EE-559 – Deep learning

5.2. Stochastic gradient descent

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To minimize a loss of the form

$$\mathscr{L}(w) = \sum_{n=1}^{N} \underbrace{\ell\big(f(x_n; w), y_n\big)}_{\ell_n(w)}$$

the standard gradient-descent algorithm update has the form

$$w_{t+1} = w_t - \eta \nabla \mathscr{L}(w_t).$$

A straight-forward implementation would be

```
for e in range(nb_epochs):
    output = model(train_input)
    loss = criterion(output, train_target)

    model.zero_grad()
    loss.backward()
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However, the memory footprint is proportional to the full set size. This can be mitigated by summing the gradient through "mini-batches":

```
for e in range(nb_epochs):
    model.zero_grad()

    for b in range(0, train_input.size(0), batch_size):
        output = model(train_input[b:b+batch_size])
        loss = criterion(output, train_target[b:b+batch_size])
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- It takes time to compute (more exactly **all our time!**).
- It is an empirical estimation of an hidden quantity, and any partial sum is also an unbiased estimate, although of greater variance.
- It is computed incrementally

$$\nabla \mathscr{L}(w_t) = \sum_{n=1}^{N} \nabla \ell_n(w_t),$$

  and when we compute ℓ_n, we have already computed $\ell_1, \ldots, \ell_{n-1}$, and we could have a better estimate of w^* than w_t.

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So instead of summing over all the samples and moving by η, we can visit only $M = N/K$ samples and move by $K\eta$, which would cut the computation by K.

Although this is an ideal case, there is redundancy in practice that results in similar behaviors.

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The stochastic behavior of this procedure helps evade local minima.

So our exact gradient descent with mini-batches

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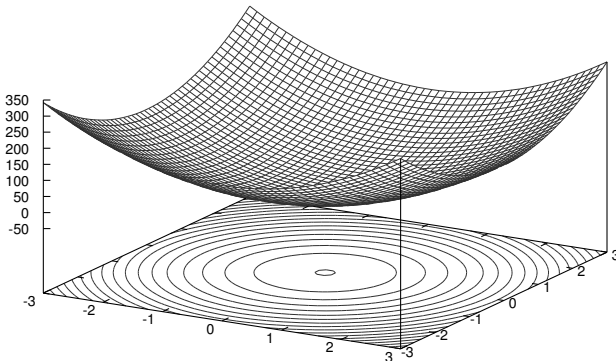
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can be modified into the mini-batch stochastic gradient descent as follows:

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for e in range(nb_epochs):
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        loss.backward()
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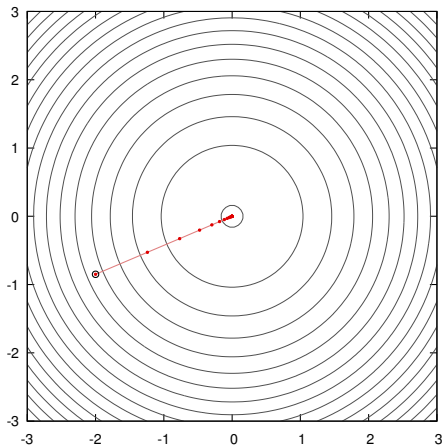Mini-batch size and loss reduction (MNIST)
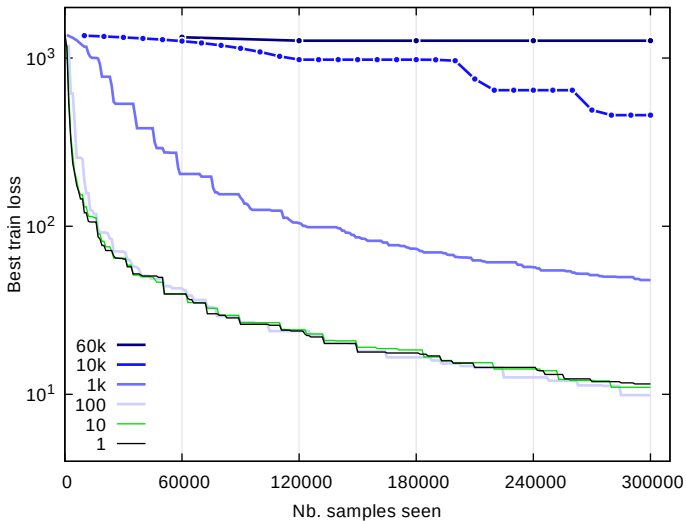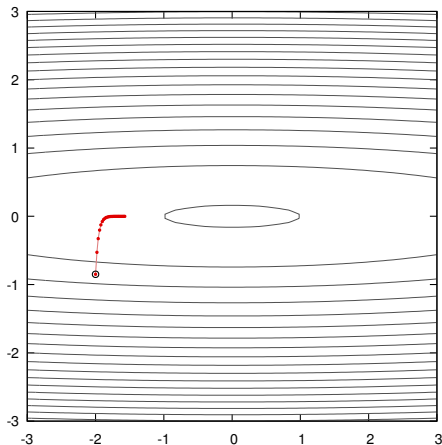
Limitation of the gradient descent

