Exam Scheduling Optimizaiton Problem

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✓ *Special thanks to Maryam Sadeghi for her great help that made this article possible.*

Abstract

Scheduling problems are among classic Mixed Integer Programming use cases. This is an attempt in solving an exam scheduling problem using Pyomo modeling package and GLPK Integer Optimization. The problem definition is thoroughly described in later sections.

We began by writing abstract mathematical representations of the constraints. The initial model consisted of no objective function. After conducting many reviews on the hard constraints and using logical complementary constraints, we moved on to the soft constraints. Having developed a good foundation on the use of logical operators, initial construction of the soft constraints was fairly easy. Then we proceeded to write the objective function and finished the model with some small tweaks in the end. We tested the solution of the model on various datasets and analyzed the results to ensure the robustness of our modeling.

Our major challenges were in the modeling part, especially towards the beginning of the work. We tended towards simplifying the model as much as possible which was challenging while defining decision variables since we had to count for all of the constraints simultaneously. Also, definition of soft constraints were not an easy task. However, we managed to propose a precise mathematical modeling of the described constraints in the end.

Introduction & Problem Definition

Suppose an educational institute has certain students each of whom are enrolled in certain courses. A final exam is going to be held for each course. All exams must take place within a specific time window. *(i.e. one weeks)* During these dates, there are specific valid time periods that exams can be held at. *(i.e. 8 AM - 10 AM, 10 AM - 12 PM, 14 PM - 16 PM)* Also, all exams must be held in available rooms which have certain capacities.

Notice

The list of students is available. In other words it is known that which student is enrolled in what courses.

Below constraints must be satisfied:

1. Each course's exam must be held on a specific day and time period.
2. Each student must have at most one exam at a single time period on a day.
3. Each course's exam must be held in at least one room.
4. Room capacities must be considered.
5. The number of used rooms at a single time period must not exceed the total number of available rooms.

Below are preferred constraints: (In order of preference)

1. Each student should preferably have no more than one exam in a day.
2. Each student should preferably have at least one day off between their exams.
3. Preferably, only one course exam should be scheduled in each room during a given time period.

Solution

Sets and Parameters

Sets

Definition	Symbol	Description
$\mathbb{C} = \{\ldots\}$	c, c'	The set of courses.
$\mathbb{D} = \{\ldots\}$	d, d'	The set of valid days to hold the exams.
$\mathbb{H} = \{\ldots\}$	h, h'	The set of valid time periods in a day. [1]
$\mathbb{R} = \{\ldots\}$	r	The set of available rooms.
$\mathbb{S} = \{\ldots\}$	s	The set of enrolled students.

Parameters

$$a_{s,c} = \begin{cases} 1 & Student\ s\ is\ enrolled\ in\ course\ c \\ 0 & o.w. \end{cases}$$

Cap_r is the capacity for each room $r \in \mathbb{R}$

Variables

Binary Variable: $\delta c, d, h, r = \begin{cases} 1 & \text{Exam for course c is held on day d at time h in room r} \\ 0 & o.w. \end{cases}$

Non Negative Integer Variable: $x_{c,d,h,r}$ is The number of enrolled students in course c whose exam is held on day d at time period h in room r

Note
The inclusion of d, h in $x_{c,d,h,r}$ will result in easier definition of constraints.

More variables will be defined as we make progress in the model. A complete list of model components can be found in the section Model Summary.

Constraints

Hard Constraints

1. Each exam must be held on a single date (d, h):

$$\sum_{d,h,r} \delta_{c,d,h,r} \geq 1 \qquad \forall c$$

This expression alone will not satisfy the constraint as we have no control over what indices the model changes in order to satisfy the expression. Therefore, it is necessary to include the following conditional expression.

$$\delta_{c,d,h,r} = 1 \implies \sum_{d',h',r':(d,h)\neq(d',h')} \delta_{c,d',h',r'} \leq 0 \qquad \forall s, c, d, h, r$$

Linearizing the above expression will result in the following:

$$\sum_{d',h',r':(d,h)\neq(d',h')} \delta_{c,d',h',r'} \leq 0 + M_0 \cdot (1 - \delta_{c,d,h,r}) \qquad \forall s, c, d, h, r$$

