Probability and Discrete Probability Distributions

Experiment, Event, Sample space, Probability, ~~Counting rules~~, Conditional probability, Bayes's rule, random variables, moments generator function
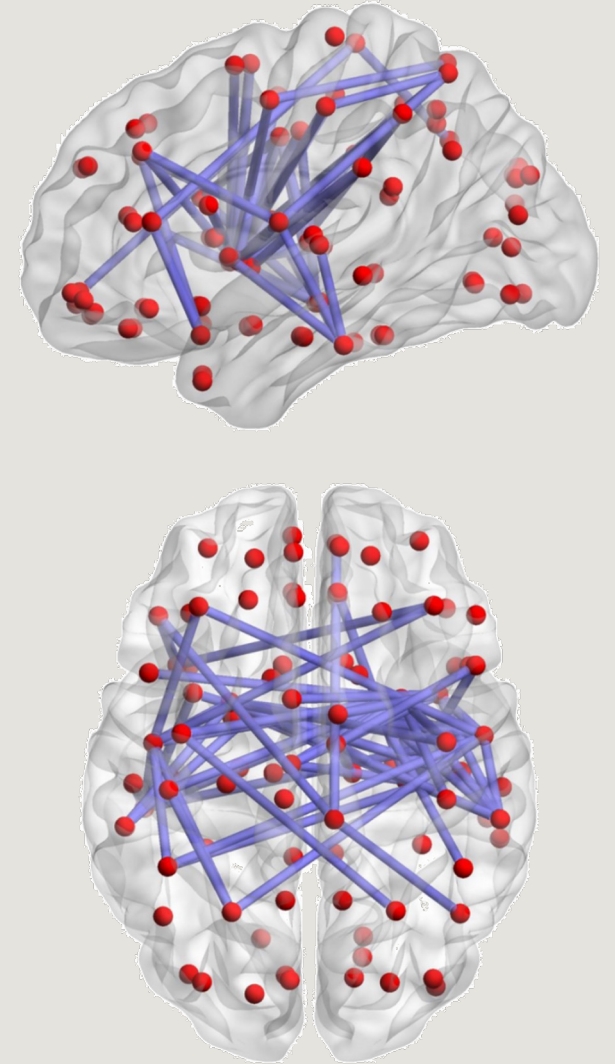
Agenda

Probability: different definitions

Experiments & Events

Conditional Probabilities

Bayes' Theorem

Random Variables

Probability Distributions

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Why Learn Probability?

- **Probability** is a way to describe **how likely** something is to happen.

  It's a measure of uncertainty — a number between **0** and **1**, where:

  - **0** means the event **cannot happen** (impossible),

  - **1** means the event **will definitely happen** (certain),

  - and numbers in between (like 0.3 or 0.75) show *degrees of likelihood*.

- For example:

  - The probability of flipping heads on a fair coin is **0.5** (50%).

  - The probability of rolling a 6 on a fair die is **1/6 ≈ 0.167** (about 17%).

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Probability: The Classical Definition (When All Outcomes Are Equally Likely)

- If an experiment has a fixed number of equally likely outcomes, then:

$$P(\text{event}) = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}}$$

- **Example:**

  - A Fair die has 6 outcomes.

  - The event "rolling an even number" has 3 favorable outcomes (2, 4, 6).

$$P(\text{even}) = \frac{3}{6}$$

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Probability: The Frequentist View (Based on Observations)

- If you repeat an experiment many times, the **relative frequency** of an event A

$$\text{Relative Frequency} = \frac{\text{Number of times event occurs}}{\text{Total number of trials}}$$

- As the number of trials n grows large (n → ∞), this relative frequency *converges* to the real probability — that's the **Law of Large Numbers**.

- Therefore, we assume that relative frequency approximates the true probability

$$p(A) = \lim_{n \to \infty} \frac{f}{n}$$

Probability: The Bayesian View (Probability as Belief)

Sometimes we talk about probability when we don't repeat experiments, like:

- "What's the probability that this satellite component will fail in orbit?"