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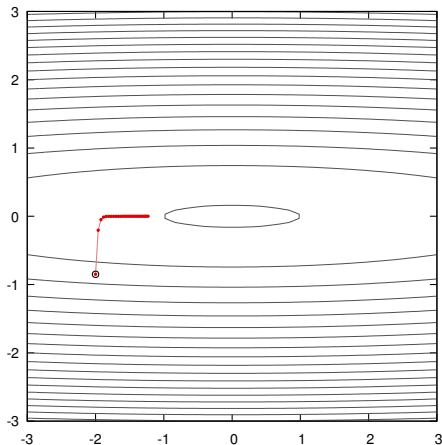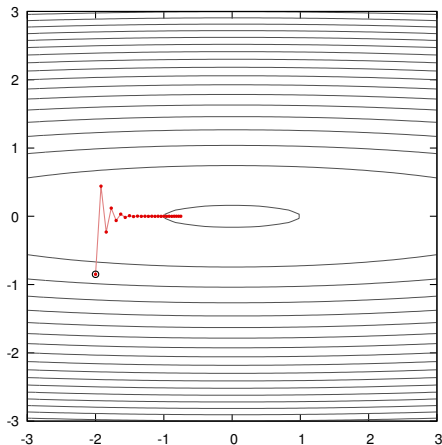
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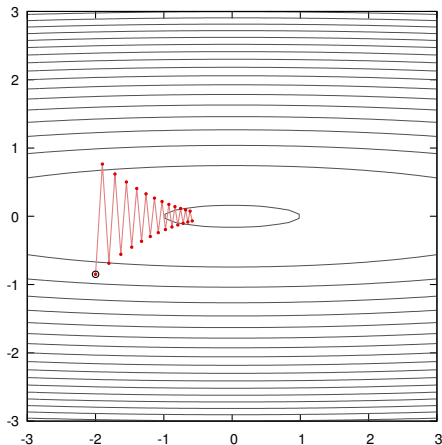
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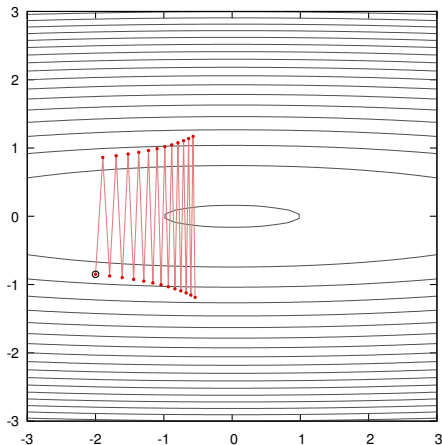
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Deep-learning generally relies on a smarter use of the gradient, using statistics over its past values to make a "smarter step" with the current one.

Momentum and moment estimation

The "vanilla" mini-batch stochastic gradient descent (SGD) consists of

$$w_{t+1} = w_t - \eta g_t,$$

where

$$g_t = \sum_{b=1}^{B} \nabla \ell_{n(t,b)}(w_t)$$

is the gradient summed over a mini-batch.

