Why are the eigenmodes of stable laser resonators structurally stable?

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Optical resonators are usually examined wave optically. We consider geometrical imaging in stable canonical resonators. We show that, with important exceptions related to eigenmode degeneracy, stable resonators generally image all transverse planes into each other. This insight leads to an intuitive understanding of important properties of the corresponding eigenmodes, most notably their well-known structural stability, i.e., the property that the eigenmodes retain their shape on propagation.

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Optical resonators are usually examined by studying the effect of one or more round trips of a light beam through the resonator, starting and finishing in the same plane. This approach has led to the definition of important concepts such as eigenmodes (beams whose amplitude cross section is unchanged after one round trip) and geometrical stability (a property of resonators with mirrors that are curved such that light rays bouncing between them stay within a finite distance of the optic axis[1]—geometrically stable resonators trap light rays).

Many lasers contain an optical resonator that satisfies the criterion for geometrical stability. One of the properties associated with these geometrically stable resonators is that their eigenmodes are structurally stable; i.e., they do not change shape on propagation. The intensity cross sections of the fundamental Gaussian eigenmode of most commercially available lasers, for example, have a different size at different positions in the beam but always the same shape. Note that structurally stable light beams must satisfy specific conditions[2]—they are the exception rather than the norm.

The standard, wave-optical explanation for structural stability involves establishing three independent properties of Hermite–Gaussian (HG) modes: They are solutions of the paraxial wave equation, they are eigenmodes of stable canonical resonators, and they do not change shape on propagation.[3] It also has to be shown that any two or more HG modes that can be resonant simultaneously add up to another structurally stable beam.

Here we consider geometrical imaging in stable resonators, simply by studying the positions of successive geometrical images of an initial transverse plane. Our earlier application of this approach to unstable canonical resonators revealed that each round trip geometrically images special self-conjugate planes back into themselves with a magnification $M \neq \pm 1$.[4] Here we show that stable canonical resonators generally image every transverse plane into every other transverse plane. This surprisingly simple result is at the heart of a ray-optical, arguably more intuitive, and yet very powerful explanation for many properties of the eigenmodes, such as degeneracy, symmetry, and structural stability, of stable laser resonators.

Figure 1 shows a simple canonical resonator consisting of two mirrors, one with focal length f, the other one planar, separated by a distance l. As far as geometrical imaging is concerned, every canonical resonator, comprising two mirrors of focal lengths f_1 and f_2, separated by a distance l_0, can be represented by an equivalent resonator of this type with parameters

$$f = \frac{f_1 f_2}{f_1 + f_2 - l_0}, \qquad l = \frac{l_0}{2}\left(1 + \frac{f}{f_1} + \frac{f}{f_2}\right). \qquad (1)$$

This works because a combination of two thin spherical mirrors can be represented by one such mirror, just as the imaging characteristics of two thin lenses are equivalent to those of one single thin lens.[5] The principal planes are generally located outside the plane of the single mirror (or lens); this is taken care of by the modified resonator length. The second, planar mirror simply completes the resonator.

During each round trip through a stable resonator of the type shown in Fig. 1, a light beam that is initially propagating to the left is first reflected off the left (concave) mirror and then off the right (planar) mirror. As far as geometric imaging is concerned, the light from any plane P_n at an object distance o_n is thereby imaged by the concave mirror into another plane, P_{n+1}, at an image distance i_n. Note that these planes are not necessarily situated inside the resonator. The object and image distances and the mirror's focal length, f, are related through the mirror equation. The image plane is also the object plane for the next round of imaging.

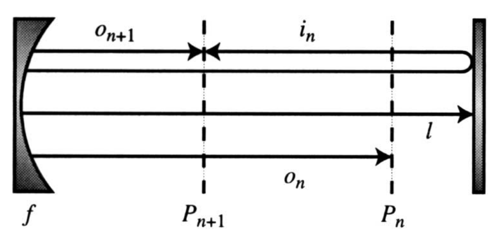

Fig. 1. Imaging in a canonical resonator with one planar and one thin mirror of focal length f. As far as geometrical imaging is concerned, resonators of this type are representative of all canonical resonators. The positions of the image and object planes are described in terms of their distance in front of the curved mirror.