When the premise is not met, the constraint must be trivial. Therefore we will solve for the maximum value of M_0. Note that when substituting $\delta_{c,d,h,r} = 0$, the left hand side of the inequality will have a maximum value of $|\mathbb{D} \times \mathbb{H} \times \mathbb{R}|$ when all δ are equal to 1.

$$\sum_{d',h',r':(d,h)\neq(d',h')} \delta_{c,d',h',r'} \leq 0 + |\mathbb{D} \times \mathbb{H} \times \mathbb{R}| \cdot (1 - \delta_{c,d,h,r}) \qquad \forall s, c, d, h, r$$

Constraint Recap: 1

$$\sum_{d,h,r} \delta_{c,d,h,r} \geq 1 \qquad \forall c$$

$$\sum_{d',h',r':(d,h)\neq(d',h')} \delta_{c,d',h',r'} \leq |\mathbb{D} \times \mathbb{H} \times \mathbb{R}| \cdot (1 - \delta_{c,d,h,r}) \qquad \forall s, c, d, h, r$$

This basically implies that all courses must hold at least one exam and if a course exam is being held on a specific date, then the exam for the same course must not be held on any other date.

2. Each student must have only one exam on a specific date (d, h). Meaning exams must not be conflicting for students:

$$\sum_r a_{s,c} \cdot \delta_{c,d,h,r} \geq 1 \implies \sum_{c',r\,:\,c\neq c'} a_{s,c'} \cdot \delta_{c',d,h,r} \leq 0 \qquad \forall s, c, d, h$$

To linearizing the above expression will define binary decision variable $\psi_{s,c,d,h}$ as the intermediate variable. Then we will separate the implication into two and linearize each.

$$\sum_r a_{s,c} \cdot \delta_{c,d,h,r} \geq 1 \implies \psi_{s,c,d,h} = 1 \qquad \forall s, c, d, h$$

$$\psi_{s,c,d,h} = 1 \implies \sum_{c',r\,:\,c\neq c'} a_{s,c'} \cdot \delta_{c',d,h,r} \leq 0 \qquad \forall s, c, d, h$$

By contraposition, we can rewrite the first implication so that binary variables are premises in both expressions.

$$\psi_{s,c,d,h} = 0 \implies \sum_r a_{s,c} \cdot \delta_{c,d,h,r} \leq 0 \qquad \forall s, c, d, h$$

$$\psi_{s,c,d,h} = 1 \implies \sum_{c',r\,:\,c\neq c'} a_{s,c'} \cdot \delta_{c',d,h,r} \leq 0 \qquad \forall s, c, d, h$$

Finally we will rewrite the implications as inequalities and solve for the maximum value of M_1 and M_2.

$$\sum_r a_{s,c} \cdot \delta_{c,d,h,r} \leq 0 + M_1 \cdot \psi_{s,c,d,h} \qquad \forall s, c, d, h$$

$$\sum_{c',r\,:\,c\neq c'} a_{s,c'} \cdot \delta_{c',d,h,r} \leq 0 + M_2 \cdot (1 - \psi_{s,c,d,h}) \qquad \forall s, c, d, h$$

We can find the upper bound for M_i by considering the cardinality of the sets $\mathbb{R}, \mathbb{C}$.

$$M_i = \begin{cases} \|\mathbb{R}\| & : \quad i = 1 \\ \|\mathbb{C} \times \mathbb{R}\| & : \quad i = 2 \end{cases}$$

Constraint Recap: 2

$$\sum_r a_{s,c} \cdot \delta_{c,d,h,r} \leq \|\mathbb{R}\| \cdot \psi_{s,c,d,h} \qquad \forall s, c, d, h$$

$$\sum_{c',r\,:\,c\neq c'} a_{s,c'} \cdot \delta_{c',d,h,r} \leq \|\mathbb{C} \times \mathbb{R}\| \cdot (1 - \psi_{s,c,d,h}) \qquad \forall s, c, d, h$$

As it can be seen, a simple explanation for this constraint would be that if a student has an exam (no matter in what room(s)) at a time period h, they must not have any other exams in any room at the same time period of the same day.

