-fitting, capacity. K -nearest-neighbors

A simple classification procedure is the “ K -nearest neighbors.”

Given

$$(x_n, y_n) \in \mathbb{R}^D \times \{1, \dots, C\}, \quad n = 1, \dots, N$$

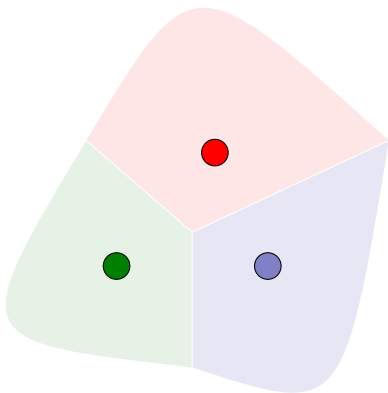
to predict the y associated to a new x , take the y_n of the closest x_n :

$$n^*(x) = \underset{n}{\operatorname{argmin}} \|x_n - x\|$$

$$f^*(x) = y_{n^*(x)}.$$

This recipe corresponds to $K = 1$, and makes the empirical training error zero.





$K = 1$

Under mild assumptions of regularities of $\mu_{X,Y}$, for $N \rightarrow \infty$ the asymptotic error rate of the 1-NN is less than twice the (optimal!) Bayes' Error rate.

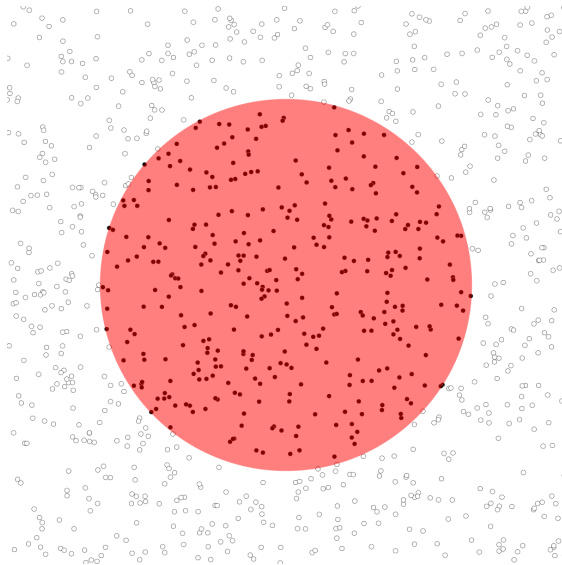
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It can be made more stable by looking at the $K > 1$ closest training points, and taking the majority vote.

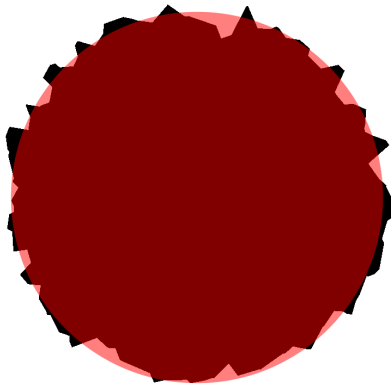
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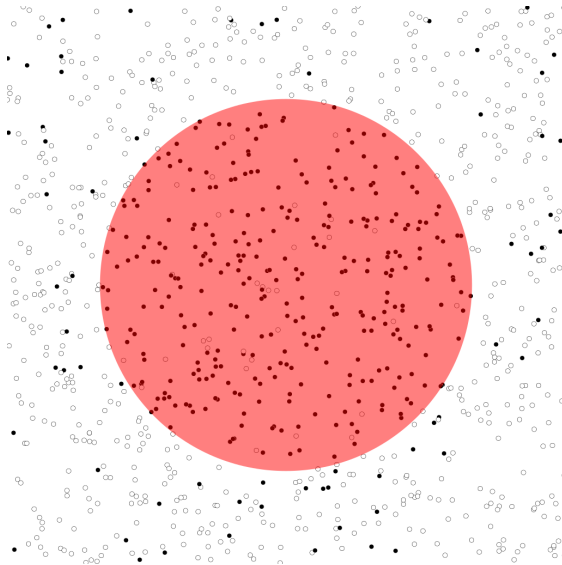
If we let also $K \rightarrow \infty$ “not too fast”, the error rate is the (optimal!) Bayes' Error rate.



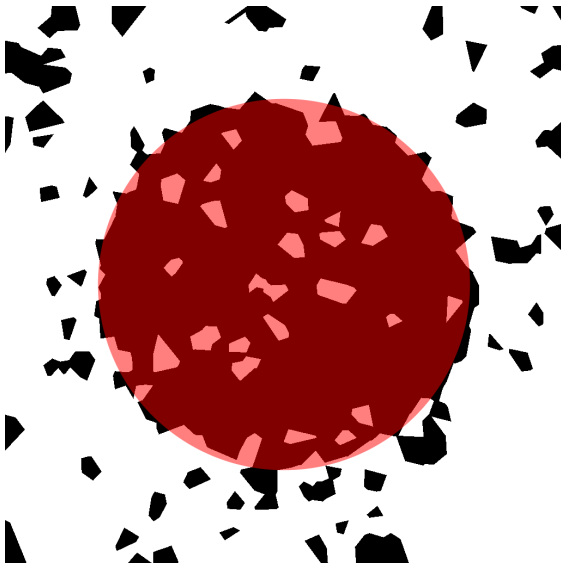
Training set



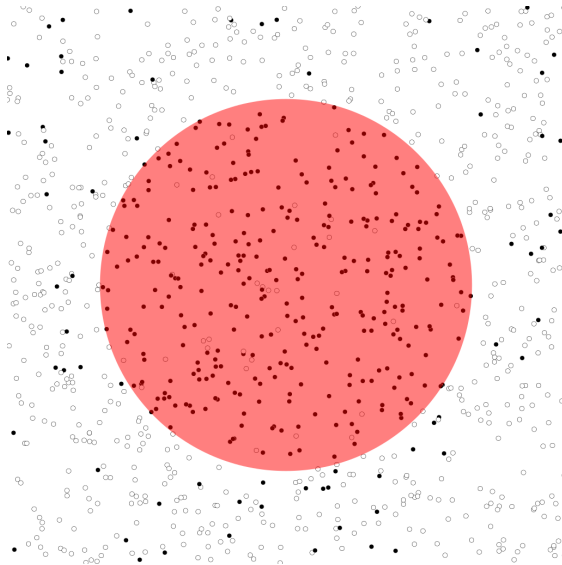
Prediction ($K=1$)



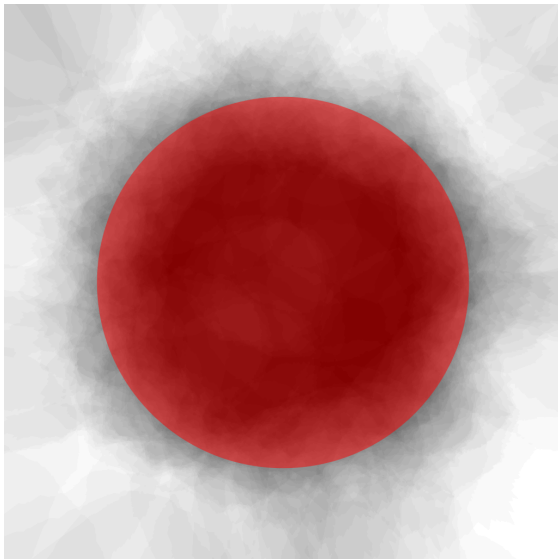
Training set



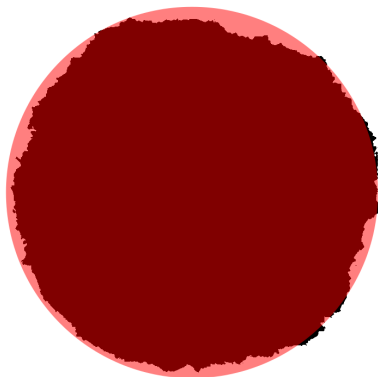
Prediction ($K=1$)



