LINEAR PROGRAMMING
SIMPLEX METHOD

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Introduction

Many people rank the development of linear programming among the most important scientific advances of the mid-twentieth century. Its seeds were sown during World War II when the military supplies and personnel had to be moved efficiently, and from that date its impact becomes extraordinary. In fact, a very major proportion of all scientific computation on computers is devoted to the use of linear programming and closely related techniques.

A *Linear Programming* problem is a special case of a *Mathematical Programming* problem. It uses a mathematical model to describe the problem of concern. The adjective "linear" means that all the mathematical functions in this model are required to be linear functions. The word "programming" is essentially a synonym for planning. Thus, linear programming involves identifying an *extreme* (i.e., minimum or maximum) point of a function $\mathbf{f(x_1, x_2, ..., x_n)}$, which furthermore satisfies a set of constraints, $\mathbf{g(x_1, x_2,...x_n) \leq b_i}$.

Even if it may seem quite theoretical in a first approach, linear optimization has a lot of practical applications in real problems, because any problem whose mathematical model fits the very general format for the linear programming model is a linear programming problem. Especially, linear programming is widely used in industry, governmental organizations, ecological sciences, transportation and business organizations to minimize objectives functions, which can be production costs, numbers of employees to hire, or quantity of pollutants released, given a set of constraints such as availability of workers, of machines, or labors time.

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Standard Form for the Linear Programming Problem

The standard form for the linear programming problem is presented below. Any situation whose mathematical formulation fits this model is a linear programming problem.

Maximize $Z = c_1*x_1 + c_2*x_2 + \ldots.. + c_n*x_n$,

Subject to the restrictions

$$
\begin{aligned}
&a_{11}*x_1 + a_{12}*x_2 + \ldots.. + a_{1n}*x_n \le b_1\\
&a_{21}*x_1 + a_{22}*x_2 + \ldots.. + a_{2n}*x_n \le b_2\\
&\qquad\qquad\vdots\\
&\qquad\qquad\vdots\\
&a_{m1}*x_1 + a_{m2}*x_2 + \ldots.. + a_{mn}*x_n \le b_m,
\end{aligned}
$$

and

$x_1 \ge 0,\ x_2 \ge 0,\ \ldots..,\ x_n \ge 0.$

<u>Terminology:</u>

- The x_j variables are **decision variables**.
- The input constants (a_{ij}, b_i, c_j) may be referred to as **parameters** of the model.
- The function being maximized, $c_1*x_1 + c_2*x_2 + \ldots.. + c_n*x_n$, is called the **objective function**.
- The restrictions normally are referred to as constraints. The first m constraints, those with a function $a_{i1}*x_1 + a_{i2}*x_2 + \ldots.. + a_{in}*x_n$, representing the total usage of resource i, are called **functional constraints**. Similarly, $x_j \ge 0$ restrictions are called **nonnegativity constraints**.

<u>The other legitimate forms are the following:</u>

- Minimizing rather than maximizing the objective function:

  Minimize $Z = c_1*x_1 + c_2*x_2 + \ldots.. + c_n*x_n$

- Some functional constraints with a greater-than-or equal-to (≥) inequality:

  $a_{i1}*x_1 + a_{i2}*x_2 + \ldots.. + a_{in}*x_n \ge b_i$, for some values of i.

- Some functional constraints in equation form:

  $$a_{i1}*x_1+ a_{i2}*x_2+.....+ a_{in}*x_n = b_i, \text{ for some values of } i.$$

- Deleting the nonnegativity constraints for some decision variables:

  x_j unrestricted in sign, for some values of j.

Any problem that mixes some or all of these forms with the remaining parts of the preceding model is still a linear programming problem as long as they are the only new forms introduced.

Terminology for Solutions of the Model:

Any specification of values for the decision variables (x_1, x_2, ….., x_n) is called a solution, regardless of whether it is desirable or even an allowable choice.

A **feasible solution** is a solution for which all the constraints are satisfied. It is possible for a problem to have no feasible solutions. Given that there are feasible solutions, the goal of the linear programming is to find which one is the best, as measured by the value of the objective function in the model.

An **optimal solution** is a feasible solution that has the most favorable value of the objective function. Most favorable value means the largest or the smallest value, depending upon whether the objective is maximization or minimization. Thus an optimal solution maximizes/minimizes the objective function over the entire feasible region. Usually a problem will have just one optimal solution. However, it is also possible to have multiple optimal solutions. The third possibility is that a problem has no optimal solutions. This occurs only if it has no feasible solutions, or the constraints do not prevent increasing the value of the objective function indefinitely in the favorable direction.

Assumptions of linear programming:

❖ Proportionality:

Proportionality is an assumption about individual activities considered independently of the others. Therefore consider the case where only one of the n activities is undertaken. Call it activity k, so that $x_j = 0$ for all j=1, 2, ….., n except j=k.

The assumption is that the measure of effectiveness Z equals (c_k * x_k) and the usage of each resource "i" equals a_{ik}*x_k; that is, both quantities are directly proportional to the level of each activity k conducted by itself (k=1, 2,….., n). This implies in particular that there is no extra start-up charge with beginning the activity and that the proportionality holds over the entire range of levels of the activity.

❖ Additivity

The proportional assumption is not enough to guarantee that the objective function and constraint functions are linear. Cross-product terms will arise if there are interactions between some of the activities that would change the total measure of effectiveness or the total usage of some resources. Additivity assumes that there are no such interactions between any of the activities. Therefore, the additivity assumption requires that, given any activity levels (x_1, x_2, ….., x_n), both the total measure of effectiveness and the total usage of each resource equal to the sum of the corresponding quantities generated by each activity conducting by itself.

❖ Divisibility

Sometimes the decision variables have physical significance only if they have integer values. However, the optimal solution obtained by linear programming is often a noninteger one. Therefore, the divisibility assumption is that activity units can be divided into any fractional levels, so that noninteger values for the decision variables are permissible.

Frequently, linear programming is still applied even when an integer solution is required. If the solution obtained is noninteger one, then the noninteger variables are merely rounded to integer values. This may be satisfactory, particularly if the decision variables are large.

- ❖ **Certainty**

The certainty assumption is that all the parameters of the model (the a_{ij}, b_i, and c_j values) are known constants. In real problems, this assumption is seldom satisfied precisely. Linear programming models usually are formulated to select some future course of reaction. Therefore, the parameters used would be based on a prediction of future conditions, which inevitably introduces some degree of uncertainty.

Steps in formulating a Linear Programming (LP) Model

To understand the necessary steps in formulation of Linear Programming Model, we have to start with a prototype example of a linear programming problem. Let us suppose that Reebok Sports manufactures two types of t-shirts: sleeveless and sleeve. They are trying to figure out how many sleeveless and how many sleeves should be produced per week, to maximize profits regarding the following constraints:

- The (profit) contribution per sleeveless is \$3.00, compared to \$4.50 per sleeve.
- Sleeves use 0.5 yards of material; sleeveless use 0.4 yards. 300 yards of material are available.
- It requires 1 hour to manufacture one sleeveless and 2 hours for one sleeve. 900 labors hours are available.
- There is unlimited demand for sleeveless but total demand for sleeves is 375 units per week.
- Each sleeveless uses 1 insignia logo and 600 insignia logos are in stock.

The first thing we should do is trying to understand the problem completely. Next we have to identify the decision variables. For this problem the decision variables are the number of sleeve and sleeveless t-shirts to be manufactured.

x_1:sleeve
x_2 :sleeveless

Next we state objective function as a linear combination of the decision variables.

– maximize (\$3.00 * sleeveless) + (\$4.50 * sleeve)
► **maximize $Z = (4.50 * x_1) + (3.00 * x_2)$**

The following step is to state the functional and nonnegativity constraints for the given problem.

- Functional constraints:
  - **Material:** $(0.5 * x_1) + (0.4 * x_2) \leq 300$ **yards**
  - **Labor:** $(2 * x_1) + (1 * x_2) \leq 900$ **hrs**
  - **Demand:** $(1 * x_1) + (0 * x_2) \leq 375$ **units**
  - **Logos:** $(0 * x_1) + (1 * x_2) \leq 600$
- Non-Negativity constraints:

  $x_1 \geq 0$
  $x_2 \geq 0$

Graphical Formulation

The problem stated above is a very small problem that has only two decision variables (x_1 : sleeves and x_2: sleeveless). Thus the problem has only two dimensions, so a graphical procedure can be used to solve it. This procedure involves constructing a two-dimensional graph with x_1 and x_2 as the axes.

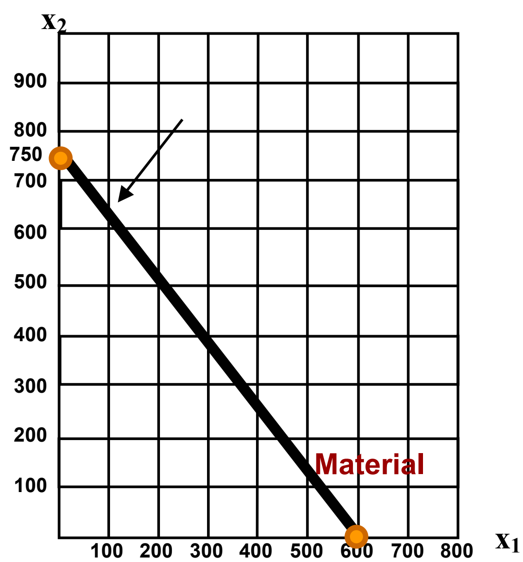

Figure1. $(0.5*x_1 + 0.4*x_2) \leq 300$

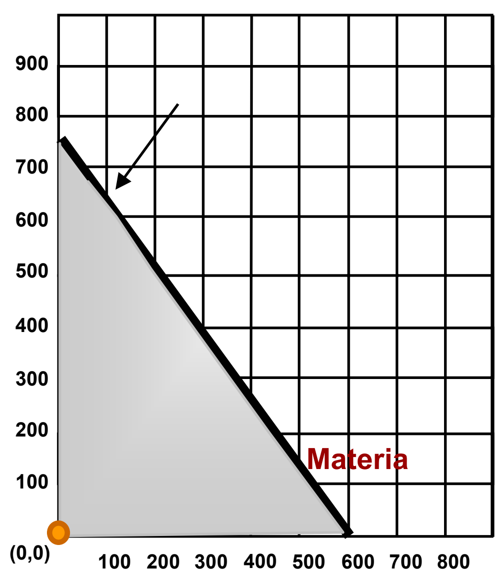

Figure2. Feasible region (I)

The first step is to identify values of $(\mathbf{x_1}, \mathbf{x_2})$ that are permitted by the restrictions. The nonnegativity restrictions require $(\mathbf{x_1}, \mathbf{x_2})$ to lie on the positive side of the axes. Next we draw the line of the equation $\mathbf{(0.5*x_1 + 0.4*x_2) \leq 300}$ which states the restriction on the material. To draw the line of this equation, we first assume that $x_1 = 0$. Thus, $0.4 * x_2 \leq 300$. This equation yields for the values of $x_2 \leq 750$. So, we mark the point (0, 750). We do the same thing to find the value of x_1, when $x_2 = 0$. As a result we find out the point (600,0). We connect this two points by a line, which defines the upper boundary of the equation $\mathbf{(0.5*x_1 + 0.4*x_2) \leq 300.}$

To determine on which side of the upper boundary line crossing from the points (0, 750) and (600,0) the feasible region lies, we take the point (0,0) and compute $(0.5*x_1 + 0.4*x_2) = 0 \leq 300$. Since the inequality holds, (0,0) must be within the feasible region, so we can shade the side of the line where (0,0) lies.

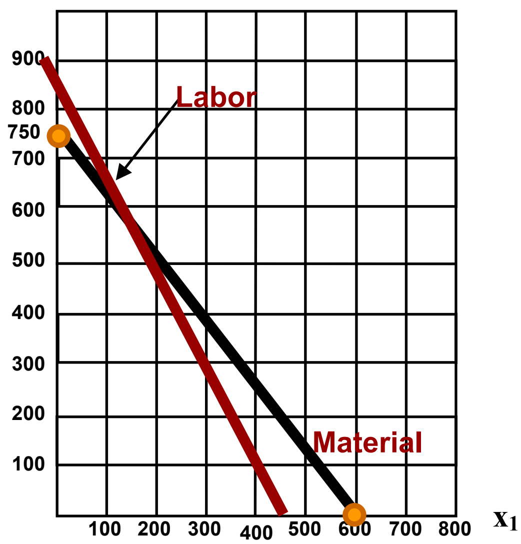

Figure 3. $(2*x_1+1*x_2)\leq 900$

Next we draw the line **$(2*x_1+1*x_2)\leq 900$** which states the restrictions on the labors hours. By drawing this line we repeat the procedure. First we set x_1 to zero and find the value of x_2= 900, and then assume that x_2 is zero and find x_1 as 450. Thus the line crossing from the points (0,900) and (600,0) is the upper bound labor constraint.

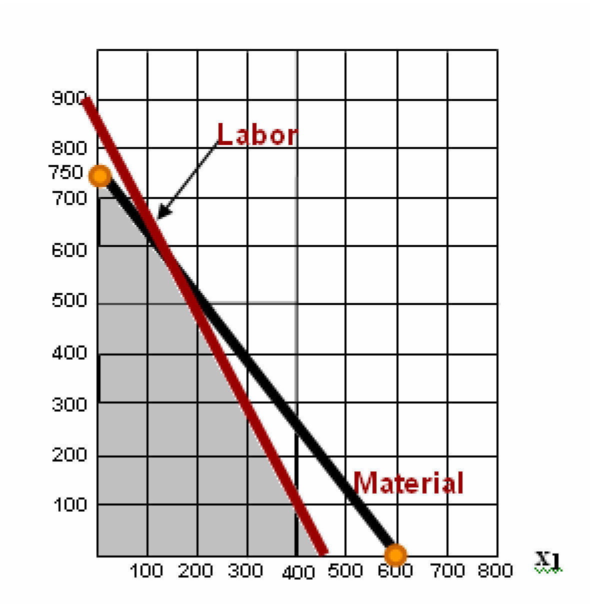

Figure 4. Feasible region (II)

To determine which side of the line is the feasible region regarding both constraints, we substitute the point (0,0) into the equation:

$$(2 * 0) + (1 * 0) = 0 \leq 900$$

Since the inequality holds, (0,0) must be within the feasible region, so we can shade the side of the line where (0,0) lies. Thus the feasible region becomes the region below the upper bounds of labors hour constraint and material constraint.

We repeat the same procedure for the demand and logos constraints. The resulting feasible region is presented in the figure below. Any point in this region is a feasible solution to the t-shirt problem of Reebok Sports Company.

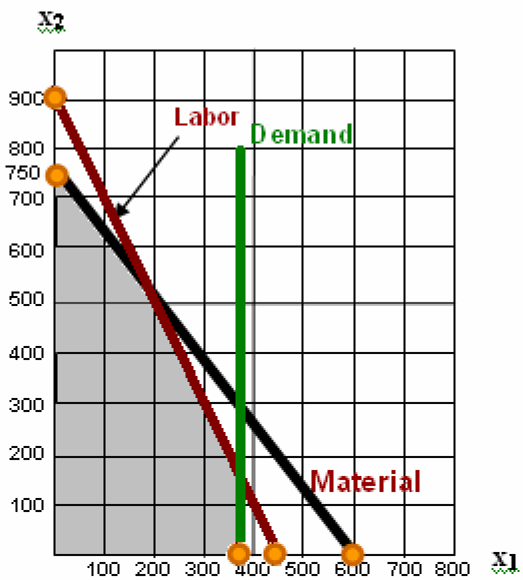

Figure 5. Feasible region (III)

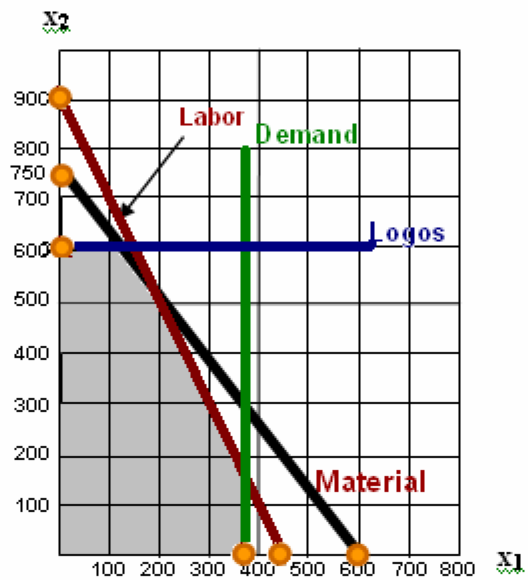

Figure 6. Feasible region of the problem

The final step is to pick out the point in this region that maximizes the value of **$Z=(4.5*x_1)+(3*x_2)$**. To find the maximum value we will proceed by trial and error. We first try **$Z= 600= (4.5*x_1) + (3*x_2)$** to see if there are in the permissible region any values of **(x_1,x_2)** that yield a value of Z as large as 600. By drawing the line (Figure 7.) we can see that there are many points on this line that lie within region. Therefore, we try a larger value of Z: **$Z= 1200= (4.5*x_1) + (3*x_2)$.** Again Figure 8 reveals that a segment of the line $(4.5*x_1) + (3*x_2) = 1200$ lies within the region, so that the maximum permissible value of Z must be at least 1200.

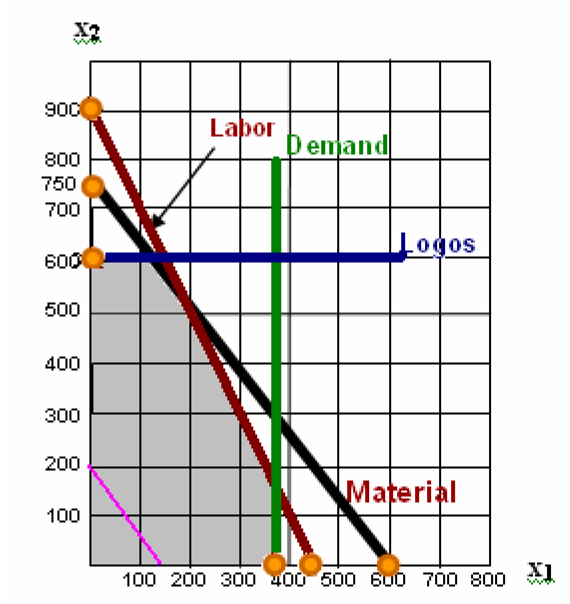

Figure 7. Z= 600= (4.5*x_1) + (3*x_2)

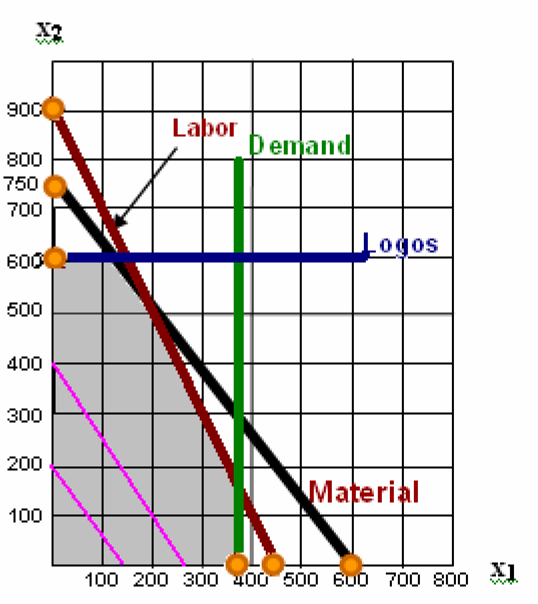

Figure 8. Z= 1200= (4.5*x_1) + (3*x_2)

The line Z= 1200= (4.5*x_1) + (3*x_2) giving a larger value of Z is farther up and away from the origin. Than the first line and that the two lines are parallel. Thus this trial-and-error procedure involves nothing more than drawing a family of parallel lines containing at least one point in the permissible region and the line that is the greatest distance from the origin (in the direction of increasing values of Z). This line passes through the point (200, 500) as indicated in the Figure 10. so that the equation is **Z= (4.5*x_1) + (3*x_2) = 2400.**

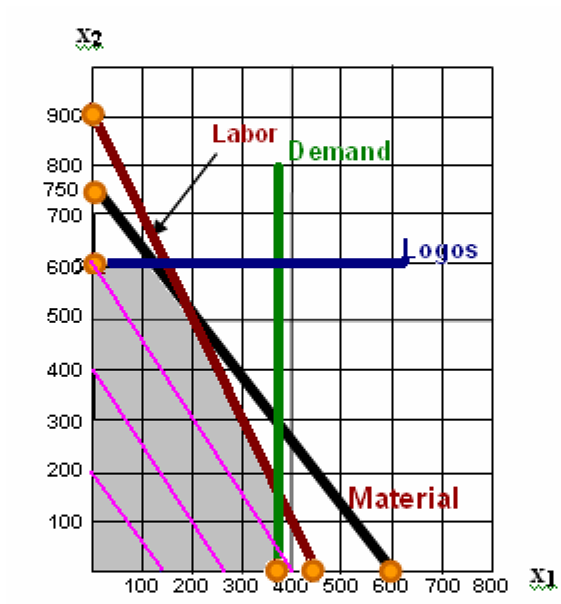

Figure 9. Z= 1800= (4.5*x_1) + (3*x_2)

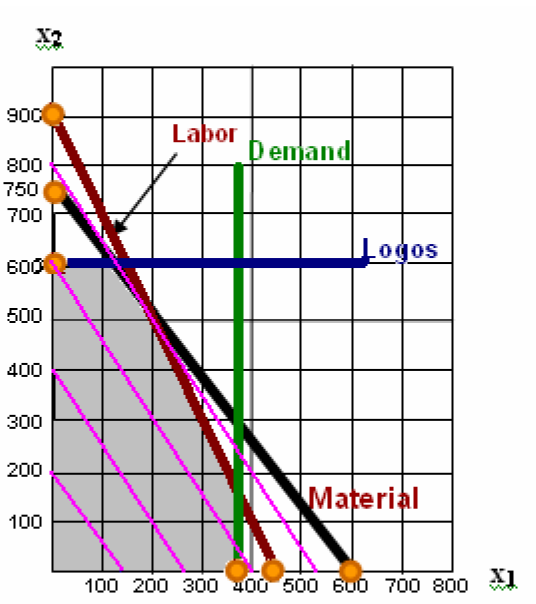

Figure 10. Z= 2400= (4.5*x_1) + (3*x_2)

To check the solution whether it is optimal or not, we can take several points in the permissible region as shown in the Figure 11. Next we calculate the total profit for these combinations of products. As can be seen from the Table 1, the maximum profit can be obtained if Reebok Sports manufactures 200 sleeves and 500 sleeveless shirts. Thus, the computation and the graph give the same optimal solution.

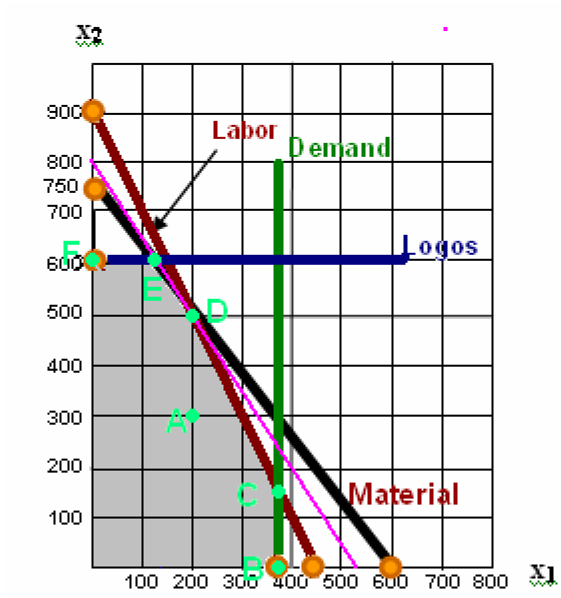

Figure 11. Points in the feasible region

	$4,50	$3,00	Total
	Sleeve	Sleeveless	Profit
A	200	300	$1.800,00
B	375	0	$1.687,50
C	375	150	$2.137,50
D	200	500	$2.400,00
E	120	600	$2.340,00

Table 1. Total Profit for Some Points

THE SIMPLEX ALGORITHM

The simplex algorithm, which was discovered in1947 by George Dantzig, is a simple, straightforward method for solving linear programming problems. The idea is to identify certain "basic" feasible points and to prove that the maximum value of the objective function *Z* occurs at one of these points. Then the algorithm proceeds roughly as follows: If a basic feasible point is at hand, a procedure is given for deciding whether it yields the maximum value of the objective function and, if not, for finding a basic feasible point that produces a larger value of the objective function. The process continues until a maximum is reached.

In an algebraic procedure, it is much more convenient to deal with equations than with inequality relationships. Therefore, the first step in the setting up the simplex method is to convert inequality constraints into equality constraints. This conversion can be succeeded by introducing **slack variables**.

Initialization Step

To illustrate consider example where the objective function $\mathbf{Z= (4.5*x_1) + (3*x_2)}$ We are trying to maximize Z regarding the restrictions $x_i \geq 0$ for $i = 1, 2$ and:

$$\mathbf{(0.5 * x_1) + (0.4 * x_2) \leq 300}$$

$$\mathbf{(2 * x_1) + (1 * x_2) \leq 900}$$

$$\mathbf{(1 * x_1) + (0 * x_2) \leq 375}$$

$$\mathbf{(0 * x_1) + (1 * x_2) \leq 600}$$

The slack variable for the constraint $\mathbf{x_1 \leq 375}$ is $\mathbf{x_5 = 375 - x_1}$, which is just the slack between two sides of the inequality. Thus :

$$\mathbf{375 = x_1 + x_5}.$$

By introducing slack variables in an identical fashion for the other functional constraints, the original linear programming model can now be replaced by the equivalent model:

Maximize Z= (4.5*x_1) + (3*x_2),

Subject to

(i)	$0.5\ x_1+0.4\ x_2+x3$	$=300$	(Material)
(ii)	$2\ x_1+\ x_2+\ x_4$	$=900$	(Labor)
(iii)	$x_1+\ x_5$	$=375$	(Demand)
(iv)	$x_2+\ x_6$	$=600$	(Logo)

and

$$x_j \geq 0 \text{ for } j = 1, 2, \dots, 6.$$

Although this problem is identical to the original, this form is much more convenient for algebraic manipulation. We call this equality form of the problem in order to obtain an **augmented solution**, which is a solution that was originally in inequality form that has been augmented by the corresponding values of the slack variables to change the problem into equality form.

In this equality form the system of functional constraints has two more variables than equations. This fact gives us two-DOF in solving the system, since any two variables can be set equal to any two arbitrary values. The variables that are currently set to zero are called **nonbasic variables**, and the others are called **basic variables**. The resulting solution is a **basic solution**. If all of the basic variables are nonnegative, the solution is called a **basic feasible solution**. In tabular form we can represent this system as follows:

Basic variable	Eq. No.	Coefficient of							Right Side
		Z	x_1	x_2	x_3	x_4	x_5	x_6	
Z	0	1	-4.5	-3	0	0	0	0	0
x_3	1	0	0.5	0.4	1	0	0	0	300
x_4	2	0	2	1	0	1	0	0	900
x_5	3	0	1	0	0	0	1	0	375
x_6	4	0	0	1	0	0	0	1	600

Table 2. Tabular form of the example

For simplicity, we select x_1 and x_2 equal to zero. Then x_3 becomes 300, x_4 becomes 900, x_5 becomes 375 and x_6 becomes 600. These basic variables give the initial basic feasible solution **(0, 0, 300, 900, 375, 600).**

Iterative Step

At each iteration, the simplex method moves from the current basic feasible solution to a better adjacent basic feasible solution. This movement involves replacing one nonbasic variable by a new one and identifying the new feasible solution.

The candidates for the entering basic variable are current nonbasic variables. The one chosen would be increased from zero to a positive number, and the other will be kept at zero. We determine the entering variable by selecting the variable with the negative coefficient having the largest absolute value (i. e. the one that has the largest coefficient, and so would increase Z at the fastest rate) and this column is called the pivot column.

$$\mathbf{Z - 4.5^*x_1 - 3^*x_2 = 0}$$

Both variables have negative coefficients, so increasing either one would increase Z, but at the different rates of 4.5 and 3 per unit increase in the variable. Since 3<4.5, we choose x_1 as entering basic variable.

The current basic variable whose nonnegativity constraint imposes the smallest upper bound on how much the entering basic variable can be increased, is the leaving basic variable. We pick out each coefficient in the pivot column that is strictly positive. The possibilities for leaving basic variable in the example are x_3, x_4, x_5 and x_6. Since x_5 imposes the smallest upper bound on x_1, the leaving basic variable is x_5, thus $x_5 = 0$ and $x_1 = 375$ in the new basic feasible solution. After identifying the entering and leaving basic variables, we need to identify the new basic feasible solution using the new values of the variables. The table is presented below.

Basic variable	Eq. No.	Coefficient of							Right Side
		Z	x_1	x_2	x_3	x_4	x_5	x_6	
Z	**0**	**1**	**-4.5**	**-3**	**0**	**0**	**0**	**0**	**0**
x_3	**1**	**0**	**0.5**	**0.4**	**1**	**0**	**0**	**0**	**300**
x_4	**2**	**0**	**2**	**1**	**0**	**1**	**0**	**0**	**900**
x_5	**3**	**0**	**1**	**0**	**0**	**0**	**1**	**0**	**375**
x_6	**4**	**0**	**0**	**1**	**0**	**0**	**0**	**1**	**600**

Table 3. First iteration pivot row and pivot column

Next we try to solve for the new basic feasible solution by using elementary row operations to construct a new simplex tableau in proper form from Gaussian elimination. The original set of the equations for the system was:

(0) $\mathbf{Z} - 4.5\ x_1 - \ 3\ x_2 = 0$

(i) $0.5\ x_1 + 0.4\ x_2 + \ x_3 = 300$ (Material)

(ii) $2\ x_1 + \ x_2 + \ x_4 = 900$ (Labor)

(iii) $x_1 + \ x_5 = 375$ (Demand)

(iv) $x_2 + \ x_6 = 600$ (Logo)

and

$$x_j \geq 0 \text{ for } j = 1, 2, \ldots, 6.$$

x_1 has replaced x_5 as the basic variable in Equation (iii). Since x_1 has a coefficient of +1, this equation would be multiplied with (1) to give its new basic variable a coefficient of +1. The resulting new Equation (iii) is:

(iii) $x_1 + x_5 = 375$

Next x_2 must be eliminated from the other equations in which it appears, to set it up for the optimality test. This elimination is done as follows:

New Equation (ii) = Old Equation (ii) + (-2)* New Equation (iii)

New Equation (i) = Old Equation (i) + (-0.5)* New Equation (iii)

New Equation (0) = Old Equation (0) +(4.5)* New Equation (iii)

Thus the second set of the equations is as follows:

0) $\mathbf{Z} - 3\ x_2 + 4.5\ x_5 = 1687.5$

(i) $0.4\ x_2 + x3 - 4.5\ x_5 = 112.5$

(ii) $x_2 + x_4 - 2\ x_5 = 150$

(iii) $x_1 + x_5 = 375$

(iv) $x_2 + x_6 = 600$

and $x_j \geq 0$ for $j = 1, 2, \dots, 6$.

We now obtain a new set of basic feasible solution: $(x_1, x_2, x_3, x_4, x_5, x_6) = (375, 0, 112.5, 150, 0, 600)$, which yields Z=1687.5.

To determine whether the current basic feasible solution is optimal, the equation (0) is used to rewrite the objective function just in terms of the current nonbasic variables.

$$\mathbf{Z} = 1687.5 + 3 * x_2 - 4.5 * x_4.$$

Because x_2 has a positive coefficient, increasing x_2 would lead toward an adjacent basic feasible solution that is better than the current basic feasible solution, so the current basic feasible solution is not optimal. In general terms, the current basic feasible solution is optimal if and only if all the nonbasic variables have nonpositive coefficients.

For the second iteration we choose x_2 as the new entering basic variable, since increasing only x_2 would increase Z. Since x_4 imposes the smallest upper bound on x_2, the leaving basic variable is x_5, thus $x_4 = 0$ and $x_2 = 150$ in the new basic feasible solution.

Basic variable	**Eq. No.**	**Coefficient of**							**Right Side**
		Z	**x_1**	**x_2**	**x_3**	**x_4**	**x_5**	**x_6**	
Z	**0**	**1**	**0**	**-3**	**0**	**0**	**4.5**	**0**	**1687.5**
x_3	**1**	**0**	**0**	**0.4**	**1**	**0**	**-0.5**	**0**	**112.5**
x_4	**2**	**0**	**0**	**1**	**0**	**1**	**-2**	**0**	**150**
x_1	**3**	**0**	**1**	**0**	**0**	**0**	**1**	**0**	**375**
x_6	**4**	**0**	**0**	**1**	**0**	**0**	**0**	**1**	**600**

Table 4. Second iteration pivot row and pivot column

We repeat the same process until we reach to a solution where all the nonbasic variables have nonpositive coefficients.

Basic variable	Eq. No.	Coefficient of							Right Side
		Z	x_1	x_2	x_3	x_4	x_5	x_6	
Z	**0**	**1**	**0**	**0**	**0**	**3**	**-1.5**	**0**	**2137.5**
x_3	**1**	**0**	**0**	**0**	**1**	**-0.4**	**0.3**	**0**	**52.5**
x_2	**2**	**0**	**0**	**1**	**0**	**1**	**-2**	**0**	**150**
x_1	**3**	**0**	**1**	**0**	**0**	**0**	**1**	**0**	**375**
x_6	**4**	**0**	**0**	**0**	**0**	**-1**	**2**	**1**	**450**

Table 5. Third iteration pivot row and pivot column

Basic variable	Eq. No.	Coefficient of							Right Side
		Z	x_1	x_2	x_3	x_4	x_5	x_6	
Z	**0**	**1**	**0**	**0**	**5**	**1**	**0**	**0**	**2400**
x_5	**1**	**0**	**0**	**0**	**10/3**	**-4/3**	**1**	**0**	**175**
x_2	**2**	**0**	**0**	**1**	**20/3**	**5/3**	**0**	**0**	**500**
x_1	**3**	**0**	**1**	**0**	**-10/3**	**4/3**	**-1**	**0**	**200**
x_6	**4**	**0**	**0**	**0**	**-20/3**	**5/3**	**0**	**1**	**100**

Table 6. Fourth iteration

As all the coefficients in the equation no=0 are positive, the solution obtained is the basic feasible solution of the problem. Thus, the maximum profit $2400 is obtained when Reebok Sports produce 500 sleeveless and 200 slevees t-shirts per week.

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