Quantum Walks and Applications to Quantum Money

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Quantum Computation - Preliminaries and Notation

In this talk, we explore the interplay between quantum computation, group theory, and cryptography through the lens of quantum walks. The presentation is organized as follows:

1. **Quantum Computation Preliminaries**
2. **Quantum Walks**
3. **Quantum Money Schemes**
4. **Verification of our New Quantum Money scheme using Quantum Walks**

Quantum Computation - Preliminaries and Notation

- Consider a finite Hilbert space $\mathcal{H}$ with an orthonormal set of basis states $\{|s_i\rangle\}$ for $s \in \mathcal{S}$. The states $s \in \mathcal{S}$ may be interpreted as the possible classical states of the system described by $\mathcal{H}$.

- In general, the state of the system, $|\alpha\rangle$, is a unit vector in the Hilbert space $\mathcal{H}$ and can be written as $|\alpha\rangle = \sum_{s \in \mathcal{S}} a_s |s\rangle$, where $\sum_{s \in \mathcal{S}} |a_s|^2 = 1$.

- $\langle\alpha|$ denotes the conjugate transpose of $|\alpha\rangle$. The expression $\langle\beta|\alpha\rangle$ denotes the inner product of $|\alpha\rangle$ and $|\beta\rangle$.

Quantum Computation - Quantum Postulates

- **Unitary evolution**: Quantum physics requires that the evolution of quantum states is unitary; that is, the state $|\alpha\rangle$ is mapped to $U|\alpha\rangle$, where U satisfies $U \cdot U^\dagger = I$, and $U^\dagger$ denotes the conjugate transpose of U.

- **Measurement**: We will describe here only a measurement in the orthonormal basis $|s\rangle$. The output of the measurement of the state $|\alpha\rangle$ is an element $s \in \mathcal{S}$, with probability $|\langle s|\alpha\rangle|^2$. Moreover, the new state of the system after the measurement is $|s$.

- **Combining two quantum systems**: If $\mathcal{H}_A$ and $\mathcal{H}_B$ are the Hilbert spaces of two systems, A and B, then the joint system is described by the tensor product of the Hilbert spaces, $\mathcal{H}_A \otimes \mathcal{H}_B$. If the basis states for $\mathcal{H}_A$ and $\mathcal{H}_B$ are $\{|a_i\rangle\}$ and $\{|v_i\rangle\}$ respectively, then the basis states of $\mathcal{H}_A \otimes \mathcal{H}_B$ are $\{|a_i\rangle \otimes |v_i\rangle\}$.

Quantum Walks

- Quantum walks are quantum analogs of classical random walks and play a fundamental role in quantum algorithms
- Two types: continuous-time and discrete-time
- Quantum walks leverage interference to explore graphs more efficiently than classical walks
- For a graph Γ, a continuous-time classical walk on Γ is:

$$\frac{d}{dt}q(t) = Lq(t)$$

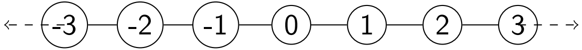
- In the quantum setting, the dynamics of the walk is given by the Schrödinger equation:

$$i\frac{d}{dt}\left|\psi(t)\right\rangle = L\left|\psi(t)\right\rangle$$

Continuous-Time Quantum Walks

- The solution to this differential equation can be written in closed form as:
$$|\psi(t)\rangle = e^{-iLt} |\psi(0)\rangle .$$
- In practice, we often (including this work) use the adjacency matrix A of Γ as the Hamiltonian of the walk

Example of a Continuous Quantum Walk



- **Graph:** Infinite line with vertices labeled by integers $n \in \mathbb{Z}$.
- **Adjacency Matrix:** Each vertex is connected to its neighbors $n \pm 1$.

$$A_{i,j} = \begin{cases} 1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

- **Hilbert Space:** Spanned by basis states $|n\rangle$, representing positions.
- **Time Evolution:** Governed by Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle = A|\psi(t)\rangle \quad \Rightarrow \quad |\psi(t)\rangle = e^{-iAt}|\psi(0)\rangle$$

Example of a Continuous Quantum Walk

- **Initial State:** Particle starts at the origin:

$$|\psi(0)\rangle = |0\rangle$$

- **Evolved State:**

$$|\psi(t)\rangle = \sum_{n\in\mathbb{Z}} \psi_n(t)|n\rangle \quad \text{where } \psi_n(t) = i^n J_n(2t)$$

