Assignment 2: Probability, Bayes Rule, and Bayes Filtering

Course: CS128 - Introduction to Robotics (Spring 2025)
Instructor: Md Alimoor Reza, Assistant Professor of Computer Science, Drake University
Due: Monday, April 07, 11:59 PM

In this assignment, you will be doing probabilistic calculations, solving problems with the Bayes rule, and predicting the belief probability of a state with Bayes Filtering.

Part 1: Discrete Probability Distribution (6 points)

As we have seen during the lecture, the discrete probability distribution is a function that will depend on a random variable eg, X that takes discrete values. $P(X)$ is a function that maps from all possible values of X to the probability of the corresponding event. For example, X is a random variable that can take one of the 2 values from $\{true, false\}$. For another example, Y is a random variable that can take one of the 6 values from $\{red, blue, green, cyan, white, black\}$. The joint probability distribution expresses the probability distribution of observing varied instances of (x, y) where some paired outcome occurs more frequently than others. Let's assume X is a random variable with three discrete values from the set $\{a, b, c\}$. Also assume that Y is a random variable with three discrete values from the set $\{\alpha, \beta, \gamma\}$.

	$X = a$	$X = b$	$X = c$
$Y = \alpha$	0.20	0.05	0.05
$Y = \beta$	0.05	0.30	0.05
$Y = \gamma$	0.05	0.05	0.20

Marginal Distribution.

The conditional probability distribution expresses the relative propensity. Conditional probability distribution of any single random variable X conditioned on another variable Y value fixed to a particular value using the following equation:

$$P(X) = \sum_{y} P(X, Y = y)$$

Find the marginal distribution $P(X = a)$, $P(X = b)$, and $P(X = c)$.

Conditional Distribution.

The marginal probability distribution expresses the probability distribution of any single random variable from a joint probability distribution by summing over all other variables as follows:

$$\begin{aligned} P(X = a|Y = \alpha) &= \frac{P(X = a, Y = \alpha)}{P(Y = \alpha)} \\ &= \frac{P(X = a, Y = \alpha)}{P(Y = \alpha, X = a) + P(Y = \alpha, X = b) + P(Y = \alpha, X = c)} \end{aligned}$$

Find the conditional distribution $P(X = a|Y = \alpha)$, $P(X = b|Y = \alpha)$, and $P(X = c|Y = \alpha)$.

Part 2: Bayes' Rule (4 points)

In this part, you are required to compute the posterior probability using Bayes' rule, which is stated as follows:

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

Coin Tossing.

Consider a scenario where two coins are available. One is a fair coin, meaning it has one head and one tail, with an equal probability of landing on either side when flipped. The second coin is biased, exhibiting a tendency to land on heads 90% of the time. I choose a coin at random, flip it, and get a head. What is the probability that it is **the biased one?**

Disease Prediction.

A doctor says that Professor X has **Bovid-19** (an illness) that afflicts 0.01% of the population. Her diagnoses are right 99% of the time. What's the probability that Professor X has **Bovid-19**? To obtain full credit, apply Bayes' rule to derive the marginal distribution and, ultimately, the posterior probability equation.

Part 3: Probabilistic Inference with Bayes Filtering (8 points)

In this part, you are required to solve a probabilistic state estimation task using the Bayes Filtering Algorithm. In this problem, a mobile robot wants to estimate the state of a door using a camera sensor and manipulators, as shown in Figure 1 below.

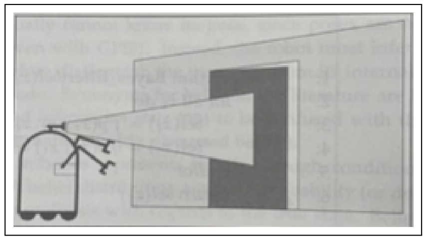

Figure 1: State of the door estimation for a mobile robot.

To make this problem simple, let's assume that the door can be in one of the two states $\{is\_open, is\_closed\}$, and only the robot can change the state of the door. Also, assume that the robot does not know the state of the door initially. Instead, it assigns equal prior probability (*ie, 0.5 each*) to the two possible door states: $bel(X_0 = is\_open) = 0.5, bel(X_0 = is\_closed) = 0.5$. As shown during the lecture, the probabilistic graphical model used in the Bayes Filtering algorithm is a Hidden Markov Model (HMM), as shown in the Figure below.