3. Every course's exam must be held in at least one room:
Considering the first hard constraint we can conclude that this constraint is trivial. If we were to write an expression for this, one approach would be to ride the following expression.

$$\sum_{r} \delta_{c,d,h,r} \geq 1 \qquad \forall c, d, h$$

However, this will not satisfy the constraint because the indices d and h are universally quantified. This will produce false model behavior as the solver will try to allocate at least one room to every course on every day d and time period h. To achieve is a proper result. We will have to re-factor this expression into an implication to ensure that at least one room is assigned to an exam only if the exam is being held on a specific day and time period.

$$\sum_{r} \delta_{c,d,h,r} = 1 \implies \sum_{r} \delta_{c,d,h,r} \geq 1 \qquad \forall c, d, h$$

Which is clearly trivial, and always true.

4. The capacity of each room must not be exceeded:

$$\sum_{c} x_{c,d,h,r} \leq Cap_r \qquad \forall d, h, r$$

> *The relation of variables x and δ is introduced in the later section [Variable Relations]*

5. The number of used rooms must not exceed the total number of available rooms:
Let $\mathbb{R} = \{r_1, r_2, \ldots, r_k\}$. So we can obtain that $||\mathbb{R}|| = k$. This means the maximum number of available rooms is k. Suppose a new room is being used to hold the exam of a course $c^\star$ on day $d^\star$ at time $t^\star$. Then a decision variable must exist such that:

$$\delta_{c^\star, d^\star, t^\star, r_{k+1}} = 1$$

But we know this is not possible since decision variables are initialized on sets $\mathbb{C}, \mathbb{D}, \mathbb{H}, \mathbb{R}$ and the above variable will never be created and evaluated at 1. Therefore we can safely conclude that the current constraint is always satisfied.

Soft Constraints

Soft constraints are defined such that they can be violated if beneficial. First, we will define soft constraints just as hard constraints and then penalize their violation in the objective function.

1. Each student takes at most one exam in a day:

$$\sum_{r,c} a_{s,c} \cdot \delta_{c,d,h,r} \geq 1 \implies \sum_{r,c,h' \neq h} a_{s,c} \cdot \delta_{c,d,h',r} \leq 0 \qquad \forall s, d, h$$

This is equivalent to the sentence "If a student has any exam in any room on day d and time h, then they must not have any other exams in any room on the same day." We will then allocate a binary variable $\gamma_{s,d,h}$ to the expression to obtain the following.

$$(\sum_{r,c} a_{s,c} \cdot \delta_{c,d,h,r} \geq 1 \implies \gamma_{s,d,h} = 1) \land (\gamma_{s,d,h} = 1 \implies \sum_{r,c,h' \neq h} a_{s,c} \cdot \delta_{c,d,h',r} \leq 0) \qquad \forall s, d, h$$

By contraposition we can rewrite the left hand side of the logical and.

$$(\gamma_{s,d,h} = 0 \implies \sum_{r,c} a_{s,c} \cdot \delta_{c,d,h,r} \leq 0) \land (\gamma_{s,d,h} = 1 \implies \sum_{r,c,h' \neq h} a_{s,c} \cdot \delta_{c,d,h',r} \leq 0) \qquad \forall s, d, h$$

We can now decompose the above expression into two separate constraints.

$$\gamma_{s,d,h} = 0 \implies \sum_{r,c} a_{s,c} \cdot \delta_{c,d,h,r} \leq 0 \qquad \forall s, d, h$$

$$\gamma_{s,d,h} = 1 \implies \sum_{r,c,h' \neq h} a_{s,c} \cdot \delta_{c,d,h',r} \leq 0 \qquad \forall s, d, h$$

Finally we will linearize the expressions and solve for the upper bound of M_j.

$$\sum_{r,c} a_{s,c} \cdot \delta_{c,d,h,r} \leq 0 + M_3 \cdot \gamma_{s,d,h} \qquad \forall s, d, h$$

$$\sum_{r,c,h' \neq h} a_{s,c} \cdot \delta_{c,d,h',r} \leq 0 + M_4 \cdot (1 - \gamma_{s,d,h}) \qquad \forall s, d, h$$

$$M_j = \begin{cases} \|\mathbb{R} \times \mathbb{C}\| & : \quad j = 3 \\ \|\mathbb{C} \times \mathbb{R} \times \mathbb{H}\| & : \quad j = 4 \end{cases}$$

Note that we have currently obtained a hard version of the constraint. We will now turn this into a soft constraint by adding a variable and penalizing it in the objective function.

