Discrete Math

Staff graded Assignment (1)

2023ebcs072

1) Take $p = \frac{m+n}{2}$

$$|m-p| = \left|m - \frac{m+n}{2}\right|$$
$$= \left|\frac{m-n}{2}\right|$$

$$|n-p| = \left|n - \frac{m+n}{2}\right|$$
$$= \left|\frac{n-m}{2}\right|$$
$$= \left|\frac{m-n}{2}\right| \quad \boxed{\because |-a| = |a|}$$

$\therefore |m-p| = |n-p| \mid p = \frac{m+n}{2}$ .... ①

Since m is odd, it can be written as $m = 2i+1$, i is an integer

Similarly, $n = 2j+1$ as n is odd, j is an integer

$\frac{m+n}{2} = \frac{2i+1+2j+1}{2} = \frac{2(i+j)+2}{2} = (i+j)+1$, which is also an integer.

$\therefore p$ is an integer $\mid p = \frac{m+n}{2}$ ..... ②

From ① and ②, $p = \frac{m+n}{2}$ is an integer, that satisfies $|m-p| = |n-p|$

So, $\exists p, p \in \mathbb{Z}, |m-p| = |n-p| \mid m$ and n are odd integers

② <u>Basis</u> <u>step</u>:

$$1^2 = \frac{n(n+1)(2n+1)}{6} \Big|_{n=1}$$

$$\frac{n(n+1)(2n+1)}{6}\Big|_{n=1} = \frac{1(2)(2+1)}{6} = \frac{2\times 3}{6} = 1$$

So the statement holds for $n=1$

<u>Inductive</u> <u>step</u>: Assume $1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$ is true for $k \in N$

$$1^2+2^2+\dots+k^2+(k+1)^2 = \frac{1}{6}\left[(k)(k+1)(2k+1)\right] + (k+1)^2$$

$$= \frac{1}{6}\left[k(k+1)(2k+1)\right] + \frac{6(k+1)^2}{6}$$

$$= \frac{1}{6}\left[(2k^3+3k^2+k)+6(k^2+2k+1)\right]$$

$$= \frac{1}{6}\left[2k^3+9k^2+13k+6\right]$$

$$= \frac{1}{6}\left[(k+1)(2k^2+7k+6)\right]$$

$$= \frac{1}{6}\left[(k+1)(k+2)(2k+3)\right]$$

$$= \frac{1}{6}\left[(k+1)((k+1)+1)(2(k+1)+1)\right]$$

$$= \frac{n(n+1)(2n+1)}{6}\Big|_{n=k+1}$$

$\Rightarrow 1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$ holds for $n=k+1$ given it is true for $n=k \,|\, k \in N$ and $n=1$

<u>Hence Proved</u>

3 · Let's consider the two sets

- Set of whole numbers, $N + \{0\}$, or $\{x-1 \mid x \in N\}$ ... a

and

- Set of non-positive integers, $Z - N$ or $\{1-x \mid x \in N\}$ ... b

Both these sets are countably infinite, as I defined them with a bijective function from N.

a is $\{0, 1, \dots\}$, b is $\{0, -1, -2, \dots\}$

clearly, $a \cap b$ is $\{0\}$, which is a finite set

- Let's consider the two sets

- N ... a
- $N + \{0\}$, the set of whole numbers or $\{x-1 \mid x \in N\}$

a is countably infinite since it is N

b is defined using a bijective function from N

a is $\{1, 2, \dots\}$, b is $\{0, 1, 2, \dots\}$

Clearly, $a \cap b$ is N, or $\{1, 2, \dots\}$

which is ~~also~~ countably infinite

4) - Let $x = y+1$ or $x = y-1$ be R

For it to be reflexive, (x,x) should be satisfied $\mid \forall x \in R$

$x = x+1 \cdots ①$, $x = x-1 \cdots ②$

Neither ① nor ② can be satisfied, so R is $\boxed{\text{not reflexive}}$

- For it to be symmetric,

$$\underbrace{(x = y+1) \vee (x = y-1)}_{①} \longrightarrow \underbrace{\underset{(x = y-1)}{(y = x+1)} \vee \underset{(x = y+1)}{(y = x-1)}}_{②}$$

Since there is a pair of trues with an element in ① and ②, the compound proposition is true,

So R is $\boxed{\text{Symmetric}}$

- For a relation to be both symmetric and antisymmetric, R should only consist of all the reflexive pairs,

but since, R is symmetric and not reflexive, R is $\boxed{\text{not antisymmetric}}$

- Let $\overset{①}{x = y+1}$, now, $\overset{②}{y = z+1}$, or $\overset{③}{y = z-1}$

Substituting ② in ① $x = z+1+1$, $x = z+2$

③ in ① $x = z-1+1$, $x = z$

neither are true

now, let $\underset{④}{x = y-1}$, $\underset{⑤}{y = z+1}$ or $\underset{⑥}{y = z-1}$

Substituting ⑤ in ④ $x = z+1-1$, $x = z$

⑥ ④ $x = z-1-1$, $x = z-2$

neither are true

So $(x,y) \wedge (y,z) \rightarrow (x,z)$ is always false

So R is $\boxed{\text{not transitive}}$

5) - one to one but not onto

$f: Z \to N \mid \begin{cases} 2x+1 & \forall x \in N \\ 2(1-x) & \forall x \in Z-N \end{cases}$

The range is $\{2, 3, 4, 5, 6, \cdot\}$

for the inputs $\{0, 1, -1, 2, -2, \dots\}$

The range doesn't include $\{1\}$, so it is not onto

- Onto, but not one-one

$f: Z \to N \mid \begin{cases} x \mid x \in N \\ 1-x \mid x \in Z-N \end{cases}$

Since for all $x \in N$, $f(x) = x$, so the range covers N, so it is onto.

For f to not be one-one, $\exists [f(a) = f(b) \mid a \neq b]$

taking $x = 1$, $f(x) = x = 1$

$x = 0$, $f(x) = 1 - x = 1$

So f is not one-to-one