- Here, probability expresses **degree of belief** — how confident we are in a statement, given what we know which could be just a rough estimate.

- This is the **Bayesian interpretation** — it uses probability to quantify uncertainty in our knowledge.

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Probability goes from theory → data
You know something about the population, so you predict what data should look like.

- Suppose I know exactly the failure rates of different components in a high-integrity avionics system during a mission is 0.001 (that's 1 in 1,000).

- Then you can use **probability** to predict or simulate outcomes.

  - "What's the probability that in a batch of 1000 flights, at least one failure occurs?"

  - "What's the expected number of failures over 10,000 flights?"

- This is forward reasoning — from the known model to possible data.

- This is probabilistic reasoning as I know the population (failure rates) and predict the performance of a specific system

Example: Statistical Reasoning — Inferring from Observed Data

- Suppose you *don't know* the component's true failure probability.

- You can monitor the performance of a random sample of avionics systems over time and analyze the data to estimate the failure rates of the components.

Example:

- You collect data from 10,000 flights and see that 8 failures occurred.From this **sample**, you can estimate the underlying (population) probability:

$$\hat{p} = \frac{8}{10{,}000} = 0.0008$$

This is statistical reasoning as I infer the population characteristics (failure rates) from sample observations. **You can also** quantify uncertainty (for example, build a confidence interval). That's **statistical inference** — going from data back to the model.

Of course, the larger your sample is the more certain you can be about your estimate.

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Why Learn Probability?

Probability underpins:

- **Statistics**: Inferring population characteristics from samples.

- **Risk Analysis:** Estimating the likelihood and cost of system failures or accidents.

- **Machine Learning:** Modeling uncertainty and predicting outcomes.

- **High-Integrity Systems:** Measuring reliability and safety margins.

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Basic terminology: Experiment and Event

- An **Experiment** is the process through which observations (or measurements) are obtained.

- An **event** is an outcome of an experiment, typically denoted by a capital letter.

- **Probability** is applied to events, assessing the likelihood of a specific event occurring.

- **Experiment**: System Integrity Check in a Secure Database

  ➢ Event A: The system passes all security compliance tests.

  ➢ Event B: A data breach is detected during the check.

- **Experiment: Monitoring Server Uptime**

  ➢ Event A: The server remains operational without downtime for 24 hours.

  ➢ Event B: The server experiences several unexpected downtime within the 24-hour period
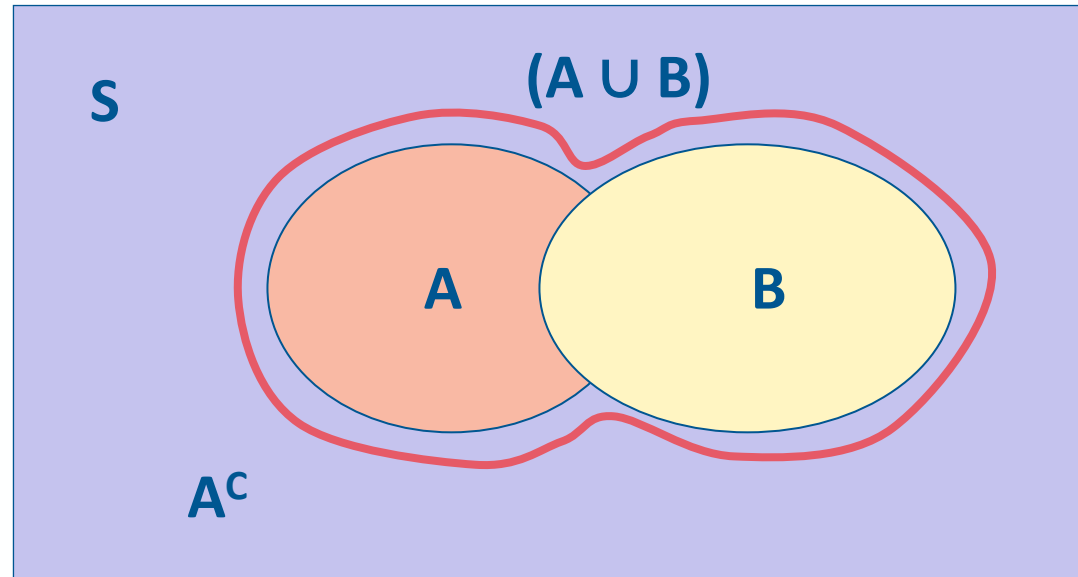
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Basic terminology: Sample Space and Simple Events