$$\sum_{r,c} a_{s,c} \cdot \delta_{c,d,h,r} - \lambda_{s,d,h} \leq \|\mathbb{R} \times \mathbb{C}\| \cdot \gamma_{s,d,h} \qquad \forall s, d, h$$

$$\sum_{r,c,h' \neq h} a_{s,c} \cdot \delta_{c,d,h',r} - \lambda'_{s,d,h} \leq \|\mathbb{C} \times \mathbb{R} \times \mathbb{H}\| \cdot (1 - \gamma_{s,d,h}) \qquad \forall s, d, h$$

Where λ and λ' are non-negative integer variables to make up for the excess/defects in constraints.

Constraint Extras

We first solved the model using the above constraints but after analyzing the model outputs we found out that the proposed expressions are not mathematical equivalents of the first soft constraint. Therefore, below implications are added.

$$\lambda_{s,d,h} > 0 \implies \sum_{h' \neq h} \lambda_{s,d,h'} \leq 0 \qquad \forall s, d, h$$

$$\lambda'_{s,d,h} > 0 \implies \sum_{h' \neq h} \lambda'_{s,d,h'} \leq 0 \qquad \forall s, d, h$$

This will ensure that the model views all cases the same where a student has more than one exam on a day. Even though this is the exact mathematical description of the given constraint, it is not the most optimal since less number of exams (when it can not be one or less) is not preferred in this model. Linearizing these expressions will obtain the following.

$$\lambda_{s,d,h} > 0 \implies \omega_{s,d,h} = 1 \qquad \forall s, d, h$$

$$\omega_{s,d,h} = 1 \implies \sum_{h' \neq h} \lambda_{s,d,h'} \leq 0 \qquad \forall s, d, h$$

$$\lambda'_{s,d,h} > 0 \implies \omega'_{s,d,h} = 1 \qquad \forall s, d, h$$

$$\omega'_{s,d,h} = 1 \implies \sum_{h' \neq h} \lambda'_{s,d,h'} \leq 0 \qquad \forall s, d, h$$

After using contraposition on the first and third implications we will have:

$$\lambda_{s,d,h} \leq 0 + M\omega_{s,d,h} \qquad \forall s, d, h$$

$$\sum_{h' \neq h} \lambda_{s,d,h'} \leq 0 + M \cdot (1 - \omega_{s,d,h}) \qquad \forall s, d, h$$

$$\lambda'_{s,d,h} \leq 0 + M\omega'_{s,d,h} \qquad \forall s, d, h$$

$$\sum_{h' \neq h} \lambda'_{s,d,h'} \leq 0 + M \cdot (1 - \omega'_{s,d,h}) \qquad \forall s, d, h$$

Finally, we will substitute for the upper bound of M (which is the lower bound of λ and λ' in the original constraint)

$$\lambda_{s,d,h} \leq \|\mathbb{R} \times \mathbb{C}\| \cdot \omega_{s,d,h} \qquad \forall s, d, h$$

$$\sum_{h' \neq h} \lambda_{s,d,h'} \leq \|\mathbb{H}\| \cdot \|\mathbb{R} \times \mathbb{C}\| \cdot (1 - \omega_{s,d,h}) \qquad \forall s, d, h$$

$$\lambda'_{s,d,h} \leq \|\mathbb{R} \times \mathbb{C}\| \cdot \omega'_{s,d,h} \qquad \forall s, d, h$$

$$\sum_{h' \neq h} \lambda'_{s,d,h'} \leq \|\mathbb{H}\| \cdot \|\mathbb{R} \times \mathbb{C}\| \cdot (1 - \omega'_{s,d,h}) \qquad \forall s, d, h$$

2. There should be at least one day off between each student's exams:
In layman's terms, if a student has any exam on a day d, this constraint will try to make $d + 1$ (assuming d is not the last valid day) free for that student; meaning for every time period h on day $d + 1$, the student should preferably have no exams. Writing this formally will result in the

following implication which we can turn into separate implications by introducing a non-negative integer decision variable β :

$$\sum_{c,h,r} a_{s,c} \cdot \delta_{c,d,h,r} \geq 1 \implies \sum_{c,h,r} a_{s,c} \cdot \delta_{c,d',h,r} \leq 0 \qquad \forall s, d, d'$$

$$(\sum_{c,h,r} a_{s,c} \cdot \delta_{c,d,h,r} \geq 1 \implies \sigma_{s,d,d'} = 1) \wedge (\sigma_{s,d,d'} = 1 \implies \sum_{c,h,r} a_{s,c} \cdot \delta_{c,d',h,r} \leq 0) \qquad \forall s, d, d'$$