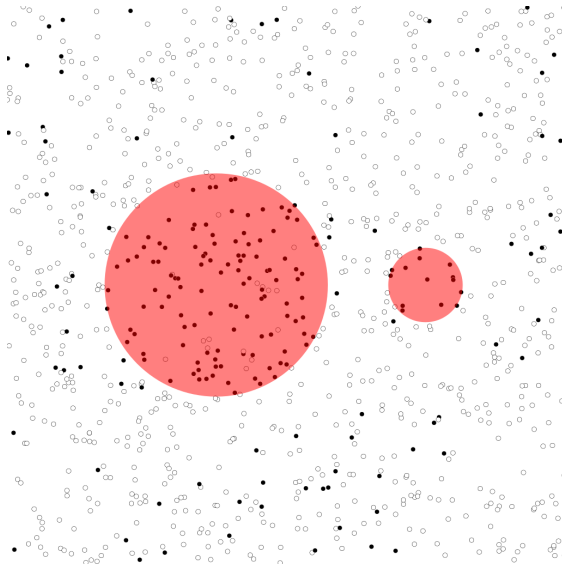
Training set



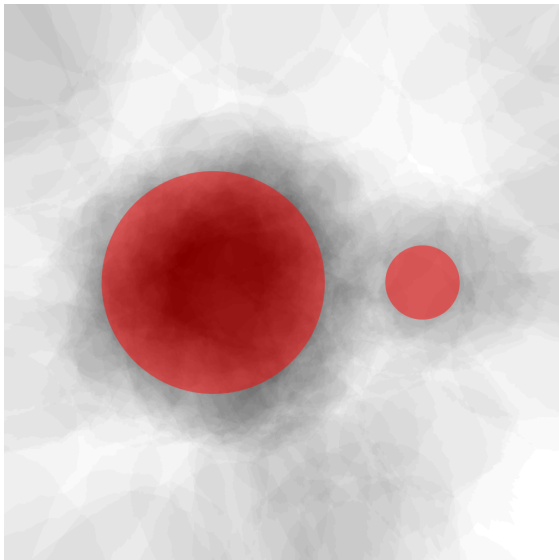
Votes ($K=51$)



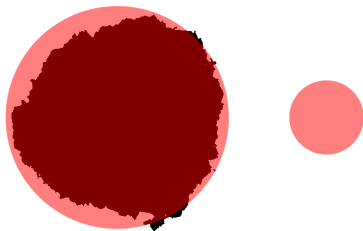
Prediction ($K=51$)



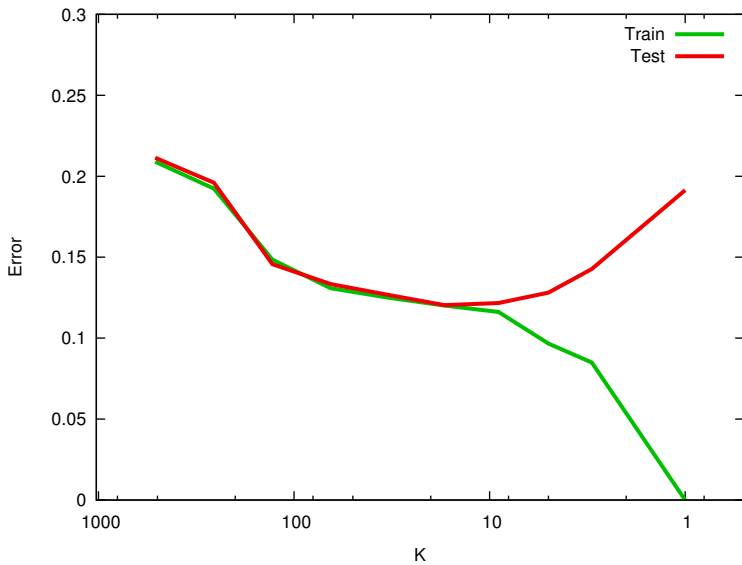
Training set

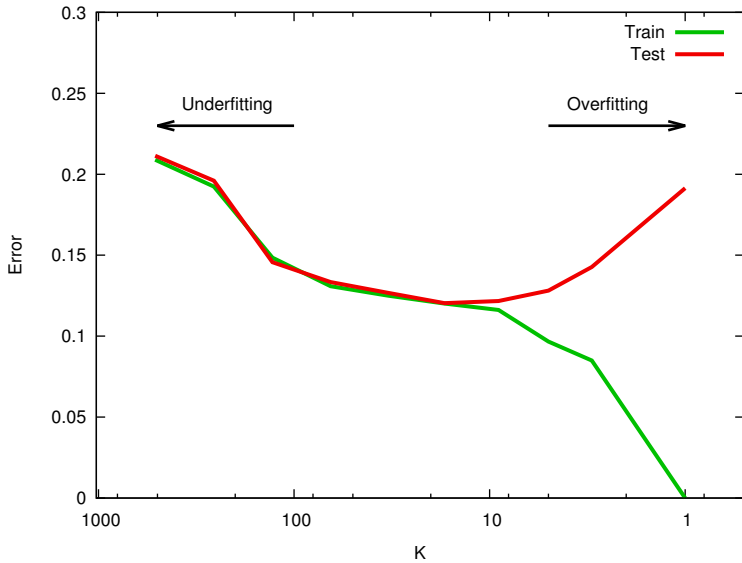


Votes ($K=51$)



Prediction ($K=51$)





Over and under-fitting, capacity, polynomials

Given a polynomial model

$$\forall x, \alpha_0, \dots, \alpha_D \in \mathbb{R}, f(x; \alpha) = \sum_{d=0}^D \alpha_d x^d.$$

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Hence, minimizing this loss is a standard quadratic problem, for which we have efficient algorithms.

$$\underset{\alpha}{\operatorname{argmin}} \left\| \begin{pmatrix} x_1^0 & \dots & x_1^D \\ \vdots & & \vdots \\ x_N^0 & \dots & x_N^D \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_D \end{pmatrix} - \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \right\|^2$$

```
def fit_polynomial(D, x, y):
    X = torch.empty(x.size(0), D + 1)
    for d in range(D + 1):
        X[:, d] = x.pow(d)

    # gels expects a matrix for target
    Y = y.view(-1, 1)

    # LAPACK's GEneralized Least-Square
    alpha, _ = torch.gels(Y, X)

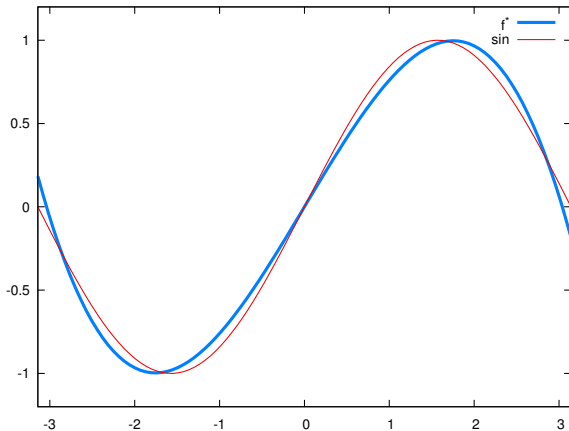
    return alpha[:D+1, 0]
```

```
D, N = 4, 100
x = torch.linspace(-math.pi, math.pi, N)
y = x.sin()
alpha = fit_polynomial(D, x, y)

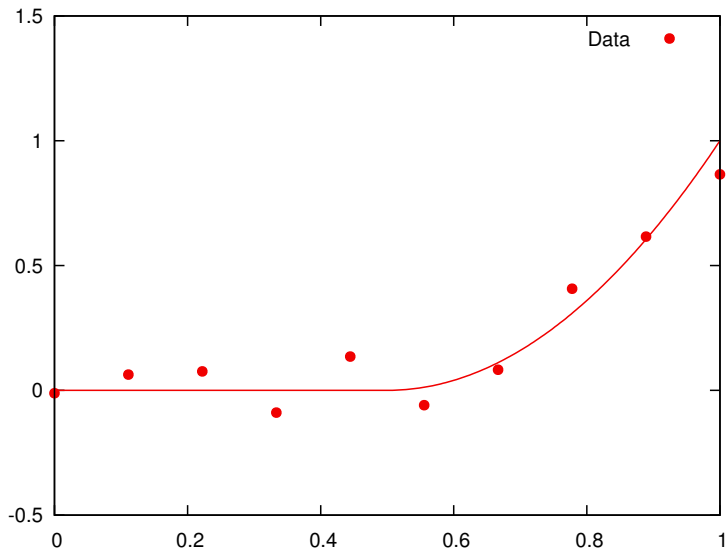
X = torch.empty(N, D + 1)
for d in range(D + 1):
    X[:, d] = x.pow(d)

yhat = X.mv(alpha)

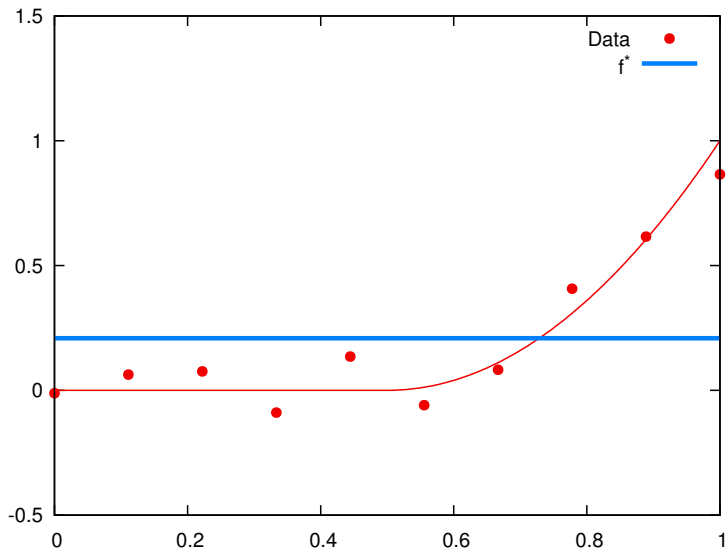
for k in range(N):
    print(x[k].item(), y[k].item(), yhat[k].item())
```



