Date: Biconditional Statements

To prove a theorem that is a biconditional statement of the form $p \leftrightarrow q$, we show that $p \to q$ & $q \to p$ are both true.
Sometimes, iff is used as an abbreviation for "if & only if"
as in "an integer n is odd iff n^2 is odd."

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

Homework. Week 5 ——! надо повторить.

P-1: Prove that e is irrational.

P-2: Use a direct proof to show that every odd integer is the difference of 2 squares.

S: The difference of 2 squares can be factored: $a^2 - b^2 = (a+b)(a-b)$. If we can arrange for our given odd integer to equal $a+b$ & for $a-b$ to equal 1, then we will be done. But we can do this by letting a & b be the integers that straddle $n/2$. For ex, if $n = 11$, then we take $a = 6$ & $b = 5$. Specifically if $n = 2k+1$, then we let $a = k+1$ & $b = k$. Here, then is our proof. Since n is odd we can write $n = 2k+1$ for some integer k. Then $(k+1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1 = n$

This expresses n as the difference of 2 squares.

P-3 Use a proof by contradiction to prove that the sum of an irrational number & a rational number is irrational

S: If r is a rational number & i is an irrat-l number, then $s = r + i$ is an irrational number. So suppose r is rational, i is irrational & s is rational. Then by ex 7. the sum of the rational numbers s & $-r$ must be rational (indeed, if $s = a/b$ & $r = c/d$ where a, b, c & d are integers, with $b \neq 0$ & $d \neq 0$ then by algebra we see that $s + (-r) = (ad - bc)/(bd)$, so that patently $s + (-r)$ is a rational number). But $s + (-r) = r + i - r = i$ forcing us to the conclusion that i is rational. This contradicts our hypothesis that i is irrational. Therefore the assumption that s was rational was incorrect, & we conclude, as desired, that s is irrational.

P4. Prove or disprove that the product of 2 irrational numbers is irrational.

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S: To disprove this proposition it is enough to find a counterexample, since the proposition has a implied universal quantification. We know from Ex.10 that $\sqrt{2}$ is irrational. If we take the product of the irrational number $\sqrt{2}$ & the irrational number $\sqrt{2}$, then we obtain the rational number 2. This counterexamle refutes the proposition.

P.5 Use a proof by contraposition to show that if $x+y\geq 2$ where x an y are real numbers, then $x\geq 1$ or $y\geq 1$.

S: We ~~recast~~ will prove the contrapositive (that if it is not true that $x\geq 1$ or $y\geq 1$ then it's not true that $x+y\geq 2$), using a direct argument. Assume that it is not true that $x\geq 1$ or $y\geq 1$. Then (by De Morgan's Law) $x<1$ & $y<1$. Adding these two inequalities, we obtain $x+y<2$. This is the the negation of $x+y\geq 2$ & our proof is complete.

P-6. Show that if n is an integer & n^3+5 is odd, then n is even using

a) a proof by contraposition

b) a proof by contradiction.

S. a) We must prove the contrapositive: if n is odd, then n^3+5 is even. Assume that n is odd. Then we can write $n=2k+1$ for some integer k. Then $n^3+5=(2k+1)^3+5=$
$8k^3+12k^2+6k+6=2(4k^3+6k^2+3k+3)$.
Thus, n^3+5 is two times so integer, so it is even.

b) Suppose that n^3+5 is odd & that n is odd. Since n is odd, & the product of odd numbers is odd, In 2 steps we see that n^3 is odd. But then subtracting we conclude that 5, being the difference of the 2 odd numbers n^3+5 & n^3 is even. This is not true. Therefore our supposition was wrong, & the proof by contradiction is complete.

P-7 The barber is the one who shaves all those men who do not shave themselves. The question is, does the barber shave himself?

S: Let U be the set all men in the community. This is the universal set. Let $S: U \to \{T, F\}$ be the propositional function taking a man to the truth value representing whether or not he shaves himself. Also we have:

all men such that

$$\forall x \in U : B(x) \iff x \text{ is shaved by barber.}$$

The initial premises can be written in formal logic as:

$$\forall x \in U : (\neg S(x)) \iff B(x)$$

all men that don't shave themselves are shaved by barber

they are shaved by barber

$$B(b) \iff S(b)$$

barber (b) is shaved by barber, so he is one of those who shave themselves.

From these it follows that:

$$\underset{p}{S(b)} \iff \underset{q}{B(b)} \iff \underset{\neg p}{(\neg S(b))}$$

However it follows from transitivity of the bi-conditional operator:

$\to q \iff \neg p$

$$\vdash ((p \iff q) \wedge (q \iff r)) \implies (p \implies r)$$

$\to \neg p$ Basically if p implies q & q implies r then p implies r.)

we have

$$S(b) \iff (\neg S(b))$$

This is a contradiction, so the initial premises are contradictory & cannot both hold. This shows that there does not exist a universe with these properties.

p	q	r	$p \leftrightarrow q$	$q \leftrightarrow r$	$(p \leftrightarrow q) \wedge (q \leftrightarrow r)$	$p \leftrightarrow r$
0	0	0	1	1	1	1
0	0	1	1	0	0	0
0	1	0	0	0	0	1
0	1	1	0	1	0	0
					0	0

$$(p \leftrightarrow q) \wedge (q \leftrightarrow r) \to (p \iff r)$$

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P-8. Show that if x & y are integers & both xy & $x+y$ are even, then both x & y are even.

S: We will use proof by contraposition, the notion of without loss of generality, & proof by case. 1st suppose the x & y are not both even. That is, assume that x is odd, or y is odd (or both). Without loss of generality, we assume that x is odd, so that $x = 2m+1$ for some integer k.

To complete we need to show that xy or $x+y$ is odd. Consider 2 cases. (i) y even, & (ii) y odd. in (i) $y = 2n$ for some integer n, so that $x+y=(2m+1)+2n=2(m+n)+1$ is odd.

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In (ii) $y=2n+1$ for integer n so that $xy=$
$(2m+1)(2n+1)=4mn+2n+2n+1=2(2mn+n+n)+1$
is odd. This complete the proof by ~~contradiction~~ contra-position.

ek 6. Set Theory. Sets 2.1