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As can be seen from Fig. 1, the object distance o_{n+1} is given in terms of the image distance, i_n, by the equation $o_{n+1} = 2l - i_n$, so the relationship between the nth and $n + 1$th object distances can be expressed as

$$\frac{1}{o_n} + \frac{1}{2l - o_{n+1}} = \frac{1}{f} \cdot \tag{2}$$

We can easily solve Eq. (2) for o_{n+1} and iterate it to find the positions of successive images of any initial plane. However, for an analytical discussion the following reformulation is more useful.

We now describe the position of the planes P_n by the variable

$$\Omega_n = \pi + 2 \arctan \frac{O_n - L}{\sqrt{1 - L^2}}, \tag{3}$$

where we use the dimensionless quantities

$$O_n = \frac{o_n}{f} - 1, \qquad L = \frac{l}{f} - 1 \tag{4}$$

(stable resonators correspond to the range $-1 < L < 1$). In terms of Ω_n, a round of imaging in the resonator can be described as

$$\Omega_{n+1} = (\Omega_n + \delta) \mathrm{mod}\, 2\pi, \tag{5}$$

where

$$\delta = \pi + 2 \arctan \frac{L}{\sqrt{1 - L^2}} \cdot \tag{6}$$

Ω_n behaves as an angle, taking values from 0 to 2π. Each new round of imaging, i.e., each iteration of Eq. (5), simply corresponds to the addition of a constant step angle δ (see Fig. 2).

The dynamics of successive iterations of Eq. (5) can be divided into two cases. In the first case the step angle δ can be written in the form

$$\delta = \frac{p}{q} 2\pi, \tag{7}$$

where p and q are both integers, i.e., if $\delta/2\pi$ is a rational number. In this case, the angle Ω has performed p full 2π rotations after q iterations. This means that the initial plane is imaged into $(q - 1)$ intermediate planes and, after q round trips through the resonator, again into the initial plane. This case is closely related to periodic focusing.[6] In the second case, δ cannot be written in the form of Eq. (7); i.e., $\delta/2\pi$ is irrational. The focusing is not periodic; no plane is ever imaged back into itself but is instead imaged into infinitely many other planes. What is more, the values of Ω_n get arbitrarily close to any value in the range 0 to 2π; mathematically speaking, the set $\{\Omega_n\}$ is dense in the interval $[0, 2\pi)$. As the interval $\Omega_n \in [0, 2\pi)$ is mapped one to one onto the intervals $O_n \in (-\infty, \infty)$ and $o_n \in (-\infty, \infty)$, the sets $\{O_n\}$ and $\{o_n\}$ are also dense everywhere. Physically speaking, every plane is imaged into every other plane.

From number theory it is known that both rational and irrational numbers are dense everywhere (see, for example, Ref. 7). This implies that every stable resonator is infinitely close in parameter space to the periodic-imaging case and to the aperiodic-imaging case. However, as there are infinitely more irrational numbers than rational numbers,[8] it can be argued that stable resonators with irrational values of $\delta/2\pi$ are the normal case. This is the justification for the statement that stable resonators generally image all transverse planes into each other.

We now consider the implications of these results for the eigenmodes of stable resonators. Eigenmodes are stationary, so all the imaging processes occur simultaneously. The intensity cross sections in conjugate planes, i.e., planes that are imaged into each other, have to be images of each other, so they have to be similar. Therefore the eigenmodes of resonators that image every plane into every other plane, i.e., resonators corresponding to irrational values of $\delta/(2\pi)$, are structurally stable. Reassuringly, it can be shown that geometrical imaging predicts the same scaling of the eigenmodes on propagation as the standard wave-optical approach, namely, matching a Gaussian beam's phase-front curvature at the mirrors to the respective curvatures of the mirrors.[9]

The above argument glosses over one subtle point: When the magnification is negative, the intensity cross sections in two conjugate planes are similar but rotated through 180°. We now show that the eigenmodes of aperiodic-imaging resonators have intensity cross sections that are invariant under 180° rotations and that, therefore, the conclusion that they are structurally stable still holds.

When an object at object distance o is geometrically imaged to an image distance i, the image is magnified by a factor of $M = -i/o$. In a stable resonator, the nth imaging iteration has a magnification

$$M_n = -i_n/o_n = -1/O_n. \tag{8}$$

This implies that the magnification is positive when the position of the object plane is described by a variable $O_n < 0$. Conversely, $M_n < 0$ whenever $O_n > 0$.