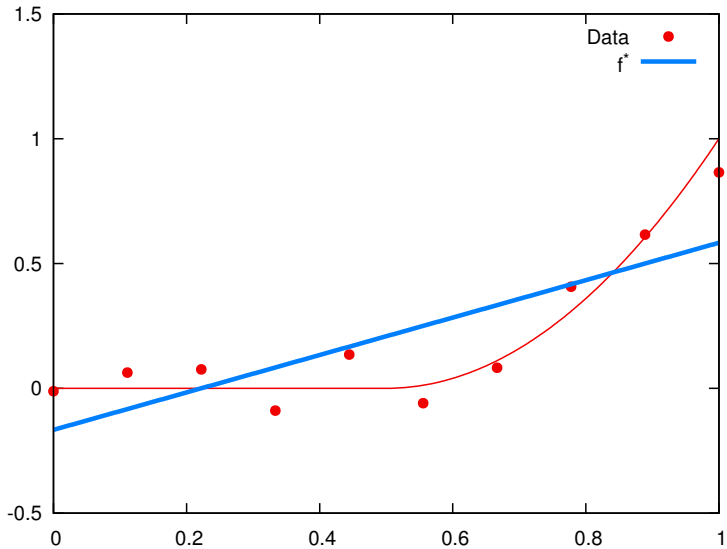
We can use that model to illustrate how the prediction changes when we increase the degree or the regularization.