- A **simple event** is a single, specific outcome from a random experiment. It's the most basic type of event because it cannot be broken down into a combination of other events.

- For example, rolling a "3" on a standard six-sided die is a simple event because it is only one possible outcome out of the six possibilities.

- An **event** can be a collection of one or more simple events.

  - A: an even number

  - B: a number < 5

- All the possible outcomes of an experiment are called **sample space**, S.

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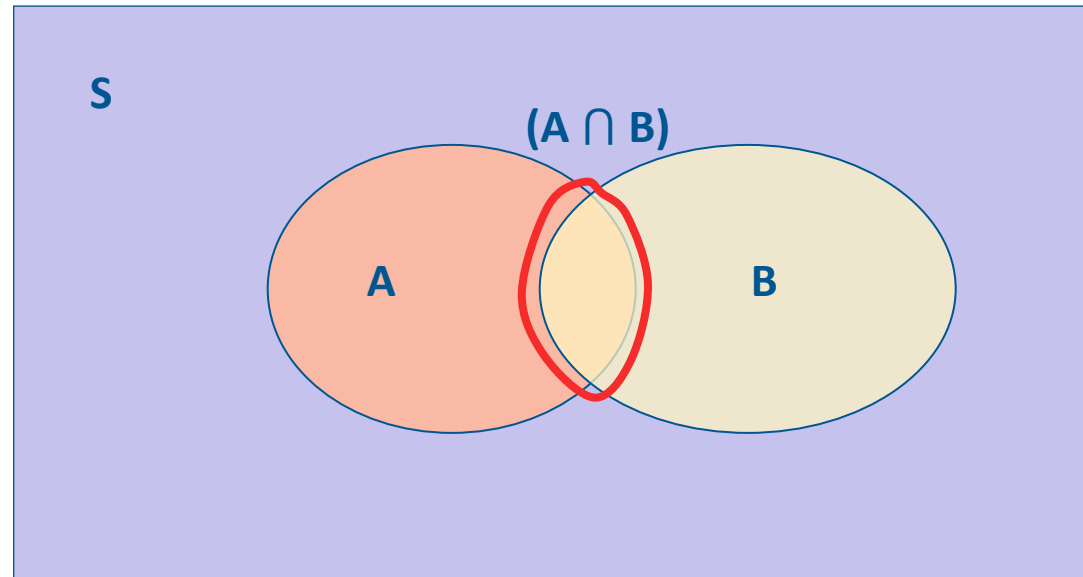OF APPLIED SCIENCES