The first improvement is the use of a "momentum" to add inertia in the choice of the step direction

$$u_t = \gamma u_{t-1} + \eta g_t$$
$$w_{t+1} = w_t - u_t.$$

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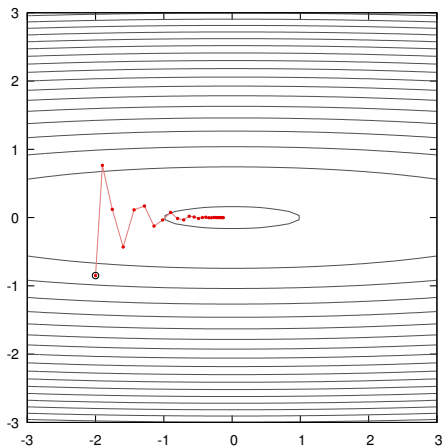- it accelerates if the gradient does not change much:

$$(u = \gamma u + \eta g) \Rightarrow \left(u = \frac{\eta}{1-\gamma} g \right),$$

- it dampens oscillations in narrow valleys.

$\eta = 5.0e-2, \gamma = 0$
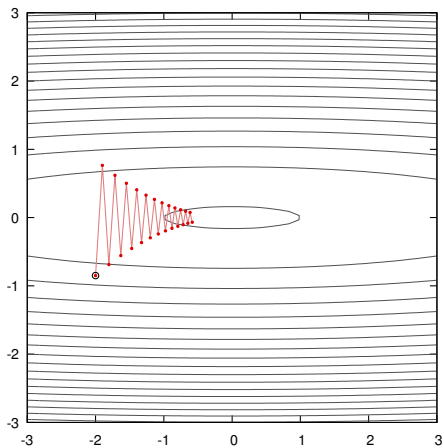
$\eta = 5.0e - 2, \gamma = 0.5$

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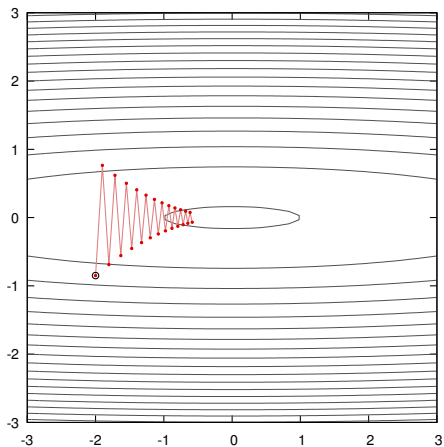
The update rule is, **on each coordinate separately**

$$m_t = \beta_1 m_{t-1} + (1 - \beta_1) g_t$$

$$\hat{m}_t = \frac{m_t}{1 - \beta_1}$$

$$v_t = \beta_2 v_{t-1} + (1 - \beta_2) g_t^2$$

$$\hat{v}_t = \frac{v_t}{1 - \beta_2}$$

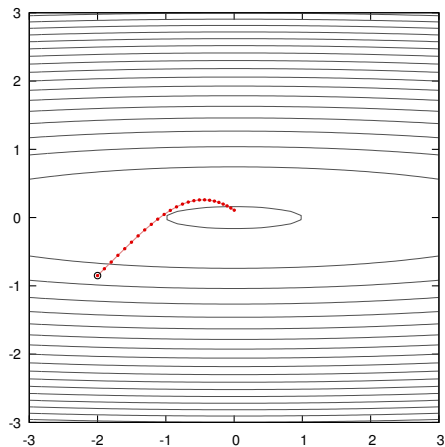$$w_{t+1} = w_t - \frac{\eta}{\sqrt{\hat{v}_t} + \epsilon} \hat{m}_t$$

(Kingma and Ba, 2014)

This can be seen as a combination of momentum, with $\hat{m}_t$, and a per-coordinate re-scaling with $\hat{v}_t$.

$\eta = 5.0e - 2$

Adam, $\beta_1 = 0.9, \beta_2 = 0.999, \epsilon = 1e - 8, \eta = 1.0e - 1$

These two core strategies have been used in multiple incarnations:

- Nesterov's accelerated gradient,
- Adagrad,
- Adadelta,
- RMSprop,
- AdaMax,
- Nadam …

The end

References

D. Kingma and J. Ba. Adam: A method for stochastic optimization. CoRR, abs/1412.6980, 2014.

D. E. Rumelhart, G. E. Hinton, and R. J. Williams. Learning representations by back-propagating errors. Nature, 323(9):533–536, 1986.