- **Key Features:**
  - Probability: $|\psi_n(t)|^2$
  - Wave-like, oscillatory distribution due to interference
  - Faster spread than classical walk: standard deviation grows linearly in t
  - No coin space needed — evolution depends only on graph structure

Discrete-Time Quantum Walks

- If the Γ has N vertices, the discrete time quantum walk on Γ is defined by a unitary operator on the finite Hilbert space $\mathbb{C}^N \times \mathbb{C}^N$ as follows:
- Define the states:

$$|\phi_j\rangle = \frac{1}{\sqrt{\deg(j)}} \sum_{k=1}^{N} \sqrt{P_{jk}}\, |j,k\rangle\,,$$

- project and swap operators:

$$\Pi = \sum_{j=1}^{N} |\phi_j\rangle \langle\phi_j|\,, \quad S = \sum_{j,k=1}^{N} |j,k\rangle \langle k,j|\,.$$

- Then, a step of the quantum walk is defined by the unitary:

$$W = S(2\Pi - 1)$$

Example of a Discrete-Time Quantum Walk

- **Hilbert Space:** $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_P$
  - $\mathcal{H}_C$: 2D coin space with basis $\{|0\rangle, |1\rangle\}$
  - $\mathcal{H}_P$: Infinite-dimensional position space with basis $\{|n\rangle : n \in \mathbb{Z}\}$
- **Coin Operator:** Apply a unitary C (e.g., Hadamard) to $\mathcal{H}_C$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- **Shift Operator:** Moves the walker based on coin state:

$$S|0\rangle|n\rangle = |0\rangle|n+1\rangle, \quad S|1\rangle|n\rangle = |1\rangle|n-1\rangle$$

- **One Step:** Apply the unitary operator

$$U = S \cdot (C \otimes I)$$

Example of a Discrete-Time Quantum Walk

- **Initial State:** $|\psi(0)\rangle = |0\rangle \otimes |0\rangle$
- **After One Step:**

$$|\psi(1)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle + |1\rangle|-1\rangle\right)$$

- **Key Properties:**
  - Superposition leads to **parallel exploration** of paths.
  - Repeated application of U creates **interference patterns**.
  - Spread is faster than classical: $\sigma(t) \sim t$ vs. $\sqrt{t}$.
- **Measurement:**
  - Measuring after each step yields a classical random walk.
  - **Quantum behavior requires delaying measurement.**

Continuous vs Discrete Quantum Walks

- **CTQW**: Easier to analyze due to direct spectral decomposition.
- **CTQW**: Harder to implement on quantum circuits (requires simulating e^{-iAt}).
- **DTQW**: More complex to analyze (involves coin and shift operators).
- **DTQW**: Easier to implement on gate-based quantum hardware.

Quantum Walks on Cayley Graphs

Cayley Graphs:

Let G be an abelian group and let $Q = \{q_1, q_2, \ldots, q_k\} \subset G$ be a symmetric set, i.e., $q \in Q$ if and only if $-q \in Q$. The Cayley graph associated to G and Q is a graph $\Gamma = (V, E)$, where the vertex set is $V = G$, and the edge set E consists of pairs $(a, b) \in G \times G$ such that there exists $q \in Q$ with $b = q + a$.

An Example of a Cayley Graph

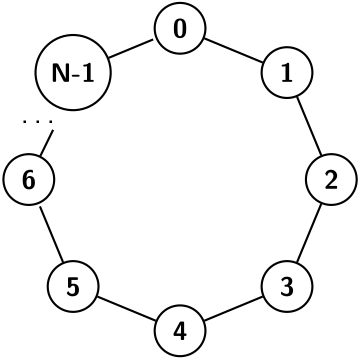


Figure: Cayley graph of $\mathbb{Z}_N$ with generators $\{\pm 1\}$

Quantum Walks on Cayley Graphs

Cayley Graphs:

The adjacency matrix of the Cayley graph $\Gamma = (V, E)$ can be expressed as:

$$A = \sum_{a \in G} \lambda_a \left|\hat{a}\right\rangle \left\langle\hat{a}\right| ,$$

where $\left|\hat{a}\right\rangle = \mathrm{QFT_G} \left|a\right\rangle$ is the *Quantum Fourier transform (QFT)* of $\left|a\right\rangle$.