Event Relations: Union of events

- The **union** of two events, **A** and **B**, is the event that either **A or B or both** occur when the experiment is performed.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
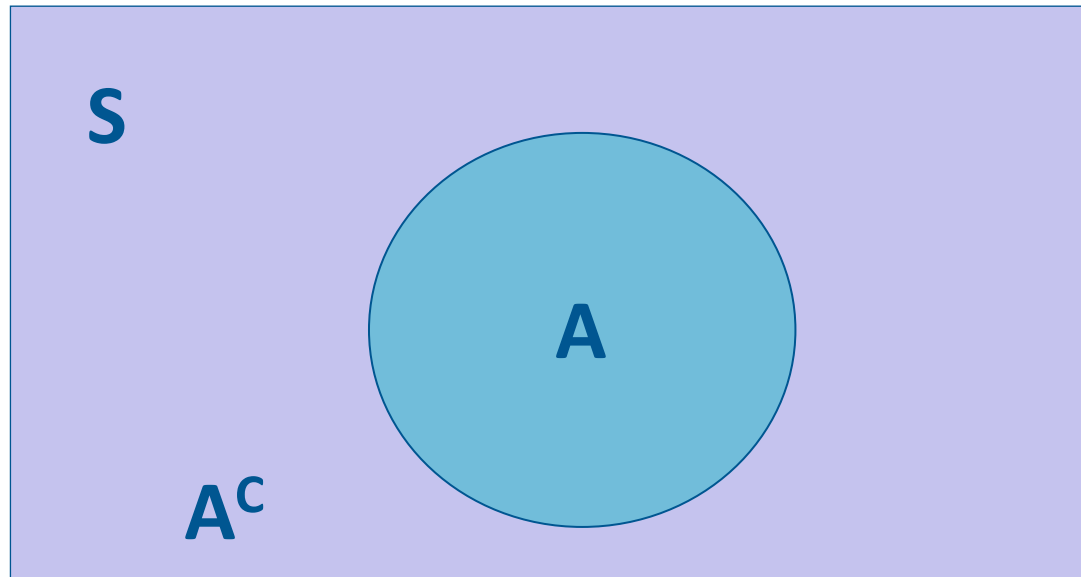
Event Relations: complementary event

- The **intersection** of two events, **A** and **B,** is the event that both A **and** B occur when the experiment is performed. We write **A ∩ B.**

Event Relations: complementary event

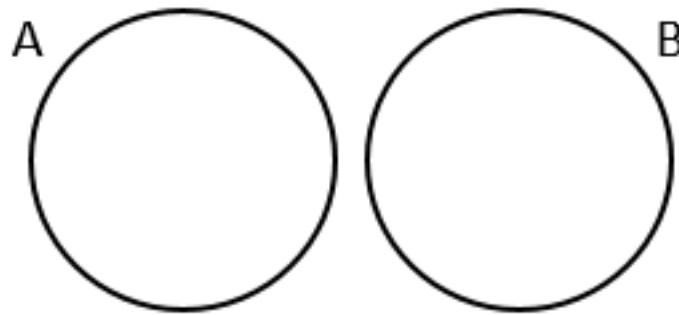- The **complement** of an event **A** consists of all outcomes of the experiment that do not result in event **A**.

- We write A^C.

$$P(A^C) = 1 - P(A)$$

Event Relations: Mutually exclusive events

- Two events are mutually exclusive if, when one event occurs, the other cannot, and vice versa.

- **Occurrence of one event will result in the non-occurrence of the other.**

- Let A, B be mutually exclusive events: $P(A \cap B) = 0$ & $P(A \cup B) = P(A) + P(B)$

$$P(A \text{ or } B) = P(A) + P(B)$$

Event Relations: Mutually exclusive events

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Experiment: stop a student at FraUAS
  - A: observe an HIS student
  - B: observe an Inclusive Design student

The probability of A and B is the 0 (it is impossible for a student to enroll in both programs):
$$P(A \text{ and } B) = P(HIS \cap ID) = 0$$

The probability of A or B is the sum of the individual probabilities:
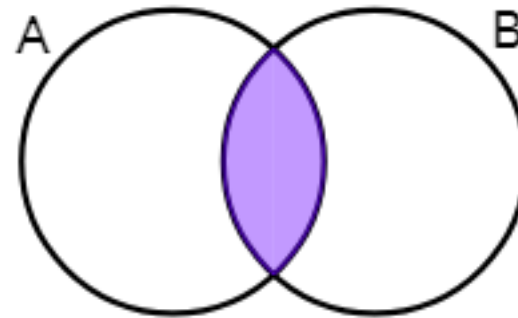$$P(A \text{ or } B) = P(HIS \cup ID) = P(HIS) + P(ID)$$

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Event Relations: Not Mutually exclusive events

- Not mutually exclusive if, when one event occurs, the other can still occur.

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \text{ and } B)$$

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Not Mutually exclusive events

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$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \text{ and } B)$$

- Altogether there are **180 HIS students**.

