"Order of

Date:

Ex. ① Let $P(x,y)$ be the statement "$x+y=y+x$". Assume that the domain is the real numbers. Then $\forall x \forall y P(x,y)$ & $\forall y \forall x P(x,y)$ are all true

② Let $Q(x,y)$ be the statement "$x+y=0$". Assume that the domain in the real numbers. Then $\forall x \exists y Q(x,y)$ is true, but $\exists y \forall x Q(x,y)$ is false

Week 4. Homework.

P-1: Acme C. — 138 billion \$, net profit — 8 billion \$.
Nadir S. — 87 billion \$ net profit — 5 billion \$
Quixote M. — 111 billion \$ net profit — 13 billion \$

false a) Quixote M. had the largest annual revenue

true (both parts) b) Nadir S. had the lowest net profit & ~~the~~ Acme C had the largest annual revenue

true (the 2nd part) c) Acme Com. had the largest net profit or Quixote M. had the largest net profit.

d) The hypothesis of this conditional statement is false & the conclusion is true, so by the truth table definition this is a true statement.
(Either of those conditions would have been enough to make the statement true).

P-2: Let p & q be the proportions.
p: It's below freezing.
q: It's snowing.

Write these props. using p & q & logical connectives (including negations)

a) ~~or~~ It's below freezing & snowing. $p \wedge q$

b) It's below freezing but not snowing
Here we have a conjunction of p with the negation of q, namely $p \wedge \neg q$. Note that "but" logically means the same thing as "and".

c) It's not below freezing & it's not snowing $\neg p \wedge \neg q$ (conjunction).

d) It's either snowing or below freezing (or both) $p \vee q$ (disjunction)

e) If it's freezing, it's also snowing. $p \to q$ (conditional)

f) Either it's below freezing or it's snowing, but it's not snowing if it's below freezing. This is a conjunction of props, both of which are compound:
$(p \vee q) \wedge (p \to \neg q)$

g) That it is below freezing is necessary & sufficient for it to be snowing.
biconditional, $p \leftrightarrow q$.

How to find how many rows

P-3. Date: How many rows appear in a truth table for each of these compound proposit-ns?

a) $p \to \neg p$ → 2 $2^1 = 2$

b) $(p \vee \neg r) \wedge (q \vee \neg s)$ $2^4 = 16$

c) $q \vee p \vee \neg s \vee \neg r \vee \neg t \vee u$ $2^6 = 64$

d) $(p \wedge r \wedge t) \leftrightarrow (q \wedge t)$ $2^4 = 16$

P-4 Construct a truth table for each of these compound propositions

a) $p \wedge \neg p$

b) $p \vee \neg p$

c) $(p \vee \neg q) \to q$

d) $(p \vee q) \to (p \wedge q)$

e) $(p \to q) \leftrightarrow (\neg q \to \neg p)$

f) $(p \to q) \to (q \to p)$

— How to construct a truth table?

To construct a truth table for each of these compound proposit-s, we work from inside out. In each case, we will show the intermediate steps. In part (d), for ex., we first construct the truth table for $p \vee q$, then the truth table for $p \wedge q$, & finally combine them to get the truth table for $(p \vee q) \to (p \wedge q)$ for parts a & b

p	$\neg p$	$p \wedge \neg p$	$p \vee \neg p$
T	F	F	T
F	T	F	T

for part c we have the following table

p	q	$\neg q$	$p \vee \neg q$	$(p \vee \neg q) \to q$
T	T	F	T	T
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

for part D we have the following table

p	q	$p \vee q$	and $(p \wedge q)$	$(p \vee q) \to (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

Date: For part (e) we have the following table. This time we have omitted the column explicitly showing the negations of p & q. Note that this true propositions is telling us that a conditional statement & its contrapositive always have the same truth value.

$\neg p$	p	q	$\neg q$	$p \to q$	$\neg q \to \neg p$	$(p\to q) \leftrightarrow (\neg q \to \neg p)$
F	T	T	F	T	T	T
F	T	F	T	F	F	T
T	F	T	F	T	T	T
T	F	F	T	T	T	T

For (f) ~~we have~~ The fact that this propositions is not always true tells us that knowing a conditional statement in one direction does not tell us that the conditional statement is true in the other direction.

how to read this?

p	q	$p \to q$	$q \to p$	$(p \to q) \to (q \to p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

P-5 ~~P~~ P1 → "I don't know" ✓

Does everyone wants coffee? P2 → I don't know ✓

P3 → No, not everyone want coffee. ✗

The hostess came back & gave coffee to P who want it. How did she figure out?

