Ex 14: Let $A, \ldots$

Date: $\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$

Solution: We have.

$\overline{A \cup (B \cap C)} = \bar{A} \cap \overline{(B \cap C)}$ by the 1st De Morgan law.

$= \bar{A} \cap (\bar{B} \cup \bar{C})$ by the 2nd De Morgan law

$= (\bar{B} \cup \bar{C}) \cap \bar{A}$ by the commutative law for intersections.

$= (\bar{C} \cup \bar{B}) \cap \bar{A}$ - by the commu-ve law for unions.

Generalized Unions & Intersections

Let $A_1, A_2 \ldots A_n$ be an indexed collection of sets. We define:

$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$ $\quad \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$

These are well defined since union & intersection are associative

Ex: For $i = 1, 2 \ldots$ let $A_i = \{i, i+1, i+2, \ldots\}$ Then,

$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2 \ldots\} = \{1, 2, 3, \ldots\}$

$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2 \ldots\} = \{n, n+1, n+2 \ldots\} = A_n$

Cardinality of sets

The cardinality of a set A is equal to the cardinality of a set B,

$|A| = |B|$

if and only if there is a bijection from A to B. In this case, we say that A & B are equipotent

If there is an injection from A to B, the cardinality of A is less than or the same as the cardinality of B & we write: $|A| \leq |B|$

When $|A| \leq |B|$ & A & B have different cardinality, we say that the cardinality of A is less than the cardinality of B & write: $|A| < |B|$

Week6 Homework.

P-1 (23) How many elements does each of these sets have where a & b are distinct elements?

a) $P(\{a, b, \{a, b\}\})$

b) $P(\{\emptyset, a, \{a\}, \{\{a\}\}\})$

c) $P(P(\emptyset))$

Date:

Sol-n: a) Since the set we are working with has 3 elements, the power set has $2^3 = 8$ elements

b) Since the set has 4 elements, the power set has $2^4 = 16$ elements

c) The power set of the empty set has $2^0 = 1$ element. The power set of this therefore has $2^1 = 2$ elements. In particular, it is $\{\emptyset, \{\emptyset\}\}$.

P-2. How many different elements does A^n have when A has m elements
37 & n is a positive integer?

The cartesian product $A \times A$ has m^2 elements. (This problem foreshadows the general discussion of counting in Chapter 6). To see that this answer is correct, note that for each $a \in A$ there are m different elements $b \in A$ (including a itself) with which to form the pair (a,b). Since there are m different elements of A, each leading to m different pairs, there must be m^2 pairs altogether. Similarly, for each of the m^2 choices of a & b, there are m choices of c with which to form the triple (a,b,c) in A^3, so A^3 has m^3 elements. Continuing in this way, we see that A^n has m^n elements.

P-3. Find the truth set of each of these predicates where the domain
48 is the set of integers.

a) $P(x): x^2 < 3$ c) $R(x): 2x+1=0$

b) $Q(x): x^2 > x$

Solution. In each case we want the set of all values of x in the domain (the set of integers) that satisfy the given equation or inequality.

a) The only integers whose squares are less than 3 are the integers whose absolute values are less than 2. So the truth set is $\{x \in \mathbb{Z} \mid x^2 < 3\} = \{-1, 0, 1\}$

b) All negative integers satisfy this inequality as do all nonnegative integers other than 0 & 1. So the truth set is $\{x \in \mathbb{Z} \mid x^2 > x\} = \mathbb{Z} - \{0,1\} = \{\ldots, -2, -1, 2, 3, 4, \ldots\}$

c) The only real number satisfying this equation is $x = -1/2$. Because this value is not in our domain, the truth set is empty: $\{x \in \mathbb{Z} \mid 2x+1=0\} = \emptyset$

P-4. (47) Describe a procedure for listing all the subsets of a finite set.

S-n: We can do this recursively, using the idea from Section 5.4 of reducing a problem to a smaller instance of the same problem. Suppose that the elements of the set in question are listed: $A = \{a_1, a_2, a_3, \ldots, a_n\}$. First we will write down all the subsets that do not involve a_n. This is just same problem we are talking about all over again, but with a smaller set — one with just $n-1$ elements. We do this by the process we are currently describing. Then we write these same subsets down again but this time adjoin a_n ~~& then all those that do.~~ to each one.

Date:

Each subset of A will have been written down, then first all those that do not include a_n, & then all those that do.

Fex:, using this procedure the subsets of $\{p, d, q\}$ would be listed in the order $\emptyset$, $\{p\}$, $\{d\}$, $\{p, d\}$, $\{q\}$, $\{p, q\}$, $\{d, q\}$, $\{p, d, q\}$

An alternative solution is given in the answer key in the back of the textbook.

P-5.(17) Show that if A, B & C are sets, then $\overline{A\cap B\cap C} = \bar{A}\cup\bar{B}\cup\bar{C}$

a) by showing each side is a subset of the other side.

b) using a membership table.

Sol-n: This exer. asks for a proof of a generalization of one of De Morgan's laws for sets from 2 sets to three.

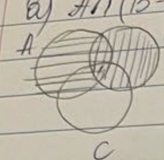
a) This proof is similar to the proof of the 2-set property, given in Ex.10. Suppose $x\in\overline{A\cap B\cap C}$. Then $x\in A\cap B\cap C$, which means that x fails to be in at least one of these 3 sets. In other words, $x\notin A$ or $x\notin B$ or $x\notin C$. This is equivalent to saying that $x\in\bar{A}$ or $x\in\bar{B}$ or $x\in\bar{C}$. Therefore $x\in\bar{A}\cup\bar{B}\cup\bar{C}$ as desired. Conversely, if $x\in\bar{A}\cup\bar{B}\cup\bar{C}$ then $x\in\bar{A}$ or $x\in\bar{B}$ or $x\in\bar{C}$. This means $x\notin A$ or $x\notin B$ or $x\notin C$, so x cannot be in the intersection of A, B and C. Since ~~$x\notin A$~~ $x\notin A\cap B\cap C$, we conclude that $x\in A\cap B\cap C$, as desired.