- Prof. Marouf supervises **17 people on "Project HIS"**

- He teaches **170 students "IDA".**

- How many students study **both IDA and Project HIS** under the supervision of Prof. Marouf?

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Quiz

How many students study **both IDA and Project HIS** under the supervision of Prof. Marouf?

- Altogether there are **180 HIS students**.

- Prof. Marouf supervises **17 people on "Project HIS"**

- He teaches **170 students "IDA".**

a. 7

b. 10

c. 18

Quiz

- How many students study **both IDA and Project HIS** under the supervision of Prof. Marouf?
  - Altogether there are **180 HIS students**.
  - Prof. Marouf supervises **17 people on "Project HIS"**
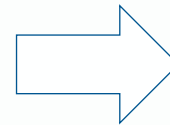 - He teaches **170 students "IDA".**

This is a case of **not** Mutually Exclusive (a student can study IDA **and** project).

Let's say **b** is how many students both subjects:
- people studying IDA Only must be 170-b
- people studying project Only must be 17-b

And we know there are **180** people, so:

$(170-b) + b + (17-b) = 180 \quad \rightarrow \quad b = 7$

So, we know all this now:
P(IDA) = 170/180
P(project) = 17/180
P(IDA Only) = 163/180
P(project Only) = 10/180
P(IDA or project) = 180/180 = 1
P(IDA and project) = 7/180

Lastly, let's check with our formula:
P(A or B) = P(A) + P(B) − P(A and B).
180/180 = 170/180 + 17/180 − 7/180

Scenario: Aircraft Flight Control System Failure

System Failure (Event A): Occurs if at least one of the following software modules fails. These modules depend on each other and failure in one module increase the chance of failure in other modules.

- **Event S1:** Failure of the autopilot software. P(S1)=0.005

- **Event S2:** Failure of the navigation software. P(S2)=0.007

- **Event S3:** Failure of the sensor data processing software. P(S3)=0.004

- **Calculate the the probability of system failure:**

$$P(A)=\ldots\ldots$$

Scenario: Aircraft Flight Control System Failure

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- **Event S2:** Failure of the navigation software. P(S2)=0.007

- **Event S3:** Failure of the sensor data processing software. P(S3)=0.004

Actually, we need more information to be able to calculate the probability. Assume:

- $P(S1 \text{ and } S2)=0.002$
- $P(S1 \text{ and } S3)=0.001$
- $P(S2 \text{ and } S3)=0.001$
- $P(S1 \text{ and } S2 \text{ and } S3)=0.0001$

So, probability of system failure:

$P(A)=P(S1)+P(S2)+P(S3)-P(S1 \cap S2)-P(S1 \cap S3)-P(S2 \cap S3)+P(S1 \cap S2 \cap S3)$

P(A)=(0.005+0.007+0.004)−(0.002+0.001+0.001)+(0.0001) = 0.0121

independent events

- Two events, **A** and **B**, are said to be **independent** if the occurrence or nonoccurrence of one of the events does not change the probability of the occurrence of another event.

- Occurrence of one event will have no influence on the occurrence of the other.

- The probability that either event **A** or **B** occurs is given by:

$$P(A \text{ and } B) = P(A \cap B) = P(A)\,P(B)$$

The probability of A or B is :

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)\,P(B)$$

Quiz: which of the following statements is true?

1. Mutually exclusive events **cannot be independent** unless one of them has probability 0

2. Mutually exclusive events **cannot be independent**

3. Mutually exclusive events **are by default independent**

Conceptually, mutually exclusive events are dependent. If one event occurs, the other cannot.

The probability of both events occurring together is zero (P(A ∩ B) = 0). However, if both events have a non-zero probability individually, then P(A) * P(B) would be greater than zero, contradicting the condition for independence, where P(A ∩ B) should equal P(A) * P(B).

The only scenario where mutually exclusive events can be considered independent is if at least one of the events has a probability of zero. In such cases, P(A) * P(B) = 0, which satisfies the condition for independence. Therefore, mutually exclusive events cannot be independent unless one of them has a probability of zero.