... But, What is QFT??

Quantum Fourier Transform (QFT)

Let G be an abelian group. The set of characters of G, denoted by $\hat{G}$, is the set of homomorphisms $\chi(a, \cdot) : G \to \mathbb{C}$ where $a \in G$. If $G \cong \mathbb{Z}_{N_1} \oplus \cdots \oplus \mathbb{Z}_{N_k}$ then the character $\chi(a, \cdot)$ can be explicitly written as

$$\chi(a, x) = \omega_{N_1}^{a_1 x_1} \cdots \omega_{N_k}^{a_k x_k}$$

where $\omega_M = \exp(2\pi i / M)$ is a primitive M-th root of unity. The Fourier transform of a function $f : G \to \mathbb{C}$ is given by

$$\hat{f}(a) = \frac{1}{\sqrt{|G|}} \sum_{x \in G} \chi(a, x) f(x).$$

The quantum Fourier transform:

$$\sum_{g \in G} f(g) |g\rangle \mapsto \sum_{x \in G} \hat{f}(x) |x\rangle$$

Quantum Walks on Cayley Graphs

Cayley Graphs:

The adjacency matrix of the Cayley graph $\Gamma = (V, E)$ can be expressed as:

$$A = \sum_{a \in G} \lambda_a |\hat{a}\rangle \langle \hat{a}|,$$

Where $|\hat{a}\rangle = \mathrm{QFT}_G(|a\rangle) = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \chi(a, g) |g\rangle$.
The eigenvalues λ_a are given by:

$$\lambda_a = \sum_{q \in Q} \chi(a, q).$$

- Note that the eigenvectors $|\hat{a}\rangle$ of A depend only on G and not on the set Q.

Quantum Walks on Cayley Graphs

Cayley Graphs:

Proof:

$$A\,|\hat{a}\rangle = A.\frac{1}{\sqrt{|G|}}\sum_{y\in G}\chi(a,y)\,|y\rangle = \frac{1}{\sqrt{|G|}}\sum_{y\in G}\chi(a,y).A\,|y\rangle$$

$$= \frac{1}{\sqrt{|G|}}\sum_{y\in G}\chi(a,y).\sum_{q\in Q}|qy\rangle$$

Consider $\beta = qy$. Then:

$$= \frac{1}{\sqrt{|G|}}\sum_{q\in Q}\chi(a,q)\sum_{\beta}\chi(a,\beta)\,|\beta\rangle = \sum_{q\in Q}\chi(a,q).\,|\hat{a}\rangle$$

$$= \lambda_a\,|\hat{a}\rangle$$

Group Actions

Cayley graphs can also be constructed using *group actions*.

Group Actions:

For a group G and a set X, we say that G *acts on* X if there is a mapping $* : G \times X \to X$ that satisfies the following properties:

1. Compatibility: for every $a, b \in G$ and every $x \in X$,
   $g * (h * x) = (gh) * x$,
2. Identity: for the identity $1 \in G$ and every $x \in X$, $1 * x = x$.

- We use the notation $(G, X, *)$ to denote a group G acting on a set X through the action $*$.
- A group action is called *regular* if for every $x, y \in X$ there exists a unique $g \in G$ such that $g * x = y$.

Cayley Graphs with Group Actions

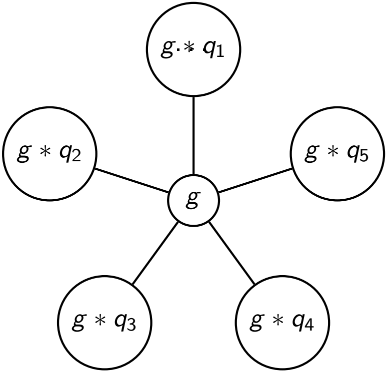


Figure: Cayley graph of a group G with generators $\{q_1, q_2, ..., q_k\}$

Cayley Graphs with Group Actions

Given a regular group action $(G, X, *)$ with a fixed element $x \in X$ and a set $Q = \{q_1, q_2, \ldots, q_k\} \subset G$, let $\Gamma = (X, E)$ be a graphs with vertex set X and edge set consisting of pairs $(x, y) \in X \times X$ such that $y = q * x$ for some $q \in Q$. The adjacency matrix of Γ is

$$A = \sum_{h \in G} \lambda_h \left| G^{(h)} * x \right\rangle \left\langle G^{(h)} * x \right|,$$

Where:

$$\left| G^{(h)} * x \right\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \chi(g, h) \left| g * x \right\rangle$$

And $\lambda_h = \sum_{q \in Q} \chi(h, q)$.