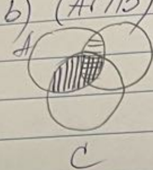
b) The following membership table gives the desired equality, since columns five & nine are identical.

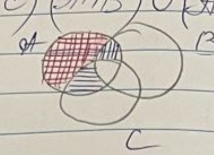
A	B	C	$A\cap B\cap C$	$\overline{A\cap B\cap C}$	$\bar{A}$	$\bar{B}$	$\bar{C}$	$\bar{A}\cup\bar{B}\cup\bar{C}$
1	1	1	1	0	0	0	0	0
1	1	0	0	1	0	0	1	1
1	0	1	0	1	0	1	0	1
1	0	0	0	1	0	1	1	1
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	0	1	1
0	0	1	0	1	1	1	0	1
0	0	0	0	1	1	1	1	1

P-6 Draw the Venn diagrams for each of these combinations ~~Date: the sets~~ A, B, C

a) $A \cap (B-C)$ b) $(A \cap B) \cup (A \cap C)$ c) $(A \cap B) \cup (A \cap \bar{C})$







P-7. What can you say about the sets A & B if we know that

a) $A \cup B = A$?
b) $A \cap B = A$?
c) $A - B = A$?
d) $A \cap B = B \cap A$?
e) $A - B = B - A$?

a) If B adds nothing new to A, then we can conclude that all the elements of B were already in A. In other words, $B \subseteq A$.

b) In this case, all elements of A are forced to be in B as well, so we conclude that $A \subseteq B$.

c). This equality hold precisely when none of the elements of A are in B (If there were any such elements then $A-B$ would not contain all the elements of A) Thus we conclude that A and B disjoint ($A \cap B = \emptyset$)

d) We can conclude nothing about A & B in this case, since this equality always holds.

e) Every element in $A-B$ must be in A, & every element in $B-A$ must ~~be~~ not be in A. Since no item can be in A & not be in A at the same time, there are no elements in both $A-B$ & $B-A$. Thus the only way for these 2 sets to be equal is if both of them are the empty set. This means that every element of A must be in B, & every elem. of B must be in A. Thus we conclude that $A = B$.

(41)

P-8. The symmetric difference of A & B, denoted by $A \oplus B$ is the set containing those elements in either A or B, but not in both A & B.

- Suppose that A, B, & C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

Solution: Yes. To show that $A = B$, we need to show that $x \in A$ implies $x \in B$ & conversely. By symmetry it will be enought to show one direction of this. So assume that $A \oplus C = B \oplus C$ & let $x \in A$ be given. There are 2 cases to consider, depending on whether $x \in C$. If $x \in C$, then by definition we can conclude that $x \notin A \oplus C$. Therefore $x \notin B \oplus C$. Now if x were not in B, then x would be in $B \oplus C$ (since $x \in C$ by assumption). Since this is not true, we conclude that $x \in B$, as desired. For the other case, assume that $x \notin C$. Then $x \in A \oplus C$

ations

Therefore $x \in B \oplus C$ as well. Again, if x were not in B, then it could not be in $B \oplus C$ (since $x \notin C$ by assumption). Once again we conclude that $x \in B$, & the proof is V.

Date:

P-9 (S1) Find $\bigcup_{i=1}^{\infty} A_i$ & $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

a) $A_i = \{-i, -i+1, \ldots -1, 0, 1, \ldots i-1, i\}$

b) $A_i = \{-i, i\}$

c) $A_i = [-i, i]$, that is, the set of real numbers x with $-i \leq x \leq i$

d) $A_i = [i, \infty)$, that is the set of real numbers x with $x \geq i$.

Solution: a) As i increases, the sets get larger: $A_1 \subset A_2 \subset A_3 \ldots$ All the sets are subsets of the set of integers & every integer is included eventually, so $\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}$. Because A_1 is a subset of each of the others, $\bigcap_{i=1}^{\infty} A_i = A_1 = \{-1, 0, 1\}$.

b) all the sets are subsets of the set of integers, & every nonzero integer is in exactly one of the sets, so $\bigcup_{i=1}^{\infty} A_i = \mathbb{Z} - \{0\}$. Each pair of these sets are disjoint, so no element is common to all of the sets. Therefore $\bigcap_{i=1}^{\infty} A_i = \emptyset$

c) This is similar to part (a), the only difference that here we are working with real numbers. Therefore $\bigcup_{i=1}^{\infty} A_i = \mathbb{R}$ (the set of all real numbers) & $\bigcap_{i=1}^{\infty} A_i = A_1 = [-1, 1]$ (the interval of all real numbers between -1 & 1, inclusive).

d) This time the sets are getting smaller as i increases: $\ldots \subset A_3 \subset A_2 \subset A_1$. Because A_1 includes all the others $\bigcup_{i=1}^{\infty} A_i = A_1 = [1, \infty)$ (all real numbers greater than or equal 1). Every number eventually gets excluded as i increases, so $\bigcap_{i=1}^{\infty} A_i = \emptyset$. Notice that ∞ is not a real number so we cannot write $\bigcap_{i=1}^{\infty} A_i = \{\infty\}$

Week 7. Growth analysis & Master theorem

Asymptotic Analysis & Master Theorem

... or data structures, or come up with our