Quiz

- In a certain factory, 10% of the products are known to have some minor production defects and needs manual improvements. Three products were randomly selected from on Monday. What is the probability that exactly one of the three has a defect?

    a. 0.1%

    b. 0.234

    c. 0.333

Define D: Defect   N: No defects

It is either the first one or second one or third one that has defects

P(exactly one has defect) = P(DNN) + P(NDN) + P(NND)

$$= P(D)P(N)P(N) + P(N)P(D)P(N) + P(N)P(N)P(D)$$

$$= (.1)(.9)(.9) + (.9)(.1)(.9) + (.9)(.9)(.1) = 3(.1)(.9)^2 = .243$$

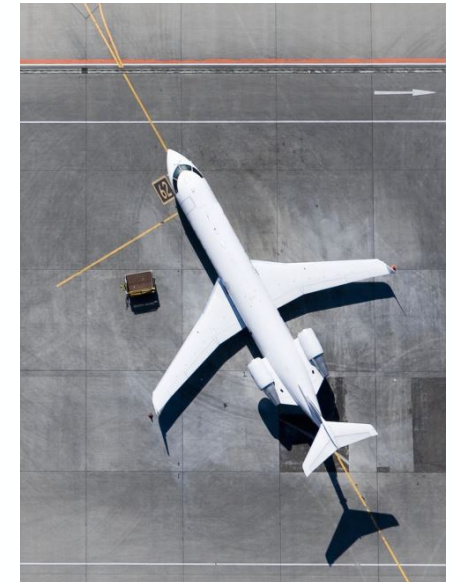# Example Scenario: Aircraft Flight Control System Failure

- Consider a high-integrity system such as an aircraft's flight control system. Suppose we are interested in the probability of a <u>system failure event</u> (Event A) due to software faults.

  The Failure scenarios are independent from each and they can be:

  - Failure of the autopilot software (Event S1)

  - Failure of the navigation software (Event S2)

  - Failure of the sensor data processing software (Event S3)

- The probability of system failure (P(A)) is the sum of the probabilities of these simple events:

  P(A) = P(S1∪S2∪S3) = P(S1)+P(S2)+P(S3)−P(S1∩S2)−P(S1∩S3)−P(S2∩S3)+P(S1∩S2∩S3)

  P(A) = P(S1)+P(S2)+P(S3)− P(S1) × P(S2) − P(S1) × P(S3) − P(S2) × P(S3) + P(S1) × P(S2) × P(S3)

Conditional Probabilities

- Conditional probability measures the likelihood of event A occurring given that event B has occurred.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) \neq 0$$

"given"

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Example: Conditional Probabilities

- Consider a flight control system with two components:

  - Event A: Failure of the autopilot system

  - Event B: degradation of the sensor system

- Suppose:

  - P(A ∩ B) = 0.0001 (degradation and failure together)

  - P(B) = 0.002 (sensor degradation)

- Then:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.0001}{0.002} = 0.05$$

Interpretation: given that the sensor has failed, there is a 5% chance that the autopilot will also fail or the probability of sensor failure increases from 0.2% to 5%.

What about independent events

We can redefine independence in terms of conditional probabilities:

Two events A and B are **independent** if and only if

$$P(A|B) = P(A) \quad \text{or} \quad P(B|A) = P(B)$$

$P(A \cap B) = P(A) \, P(B \text{ given that A occurred}) = P(A)P(B|A) = P(A) \, P(B)$

Otherwise, they are **dependent**.

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Quiz

- In a factory, 49% of the products come from a new, fully automated production line, while the remaining 51% come from the old production line. Products from the new line have a defect rate of 8%, and those from the old line have a defect rate of 10%.

- If one product is randomly selected by the quality team, what is the probability that it **is defective and comes from the new production line**?