P-6 Use De Morgan's laws to find the negation of each of the following statements.

a) Jan is rich & happy.
b) Carlos will bicycle or run tomorrow
c) Mei walks or takes the bus to class.
d) Ibrahim is smart & hard working

De Morgan's law tell us that to negate a conjunction we form the disjunction of the negations, & to negate a disjunction we form the conjunction of the negations.

a) conjunction so the negation is "Jan is not rich, or Jan is not happy"
b) disjunction. So the negation is "C. will not bicycle or run tomorrow".
c) disjunc. So the negation is "Mei doesn't walk and Mei doesn't take the bus
d) conjunction ~~¬~~ Ibrahim is not smart, or Ibrahim is not hard working."

we can also use neither.

P-7 Show that each of these conditional statements is a tautology by using truth tables.

Date:

a) $(p \wedge q) \to p$ b) $p \to (p \vee q)$
c) $\neg p \to (p \to q)$ d) $(p \wedge q) \to (p \to q)$
e) $\neg(p \to q) \to p$ f) $\neg(p \to q) \to \neg q$

a/b.

p	q	$p \wedge q$	$(p\wedge q) \to p$	$p \vee q$	$p \to (p \vee q)$
T	T	T	T	T	T
F	F	F	T	T	T
F	T	F	T	T	T
F	F	F	T	F	T

c/d

p	q	$\neg p$	$p \to q$	$\neg p \to (p \to q)$	$p \wedge q$	$(p \wedge q) \to (p \to q)$
T	T	F	T	T	T	T
T	F	F	F	T	F	T
F	T	T	T	T	F	T
F	F	T	T	T	F	T

e/f.

p	q	$p \to q$	$\neg(p \to q)$	$\neg(p \to q) \to p$	$\neg q$	$\neg(p \to q) \to \neg q$
T	T	T	F	T	F	T
T	F	F	T	T	T	T
F	T	T	F	T	F	T
F	F	T	F	T	T	T

P-8 Show that each conditional statement in Exercise 9(7) is a tautology without using truth table.

Recall the only way a conditional statement can be false is for the hypothesis to be true & the conclusion to be false, hence it's sufficient to show that the conclusion must be true whenever the hypothesis is true. An alternative approach that works for some of these tautologies is to use the equivalences given in this section & prove these "algebraically". We will demonstrate this 2nd method in some of the solutions.

a) If the hypothesis is true, then by the definition of $\wedge$ we know that p is true. Hence the conclusion is also true. For an algebraic proof, we exhibit the following string of equivalences, each one following from one of the laws in this section:

$$(p \wedge q) \to p \equiv \neg(p \wedge q) \vee p \equiv (\neg p \vee \neg q) \vee p \equiv (\neg q \vee \neg p) \vee p \equiv \neg q \vee (\neg p \vee p) \equiv q \vee T \equiv T$$

The 1st logical equivalence is the first equivalence in Table 7 (with $p \wedge q$ playing the role of p, & p playing the role of q); the 2nd is De Morgan's law; the 3rd is the commutative law, the 4th is the associative law, the fifth is the negation law (with the commutative law); & the 6th is the domination law.

b) If the hypothesis p is true, then by the definition of $\vee$, the conclusion $p \vee q$ must also be true.

c) If the hypothesis is true, then ~~by the definition~~ p must be false, hence the conclusion $p \to q$ is true since its hypothesis is false. Symbolically we have $\neg p \to (p \to q) \equiv \neg\neg p \vee (\neg p \vee q) \equiv p \vee (\neg p \vee q) \equiv (p \vee \neg p) \vee q \equiv T \vee q \equiv T$

d) If the h. is true, then by the definition of $\wedge$ we know that q must be true. This makes the conclusion $p \to q$ true, since its conclusion is true.

e) If the h. is true, then $p \to q$ must be false. But this can happen only if p is true, which is precisely what we wanted to show.

f) If the h. is true, then $p \to q$ must be false. But this can happen only if q is false, which is precisely what we wanted to show.

P-9 Show that $(p \to r) \wedge (q \to r)$ and $(p \vee q) \to r$ are logically $\equiv$ equivalent.

We will find exactly which rows of the truth table will have F as their entries. In order for $\boxed{(p \to r) \wedge (q \to r)}$ to be false, we must have at least one of the 2 conditional statements false, which happens exactly when r is false & at least one of p & q is true. But these are precisely the cases in which $p \vee q$ is true & r is false, which is precisely when $(p \vee q) \to r$ is false. Since 2 propositions are false in exactly the same situations, they are logically equivalent.

