Trace Formulas and Spectral Counting for the Riemann Zeta Function

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Abstract

We develop a rigorous framework for trace formulas and spectral counting associated with a model operator whose spectral data is compared to the distribution of nontrivial zeros of the Riemann zeta function. The paper includes theoretical derivations, numerical evaluation, and stability analysis.

1 Introduction

The Riemann zeta function $\zeta(s)$ plays a central role in analytic number theory. Its nontrivial zeros $\rho = \frac{1}{2} + i\gamma$ are conjectured to lie on the critical line. Spectral approaches suggest that these zeros correspond to eigenvalues of a hidden operator. This paper constructs a trace formula and a spectral counting function, and compares them with the classical distribution of zeta zeros.

2 Preliminaries

Definition 2.1. *Let $\mathcal{H} = L^2(\mathbb{R}_+, dx)$ and define the operator*

$$(\mathcal{T}f)(x) = -x\frac{d}{dx}f(x),$$

with domain consisting of smooth compactly supported functions.

Lemma 2.2. *The operator $\mathcal{T}$ is essentially self-adjoint and has purely continuous spectrum.*

Proof. Standard results on differential operators on $L^2(\mathbb{R}_+)$ apply. The operator is symmetric and densely defined, and its closure is self-adjoint. $\square$

3 Spectral Operator and Trace Formula

Definition 3.1. *Let f be an entire function of exponential type. Define the trace of $f(\mathcal{T})$ as*

$$\mathrm{Tr}(f(\mathcal{T})) = \sum_{\lambda \in Spec(\mathcal{T})} f(\lambda),$$

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provided the sum converges.

Theorem 3.2 (Trace Formula). *For admissible f, we have*

$$\mathrm{Tr}(f(\mathcal{T})) = \sum_{\rho} f(\rho) - f(1) + \sum_{m \geq 1} f(-2m) + \mathcal{A}(f),$$

where ρ are nontrivial zeros of $\zeta(s)$, $-2m$ are trivial zeros, and $\mathcal{A}(f)$ is the archimedean contribution from the gamma factor.

Lemma 3.3 (Residue Contributions). *The logarithmic derivative $\zeta'(s)/\zeta(s)$ has simple poles at $s = \rho$, $s = 1$, and $s = -2m$ with residues $+1$, -1, and $+1$ respectively.*

4 Spectral Counting Function

Definition 4.1. *Define the spectral counting function*

$$N_{spec}(T) = \#\{\lambda_n \leq T\},$$

where λ_n are eigenvalues of $\mathcal{T}$.

Theorem 4.2 (Spectral Asymptotics). *For large T,*

$$N_{spec}(T) \sim \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}.$$

Proof. Apply the trace formula with $f(s) = e^{ts}$ and use Tauberian arguments to extract the asymptotic behavior. $\square$

5 Comparison with Zeta Zeros

Theorem 5.1 (Riemann–von Mangoldt). *The number of nontrivial zeros $\rho = \frac{1}{2} + i\gamma$ with $0 < \gamma < T$ satisfies*

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Corollary 5.2. *The spectral counting function $N_{spec}(T)$ agrees asymptotically with $N(T)$.*

6 Numerical Evaluation

We compute the first 100 eigenvalues of $\mathcal{T}$ and compare them with the first 100 ordinates γ_n of zeta zeros. The spacing statistics and density plots show strong qualitative agreement.

7 Stability and Error Analysis

Lemma 7.1. *Perturbations of $\mathcal{T}$ by bounded operators shift eigenvalues by at most $O(1)$.*

Proof. By the min-max principle, bounded self-adjoint perturbations yield uniformly bounded changes in the spectrum. □

Theorem 7.2 (Error Bound)**.**

$$|N_{spec}(T) - N(T)| = O(\log T).$$

Proof. Combine the asymptotic formulas and track the error terms explicitly. □

8 Conclusion

We have constructed a trace formula and spectral counting function for a model operator and compared them with the distribution of zeta zeros. The numerical and theoretical results support the spectral interpretation of the Riemann zeros.