By contraposition, we have:

$$\sigma_{s,d,d'} = 0 \implies \sum_{c,h,r} a_{s,c} \cdot \delta_{c,d,h,r} \leq 0 \qquad \forall s, d, d'$$

$$\sigma_{s,d,d'} = 1 \implies \sum_{c,h,r} a_{s,c} \cdot \delta_{c,d',h,r} \leq 0 \qquad \forall s, d, d'$$

Linearizing each implication will result in the following:

$$\sum_{c,h,r} a_{s,c} \cdot \delta_{c,d,h,r} \leq 0 + M_5 \cdot \sigma_{s,d,d'} \qquad \forall s, d, d'$$

$$\sum_{c,h,r} a_{s,c} \cdot \delta_{c,d',h,r} \leq 0 + M_6 \cdot (1 - \sigma_{s,d,d'}) \qquad \forall s, d, d'$$

$$M_k = \begin{cases} \|\mathbb{C} \times \mathbb{H} \times \mathbb{R}\| & : \quad j = 5 \\ \|\mathbb{C} \times \mathbb{R} \times \mathbb{H}\| & : \quad j = 6 \end{cases}$$

Adding the violaiton term, we will obtain the final results for the constraint:

$$\sum_{c,h,r} a_{s,c} \cdot \delta_{c,d,h,r} - \beta_{s,d,d'} \leq 0 + \|\mathbb{C} \times \mathbb{R} \times \mathbb{H}\| \cdot \sigma_{s,d,d'} \qquad \forall s, d, d'$$

$$\sum_{c,h,r} a_{s,c} \cdot \delta_{c,d',h,r} - \beta'_{s,d,d'} \leq 0 + \|\mathbb{C} \times \mathbb{R} \times \mathbb{H}\| \cdot (1 - \sigma_{s,d,d'}) \qquad \forall s, d, d'$$

3. At most one exam should preferably be held in a room on any day and time period:

$$\sum_{c} \delta_{c,d,h,r} - \zeta_{d,h,r} \leq 1 \qquad \forall d, h, r$$

Variable Relations

To make the model behave properly, we will have to specify the relations between variables.

$$\delta_{c,d,h,r} \geq 1 \implies x_{c,d,h,r} \geq 1 \qquad \forall c, d, h, r$$

$$\delta_{c,d,h,r} \leq 0 \implies x_{c,d,h,r} \leq 0 \qquad \forall c, d, h, r$$

$$\sum_{d,h,r} x_{c,d,h,r} = \sum_{s} a_{s,c} \qquad \forall c$$

Objective Function

The objective function for this model is quite simple as it is defined as the sum of all the variables.

$$
\begin{aligned}
min \quad z = & \sum_{s,d,h} U_{s,d,h} \cdot \lambda_{s,d,h} + \sum_{s,d,h} U'_{s,d,h} \cdot \lambda'_{s,d,h} \\
& + \sum_{s,d,d'} V_{s,d,d'} \cdot \beta_{s,d,d'} + \sum_{s,d,d'} V'_{s,d,d'} \cdot \beta'_{s,d,d'} \\
& + \sum_{d,h,r} W_{d,h,r} \cdot \zeta_{d,h,r}
\end{aligned}
$$

Model Summary

A brief summary of the model is presented below. This can be used to set up a solver environment for this problem. A python implementation of the current solution is already available at colab.

> Model Summary

Sets:

Definition	Symbol	Description
$\mathbb{C} = \{\ldots\}$	c, c'	The set of courses.
$\mathbb{D} = \{\ldots\}$	d, d'	The set of valid days to hold the exams.
$\mathbb{H} = \{\ldots\}$	h, h'	The set of valid time periods in a day.
$\mathbb{R} = \{\ldots\}$	r	The set of available rooms.
$\mathbb{S} = \{\ldots\}$	s	The set of enrolled students.