$p \to r$ $q \to r$

T   F   T   F

P-10 $P(x) \to$ x can speak Russian

$Q(x) \to$ x knows the computer lan. C++.

Express each of these sentences in term of $P(x)$, $Q(x)$ quantifiers, & logical connectives. The domain for quantifiers consists of all students at your school.

a) There is a student at your school who can speak Russian & who knows C++ $\exists x (Px \wedge Q(x))$

b) There is a student at your school who can speak Russian but who doesn't know C++ $\exists x (Px \wedge \neg Q(x))$

c) Every student at your school either can speak Russian
Date: or knows C++ $\forall x(P(x) \vee Q(x))$

d) No student at your school can speak Russian or knows C++.
$\forall x \neg P(x) \vee Q(x)$
→ We can also write it as $\forall x((\neg P(x)) \wedge (\neg Q(x)))$. Note that it would not be correct to write $\forall x((\neg P(x)) \vee (\neg Q(x)))$ nor to write $\forall x \neg(P(x) \wedge Q(x))$. (by de Morgan's Law)

P-11 ? Determine the truth value of each of these statements if the domain for all variables consists of all integers

a) $\forall n(n^2 \geq 0)$ → This is well-known true fact that the square of a real number cannot be negative.

b) $\forall n(n^2 \geq n)$ → if n is a positive integer, then $n^2 \geq n$ is certainly true. It's also true for $n=0$; & it's trivially true if n is negative. Therefore the univ. q. is T.

c) $\exists n(n^2 = 2)$ - there are 2 real numbers that satisfy $n^2 = 2$, namely $\pm\sqrt{2}$ but there do not exist any integers with this property, so the statement is false

d) $\exists n(n^2 < 0)$ Squares can never be negative, therefore this statement is false

P-12. Suppose that the domain of the prop-l function $P(x)$ consists of the integers 0, 1, 2, 3 & 4. Write out each of these propositions using disjunctions, conjunctions & negations.

a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\exists x \neg P(x)$
d) $\forall x \neg P(x)$ e) $\neg \exists x P(x)$ f) $\neg \forall x P(x)$

Existential quantifiers are like disjunctions, & universal quantifiers are like conjunctions.

a) We want to assert that $P(x)$ is true for some x in the universe, so either $P(0)$ is true or $P(1)$ is true or $P(2)$ is true or $P(3)$ is true or $P(4)$ is true.
Thus the answer $P(0) \vee P(1) \vee P(2) \vee P(3) \vee P(4)$.

b) $P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge P(4)$

c) $\neg P(0) \vee \neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4)$

d) $\neg P(0) \wedge \neg P(1) \wedge \neg P(2) \wedge \neg P(3) \wedge \neg P(4)$

c logically ≡ to f
d ≡ e
according to De Morgan's laws

e) This is a negation of part (a): $\neg P(0) \vee P(1) \vee P(2) \vee P(3) \vee P(4)$

f) neg. of part (b): $\neg P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge P(4)$

P-13. Translate in 2 ways each of these statements into logical expressions using predicates, quantifiers, & logical connective. First, let the domain consist of the students in your class & second let it consist of all people.

a) Someone in your class can speak Hindi. $\exists$
b) Everyone in your class is friendly $\forall$
c) There is a person in your class who wasn't born in California.
d) A student in your class has been in a movie
e) No student in your class has taken a course in logic prog.

In order to do the translation the second way, we let $C(x)$ be the propositional function. "x is in your class". Note that for the 2nd way we always want to use conditional statements with universal quantifiers & conjunctions with existential quantifiers

a) Let $H(x)$ be "x can speak Hindi". 1st way. $\exists x H(x)$
? — 2. way. $\exists x (C(x) \land H(x))$ "I didn't understand about 2nd way"

b) Let $F(x)$ "x is friendly) ① $\forall x F(x)$
2. $\forall x (C(x) \to F(x))$

c) Let $B(x)$ "x was born in California 1. $\exists x \neg B(x)$
2. $\exists x (C(x) \land \neg B(x))$

d) Let $M(x)$ be "x has been in a movie" ① $\exists x M(x)$
② $\exists x (C(x) \land M(x))$

e) Everyone has failed to take the course.
$L(x)$ "x has taken the course in prog. lang".
① $\forall x \neg L(x)$
② $\forall x (C(x) \to \neg L(x))$

