Homework Week 15. 1.7–8

(25) Write the numbers $1, 2 \dots 2n$ on a blackboard, where n is an odd integer. Pick any 2 of the numbers, j & k write $|j-k|$ on the board & erase j & k. Continue this process untill only one integer is written on the board. Prove that this integer must be odd.
A: The key point here is that the parity (oddness or evenness) of the sum of the numbers written on the board never changes. If j & k are both odd, then their sum & their difference are both even, & we are replacing the even sum $j+k$ by the even difference $|j-k|$ leaving the parity of the total unchanged. If j & k have different parities, then erasing them changes the parity of the total, but their difference $|j-k|$ is odd so adding this difference restores the parity of the total. Therefore the integer we end up with at the end of the process must have the same parity as $1+2+\dots+(2n)$. It is easy to compute this sum. If we add the first & last terms we get $2n+1$, if we add the 2nd & next-to-last terms we get $2+(2n-1)=2n+1$ & so on. In all we get the sums of $2n+1$, so the total sum is $n(2n+1)$. If n is odd, this is the product of 2 odd numbers & therefore is odd, as desired

(27) Formulate a conjecture about the decimal digits that appear as the final decimal digit of the fourth power of an integer. Prove your conjecture using a proof by cases.
A: Without loss of generality we can assume that n is nonnegative since the fourth power of an integer & the 4th power of its ~~negati~~ negative are the same. To get a handle on the last digit of n, we can divide n by 10, obtaining a quotient k & remainder l, whence $n = 10k + l$, & l is an integer between 0 & 9, inclusive. Then we compute n^4 in each of these ten cases. We get the following values, where ?? is some integer that is a multiple of 10, whose exact value we do not care about.

$$(10k+0)^4 = 10000k^4 + 0$$
$$(10k+1)^4 = 10000k^4 + ??\cdot k^3 + ??\cdot k^2 + ?? + k + 1$$
$$(10k+2)^4 = 10000k^4 + ??\cdot k^3 + ?? + k^2 + ?? + k + 16$$

$(10k \cdot 3)^4 = 10000k^4 + ?? \cdot k^3 + ?? + k^2 + ?? \cdot k + 81$

4 + 256

5 + 625

6 + 1296

7 + 2401

8 + 4096

9 + 6561

Since each coef-t indicated by ?? is a multiple of 10^t, the corresponding term has no effect on the ones digit of the answer. Therefore the ones digits are 0, 1, 6, 1, 6, 5, 6, 1, 6, 1 respectively, so it is always 0, 1, 5, or 6.

Expansion of — $(t+r)^4 = t^4 + \underset{\times 1000}{4t^3}r^1 + \underset{\times 100}{6t^2r^2} + \underset{10}{4t^1r^3} + r^4$

35. Prove that between every 2 rational numbers there is an irrational number.

A: The idea is to find a small irrational number to add to the smaller of the 2 given rational numbers. Because we know that $\sqrt{2}$ is irrational, we can use a small multiple of $\sqrt{2}$.

Here is our proof: By finding a common denominator, we can assume that the given rational numbers are a/b & c/b where b is a positive integer & a & c are integers with $a<c$. In particular $(a+1)/b \leq c/b$. Thus $x = (a + \frac{1}{2}\sqrt{2})/b$ is between the 2 given rational numbers, because $0 < \sqrt{2} < 2$.

Furthermore, x is irrational, ~~because~~ if x were rational, then $2(bx-a) = \sqrt{2}$ would be as well, in violation of Ex. 10 in Sec 1.7.

Cardinality of Sets 2.5.

For finite sets, cardinality = # of elements in a set,

e.g. $|A| = 3$, where $A = \{3, 10, 7\}$

or $|B| = 0$ where $B = \emptyset$ (empty set).

Two sets have ~~the~~ same cardinality if there is a one-to-one correspondence (bijection).

E.g: $A = \{1, 2, 3, 4, 5\}$ has the same cardinality as

$B = \{41, 42, 43, 44, 45\}$ because there is bijection.

$f: A \to B \quad f(x) = x + 40 \quad x \in A \wedge (x+40) \in B$

$1 \longleftrightarrow 41 \quad 4 \longleftrightarrow 44$

$2 \longleftrightarrow 42 \quad 5 \longleftrightarrow 45$

$3 \longleftrightarrow 43$

We can compare cardinalities of 2 sets

e.g $|A| = |B|$, $|A| \leq |B|$, $|A| < |B|$

Infinite set is a set that is not finite. Even though ∞ (infinity) is indescribeable, mathematicians found ways to describe cardinalities of infinite sets.

cardinalities of 2 infinite sets ~~are~~ say $|A|$ & $|B|$ equal if there exist one-to-one correspondences

$f: A \to B$ & $g: B \to A$.

↳ SCHRODER – Bernstein Theorem.

($f: A \to B$ tells us that $|A| \leq |B|$ &

$g: B \to A$ tell us that $|B| \leq |A|$ so

it should be that $|A| = |B|$)

$Z^+ = \{1, 2, 3, 4 \ldots \infty\}$ is a countable infinity.

If you can find one-to-one correspondence from set A to Z^+

then $|A| \leq |Z^+|$ so it should countably infinite numbers

We write $|Z^+| = \aleph_0$ (Aleph null)

{ If you can predictably list out all numbers of set A
{ in countable manner, it is a countably infinite

show that set odd numbers is countably infinite set.

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 3 & 5 & 7 & 9 \end{array} \begin{array}{l} - Z^+ \\ \\ \rightarrow A \end{array}$$

$\rightarrow$ Listable! (focus on this type of proof)

$\rightarrow$ Let $A = \{1, 3, 5, 7 \dots\}$

$f: A \rightarrow Z^+$, $f(x) = \frac{x+1}{2}$ $\quad x \in A \wedge \frac{x+1}{2} \in Z^+$

$$1 \longleftrightarrow \frac{1+1}{2} = 1$$

$$3 \longleftrightarrow \frac{3+1}{2} = 2$$

$$5 \longleftrightarrow \frac{5+1}{2} = 3$$

$$7 \longleftrightarrow \frac{7+1}{2} = 4$$

$$\downarrow \qquad\qquad \downarrow$$

$$A \qquad\qquad Z^+$$

Similarly, we can show that for even numbers.

1	2	3	4	5	6	7	8	$\rightarrow Z^+$
2	4	6	8	10	12	14	16	$\rightarrow A$.

set of positive rational numbers is countable

$$Q^+ = \{\frac{p}{q} : p \cdot q \in Z^+\}$$