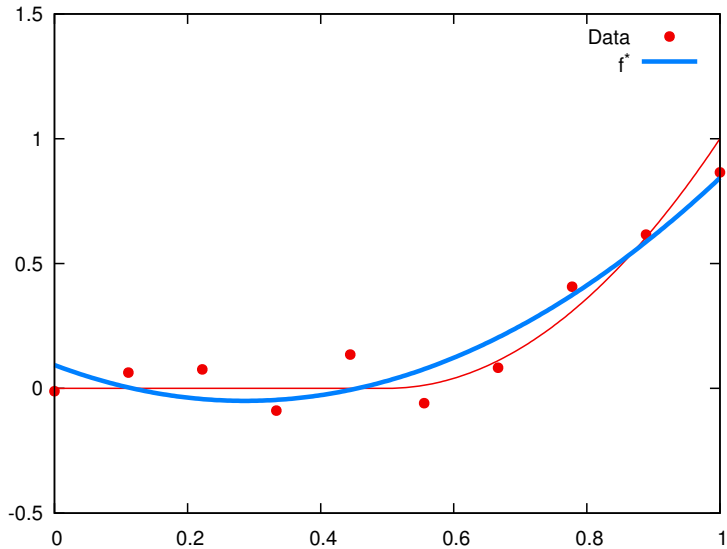
Degree D=0



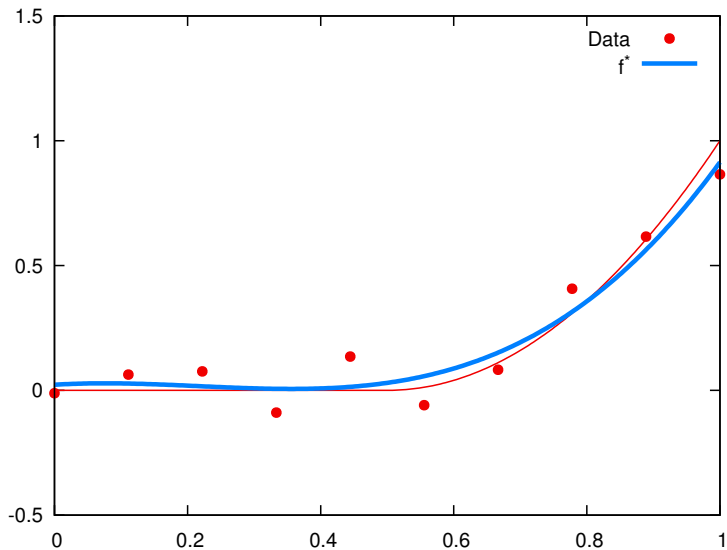
Degree D=1



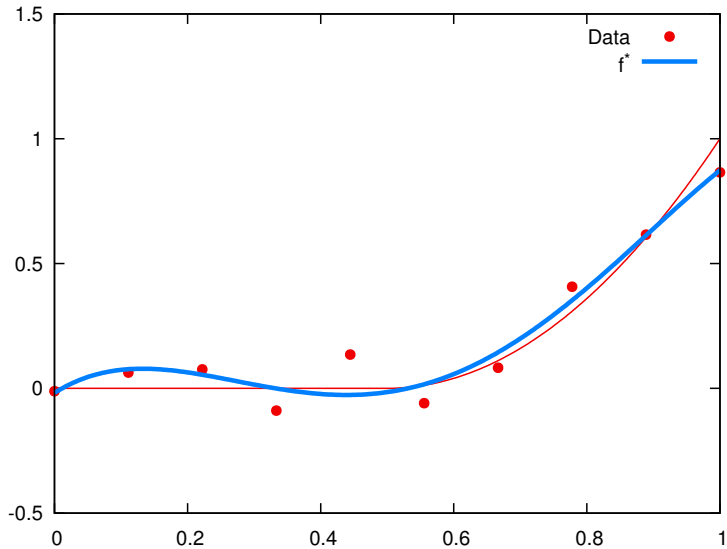
Degree D=2



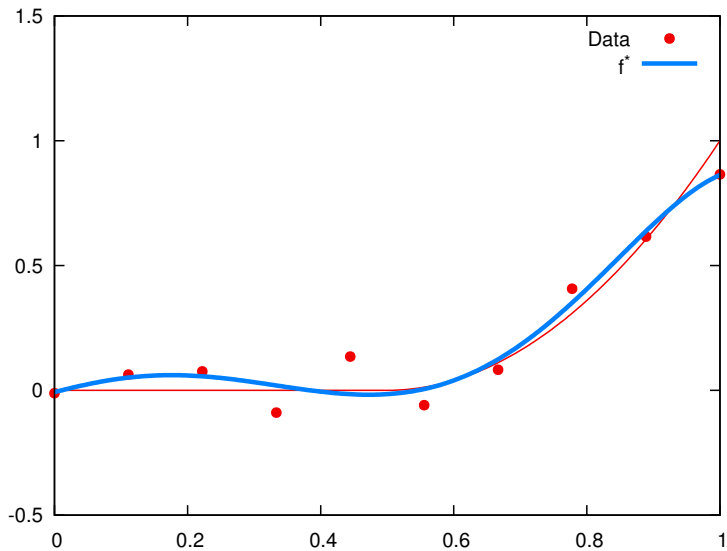
Degree D=3



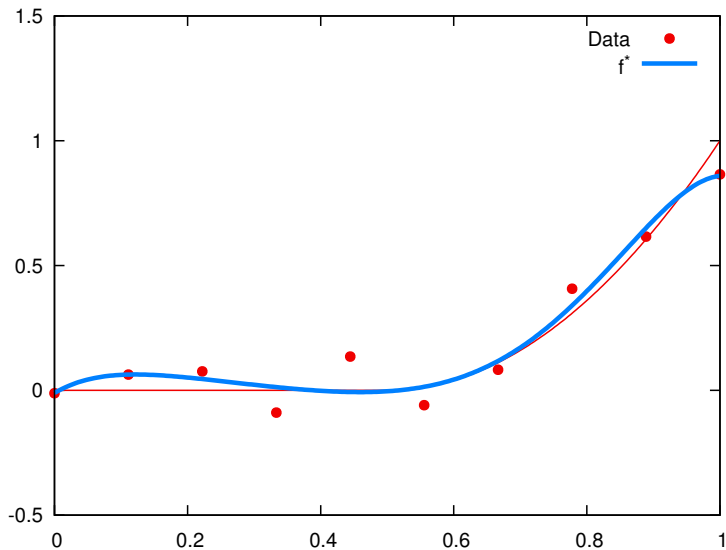
Degree D=4



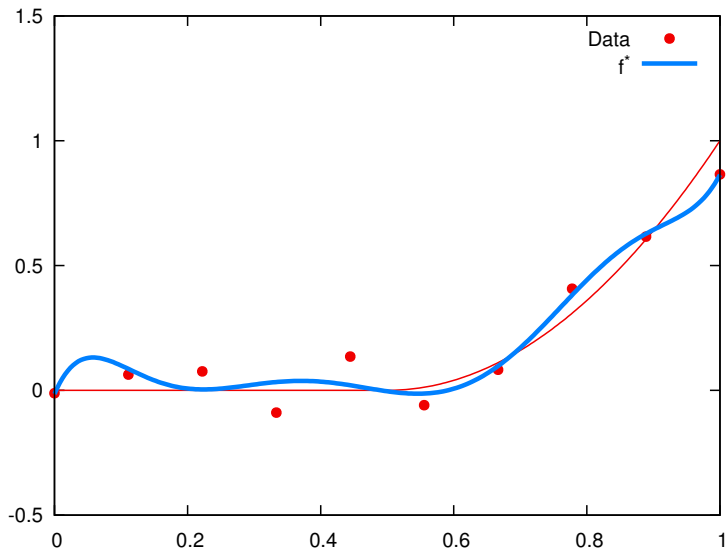
Degree D=5



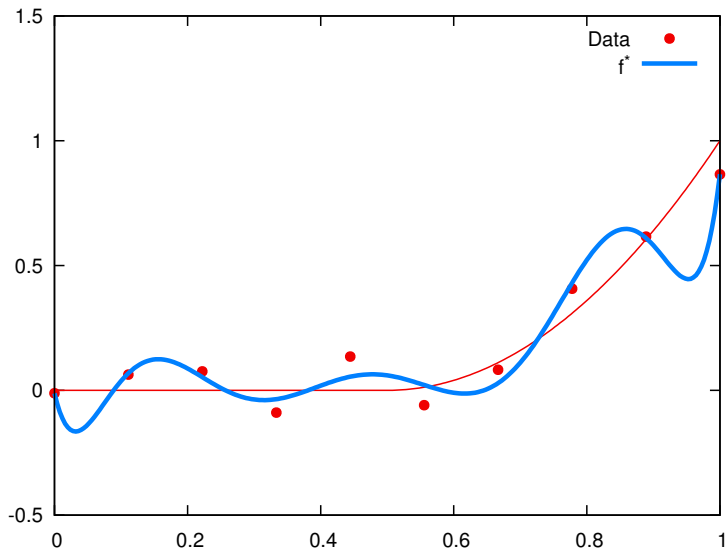
Degree D=6



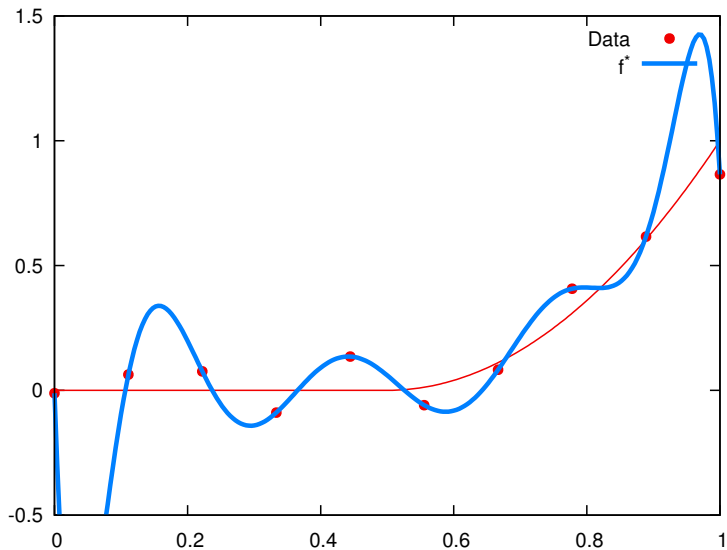
Degree D=7

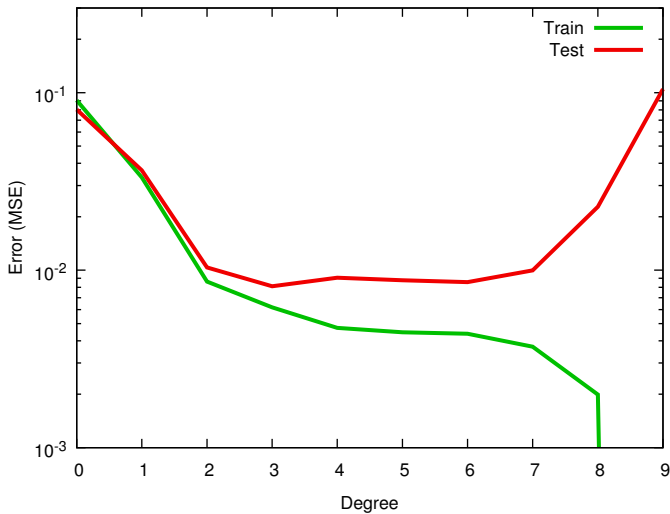


Degree D=8

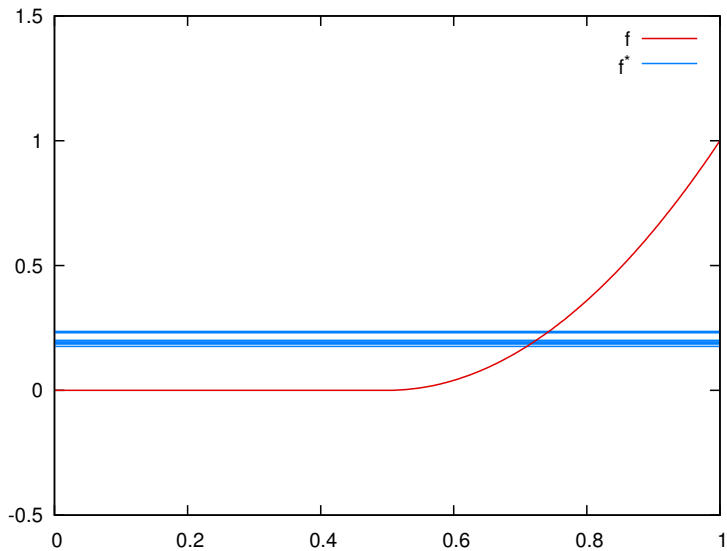


Degree D=9

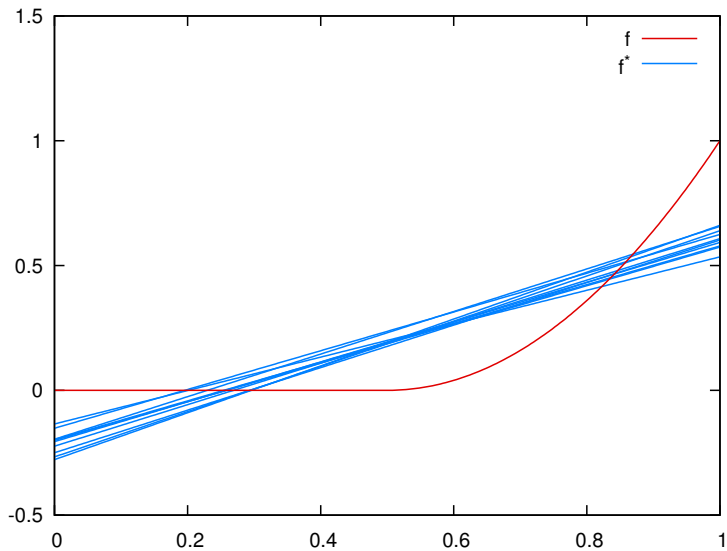




