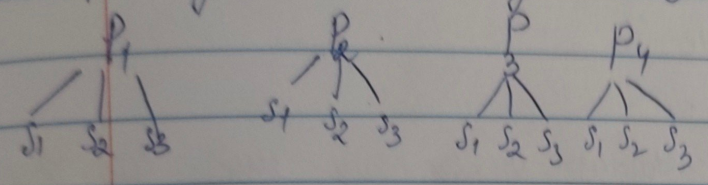
get to this place

The Binomial theorem: for any positive integer n, the coefficient of the $x^{n-k}y^k$ term of $(x+y)^n$ is $\binom{n}{k}$ in other words,

$$(x+y)^n=\binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\dots+\binom{n}{k}x^{n-k}y^k+\dots\binom{n}{n}y^n$$

V-9 Homework

P-1.9 You have 3 shirts & 4 pairs of pants. How many outfits consisting of one shirt & 1 pair of pants can you make?



S	P_1	P_2	P_3	P_4
S_1	S_1P_1	S_1P_2	S_1P_3	S_1P_4
S_2	S_2P_1	S_2P_2	S_2P_3	S_2P_4
S_3	S_3P_1	S_3P_2	S_3P_3	S_3P_4

S: For both of these pictures we can use the following reasoning: we have four choices for the pants, & for each of these 4 choices of pants, we have 3 choices for the shirt. Therefore, there are $4\cdot3=12$ outfits.

P.1.11 How many ways we can form a license plate if there are 7 characters, none of which is letter O, the 1st of which is a numerical digit (0-9), the second of which is a letter, & the remaining 5 of which can be either a digit or a letter (but not "O")?

S: Since each character does not depend on any of the other characters, our choices are independent. There are 10 choices for the 1st char. (any digit from 0 to 9), 25 choices for the 2nd char. (any letter A-Z, except for "O"), & 35 choices for each of the other 5 char. (any digit 0-9, or any letter A-Z, except O)

Therefore, since the choices are independent we have

$$10\times25\times35\times35\times35\times35\times35=10\cdot25\cdot35^5=13.130.468.750$$

P.1.16 Lottery. In the lottery 25 balls numbered 1 to 25 are placed in a bin. 4 balls are drawn one at a time & their numbers are recorded. The winning combination consists of the 4 selected numbers in the order they are selected. How many winning combinations are there, if:

a) each ball is discarded after it is removed?

b) each ball is replaced in the bin after it is removed & before the next ball is drawn?

S-n: a) We have 25 choices for the first ball. After ~~all~~ the 1st ball is drawn & discarded, There are 24 balls left in the bin, so there are 24 choices for the 2nd, 23 for the 3rd, 22 for the 4th,

Hence, the number of winning comb-n is $25 \times 24 \times 23 \times 22 = 303{,}600$

b) We have 25 choices of each of the 4 balls, since all balls are in the bin

Hence: $25 \times 25 \times 25 \times 25 = 25^4 = 390625$.

P.2.2: On the island of Mumble, the Mumblian alphabet has only 5 letters, & every word in the Mumblian lan. has no more than 3 letters in it. How many words are possible. (A word can use a letter more than once, but 0 letters does not count as a word.

S-n: Often, the tricky part of casework problems is deciding what the cases should be. For this problem, it makes sense to use as our cases the number of letters in each word.

Case-1: 1-letter words

There are 5 1-letter words.

Case-2. 2 letter words. To form a 2 letter word, we have 5 choices for 1st letter, & 5 choice for 2nd. Thus There $5 \times 5 = 25$ 2-letter words

Case 3: 3 letter word. 5 choices for the 1st letter, 5 choices for the 2nd, 5 for 3rd.

So, $5 \times 5 \times 5 = 125$ 3 letter words possible.

So the total is $5 + 25 + 125 = 155$

P2.7: The Smith family has 4 sons & 3 dauters. In how many ways can they be seated in a row of 7 chairs such that at least 2 boys are next to each other.

S-n: This problem is a perfect candidate for complementary counting. It will be fairly difficult to try to count this directly, since there are lots of possible cases (just 2 are BBBBGGG & BGGBBGB, where B is a boy & G is a girl). But there is only one way to assign genders to the seatings so that no 2 boys are next to each other, & that is BGBGBGB.

If we seat the children as BGBGBGB, then there are 4! orderings for the 4 boys, & 3! ordering for the 3 girls, giving a total of $4! \times 3! = 144$ seating for the 7 children.

These are the seatings that we don't want, so to count the seating that we do want, we need to subtract these seatings from the total number of seatings without any restrictions. Since there are 7 kids, there are 7! ways to seat them.

So, $7! - (4! \times 3!) = 504 - 144 = 4896$.

P.2.9: How many sequences $x_1, x_2, x_3 \dots x_7$ can be formed in which all the x_i are integers greater than 0 & less than 6, & no 2 adjacent x_i are equal?

S-n: This is a fairly hard problem to attack using casework or complementary counting. Counting all 7 numbers sequences of digits from 1 to 5 is easy: we have 5 choices for each digit, & there are 7 digits, so there are 5^7 of them. However, counting the ones in which there are adjacent x_i that are equal is no easy task to tackle directly. We have all sorts of cases to consider. We have multiple numbers repeated as in (1123344) we have a single

number repeated many times (as in 1114411) & so on.
So instead, we think about how we might construct such a sequence.
We have 5 choices for the 1st number x_1, namely 1, 2, 3, 4 or 5.
Once we've chosen the 1st number, regardless of what we've chosen, we have 4 choices for the 2 number x_2. We can choose x_2 to be any of the numbers 1, 2, 3, 4 or 5 except for the number that we already chose for x_1.
Similarly, once we've chosen the 2nd number, we have 4 choices for the 3rd number x_3 (any number except the number we chose for x_2) & so on down the line.
So we have 5 choices for the 1st number, & 4 choices for each of the subsequent 6 numbers.
Hence, there are $5 \times 4^6 = 20,480$ possible sequences.

P2.14 Our math club has 20 members & 3 officers: President, Vice President, & Treasurer. However, one member, Ali, hates another member, Brenda. How many ways can we fill the offices if Ali refuses to serve as an officer if Brenda is also an officer?

P-.3.4 How many distinct arrangements are ther of PAPA?
Sol: Again we pretend that the letters are all different, & we have $P_1 A_1 P_2 A_2$. We have 4! permutations (since all the letters are different) (since all the P's & A's are considered different).
But how many arrangements of $P_1 A_1 P_2 A_2$ (where the P's & A's are identical)? correspond to a single arrangement of PAPA (where the P's & A's are identical)? PAPA is counted in 4 differ. ways: as $P_1 A_1 P_2 A_2$, $P_1 A_2 P_2 A_1$, $P_2 A_1 P_1 A_2$, $P_2 A_2 P_1 A_1$.
Rather than list out the possibilities, we could have reasoned as follows: For the 2 P's each possibility is counted $2! = 2$ times & for the 2 A's, each of these 2 possibilities is counted $2! = 2$ times, for a total of $2 \times 2 = 4$ ways. $2 \times 2! = 4$ times, so we have $4!$ ways to arrange the 4 letters $P_1 A_1 P_2 A_2$; this counts each arrangement of PAPA. Therefore there are $4!/(2! \times 2!) = 6$ ways to arrange PAPA.

P3.8: A convex polygon is a polygon in which every interior angle is less than 180 degrees. A diagonal of a convex polygon is a line segment which connects 2 non-adjacent vertices. Find a formula for the number of diagonals of a convex polygon with n sides, where n is any positive integer greater than 2.

S-n: A polygon with n sides has n vertices. Every diagonal of the polygon corresponds to a pair of vertices (namely, the diagonal's 2 endpoints). By reasoning similar to that of Prob. 3.7. there are $\frac{n(n-1)}{2}$ pairs of vertices (in this case, instead of "tennis players", we have "vertices" but the argument is exactly the same).

However, n of these pairs correspond to edges of the polygon rather than diagonals, so we have to substract these from our count, & thus the number of diagonals in a convex polygon with n sides is $\frac{n(n-1)}{2} - n$.

Note that this solution combines our 2 methods for correcting for overcounting. We divide $n(n-1)$ by 2 because every diagonal that we wanted was counted twice. Then we substract n, because some of the "diagonals" that we counted aren't really diagonals, so we have to remove them from our count.