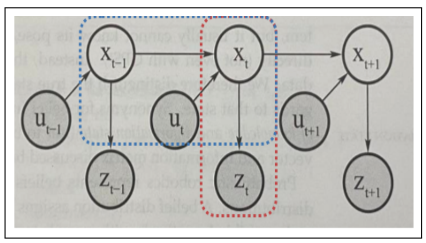

Figure 2: The measurement probability and state transition probability in HMM.

Let's assume that the camera sensor of the robot is noisy. The noise is characterized by the following measurement probabilities as shown in Table 1.

Measurement Probability	Value
$\mathrm{p}(\mathrm{Z}_t = sense\_open \mid X_t = is\_open)$	0.6
$\mathrm{p}(\mathrm{Z}_t = sense\_closed \mid X_t = is\_open)$	0.4
$\mathrm{p}(\mathrm{Z}_t = sense\_open \mid X_t = is\_closed)$	0.2
$\mathrm{p}(\mathrm{Z}_t = sense\_closed \mid X_t = is\_closed)$	0.8

Finally, let's assume that the robot uses its manipulator to push the door open (*ie, as shown in Figure 1*). The manipulator can take two values $\{push, do\_nothing\}$. The state transition probabilities are given as follows:

State Transition Probability	Value
$\mathrm{p}(\mathrm{X}_t = is\_open \mid X_{t-1} = is\_open, U_t = push)$	1
$\mathrm{p}(\mathrm{X}_t = is\_closed \mid X_{t-1} = is\_open, U_t = push)$	0
$\mathrm{p}(\mathrm{X}_t = is\_open \mid X_{t-1} = is\_closed, U_t = push)$	0.8
$\mathrm{p}(\mathrm{X}_t = is\_closed \mid X_{t-1} = is\_closed, U_t = push)$	0.2
$\mathrm{p}(\mathrm{X}_t = is\_open \mid X_{t-1} = is\_open, U_t = do\_nothing)$	1
$\mathrm{p}(\mathrm{X}_t = is\_closed \mid X_{t-1} = is\_open, U_t = do\_nothing)$	0
$\mathrm{p}(\mathrm{X}_t = is\_open \mid X_{t-1} = is\_closed, U_t = do\_nothing)$	0
$\mathrm{p}(\mathrm{X}_t = is\_closed \mid X_{t-1} = is\_closed, U_t = do\_nothing)$	1

Task#1 (4 points):
Suppose at time $t = 1$, the robot takes no control action but senses an open door. The resulting posterior belief can be calculated by the Bayes Filtering using the prior belief $bel(X_0 = is\_open) = 0.5, bel(X_0 = is\_closed) = 0.5$, the control $U_1 = do\_nothing$, and measurement $Z_1 = sense\_open$ as input.

- Calculate the resulting posterior belief $bel(X_1)$ is calculated by Bayes Filter Algorithm. i.e., the value of $bel(X_1 = is\_open) =?, bel(X_1 = is\_closed) =?$. As demonstrated in the lecture, the slides provide a structured approach to computing values using Bayes Filtering. It is recommended that you utilize the provided notebook to perform your calculations.

- According to its posterior belief, what does the robot infer about whether the **door remains open or closes**?

Task#2 (4 points):
Once you find the values of the above two quantities, you can follow the Bayes Filtering for the next step $t = 2$. For this, you can assume that the control $U_2 = push$, and measurement $Z_2 = sense\_open$ as input.

- Calculate the resulting posterior belief $bel(X_2)$ is calculated by Bayes Filter Algorithm. i.e., the value of $bel(X_2 = is\_open) =?, bel(X_2 = is\_closed) =?$. It is recommended that you utilize the provided notebook to perform your calculations.

- According to its posterior belief, what does the robot infer about whether the **door remains open or closes**?

What to turn in:

Please submit a zip file containing your document. You could write your solution in either *.docx, *.doc, or *.pdf file. if you prefer to write your answers on paper, you could do so; in that case, please take a photo of your paper or upload a scanned copy of your paper. Again, if you have any questions, please ask or email me.