Define D: Defect   NL: New production line
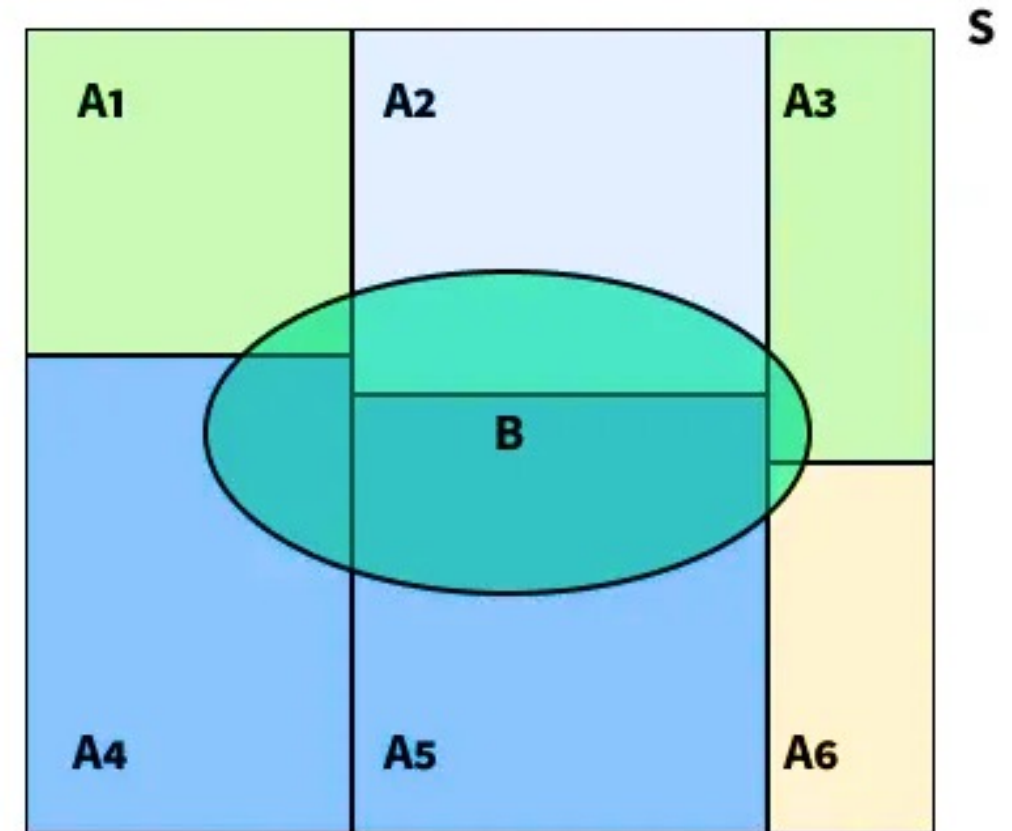
From the example, P(NL) = .49 and P(D| NL) = .08.

Use the Multiplicative Rule to calculate the joint probability:

P(D and new line) = P(D and NL) = P(NL)P(D|NL) = .49(.08) = .0392

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The Law of Total Probability

- The Law of Total Probability is used to find the probability of an event A by considering all possible ways A can happen through a partition of the sample space.

- Let A_1, A_2, A_3,..., A_k be mutually exclusive and exhaustive events (that is, one and only one must happen). Then the probability of any event A can be written as:

  $$P(B) = P(B \cap S_1) + P(B \cap S_2) + \ldots + P(B \cap S_k)$$
  $$= P(S_1)P(B|S_1) + P(S_2)P(B|S_2) + \ldots + P(S_k)P(B|S_k)$$

Example: The Law of Total Probability

- Let A = 'System Failure'

- Partition the system into:

  - B_1: Sensor normal Operation ($P(B_1) = 0.9$)

  - $B_2 = B_1'$: Sensor degraded ($P(B_2) = 0.1$)

- Given:

  - System failure upon normal operation $P(A \mid B_1) = 0.001$

  - System failure upon sensor degradation $P(A \mid B_2) = 0.01$

- Then:

  - $P(A) = (0.001 \times 0.9) + (0.01 \times 0.1) = 0.0009 + 0.001 = 0.0019$

  - $\rightarrow$ Total system failure probability is 0.0019