There is also an alternate solution method. For each vertex, there are $n-3$ diagonals which have that vertex as an endpoint: there are $n-1$ other vertices, but the segments to the 2 adjacent vertices are edges. So, counting $n-3$ diagonals for each vertex, we get a count of $n(n-3)$ diagonals. But this counts each diagonal twice — once for each endpoint vertex. Thus we must divide by 2, giving a final answer of $\frac{n(n-3)}{2}$ diagonals.

P4.3: In my state's lottery, 48 balls are numbered from 1 to 48, & 6 are chosen. How many different sets of winning numbers are there? (in this lottery, the order in which the numbers are chosen does not matter)

S-n: $48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43 =$ this is way to choose the 6 balls when we do care the order.

But of course we really don't care about the order, so we have to correct for our overcounting. For any given set of 6 balls, there are 6! different ways to arrange them. So for every set of 6 balls that we can choose without worrying about order, there are 6! ways that we can choose them while worrying about order. Hence, our count of ordered ways to choose 6 balls counts each unordered way to choose 6 balls exactly 6! times. So we need to divide the number of ordered ways to choose 6 balls by 6! to get the number of unordered ways to choose 6 balls.

Therefore our answer is $$\frac{48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43}{6!} = 12{,}271{,}512$$

(This number is also $\binom{48}{6}$.) This means that your chance of winning the lottery is worse than 1 in 12 million.

P.5.4: Coach Grumpf's rival team is the Screamers, coached by Coach Yellsalot. The Screamers also have 12 players, but 2 of them, Bob & Yogi, refuse to play together. How many starting lineups (of 5 players) can Coach Yellsalot make, if the starting lineup can't contain both Bob & Yogi?

S-n: We can solve this problem in (at least) 2 ways, by using casework or by using complementary counting. Let's do casework 1st.

Date: There are 3 ~~different~~ cases for the starting lineup.

Case 1: Bob starts (& Yogi doesn't)

In this case, the coach must choose 4 more players from the 10 remaining players. Then there are $\binom{10}{4}$ lineups that the coach can choose.

Case 2: Yogi starts (& Bob doesn't)

As in case 1, the coach must choose 4 more players from the 10 remaining players. So there are $\binom{10}{4}$ lineups in this case.

Case 3: Neither Bob nor Yogi starts.

In this case coach must choose all 5 players in the lineups from the 10 remaining players. Hence there are $\binom{10}{5}$ lineups in the case.

Total

$$\binom{10}{4}+\binom{10}{4}\binom{10}{5}=210+210+252=672$$

We can also solve this problem by complementary counting. If we have no restrictions then the coach would need to choose 5 players from the entire roster of 12 players which he can do in $\binom{12}{5}$ ways.

But then we have to subtract the lineups that are not allowed, which are the lineups in which both Bob & Yogi start. We already counted them in P.5.2. The coach must choose 3 more players from the remaining 10 players to complete the lineup, so he can do this in $\binom{10}{3}$ ways.

So the total number of lineups (without restrictions), minus the number of lineups which are not allowed. This gives us

$$\binom{12}{5}-\binom{10}{3}=792-120=672.$$

P14.6: What is the coefficient of the term of $(x+2y^2)^6$ with a y^8 in it?

S-n: This problem more than the previous 3 problems really tests our understanding of the Binomial theorem. How do we get a term in the expansion of $(x+2y^2)^6$ with a y^8 in it? The answer is that the term with a y^8 in it is the term in the expansion of $(x+2y^2)^6$ with a y^8 in it? The answer is that the term with a y^8 in it it is the term that has a $(2y^2)^4$ in it.

We know that when expanding $(x+2y^2)^6$ we have to choose 4 copies of $2y^2$ from the six $(x+2y^2)$ term in order to get a term with y^8 in it. This can be done in $\binom{6}{4}$ ways. We can take an x from each of the remaining 2 $(x+2y^2)$ terms that didn't contribute a $2y^2$. Therefore, the relevant term in the expansion of $(x+2y^2)^6$ is $\binom{6}{4}x^2(2y^2)^4 = 240x^2y^8$. The answer is 240.