Homework 4.3 5, 11, 19, 25, 37.

(5) Find the prime factorization of 10!

$10! = 2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10 = 2\cdot3\cdot2^2\cdot5\cdot(2\cdot3)\cdot$
$7\cdot2^3\cdot3^2\cdot(2\cdot5) = 2^8\cdot3^4\cdot5^2\cdot7$

3) Count the prime factors, let's group the prime factors

Twos: $2,2,2,2,2,2,2,2 \Rightarrow 2^8$

Threes: $3,3,3,3 \Rightarrow 3^4$

Fives $5,5 \Rightarrow 5^2$

sevens $7 \Rightarrow 7^1$ Final answer: $10! = 2^8 \times 3^4\cdot5^2\cdot7$

(11) Show that $\log_2 3$ is an irrational number. Recall that that an irrational number is a real number x that cannot be written as the ratio of 2 integers.

Step 1. Assume the Opposite (Proof by contradiction)

We will assume that $\log_2 3$ is rational & show that this leads to a contradiction.

• If $\log_2 3$ were a rational number, then it could be written as a fraction of 2 integers:

$\log_2 3 = \frac{p}{q}$ where p & q are integers, & $q \neq 0$

Step 2. Rewrite the logarithm as an Exponential Equation

By the definition of logarithms: $\log_2 3 = \frac{p}{q}$ means $2^{p/q} = 3$

Step 3. Eliminate the Fraction by Raising to a Power

To get rid of the fraction in the exponent, raise both sides to the power of q: $(2^{p/q})^q = 3^q$

since exponents multiply: $2^p = 3^q$

Step 4: Analyze the Equation $2^p = 3^q$

• The left side $(2^p)$ is a power of 2, meaning it is a product of only the prime number 2

• The right side $(3^q)$ is a power of 3, meaning it is a product of only the prime number 3

This equation says that a power of 2 equals a power of 3 However:

• The number $2^p$ is divisible only by 2

• The number $3^q$ is divisible only by 3 since exponents multiply: $2^p = 3^q$

Step 4: Analyze the equation $2^p = 3^q$

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This equation says that a power of 2 equals a power of 3. However:

- The number $2^p$ is divisible only by 2
- The number $3^q$ is divisible only by 3

The only way these 2 numbers could be equal is if they were both 1, which happens only when $p=0$ & $q=0$

But $q=0$ is not allowed (since fractions must have a nonzero denominator)

Step 5. Conclusion - contradiction

Since we reached an impossible equation, our assumption that $\log_2 3$ is rational must be false.

Thus, $\log_2 3$ is irrational.

19 Show that if $2^n-1$ is prime, then $n$ is prime.

Step 1. Understanding what we need to prove

- The statement tells us that $2^n-1$ is prime, meaning it has only 2 factors: 1 & itself.
- We need to prove that $n$ itself must also be prime.

Step 2 Assume $n$ is not prime & see what happens

Suppose $n$ is not prime.

That means $n$ can be written as a product of 2 smaller numbers, say:

$n = a \cdot b$ where $a, b > 1$

Step 3. Use the given identity

The problem gives us this useful factorization identity.

$$2^{ab}-1 = (2^a-1)\cdot(2^{a(b-1)} + 2^{a(b-2)} + \ldots + 2^a + 1$$

Now, apply this to our assumption that $n = ab$:

$$2^n - 1 = 2^{ab} - 1$$

Using the identity, we can factor $2^n-1$

$$2^n-1 = (2^a-1)\cdot(2^{a(b-1)} + 2^{a(b-2)} + \ldots + 2^a + 1)$$

Step 4. Check for a contradiction

Since we assumed $n$ is not prime, we just factored $2^n-1$ into 2 factors

1) $2^a-1$
2) $2^{a(b-1)} + 2^{a(b-2)} + \ldots + 2^a + 1$

But we were given that $2^n-1$ is prime! (it has only 2 factors: 1 & itself)

- A prime can not be factored like this
- This contradicts our assumption that $n$ is composite.

Step 5. Conclusion

Since assuming that $n$ is composite led to a contradiction, $n$ must be prime.

Thus, we have proved that if $2^n-1$ is prime, then $n$ must be prime.

A: if $2^n-1$ is prime, then $n$ must be prime.

**Ex25** GCD of: we just need to take the smaller exponent for each prime

a) $3^7 \cdot 5^3 \cdot 7^3$ $\quad 2^{11} \cdot 3^5 \cdot 5^9$ $\quad$ GCD $= 3^5 \cdot 5^3$

b) $11 \cdot 13 \cdot 17$ $\quad 2^9 \cdot 3^7 \cdot 5^5 \cdot 7^3$
These numbers have no common Prime factors, so the gcd is 1.

c) $23^{31}$ $\quad 23^{17}$ $\quad$ GCD $= 23^{17}$

d) $41 \cdot 43 \cdot 53$ $\quad 41 \cdot 43 \cdot 53$ $\quad$ GCD $= 41 \cdot 43 \cdot 53$

e) $3^{13} \cdot 5^{17}$ $\quad 2^{12} \cdot 7^{21}$ – These numbers have no common prime factors, so gcd is 1.

f) 1111, 0 - The gcd of any positive int. and 0 is that integer so the answer is 1111.

Rule: $\gcd(a, 0) = |a|$ (for integer $a$).

ex. 37. We need to prove that $\gcd(2^a - 1, 2^b - 1) = 2^{\gcd(a,b)} - 1$ for any positive integers $a$ and $b$.

ex36 tells us $(2^a - 1) \bmod (2^b - 1) = 2^{a \bmod b} - 1$

step 2. Apply the Euclidean algorithm
To compute $\gcd(2^a - 1, 2^b - 1)$. we use the Euclidean algorithm,
$$\gcd(A, B) = \gcd(B, A \bmod B)$$
setting $A = 2^a - 1$ & $B = 2^b - 1$ we apply modular reduction:
$$(2^a - 1) \bmod (2^b - 1) = 2^{a \bmod b} - 1$$
Thus $\gcd(2^a - 1, 2^b - 1) = \gcd(2^b - 1, 2^{a \bmod b} - 1)$

Repeating this process, we see that at each step, the exponents follow the same sequence as in the Euclidean algorithm applied to $\gcd(a,b)$. Eventually, we reach:

A: $\gcd(2^a - 1, 2^b - 1) = 2^{\gcd(a,b)} - 1$