- Again, the eigenvectors $\left| G^{(h)} * x \right\rangle$ depend only on G.

Cayley Graphs with Group Actions

Proof :

$$A\left|G^{(h)} * x\right\rangle = \frac{1}{\sqrt{|G|}}. \sum_{g \in G} \chi(g,h) A. \left|g * x\right\rangle$$

$$= A\left|G^{(h)} * x\right\rangle = \frac{1}{\sqrt{|G|}}. \sum_{g \in G} \chi(g,h) \sum_{q \in Q} \left|q * (g * x)\right\rangle$$

Consider $\beta = qg$. Then:

$$== \frac{1}{\sqrt{|G|}} \sum_{q \in Q} \sum_{\beta \in G} \chi(\beta,h)\chi(q,h) \left|\beta * x\right\rangle$$

$$= \sum_{q \in Q} \chi(q,h). \left|G^{(h)} * x\right\rangle = \lambda_h \left|G^{(h)} * x\right\rangle$$

Simulating continuous-time walks

Abelian Groups:

- $e^{-iAt} = \mathrm{QFT}_G \left(\sum_{a \in G} e^{-i\lambda_a t} |a\rangle\langle a| \right) \mathrm{QFT}_G^*$
- Efficient since:
  - QFT_G and QFT_G^* run in $\mathrm{poly}(\log |G|)$ time.
  - λ_a computable to high precision classically.

Group Actions:

- No efficient QFT-like decomposition.
- Can express:

$$e^{-iAt} = \sum_{h \in G} e^{-i\lambda_h t} |G^{(h)} * x\rangle\langle G^{(h)} * x|$$

- But still approximable for $t = \mathrm{poly}(\log |G|)$ due to structure in A.

Simulating group action quantum walks

Goal: Efficiently simulate $W = e^{-iAt}$ for $t = \text{poly}(\log |G|)$.

Approach: Use discrete-time quantum walk framework:

$$W = iS(2TT^* - 1)$$

- Isometry T and its adjoint T^* are efficiently implemented via simple unitaries U_0, U_1.
- These unitaries prepare/refactor states $|\varphi_{yb}\rangle$ using a small symmetric set $Q \subset G$.
- Efficient due to:
  - Bounded degree Cayley graph.
  - Polynomial-size Q allows brute-force mapping.
- Full walk operator W simulated via reflection trick using T, T^*, and ancilla control.

Conclusion: Group-action quantum walks can be simulated efficiently despite lack of QFT-like structure.

Application: Quantum Money

A public-key quantum money scheme consists of two QPT algorithms:

- $\text{Gen}(1^\lambda)$: This algorithm takes a security parameter λ as input and outputs a pair (s, ρ_s), where s is a binary string called the serial number, and ρ_s is a quantum state called the banknote. The pair (s, ρ_s), or simply ρ_s, is sometimes denoted by \$.
- $\text{Ver}(s, \rho_s)$: This algorithm takes a serial number and an alleged banknote as input and outputs either 1 (accept) or 0 (reject).

Quantum Money From Group Actions

- Gen(1^λ). Begin with the state $|0\rangle |x_\lambda\rangle$, and apply the quantum Fourier transform over G_λ to the first register producing the superposition
$$\frac{1}{\sqrt{|X_\lambda|}} \sum_{g \in G_\lambda} |g\rangle |x_\lambda\rangle .$$
Next, apply the unitary transformation $|h\rangle |y\rangle \mapsto |h\rangle |h * y\rangle$ to this state, followed by the quantum Fourier transform on the first register. This results in
$$\frac{1}{|G_\lambda|} \sum_{h \in G_\lambda} \sum_{g \in G_\lambda} \chi(g, h) |h\rangle |g * x_\lambda\rangle = \frac{1}{\sqrt{|G_\lambda|}} \sum_{h \in G_\lambda} |h\rangle |G^{(h)} * x_\lambda\rangle$$

Quantum Money From Group Actions

- Ver$(h, |\psi\rangle)$. First, check whether $|\psi\rangle$ has support in X_λ. If not, return 0. Then, apply *cmpIndex* to the state $|\psi\rangle |0\rangle$, and measure the second register to obtain some $h' \in G_\lambda$. If $h' = h$, return 1; otherwise return 0.

cmpIndex Algorithm:

- Given a state $|G^{(h)} * x\rangle$, there is an efficient algorithm for computing h. Specifically, there is a unitary operator that performs the transformation $|G^{(h)} * x\rangle |0\rangle \mapsto |G^{(h)} * x\rangle |h\rangle$:

Quantum Money With The Hartley Transform

Hartley Transform:

item Let N be a positive integer, and let $\mathbb{Z}_N$ be the additive cyclic group of integers modulo N. The Hartley transform of a function $f : \mathbb{Z}_N \to \mathbb{R}$ is the function $\mathrm{H}_N(f) : \mathbb{Z}_N \to \mathbb{R}$ defined by

$$\mathrm{H}_N(f)(a) = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \operatorname{cas}\left(\frac{2\pi a y}{N}\right) f(y),$$

where $\operatorname{cas}(x) = \cos(x) + \sin(x)$

For a single basis element of the cyclic group $\mathbb{Z}_N$, the quantum Hartly transform simplifies to

$$\mathrm{QHT}_N : |a\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \operatorname{cas}\left(\frac{2\pi a y}{N}\right) |y\rangle \,. \tag{1}$$

Quantum Money With The Hartley Transform

- Gen. Begin with the state $|0\rangle|x\rangle$, and apply the quantum Hartley transform over $\mathbb{Z}_N$ to the first register producing the superposition

$$\frac{1}{\sqrt{N}} \sum_{g \in \mathbb{Z}_N} |g\rangle |x\rangle .$$

Next, apply the unitary $|h\rangle|y\rangle \mapsto |h\rangle|h * y\rangle$ to this state, followed by a QHT_N on the first register. This results in

$$\frac{1}{N} \sum_{h \in \mathbb{Z}_N} \sum_{g \in \mathbb{Z}_N} \operatorname{cas}\left(\frac{2\pi g h}{N}\right) |h\rangle |g * x\rangle = \frac{1}{\sqrt{N}} \sum_{h \in \mathbb{Z}_N} |h\rangle |\mathbb{Z}_N^{(h)} * x\rangle_H$$

Measure the first register to obtain a random $h \in \mathbb{Z}_N$, collapsing the state to $|\mathbb{Z}_N^{(h)} * x\rangle_H$. Return the pair $(h, |\mathbb{Z}_N^{(h)} * x\rangle_H)$.

Quantum Money With The Hartley Transform

- In the original scheme, using the quantum Fourier transform, we could directly obtain h from the money state $|\mathbb{Z}_N^{(h)} * x\rangle$ and compare it to the given h. However, this approach does not work when we use the Hartley transform.

- To address this, we design an algorithm for computing h that utilizes quantum walks.

Computing the serial Number

- Given a state $|\mathbb{Z}_N^{(h)} * x\rangle_H$, we show how to compute h using continuous-time quantum walks.
- For any $q \in \mathbb{Z}_N$, define a Cayley graph $\Gamma = (\mathbb{Z}_N, E)$ with the generating set $Q = \{-u, u\}$.
- Let A denote the adjacency matrix of Γ. The eigenvectors and corresponding eigenvalues of A are $|\mathbb{Z}_N^{(h)} * x\rangle$ and $\lambda_h = 2\cos(2\pi uh/N)$, respectively, for $h \in \mathbb{Z}_N$.
- the unitary $W = e^{iAt}$ can be efficiently simulated to exponential accuracy.

Computing the serial Number

Lemma: The money state $|\mathbb{Z}_N^{(h)} * x\rangle_H$ is an eigenstate of W with eigenvalue $e^{i\lambda_h t}$.

$$e^{iAt}\,|\mathbb{Z}_N^{(h)} * x\rangle_H = e^{i\lambda_h t}\,|\mathbb{Z}_N^{(h)} * x\rangle_H$$

If we choose $t = \text{poly}(\log N)$, it follows from Lemma that we can run the *phase estimation algorithm* with the unitary W and the eigenstate $|\mathbb{Z}_N^{(h)} * x\rangle_H$ to compute an estimate $\tilde{\lambda}_h$ of λ_h such that

$$|\tilde{\lambda}_h - \lambda_h| \leq \frac{1}{\text{poly}(\log N)}$$

An efficient new algorithm for QHT

Now, let us briefly explain how the algorithm for QFT_N works:

$$
\begin{aligned}
\mathrm{QFT}_N |a\rangle &= \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \omega_N^{ay} |y\rangle \\
&= \frac{1}{\sqrt{N}} \sum_{y=0}^{N/2-1} \omega_N^{ay} |y\rangle + (-1)^a \sum_{y=0}^{N/2-1} \omega_N^{ay} |y + N/2\rangle \\
&= \frac{1}{\sqrt{N/2}} \sum_{y=0}^{N/2-1} \omega_N^{ay} \frac{1}{\sqrt{2}} (|0\rangle + (-1)^a |1\rangle) |y\rangle , \qquad (2)
\end{aligned}
$$

An efficient new algorithm for QHT

$$\frac{1}{\sqrt{N}}\sum_{y=0}^{N-1}\mathrm{cas}\left(\frac{2\pi ay}{N}\right)|y\rangle \tag{3}$$

$$= \frac{1}{\sqrt{N}}\sum_{y=0}^{N/2-1}\mathrm{cas}\left(\frac{2\pi ay}{N}\right)|y\rangle + \frac{1}{\sqrt{N}}\sum_{y=N/2}^{N-1}\mathrm{cas}\left(\frac{2\pi ay}{N}\right)|y\rangle\,. \tag{4}$$

The second sum in the right-hand side can be written as

$$\begin{aligned}\sum_{y=N/2}^{N-1}\mathrm{cas}\left(\frac{2\pi ay}{N}\right)|y\rangle &= \sum_{y=0}^{N/2-1}\mathrm{cas}\left(\frac{2\pi ay}{N}+\pi a\right)|y+N/2\rangle \\ &= (-1)^a\sum_{y=0}^{N/2-1}\mathrm{cas}\left(\frac{2\pi ay}{N}\right)|y+N/2\rangle\,,\end{aligned}$$

An efficient new algorithm for QHT

$$= \frac{1}{\sqrt{N/2}} \sum_{y=0}^{N/2-1} \operatorname{cas}\left(\frac{2\pi a y}{N}\right) \frac{1}{\sqrt{2}} (|0\rangle + (-1)^a |1\rangle) |y\rangle , \qquad (5)$$

We now show how to compute QHT_N recursively.

$$\begin{aligned} |0\rangle |a\rangle = |0\rangle |t\rangle |b\rangle &\mapsto \frac{1}{\sqrt{N/2}} \sum_{y=0}^{N/2-1} \operatorname{cas}\left(\frac{2\pi t y}{N/2}\right) |0\rangle |y\rangle |b\rangle \\ &= \frac{1}{\sqrt{N/2}} \sum_{y=0}^{N/2-1} \operatorname{cas}\left(\frac{4\pi t y}{N}\right) |0\rangle |y\rangle |b\rangle \\ &\mapsto \frac{1}{\sqrt{N}} \sum_{y=0}^{N/2-1} \operatorname{cas}\left(\frac{4\pi t y}{N}\right) (|0\rangle + |1\rangle) |y\rangle |b\rangle . \end{aligned}$$

An efficient new algorithm for QHT

Algorithm (QHT_N)

- Input: quantum state $|\psi\rangle \in \mathbb{C}^N$, where $N = 2^n$
- Output: quantum state $\mathrm{QHT}_N |\psi\rangle$

1- Initialize an ancilla qubit to 0 to obtain the state $|0\rangle |\psi\rangle$
2- Compute $1 \otimes \mathrm{QHT}_{N/2} \otimes 1$ recursively.
3- Apply $H \otimes 1$.
4- Apply the controlled negation $|0\rangle |y\rangle \mapsto |0\rangle |y\rangle, |1\rangle |y\rangle \mapsto |1\rangle |N/2 - y\rangle$ to the first two registers.
5- Apply the unitary U_R.
6- Apply $H \otimes 1$
7- Apply CNOT to the first and last qubits.
8- Apply $1 \otimes H$.
9- Trace out the first qubit

References