In terms of the variable Ω these regions of positive and negative magnification translate into $0 < \Omega_n < 2\pi - \delta$ $(M_n > 0)$ and $2\pi - \delta < \Omega_n < 2\pi$ $(M_n > 0)$. Note that the region of negative magnification has

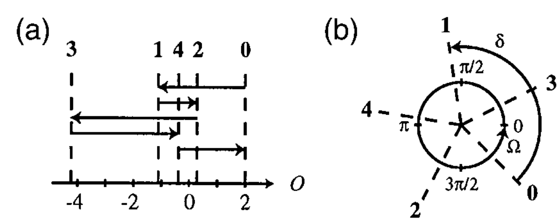

Fig. 2. Examples of periodic imaging in terms of (a) O_ns and (b) Ω_ns, represented as angles. The bold numbers indicate the round-trip numbers, n. In the latter representation, each round trip simply advances the angle Ω by δ. The figure is drawn for $\delta = (2/5)\, 2\pi$.

width δ, the step angle of each iteration. It is therefore not possible to iterate around the Ω circle without stepping into this region of negative magnification.

In general, the fact that the width of the region of negative magnification is the step angle means that, starting with any plane in any stable resonator, sooner or later planes are encountered that are imaged with negative magnification. In the case in which $\delta/(2\pi)$ is irrational, in which all the eigenmode's transverse intensity cross sections are similar, this implies that the intensity cross sections in some planes are rotated through 180° with respect to the others. In at least one position the rotated and nonrotated intensity cross sections must be immediately adjacent to each other; physically, this is possible only if the rotated and nonrotated intensity cross sections are identical, i.e., if the eigenmode's transverse intensity cross sections are invariant under 180° rotations. The eigenmode's transverse intensity cross sections in all planes are therefore similar and not rotated with respect to each other, which means that the eigenmode is structurally stable.

Do these results have any connection with wave optics? Like every light beam, the eigenmodes of stable optical resonators can be described as superpositions of HG modes with the same waist position and Rayleigh range. Under what circumstances can a superposition of HG modes be structurally unstable and at the same time be an eigenmode of a stable optical resonator? For their superposition to be structurally unstable, HG eigenmodes have to be of different mode orders N. For their superposition to be another eigenmode, individual eigenmodes have to be degenerate, which is a wave-optical property.

In terms of the frequency $\omega_{N',n'}$ of resonant modes of order N' that pass through n' full 2π phase cycles during one round trip through the resonator,[10] the condition for degeneracy can be written in the form

$$\omega_{N,n+r} = \omega_{N+s,n}\,, \tag{9}$$

where r and s are nonzero integers. Equation (9) holds if two modes with the same frequency but with different mode orders, N and $N + s$, are simultaneously resonant. This is possible because the phase difference that is due to the difference in mode orders, s, is exactly compensated for by one of the modes passing through r additional 2π phase cycles during each round trip.

On substitution of the equation for $\omega_{N,n}$,[10] the degeneracy condition, Eq. (9), can be expressed as

$$\epsilon = \frac{r}{s}\,\pi\,, \tag{10}$$

with

$$\epsilon = \arctan\frac{z_2}{z_R} - \arctan\frac{z_1}{z_R}\,\cdot \tag{11}$$

z_1 and z_2 are the mirror positions with respect to the beam waist and z_R is the beam's Rayleigh range. It can be shown that ϵ is closely related to the step angle δ, which determines whether the geometric imaging in the resonator is periodic, through the equation

$$\delta = 4\left|\left[\left(\epsilon + \frac{\pi}{2}\right)\mathrm{mod}\ \pi\right] - \frac{\pi}{2}\right|\cdot \tag{12}$$

In particular, $\delta/(2\pi)$ is rational if and only if ϵ/π is rational, so δ satisfies the periodic-imaging condition if and only if ϵ satisfies the eigenmode-degeneracy condition. Eigenmode degeneracy and periodic imaging are therefore equivalent.

In conclusion, this Letter has provided a simple explanation, purely in terms of geometrical optics, for important properties of the eigenmodes of stable resonators, most notably degeneracy and structural stability. The details of the mathematical proof are not straightforward and occasionally quite subtle; however, the principal result of our analysis, namely, that in almost all stable resonators every plane is imaged into every other plane, is easy to understand and provides fundamental insights into laser resonators.

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