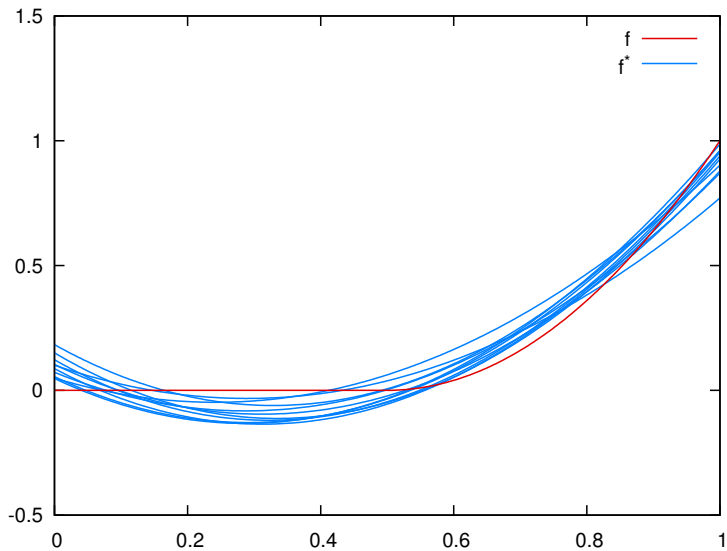
Degree $D=0$



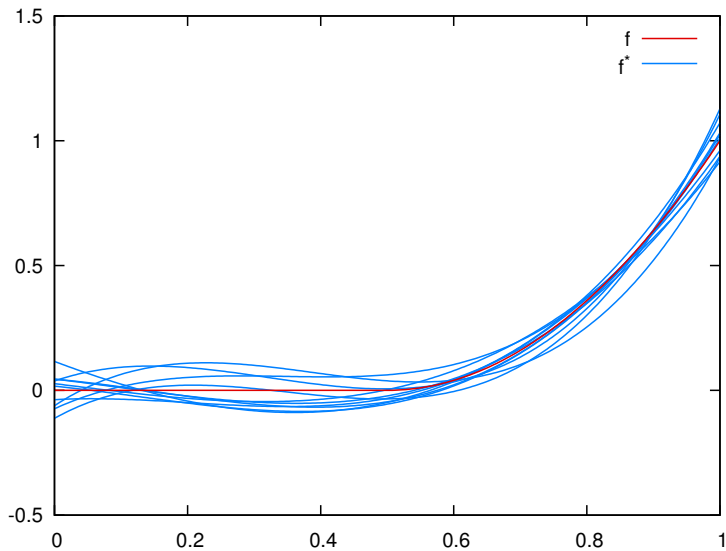
Degree $D=1$



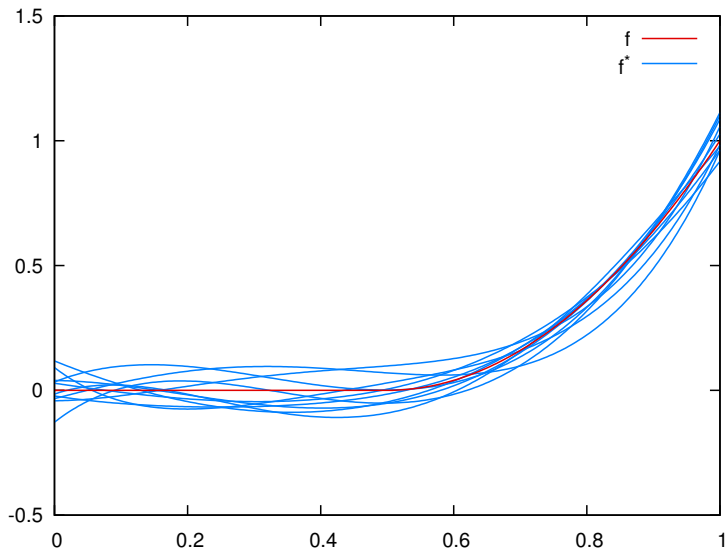
Degree D=2



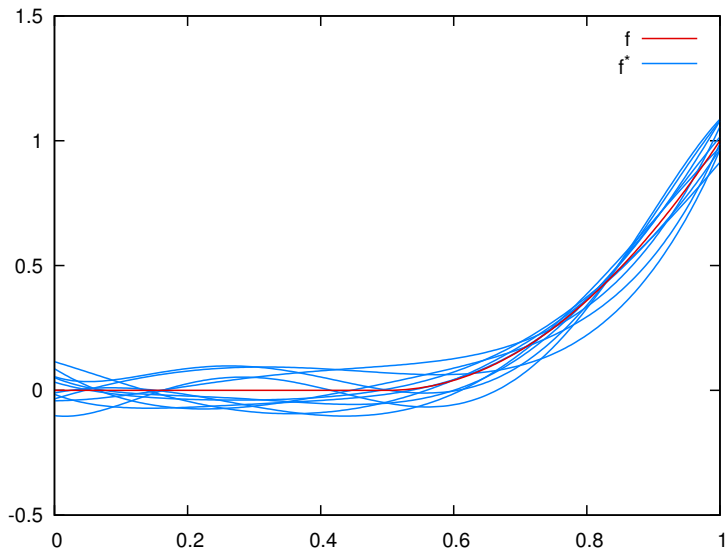
Degree D=3



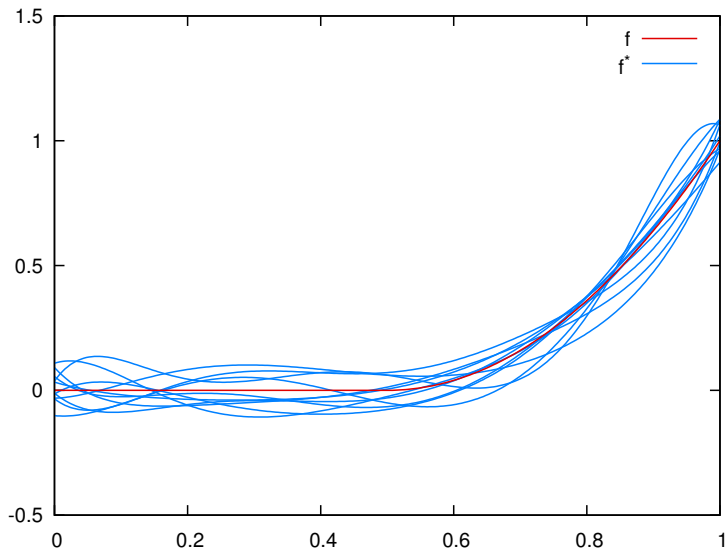
Degree D=4



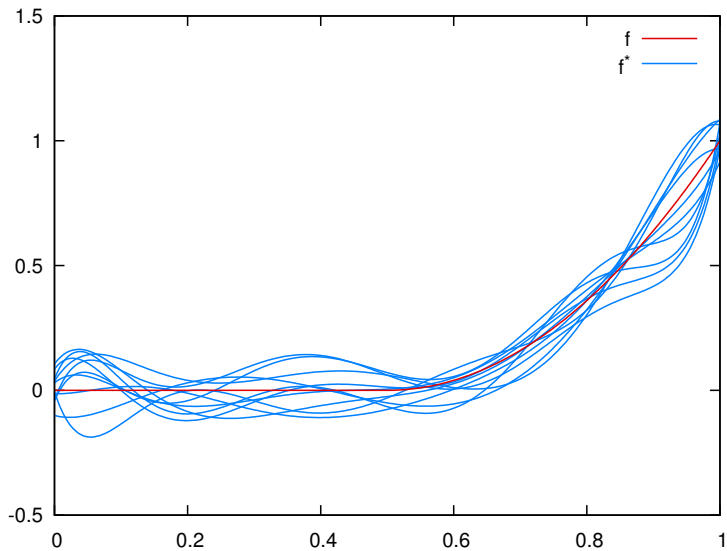
Degree D=5



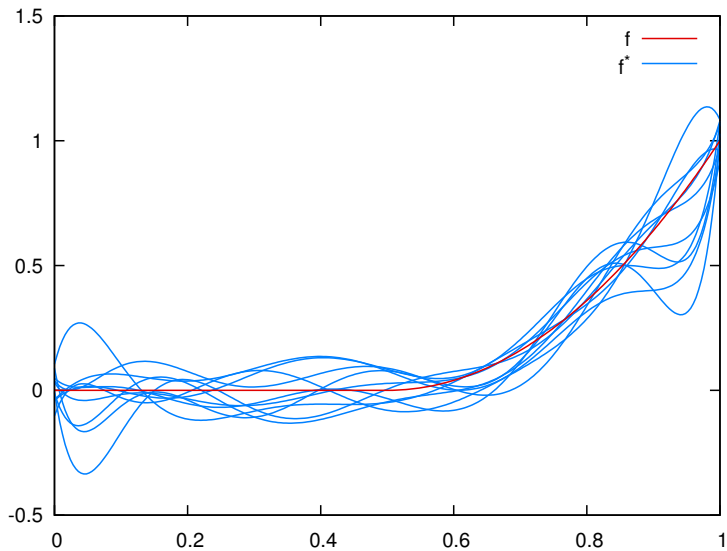
Degree D=6



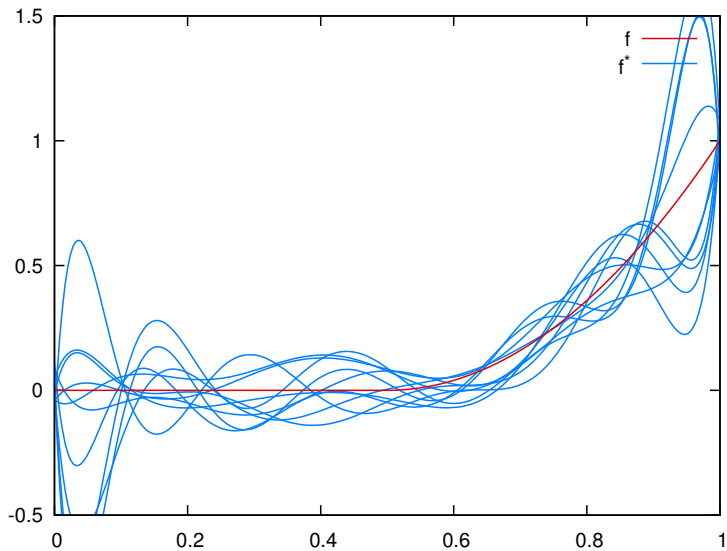
Degree D=7



Degree D=8



Degree D=9



We can reformulate this control of the degree with a penalty

$$\mathcal{L}(\alpha) = \sum_n (f(x_n; \alpha) - y_n)^2 + \sum_d l_d(\alpha_d)$$

where

$$l_d(\alpha) = \begin{cases} 0 & \text{if } d \leq D \text{ or } \alpha = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Such a penalty kills any term of degree $> D$.

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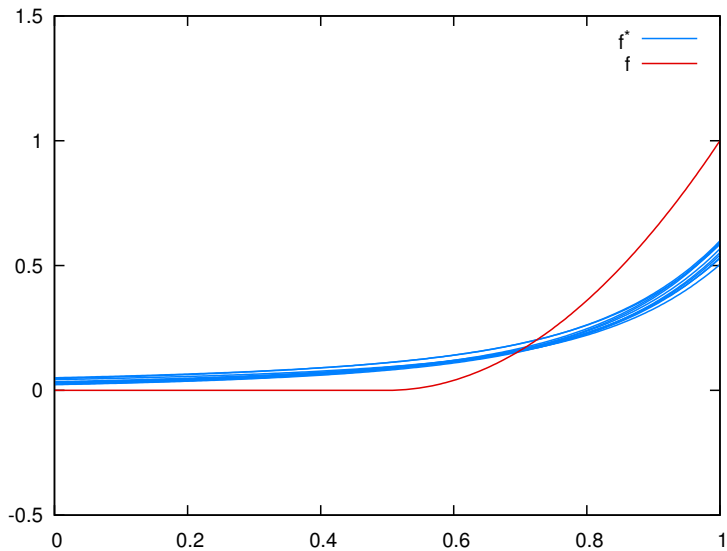
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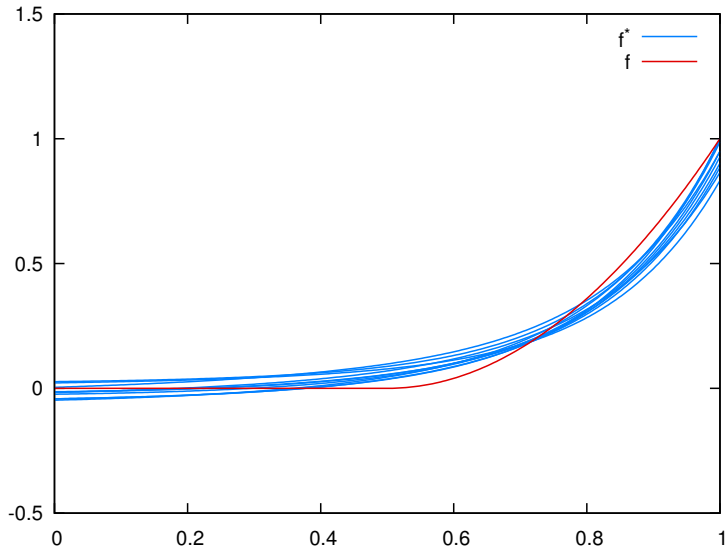
This motivates the use of more subtle variants. For instance, to keep all this quadratic

$$\mathcal{L}(\alpha) = \sum_n (f(x_n; \alpha) - y_n)^2 + \rho \sum_d \alpha_d^2.$$

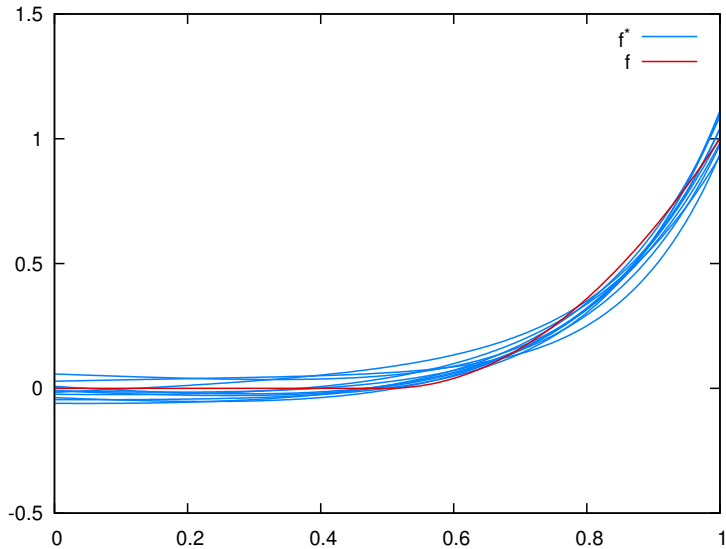
$D=9, \rho=1e1$



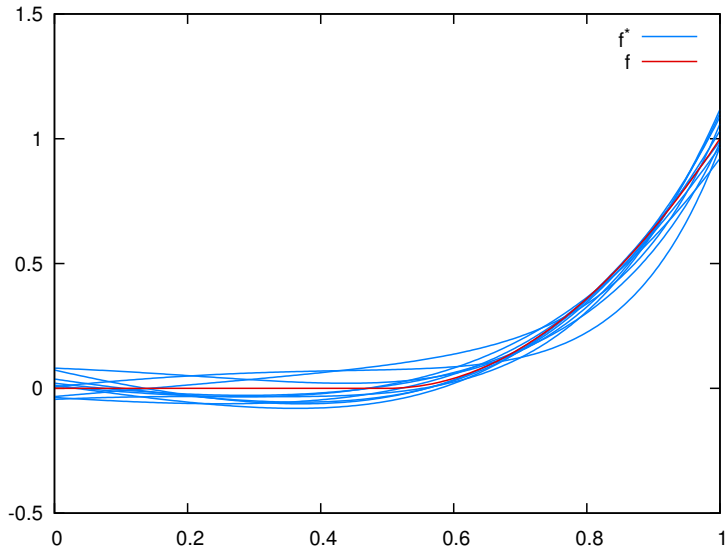
$D=9, \rho=1e0$



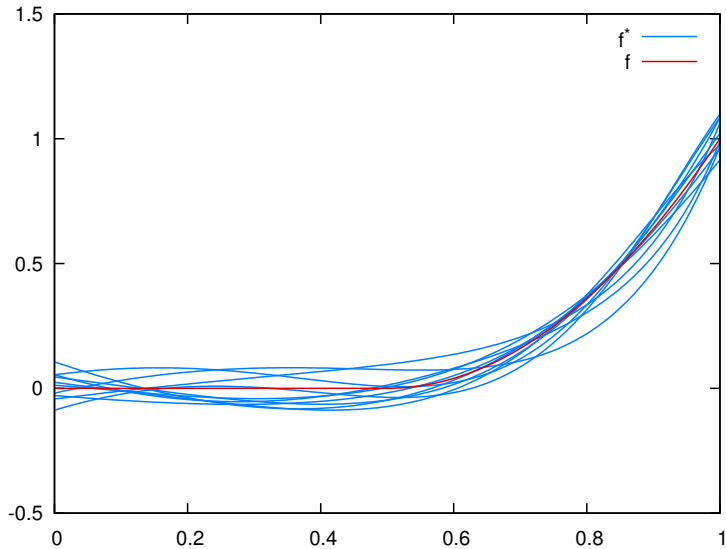
$D=9, \rho=1e-1$



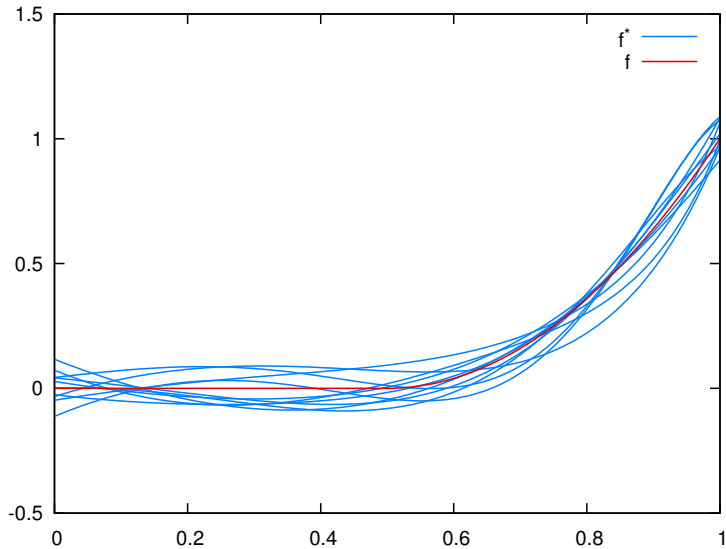
$D=9, \rho=1e-2$



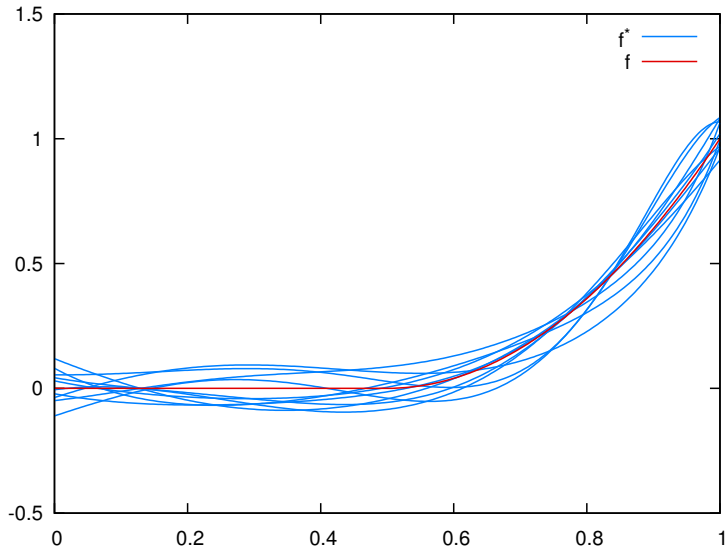
$D=9, \rho=1e-3$



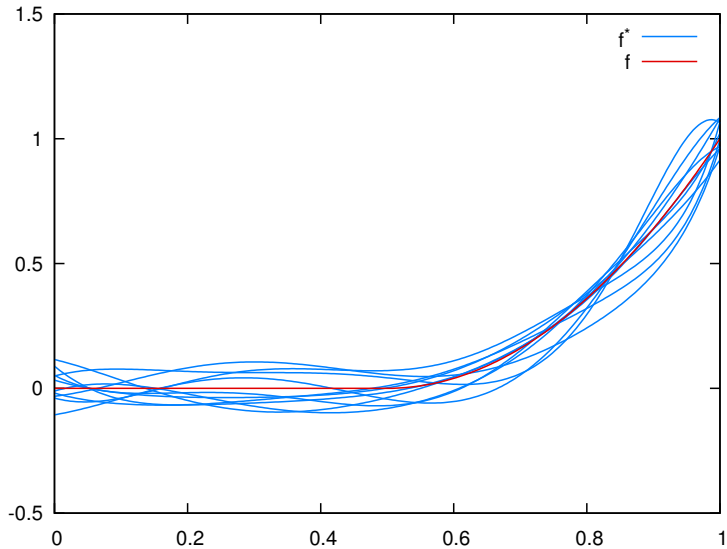
$D=9, \rho=1e-4$



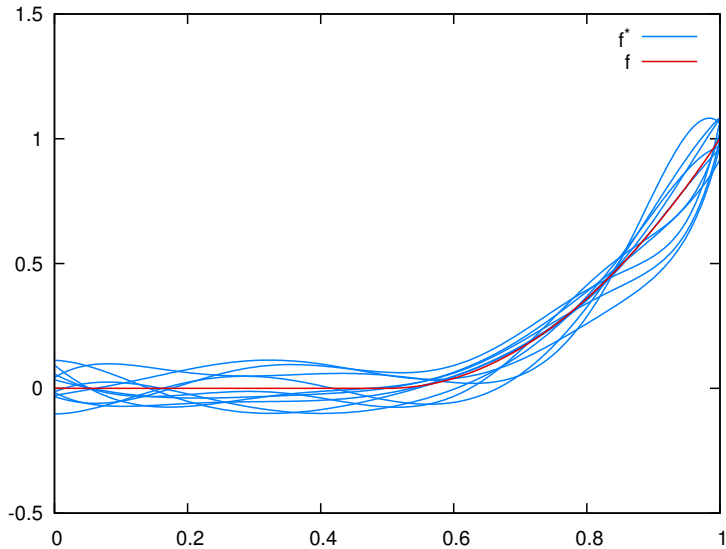
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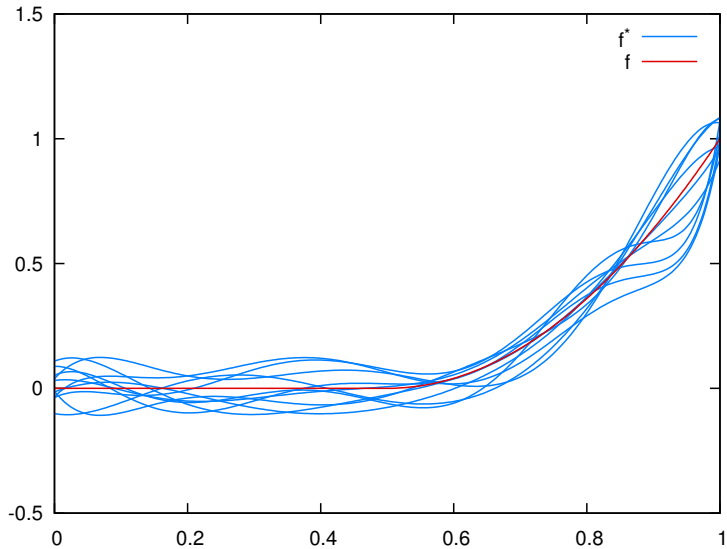
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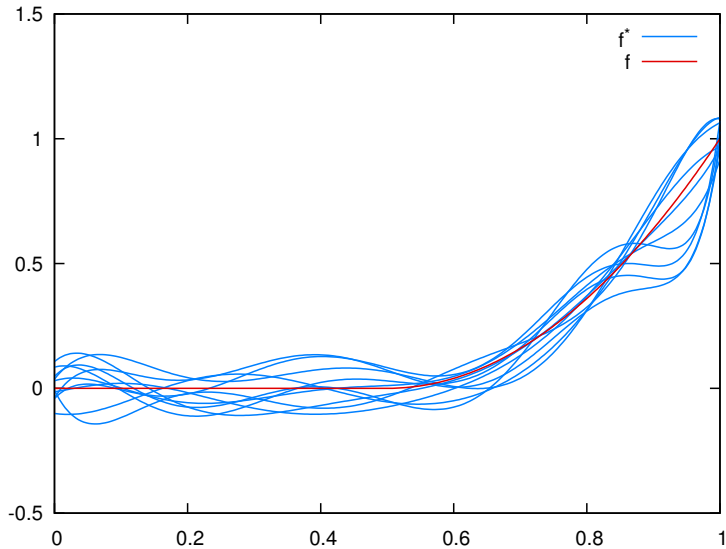
$D=9, \rho=1e-7$