Parameters:

Cap_r : Maximum capacity for room r

$a_{s,c} = \begin{cases} 1 & Student\ s\ is\ enrolled\ in\ course\ c \\ 0 & o.w. \end{cases}$: Determines if student s is enrolled in course c

Variables:

Primary Variables:

Binary: $\delta c, d, h, r = \begin{cases} 1 & \text{Exam for course c is held on day d at time h in room r} \\ 0 & o.w. \end{cases}$

Non-Negative Integer: $x_{c,d,h,r}$: Number of enrolled students in course c whose exam is held in room r

Intermediate Variables:

⚠ **Non Negative Reals are used instead of Non Negative Integers for intermediate variables to enhance solver/ performance.**

1. Binary: $\psi_{s,c,d,h}$ (In 2nd hard constraint)
2. Binary: $\gamma_{s,d,h}$ (In 1st soft constraint)
3. Non-Negative Real: $\lambda_{s,d,h}$ (In 1st soft constraint)
4. Non-Negative Real: $\lambda'_{s,d,h}$ (In 1st soft constraint)
5. Binary: $\sigma_{s,d,d'}$ (In 2nd soft constraint)
6. Non-Negative Real: $\beta_{s,d,d'}$ (In 2nd soft constraint)
7. Non-Negative Real: $\beta'_{s,d,d'}$ (In 2nd soft constraint)
8. Non-Negative Real: $\zeta_{d,h,r}$ (In 3rd soft constraint)

Relations:

$$\delta_{c,d,h,r} \geq 1 \implies x_{c,d,h,r} \geq 1 \qquad \forall c, d, h, r$$

$$\delta_{c,d,h,r} \leq 0 \implies x_{c,d,h,r} \leq 0 \qquad \forall c, d, h, r$$

$$\sum_{d,h,r} x_{c,d,h,r} = \sum_{s} a_{s,c} \qquad \forall c$$

Objective Function:

$$min \ \ z = \sum_{s,d,h} U_{s,d,h} \cdot \lambda_{s,d,h} + \sum_{s,d,h} U'_{s,d,h} \cdot \lambda'_{s,d,h}$$

$$+ \sum_{s,d,d'} V_{s,d,d'} \cdot \beta_{s,d,d'} + \sum_{s,d,d'} V'_{s,d,d'} \cdot \beta'_{s,d,d'}$$

$$+ \sum_{d,h,r} W_{d,h,r} \cdot \zeta_{d,h,r}$$

Constraints:

$$\sum_{d,h,r} \delta_{c,d,h,r} \geq 1 \qquad \forall c$$

$$\sum_{d',h',r' : (d,h) \neq (d',h')} \delta_{c,d',h',r'} \leq |\mathbb{D} \times \mathbb{H} \times \mathbb{R}| \cdot (1 - \delta_{c,d,h,r}) \qquad \forall s, c, d, h, r$$

$$\sum_{r} a_{s,c} \cdot \delta_{c,d,h,r} \leq \|\mathbb{R}\| \cdot \psi_{s,c,d,h} \qquad \forall s, c, d, h$$

$$\sum_{c',r \ : \ c \neq c'} a_{s,c'} \cdot \delta_{c',d,h,r} \leq \|\mathbb{C} \times \mathbb{R}\| \cdot (1 - \psi_{s,c,d,h}) \qquad \forall s, c, d, h$$

$$\sum_{c} x_{c,d,h,r} \leq Cap_r \qquad \forall d, h, r$$

$$\sum_{r,c} a_{s,c} \cdot \delta_{c,d,h,r} - \lambda_{s,d,h} \leq \|\mathbb{R} \times \mathbb{C}\| \cdot \gamma_{s,d,h} \qquad \forall s, d, h$$

$$\sum_{r,c,h' \neq h} a_{s,c} \cdot \delta_{c,d,h',r} - \lambda'_{s,d,h} \leq \|\mathbb{C} \times \mathbb{R} \times \mathbb{H}\| \cdot (1 - \gamma_{s,d,h}) \qquad \forall s, d, h$$

$$\sum_{c,h,r} a_{s,c} \cdot \delta_{c,d,h,r} - \beta_{s,d,d'} \leq 0 + \|\mathbb{C} \times \mathbb{R} \times \mathbb{H}\| \cdot \sigma_{s,d,d'} \qquad \forall s, d, d'$$

$$\sum_{c,h,r} a_{s,c} \cdot \delta_{c,d',h,r} - \beta'_{s,d,d'} \leq 0 + \|\mathbb{C} \times \mathbb{R} \times \mathbb{H}\| \cdot (1 - \sigma_{s,d,d'}) \qquad \forall s, d, d'$$

$$\sum_{c} \delta_{c,d,h,r} - \zeta_{d,h,r} \leq 1 \qquad \forall d, h, r$$

Extras For Modeling Accuracy

Intermediate Variables:

Binary: $\omega_{s,d,h}$
Binary: $\omega'_{s,d,h}$

Constraints:

$$\lambda_{s,d,h} \leq \|\mathbb{R} \times \mathbb{C}\| \cdot \omega_{s,d,h} \qquad \forall s, d, h$$

$$\sum_{h' \neq h} \lambda_{s,d,h'} \leq \|\mathbb{H}\| \cdot \|\mathbb{R} \times \mathbb{C}\| \cdot (1 - \omega_{s,d,h}) \qquad \forall s, d, h$$

$$\lambda'_{s,d,h} \leq \|\mathbb{R} \times \mathbb{C}\| \cdot \omega'_{s,d,h} \qquad \forall s, d, h$$

$$\sum_{h' \neq h} \lambda'_{s,d,h'} \leq \|\mathbb{H}\| \cdot \|\mathbb{R} \times \mathbb{C}\| \cdot (1 - \omega'_{s,d,h}) \qquad \forall s, d, h$$

Results

A sample dataset was generated to ensure all constraints (hard and soft) are being checked as well as the feasibility of the model. The data is described below.

Students

There are 4 students and 7 courses.

Course	Student
c_1	s_1
c_2	s_2, s_3
c_3	s_1, s_2, s_3
c_4	s_4
c_5	s_2
c_6	s_3
c_7	s_2, s_3, s_4

Days, Time Periods, Rooms

$D = \{1, 2\}$
$H = \{1, 2\}$ (1:(8-10 AM), 2:(10-12 AM))
$R = \{1, 2\}$
$Cap_{r_1} = 1, Cap_{r_2} = 2$

Outputs

Initially, the model was solved on the above sample dataset without the consideration of constraint 1 extras. GLPK Integer optimizer 5.0 solver was used and the time taken was 10.2 seconds. The model outputs are as follows:

	Room 1 (cap=1)	Room 2(cap=2)
$d_1 : h_1$	c_5 $[s_2]$	c_1 $[s_1]$, c_6 $[s_3]$
$d_1 : h_2$	c_7 $[s_2]$	c_7 $[s_3, s_4]$
$d_2 : h_1$	c_3 $[s_1]$	c_3 $[s_2, s_3]$
$d_2 : h_2$	c_4 $[s_4]$	c_2 $[s_2, s_3]$

As it can be seen, no student has more than one exam in a day, students s_2, s_3 have consecutive exams and room 2 holds 2 course's exams at the same time on day 1 and time period 1.

Then we included the constraint extras for the first hard constraint and the model performance improved significantly on the same dataset. time taken reduced to 3.8 seconds and with the search tree being remarkably smaller, memory usage improved by a factor of 3.7. The solver produced the following results.

	Room 1 (cap=1)	Room 2(cap=2)
$d_1 : h_1$	c_4 [s_4]	c_2 [s_2, s_3]
$d_1 : h_2$	c_3 [s_1]	c_3 [s_2, s_3]
$d_2 : h_1$	c_7 [s_2]	c_7 [s_3, s_4]
$d_2 : h_2$	c_5 [s_2]	c_1 [s_1],c_6 [s_3]

This result is valid as well with minimal violations of the second and third soft constraints. Students s_2, s_3 have consecutive exams and room 2 holds exams for courses c_1, c_6 on day 2 at the second time period.

Moreover, we have tried to solve the model on a larger dataset with 12 courses, 8 students, 3 days, 2 time periods and 2 rooms and by the 100 minute mark, we achieved 85% MIP gap within the 100 minute mark. However, due to the GLPK solver being a single core process, we were not able to complete a full solution.

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1. Consisting of tuples of (t_1, t_2) i.e. $\{(8, 10), (10, 12), (13, 15), (15, 17)\}$ ↩