# MATH 189 Warm-up: Matrix Algebra

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Time: 2:00—3:20 & 3:30—4:50pm TueThur
Location: CENTR 115

# Multivariate Data in Matrix Form

- In multivariate analysis, we are interested in jointly analyzing multiple dependent variables.

- These variables can be represented using matrices and vectors.

- The benefits of matrix form:
  1. Simplicity in notation
  2. Expressing important formulas in readable format

# Example 1: Multivariate Data

Suppose a dataset records the characteristics of a person by three variables: $x_1 = height\ (ft)$, $x_2 = weight\ (lb)$ and $x_3 = age\ (year)$.

Then, for every individual in the dataset, his/her characteristics can be recorded in the following *column vector:*

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$$

# Example 1: Multivariate Data

Suppose the $i$-th individual in the dataset has $height = 5'8$ $(78\ inches)$, $weight = 155\ lbs$ and $age = 40$.

In vector notation, these observed data can be expressed as:

$$\mathbf{x_i} = \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix} = \begin{pmatrix} 78 \\ 155 \\ 40 \end{pmatrix}$$

# Example 1: Multivariate Data

The collection of information from multiple individuals naturally forms a matrix.

For example, the observed data of 3 individuals in the dataset can be expressed as the matrix:

$$\mathbf{X} = \begin{pmatrix} 75 & 83 & 69 \\ 155 & 198 & 150 \\ 40 & 25 & 51 \end{pmatrix} \in \mathbb{R}^{3\times 3}$$

Each column is the observed data for one individual.

# Definition of Matrix and Vector

- A matrix is a two-dimensional array of numbers, symbols, or expressions, arranged in rows and columns.
- A vector is a matrix with either one row or one column.
- The dimension of a matrix is expressed as $number\ of\ rows \times number\ of\ columns$. For instance, the matrix in the previous example with 3 rows and 3 columns is said to be a $3 \times 3$ matrix. The vectors in the previous example are $3 \times 1$ matrices.
- A square matrix is one for which the numbers of rows and columns are equal. For instance, a $4 \times 4$ matrix is a square matrix.

# Notations of Matrix and Vector

- Most of the time in this course, we only consider matrices and vectors whose entries are real numbers.

- A real-valued matrix $\mathbf{M}$ of size $n \times m$ can be denoted as $\mathbf{M} \in \mathbb{R}^{n \times m}$.

- A real vector $\mathbf{v}$ of size $n \times 1$ or $1 \times n$ can be denoted as $\mathbf{v} \in \mathbb{R}^{n \times 1}$ or $\mathbf{v} \in \mathbb{R}^{1 \times n}$. By default, every vector $\mathbf{v}$ is implicitly assumed to be a column vector, i.e., $\mathbf{v} \in \mathbb{R}^{n}$.

# Transpose of a Matrix

- **Definition**: The transpose of a matrix $\mathbf{A}$ is a matrix whose rows are the transpose of the columns of $\mathbf{A}$.
- The transpose of $\mathbf{A}$ is denoted as $\mathbf{A}'$ or $\mathbf{A}^{\mathrm{T}}$.

Example 2: For a 3×2 matrix $\mathbf{A} = \begin{pmatrix} \mathbf{3} & \mathbf{7} \\ \mathbf{4} & \mathbf{6} \\ \mathbf{2} & \mathbf{8} \end{pmatrix}$, its transpose is a 2×3 matrix $\mathbf{A}' = \begin{pmatrix} \mathbf{3} & \mathbf{4} & \mathbf{2} \\ \mathbf{7} & \mathbf{6} & \mathbf{8} \end{pmatrix}$.

# Symmetric Matrices

- **Definition**: A squared matrix $\mathbf{A}$ is symmetric if $\mathbf{A} = \mathbf{A}'$.

- For a symmetric matrix, the element in the $i$-th row and $j$-th column equals the element in the $j$-th row and $i$-th column, i.e.,

$$a_{ij} = a_{ji}.$$

- Important examples of symmetric matrices in multivariate statistics include the variance-covariance matrix and the correlation matrix.

# Linear Combination of Two Matrices

- Two matrices may be added if and only if they have the same dimensions. To add two matrices, add corresponding elements.
- Example 3: $\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 11 & 15 \end{pmatrix}$
- Example 4: If $\mathrm{A}$ and $\mathrm{B}$ are two $n \times n$ matrices, $\alpha$ and $\beta$ are two scalars, then $\mathrm{C} = \alpha \mathrm{A} + \beta \mathrm{B}$ is also an $n \times n$ matrix whose $(i, j)th$ element is
$$c_{ij} = \alpha a_{ij} + \beta b_{ij}$$

# $\times$ Multiplication of Two Matrices

- To be able to perform the matrix multiplication $\mathbf{C} = \mathbf{AB}$, the number of columns in matrix $\mathbf{A}$ must equal the number of rows in matrix $\mathbf{B}$.

- The $(i, j)$-th entry of the answer matrix $\mathbf{C}$, is the cross-product sum of the $i$-th row of the matrix $\mathbf{A}$ and the $j$-th column of the matrix $\mathbf{B}$.

- If $\mathbf{A}$ is of size $n \times m$ and $\mathbf{B}$ is of size $m \times k$, then $\mathbf{C} = \mathbf{AB}$ is of size $n \times k$.

# Identity Matrix

- **Definition**: An identity matrix is a square matrix that has 1's on its diagonal (from upper left to bottom right) and has 0's elsewhere.

- Example 5: The $3\times3$ identity matrix is $\mathbf{I} = \mathbf{I_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

- $\mathbf{I}$ is called the identity matrix because $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$ for any squared matrix $\mathbf{A}$ (of the same size as $\mathbf{I}$).

# Matrix Inverse and Trace

- **Definition**: The inverse of a square matrix $\mathbf{A}$, if it exists, is denoted by $\mathbf{A}^{-1}$, such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

- **Definition**: The trace of an $n \times n$ matrix $\mathbf{A} = (a_{ij})$ is the sum of its diagonal elements:

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}.$$

For two matrices $\mathbf{A}$ of size $m \times k$ and $\mathbf{B}$ of size $k \times m$,

$$\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA}).$$

# Eigenvalues and Eigenvectors

- Given a symmetric matrix $\mathbf{A}$, a scalar $\lambda$ and a vector $\boldsymbol{v}$, which satisfy the following linear equation

$$\mathbf{A}\boldsymbol{v} = \lambda\boldsymbol{v},$$

  then $\lambda$ is an eigenvalue of $\mathbf{A}$ and $\boldsymbol{v}$ is the corresponding eigenvector.

- A symmetric matrix of size $n \times n$ has no more than $n$ eigenvalues.

- If $\mathbf{A}$ has $n$ eigenvalues $\lambda_1, \dots, \lambda_n$, then $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$.

# Positive Definite Matrix

- A symmetric matrix $\mathbf{A}$ of size $n \times n$ is said to be positive definite, if $\boldsymbol{z}^{\mathrm{T}}\mathbf{A}\mathbf{z} > 0$ for every non-zero vector $\mathbf{z}$ of size $n$. In short, $\mathbf{A} > 0$.

- Positive Semi-Definite: if $\boldsymbol{z}^{\mathrm{T}}\mathbf{A}\mathbf{z} \geq 0$. In short, $\mathbf{A} \geq 0$.

- A symmetric matrix of size $n \times n$ is positive definite if and only if all of its eigenvalues are positive.

- A positive definite matrix is invertible.