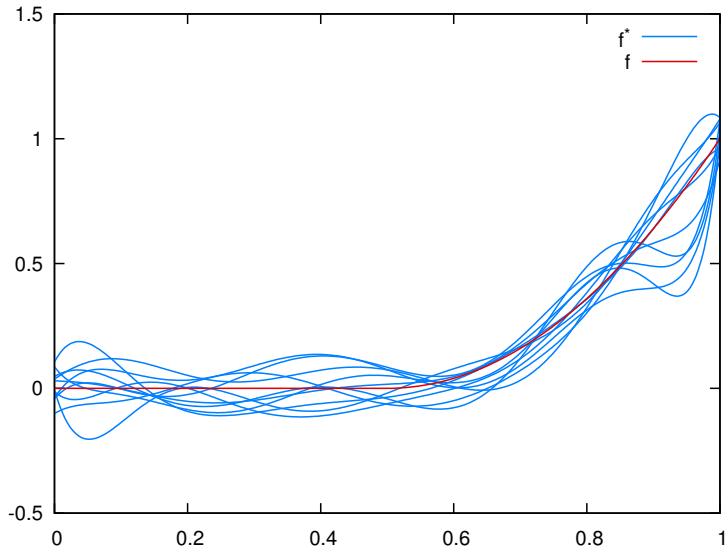
$D=9, \rho=1e-8$



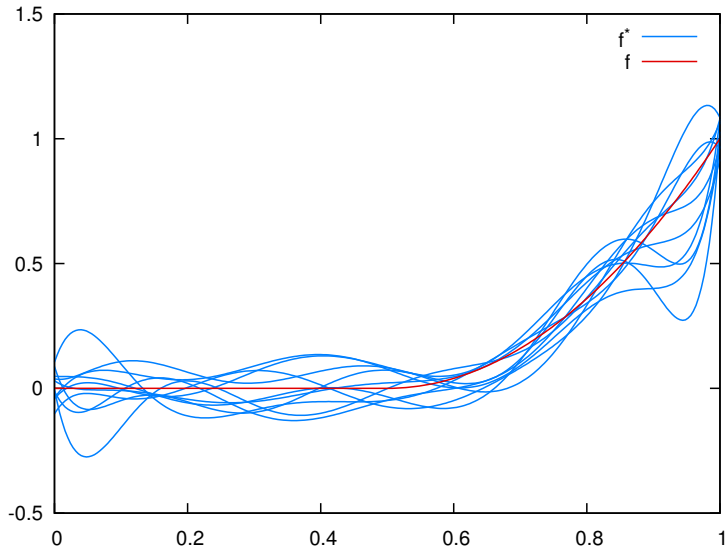
$D=9, \rho=1e-9$



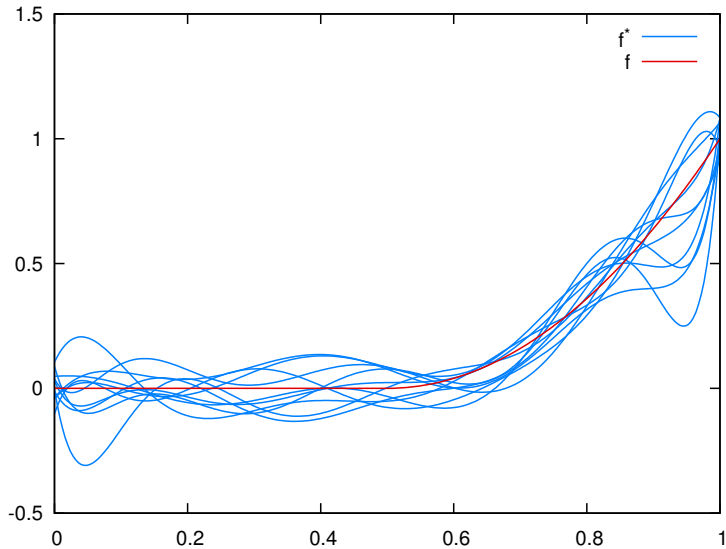
$D=9, \rho=1e-10$



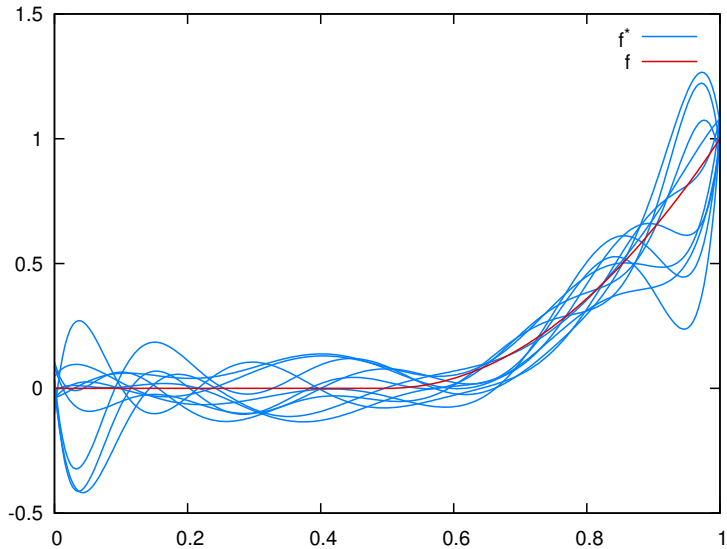
$D=9, \rho=1e-11$



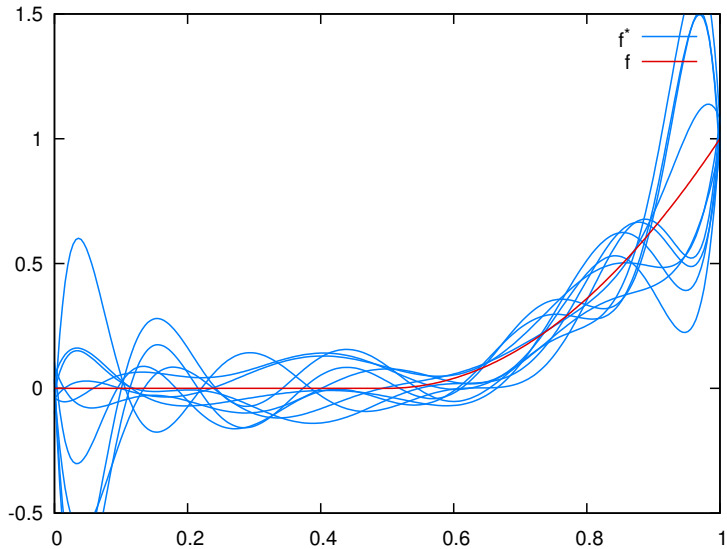
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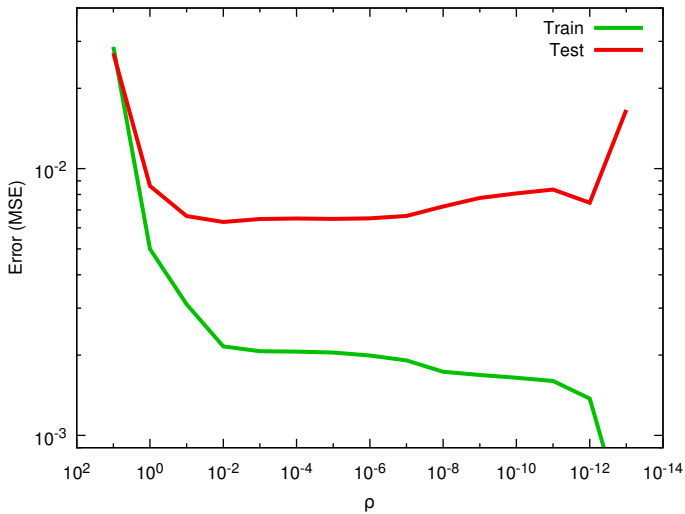


$D=9, \rho=1e-13$



$D=9, \rho=0.0$





We define the **capacity** of a set of predictors as its ability to model an arbitrary functional. This is a vague definition, difficult to make formal.

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A mathematically precise notion is the Vapnik–Chervonenkis dimension of a set of functions, which, in the Binary classification case, is the cardinality of the largest set that can be labeled arbitrarily (Vapnik, 1995).

It is a very powerful concept, but is poorly adapted to neural networks. We will not say more about it in this course.

Although the capacity is hard to define precisely, it is quite clear in practice how to modulate it for a given class of models.

In particular one can control over-fitting either by

- Impoverishing the space \mathcal{F} (less functionals, constrained or degraded optimization).
- Make the choice of f^* less dependent on data (penalty on coefficients, margin maximization, ensemble methods).

The end

References

V. N. Vapnik. *The Nature of Statistical Learning Theory*. Springer-Verlag, New York, 1995.