P-14. Let $M(x,y)$ be "x has sent y a e-mail message" & $T(x,y)$ be "x has telephoned y" where the domain consists of all student in your class. Use quantifiers to express each of these statements. (Assume that all e-mail messages that were sent are recieved, which is not the way things often work.

a) Chou has never sent e-mail to Koko.
b) Arlene has never sent an e-mail message ~~from Deborah~~ to or telephoned Sarah.
c) Jose has never received an e-mail message from Deborah
d) Every student in your class has sent an e-mail message to Ken
e) No one in your class has telephoned Nina.

f) Everyon... him
g) There is ... class an...
h) There is ... message ...
i) There ... other ...
j) There i...
k) There is ... from ... by ar...
l) Every ... messag... the e...
m) There ... stude... telep...
n) There ... sent ... class...
Solution
a)
b) 1) 2)
c)
d)
e)
f)
?— g)
h)
i)
j)
k)
l)

f) Everyone in your class has either telephoned Avi or sent him an e-mail message.

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g) There is a student in your class who has sent everyone else in your class an e-mail message.

h) There is someone in your class who has either sent an e-mail message or telephoned everyone else in your class.

i) There are 2 diff. students in your class who have sent each other email messages.

j) There is a student who has sent himself or herself an email message.

k) There is a student in your class who has not received an e-mail message from anyone else in the class & who has not been called by any other student in the class.

l) Every student in the class has either received an e-mail message or received a telephone call from an other student in the clas.

m) There are at least 2 students in your class such that one student has sent the other email & the 2nd student has telephoned the 1st student.

n) There are 2 diff. students in your class who between them have sent an e-mail message to or telephoned everyone else in the class.

Не понимаю как опред-ть что ставить между (∧) (→)

Solutions:

a) $\neg M(\text{Chou}, \text{Koko})$

b) 1) $\neg(M(\text{Arlene}, \text{Sarah}) \lor T(\text{Arlene}, \text{Sarah}))$ or

2) $\neg M(\text{Arlene}, \text{Sarah}) \land \neg T(\text{Arlene}, \text{Sarah})$

c) $\neg M(\text{Debora}, \text{Jose})$

d) $\forall x M(x, \text{Ken})$ (Ken has sent himself a message as well)

e) $\neg\forall x \neg T(x, \text{Nina})$ or $\neg\exists x T(x, \text{Nina})$

f) $\forall x (T(x, \text{Avi}) \lor M(x, \text{Avi}))$

?- g) To get the "else" in there we have to make sure that y is different from x in our answer: $\exists x \forall y (y \neq x \rightarrow M(x, y))$

h) $\exists x \forall y (y \neq x \rightarrow (M(x, y) \lor T(x, y)))$

i) $\exists x \exists y (x \neq y \land M(x, y) \land M(y, x))$

j) $\exists x M(x, x)$

k) $\exists x \forall y (x \neq y \rightarrow (\neg M(y, x) \land \neg T(y, x)))$

l) $\forall x \exists y (x \neq y \land (M(y, x) \lor T(y, x))$ → Here "y" is "another student" m). Almost identical to part i.

Date:

$\exists x \exists y\, (x \neq y) \land M(x,y) \land T(y,x))$

n) Note how the "everyone else" someone diff. from both x & y in our expression & note that there are 4 possibilities for how each such person z might be contacted):

$\exists x \exists y\, (x \neq y \land \forall z((z \neq x \land z \neq y) \rightarrow (M(x,z) \lor M(y,z) \lor T(x,z) \lor T(y,z))))$

Week 5. Introduction to Proofs S.1.8

L-1 Many theorems have the form:

$\forall x (P(x) \rightarrow Q(x))$

To prove them, we can show that $P(c) \rightarrow Q(c)$ for an arbitrary element c of the domain, and apply the rule of universal generalization.

So, we must prove something of the form: $\boxed{p \rightarrow q}$

Proving Conditional Statements: $\boxed{p \rightarrow q}$

Trivial Proof: If we know q is true, then $p \rightarrow q$ is true as well.

hypothesis conclusion

"If it is raining then 1=1"

Vacuous Proof: If we know p is false then $p \rightarrow q$ is true as well

"If I am both rich & poor, then 2+2=5"

p	q	$p \rightarrow q$
0	0	1 T
0	$\boxed{1}$ T	1 T
1	0	0
1	$\boxed{1}$ T	1 T

Proving Conditional statements:

Direct Proof: