Periodic Inscription of Isosceles Trapezoids

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Abstract

We prove that a pair of continuous disjoint periodic curves in $\mathbb{C}$ inscribes an isosceles trapezoid with any similarity type. The case of smooth curves can be identified with a Lagrangian intersection problem for a pair of Lagrangian cylinders in $\mathbb{R} \times S^1 \times \mathbb{C}$, and the continuous case follows from the smooth one by a standard convergence argument.

1 Introduction

Let $\gamma_1, \gamma_2 \colon \mathbb{R} \to \mathbb{C}$ be two continuous embeddings of the real line into $\mathbb{C}$ that satisfy the periodicity condition

$$\gamma_i(t+1) = \gamma_i(t) + \sqrt{-1}$$

for every t and $i = 1, 2$. Furthermore, assume the images of γ_1 and γ_2 are disjoint. Tao conjectured in [10] that there exist four points in $\gamma_1(\mathbb{R}) \cup \gamma_2(\mathbb{R})$ which are vertices of a square; this is a variation of the Toeplitz square peg problem for periodic curves, and Hugelmeyer proved it in [5].

For any given isosceles trapezoid Q, we show there are four points in $\gamma_1(\mathbb{R}) \cup \gamma_2(\mathbb{R})$ that are vertices of a quadrilateral similar to Q. The approach of [5] does not directly generalize even to the case of rectangles. By contrast, in this article, we use a different approach to prove not only that every pair of periodic curves inscribes every similarity type of rectangles, but also every similarity type of isosceles trapezoids.

Definition 1.1. *Assume Q is an isosceles trapezoid. We say that the pair (γ_1, γ_2) admits a balanced inscription of Q if there exist $p_1, p_2 \in \gamma_1(\mathbb{R})$ and $p_3, p_4 \in \gamma_2(\mathbb{R})$ such that the quadrilateral formed by p_1, p_2, p_3, p_4 is similar to Q, the line segments $\overline{p_1p_2}$ and $\overline{p_3p_4}$ are parallel, and $|\overline{p_1p_2}| \leq |\overline{p_3p_4}|$.*

Note that our definition depends on the order of the pair $(\gamma_1,\ \gamma_2)$ unless Q is a rectangle.

Theorem 1.2. *Suppose γ_1 and γ_2 are two continuous disjoint periodic embeddings of the real line into the plane, and suppose Q is an isosceles trapezoid. Then $(\gamma_1,\ \gamma_2)$ admits a balanced inscription of Q. Furthermore, there is a generic subset of smooth disjoint periodic pairs such that each pair in this set admits at least two balanced inscriptions of Q that are not related under translation by $\sqrt{-1}$.*

Corollary 1.3. *Let $\theta \in (0, \frac{\pi}{2}]$; then every pair of continuous disjoint periodic curves in the plane inscribes a rectangle with angle θ between its two diagonals.*

We conjecture that Theorem 1.2 is optimal, in the following sense.

Conjecture 1.4. *Let Q be a quadrilateral that admits an inscription in any pair of disjoint periodic curves in $\mathbb{C}$. Then Q is an isosceles trapezoid.*

In contrast to peg problems for closed curves, we can deduce the periodic peg problem for continuous curves from the case of smooth curves using a standard convergence argument, so it suffices to prove the result for a pair of smooth periodic curves. We prove Theorem 1.2 for smooth curves using symplectic geometry. In particular, we will use Floer homology for a pair of non-compact Lagrangian cylinders in $(\mathbb{R} \times S^1 \times \mathbb{C}, \omega)$, where the symplectic form depends on the isosceles trapezoid Q. Our approach draws inspiration from the ideas in [3, 4].

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2 Symplectic Setting

Consider two disjoint smooth periodic curves $\gamma_1 \colon \mathbb{R} \to \mathbb{C}$ and $\gamma_2 \colon \mathbb{R} \to \mathbb{C}$ in the plane; we will identify each γ_i with its image in the following. Fix an isosceles trapezoid Q. The similarity type of Q is determined by two pieces of information, the angle between its diagonals and the ratio its two diagonals intersect each other. Let θ in $(0, \pi)$ be the angle, and assume the diagonals intersect each other with ratio $\frac{c}{1-c}$ for some fixed c in $(0, \frac{1}{2}]$; see Figure 1.

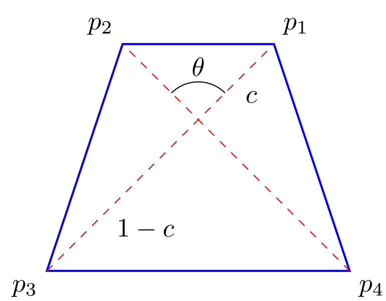

Figure 1: An annotated isosceles trapezoid Q.

We always assume the pair (c, θ) is fixed in what follows. Define a map $\psi_c \colon \mathbb{C}^2 \to \mathbb{C}^2$ by

$$\psi_c(z, w) = (z + cw, z + (c-1)w).$$

Let $R_\theta \colon \mathbb{C}^2 \to \mathbb{C}^2$ denote the map

$$(z, w) \mapsto (z, e^{\sqrt{-1}\theta} w).$$

Observe that p_1, p_2, p_3, p_4 cyclically label the vertices of an isosceles trapezoid similar to Q, with $\overline{p_1 p_2}$ parallel to $\overline{p_3 p_4}$ and $|\overline{p_1 p_2}| \leq |\overline{p_3 p_4}|$, iff there exists $(z, w) \in \mathbb{C}^2$ such that $\psi_c(z, w) = (p_1, p_3)$ and $\psi_c \circ R_\theta(z, w) = (p_2, p_4)$. Therefore, the balanced inscriptions of Q in (γ_1, γ_2) are in one-to-one correspondence with the set

$$\psi_c^{-1}(\gamma_1 \times \gamma_2) \cap R_\theta(\psi_c^{-1}(\gamma_1 \times \gamma_2)).$$

We define a pair of symplectic forms on $\mathbb{C}^2$ by

$$\omega_Q \coloneqq dx_1 \wedge dy_1 + c(1-c)dx_2 \wedge dy_2,$$
$$\omega_Q' \coloneqq (1-c)dx_1 \wedge dy_1 + cdx_2 \wedge dy_2.$$

Define a Hamiltonian H on $(\mathbb{C}^2, \omega_Q)$ by $H(z,w) = \frac{\|w\|^2}{2c(1-c)}$; R_θ is the flow of H at time θ. A quick computation shows that $\psi_c^*(\omega_Q') = \omega_Q$. The map ψ_c is a diffeomorphism, and since $\gamma_1 \times \gamma_2$ is a product Lagrangian in $(\mathbb{C}^2, \omega_Q')$, we conclude that both $\psi_c^{-1}(\gamma_1 \times \gamma_2)$ and its image under R_θ are Lagrangian submanifolds in $(\mathbb{C}^2, \omega_Q)$. We consider two actions of $\mathbb{Z}$ on $\mathbb{C}^2$, one generated by $T_1(z,w) = (z + \sqrt{-1}, w + \sqrt{-1})$ and one generated by $T_2(z,w) = (z + \sqrt{-1}, w)$. We have

$$\psi_c \circ T_2 = T_1 \circ \psi_c.$$

Notice that T_2 is a symplectomorphism of $(\mathbb{C}^2, \omega_Q)$, and $\gamma_1 \times \gamma_2$ is invariant under T_1 because we assumed both curves are periodic. Hence, $\psi_c^{-1}(\gamma_1 \times \gamma_2)$ is invariant under T_2. Furthermore, we have $H \circ T_2 = H$; thus the flow of H is invariant under T_2, and this shows $R_\theta(\psi_c^{-1}(\gamma_1 \times \gamma_2))$ is also invariant under T_2. We consider the quotient of $(\mathbb{C}^2, \omega_Q)$ under the action by T_2. Denote the resulting symplectic manifold by (X, ω), and let $\pi\colon \mathbb{C}^2 \to X$ be the quotient map; this manifold is diffeomorphic to $\mathbb{R} \times S^1 \times \mathbb{C}$, and the form ω is given by $dx_1 \wedge d\theta_1 + c(1-c)dx_2 \wedge dy_2$. Consider the vector field $V(x_1, \theta_1, x_2, y_2) = \partial_{x_1} + \partial_{x_2}$ on X; this vector field has complete flow, and it is symplectically dilating. The projection map $\pi\colon \mathbb{C}^2 \to X$ endows X with an ω-compatible complex structure which we denote by j, and the function $g(x_1, \theta_1, x_2, y_2) = x_1^2 + x_2^2 + y_2^2$ is a subharmonic exhaustion of (X, j). Thus (X, ω) is Weinstein at infinity; see [2] for the definition. The Hamiltonian H reduces to a Hamiltonian on X; by an abuse of notation, we also denote its flow by R_θ. The Lagrangian $\psi_c^{-1}(\gamma_1 \times \gamma_2)$ projects to a Lagrangian infinite cylinder L_γ in (X, ω). We denote $R_\theta(L_\gamma)$ by L_γ^θ.

Our goal is to define a Lagrangian Floer homology for the pair $(L_\gamma, L_\gamma^\theta)$ and prove it is non-trivial. This will show the intersection $L_\gamma \cap L_\gamma^\theta$ is non-empty. As a result, we must have an intersection point between $\pi^{-1}(L_\gamma) = \psi_c^{-1}(\gamma_1 \times \gamma_2)$ and $\pi^{-1}(L_\gamma^\theta) = R_\theta(\psi_c^{-1}(\gamma_1 \times \gamma_2))$. Hence, we need to show Floer homology for the pair $(L_\gamma, L_\gamma^\theta)$ is well-defined. Since the two Lagrangians are non-compact, we have to show their intersection is compact in order to establish their Floer homology is well-defined and invariant under compactly supported Hamiltonian isotopies.

Remark. Notice that the intersection $L_\gamma \cap L_\gamma^\theta$ parametrizes the orbits of balanced inscriptions of Q in (γ_1, γ_2) under translation by $\sqrt{-1}$.

The following lemma is the main result we will need for later compactness arguments. Its proof follows easily from the notion of the width of a compact set $K \subset \mathbb{C}$, i.e. the infimal width of an infinite strip containing K. One simply notes that width scales linearly under similarity transformations and is positive when K is the vertex set of a triangle.

Lemma 2.1. *Consider a positive number N and $(z,w) \in \mathbb{C}^2$. Assume z, $z + cw$, and $z + ce^{\sqrt{-1}\theta}w$ are inside the strip $[-N, N] \times \mathbb{R}$. Then there is a constant $b(N, c, \theta)$ depending only on N, c, and θ such that $\|w\| < b(N, c, \theta)$.* ■

Corollary 2.2. *The intersection $L_\gamma \cap L_\gamma^\theta$ is compact.*

Proof. We can find a positive number N such that both γ_1 and γ_2 are inside $[-N, N] \times \mathbb{R}$. Consider $\pi(z,w)$ in $L_\gamma \cap L_\gamma^\theta$. Then z, $z + cw$, and $z + ce^{-\sqrt{-1}\theta}w$ are all inside this strip, so we can apply Lemma 2.1. Thus

$\|w\|$ is bounded, and this shows $\pi(z,w)$ is in $[-N,N]\times S^1\times D_{b(N,c,\theta)}$, where $D_{b(N,c,\theta)}$ is the closed disk with radius $b(N,c,\theta)$ around the origin in $\mathbb{C}$. ■

We will work with mod 2 coefficients for singular homology and Floer homology in the following. Define the path space $\mathcal{P}(L_\gamma, L_\gamma^\theta)$ by

$$\{x \in W^{1,2}([0,1],X) \mid x(0)\in L_\gamma,\ x(1)\in L_\gamma^\theta\}.$$

Let $\lambda = x_1 d\theta_1 + c(1-c)x_2 dy_2$ be a primitive for ω on X, and consider a compactly supported Hamiltonian $\widehat{H}\colon X\times[0,1]\to\mathbb{R}$. We define the symplectic action one form $\omega_{\widehat{H}}$ on the path space $\mathcal{P}(L_\gamma, L_\gamma^\theta)$ by

$$\omega_{\widehat{H}}(\xi) = \int_0^1 \omega(\xi, \dot{x}(t) - X_{\widehat{H}}(x(t),t))dt$$

for every $x\in\mathcal{P}(L_\gamma, L_\gamma^\theta)$ and $\xi\in T_x\mathcal{P}(L_\gamma, L_\gamma^\theta)$. In general, this defines a closed one-form on the path space between two Lagrangians, but it is not necessarily exact; in this case, we have the following.

Lemma 2.3. *The form $\omega_{\widehat{H}}$ on $\mathcal{P}(L_\gamma, L_\gamma^\theta)$ is exact for every Hamiltonian $\widehat{H}$.*

Proof. It suffices to show if $u\colon[0,1]\to\mathcal{P}(L_\gamma, L_\gamma^\theta)$ is a closed curve, then $\omega_{\widehat{H}}[u]=0$. We can view u as a map from $S^1\times[0,1]$ to X, where $\beta_0 \coloneqq u(,0)$ is in L_γ and $\beta_1\coloneqq u(,1)$ is in L_γ^θ. We get

$$\omega_{\widehat{H}}[u] = \int_{S^1}\int_0^1 \omega(u_s, u_t - X_{\widehat{H}}(u,t))dtds = \int_u \omega - \int_0^1\int_{S^1}\frac{\partial \widehat{H}(u,t)}{\partial s}dsdt = \int_u \omega = \lambda(\beta_1)-\lambda(\beta_0),$$

where in the last step, we applied Stokes' theorem. Firstly, note that β_1 and $R_\theta(\beta_0)$ are homotopic in L_γ^θ since β_0 and $R_\theta(\beta_0)$ are homotopic in X, u gives a homotopy between β_0 and β_1 in X, and the inclusion map from L_γ^θ to X induces an isomorphism on π_1. Secondly, we note that λ is a closed one form on L_γ^θ, so we must have $\lambda(R_\theta(\beta_0))=\lambda(\beta_1)$. Finally, since R_θ is a Hamiltonian diffeomorphism, the form $R_\theta^*(\lambda)-\lambda$ is exact; see [6, Chapter 3] for more details. Hence, we must have $\lambda(\beta_0)=\lambda(R_\theta(\beta_0))$. ■

Consider a primitive for $\omega_{\widehat{H}}$ on $\mathcal{P}(L_\gamma, L_\gamma^\theta)$, and denote it by $\mathcal{A}_{\widehat{H}}$; we call this the symplectic action functional. Critical points of $\mathcal{A}_{\widehat{H}}$ are in one-to-one correspondence with the set $\phi^1_{\widehat{H}}(L_\gamma)\cap L_\gamma^\theta$ where $\phi^1_{\widehat{H}}$ denotes the time one flow of $\widehat{H}$. This set is compact because $\widehat{H}$ is compactly supported, and $L_\gamma\cap L_\gamma^\theta$ is compact. In particular, this proves the critical values of $\mathcal{A}_{\widehat{H}}$ are bounded. Now consider a family of ω-compatible time dependent almost complex structures $\{J_t\}_{0\le t\le 1}$ on X that agree with j at infinity. This family induces a metric on $\mathcal{P}(L_\gamma, L_\gamma^\theta)$, and the gradient flow lines of $\mathcal{A}_{\widehat{H}}$ are in one-to-one correspondence with the solutions of the perturbed Cauchy-Riemann equation (with respect to $\{J_t\}_{0\le t\le 1}$ and $\widehat{H}$) with boundary conditions on the pair $(L_\gamma, L_\gamma^\theta)$; we refer the unfamiliar reader to [1, Section 1.3].

For a brief discussion of when Lagrangian-Floer homology is well-defined, see [8]. The main issues are to show that a moduli space of solutions to a (perturbed) Cauchy-Riemann equation on the strip is precompact in the moduli space of broken trajectories with sphere and disk bubbles, and then that there are no sphere or disk bubbles.

Lemma 2.4. *Floer homology for the pair $(L_\gamma, L_\gamma^\theta)$ is well-defined.*

Proof. Choose a compactly supported Hamiltonian $\widehat{H}$ such that $\phi^1_{\widehat{H}}(L_\gamma)$ intersects L^θ_γ transversely and a family of ω-compatible time dependent almost complex structures $\{J_t\}_{0\le t\le 1}$ on X that agree with j at infinity. Consider a solution of the perturbed Cauchy-Riemann equation u that converges to x and y on its two ends; we have $E(u) = \mathcal{A}_{\widehat{H}}(x) - \mathcal{A}_{\widehat{H}}(y)$, where E is the energy of u. Since the critical values of $\mathcal{A}_{\widehat{H}}$ are bounded, we conclude the energy is uniformly bounded for every solution u. It follows from [7, Theorem 2.1] that any solution u must have bounded image inside a compact set depending on $\{J_t\}_{0\le t\le 1}$ and $\widehat{H}$. This proves the moduli space of solutions is precompact in the moduli space of broken trajectories with sphere and disk bubbles. The sphere bubbles cannot happen since (X,ω) is exact, and the disk bubbles cannot happen because both $\pi_2(X, L_\gamma)$ and $\pi_2(X, L^\theta_\gamma)$ are trivial; this follows from the fact that the inclusion map for each of these Lagrangians induces an isomorphism on the fundamental group. ∎

Our next goal is to show this Floer homology is invariant under a certain Hamiltonian isotopy that is not necessarily compactly supported; this will reduce the computation of $HF(L_\gamma, L^\theta_\gamma)$ to the case $HF(L_\delta, L^\theta_\delta)$, where δ_1 and δ_2 are two disjoint vertical lines.

Definition 2.5. *Let L_0, L_1 be two Lagrangians in X, and consider a Hamiltonian isotopy $\phi\colon X\times[0,1]\to X$. We say this Hamiltonian isotopy does not escape to infinity with respect to the pair (L_0, L_1) if the intersection $L_0 \cap \phi(L_1\times[0,1])$ is compact.*

It follows from [7, Theorem I] that if ϕ is a Hamiltonian isotopy that does not escape to infinity with respect to the pair (L_0, L_1), then the continuation map from $CF(L_0, L_1)$ to $CF(L_0, \phi_1(L_1))$ is well-defined, and it induces an isomorphism between $HF(L_0, L_1)$ and $HF(L_0, \phi_1(L_1))$.

Let N be a positive number such that each γ_i is inside $[-N, N]\times\mathbb{R}$. The quotient of $\mathbb{C}$ under translation by $\sqrt{-1}$ is an infinite cylinder, and we denote the quotient map by q. The pair (γ_1, γ_2) is projected to a pair of disjoint simple closed curves in $[-N, N]\times S^1$. We can find two distinct numbers α_1 and α_2 in $(-N, N)$ so that the signed area between $q(\gamma_i)$ and $\{\alpha_i\}\times S^1$ is zero for each i. Therefore, there are Hamiltonian functions H_1 and H_2 supported in $[-N, N]\times S^1$ such that the time one flow of H_i takes $q(\gamma_i)$ to $\{\alpha_i\}\times S^1$ for each i. Define a Hamiltonian function F on $(\mathbb{C}^2, \omega'_Q)$ by

$$F(z,w) = (1-c)H_1(q(z)) + cH_2(q(w)).$$

This Hamiltonian is invariant under the action by T_1, and it takes $\gamma_1\times\gamma_2$ to $\delta_1\times\delta_2$ where $\delta_i = \{\alpha_i\}\times\mathbb{R}$. In particular, this induces a Hamiltonian isotopy ϕ on $X\times[0,1]$ such that $\phi_1(L_\gamma) = L_\delta$.

Proposition 2.6. *We have*

$$HF(L_\gamma, L^\theta_\gamma) \cong HF(\phi_1(L_\gamma), L^\theta_\gamma) \cong HF(L_\delta, L^\theta_\delta). \tag{1}$$

Proof. We have a Hamiltonian isotopy ϕ from L_γ to L_δ and a Hamiltonian isotopy from L^θ_γ to L^θ_δ defined by $u_s := R_\theta\circ\phi_s\circ R_\theta^{-1}$. Choose an arbitrary time $s\in[0,1]$, and suppose $\pi(z,w)$ is in $\phi_s(L_\gamma)\cap L^\theta_\gamma$. Then z, $z+cw$, and $z+ce^{-\sqrt{-1}\theta}w$ are all inside the strip $[-N,N]\times\mathbb{R}$; hence, $\|w\|$ is bounded by Lemma 2.1. This proves the intersection $\phi([0,1]\times L_\gamma)\cap L^\theta_\gamma$ is inside $[-N,N]\times S^1\times D_{b(N,c,\theta)}$, so it is compact. Similarly, one can show $L_\delta\cap u([0,1]\times L^\theta_\gamma)$ is compact. Therefore, both of these Hamiltonian isotopies do not escape to infinity, and we get the claim. ∎

Proposition 2.7. *Let $\delta_1 = \{\alpha_1\}\times\mathbb{R}$ and $\delta_2 = \{\alpha_2\}\times\mathbb{R}$ be two distinct vertical lines in the plane. Then $HF(L_\delta, L^\theta_\delta)\cong\mathbb{F}_2^2$.*

Proof. Assume $\pi(z,w)$ is in $L_\delta \cap L_\delta^\theta$. Then we must have $z+cw \in \delta_1$ and $z+(c-1)w \in \delta_2$. Thus z lies on the line $\{x=(1-c)\alpha_1 + c\alpha_2\}$. We also have $z + ce^{-\sqrt{-1}\theta} w \in \delta_1$ and $z+(c-1)e^{-\sqrt{-1}\theta}w \in \delta_2$. Hence, we get

$$v = (\alpha_1-\alpha_2) - (\alpha_1-\alpha_2)\tan(\frac{\theta}{2})\sqrt{-1}.$$

This shows the intersection of $\psi_c^{-1}(\delta_1\times\delta_2)$ and $R_\theta(\psi_c^{-1}(\delta_1\times\delta_2))$ is diffeomorphic to the real line, and can be identified with the image of the embedding

$$s\in\mathbb{R}\mapsto ((1-c)\alpha_1 + c\alpha_2 + s\sqrt{-1}, (\alpha_1-\alpha_2)-(\alpha_1-\alpha_2)\tan(\frac{\theta}{2})\sqrt{-1}) \in \mathbb{C}^2.$$

We claim this intersection is clean. Identify $\mathbb{C}^2$ with $\mathbb{R}^4$, and consider the coordinates (x_1,y_1,x_2,y_2). At a point (u_1,u_2) in $\psi_c^{-1}(\delta_1\times\delta_2)$, we have

$$T_{(u_1,u_2)}\psi_c^{-1}(\delta_1\times\delta_2) = \langle \partial_{y_1}, \partial_{y_2}\rangle,$$

and for a point (u_1,u_2) in $R_\theta(\psi_c^{-1}(\delta_1\times\delta_2))$, we have

$$T_{(u_1,u_2)}R_\theta(\psi_c^{-1}(\delta_1\times\delta_2)) = \langle \partial_{y_1}, \cos(\theta)\partial_{y_2} - \sin(\theta)\partial_{x_2}\rangle.$$

We conclude the intersection is clean because $\theta\in(0,\pi)$. This shows the intersection between L_δ and L_δ^θ is also clean, since $\pi\colon \mathbb{C}^2\to X$ is a local diffeomorphism. Furthermore, $L_\delta\cap L_\delta^\theta$ becomes a circle after applying π to the line of intersection in $\mathbb{C}^2$. Consider the symplectic action one form $\omega_{\widehat{H}}$ on $\mathcal{P}(L_\delta, L_\delta^\gamma)$ with $\widehat{H}=0$; this form is exact by Lemma 2.3, so the symplectic action functional is defined on the whole of $\mathcal{P}(L_\gamma, L_\gamma^\theta)$. Hence, there is an isomorphism between $H^*(S^1)\cong\mathbb{F}_2^2$ and $HF(L_\delta, L_\delta^\theta)$ according to [9, Theorem 3.4.11]. ■

3 Proof of the Main Theorem

Proof of Theorem 1.2. We can find a positive number N such that γ_1 and γ_2 are inside $[-N,N]\times\mathbb{R}$. First assume γ_1 and γ_2 are smooth. By Propositions 2.6 and 2.7, we get $HF(L_\gamma, L_\gamma^\theta)\cong HF(L_\delta, L_\delta^\theta)\cong\mathbb{F}_2^2$ where (δ_1,δ_2) is a pair of distinct vertical lines in $[-N,N]\times\mathbb{R}$. Hence, the intersection $L_\gamma\cap L_\gamma^\theta$ must be non-empty, and (γ_1,γ_2) admits a balanced inscription of Q. Moreover, L_γ and L_γ^θ intersect transversely for a pair of generic smooth curves; thus in this case, we must have

$$|L_\gamma\cap L_\gamma^\theta| \geq \dim(HF(L_\gamma, L_\gamma^\theta)) = 2.$$

We conclude for a pair of generic smooth curves, there are at least two balanced inscriptions of Q that are not related under translation by $\sqrt{-1}$. Now suppose γ_1 and γ_2 are continuous, and approximate each one by a sequence of smooth periodic curves γ_i^n inside the strip $[-N,N]\times\mathbb{R}$ with γ_1^n and γ_2^n disjoint for every n. Let Q_n denote an inscription of Q inside the pair (γ_1^n,γ_2^n). One can assume the intersection point between the diagonals of Q_n lies in $[-N,N]\times[0,1]$ for every n. This can be done because every translation of Q_n by an integer multiple of $\sqrt{-1}$ is a also an inscription of Q in (γ_1^n,γ_2^n). Moreover, the diameter length of each Q_n is bounded by a universal constant depending only on N,θ, and c according to Lemma 2.1. We conclude that the sequence Q_n must have a limit point $\tilde{Q}$ inscribed by the pair (γ_1,γ_2); this limit point is a non-degenerate isosceles trapezoid because γ_1 and γ_2 are disjoint. ■

Proof of Corollary 1.3. This follows from Theorem 1.2 applied to the case $(c = \frac{1}{2}, \theta)$. ■

References

[1] Alberto Abbondandolo and Matthias Schwarz, *On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math. **59** (2006), no. 2, 254–316.

[2] Yakov Eliashberg and Mikhael Gromov, *Convex symplectic manifolds*, Several complex variables and complex geometry, Part 2, Amer. Math. Soc., Providence, RI, 1991, pp. 135–162.

[3] Joshua Evan Greene and Andrew Lobb, *Cyclic quadrilaterals and smooth Jordan curves*, Invent. Math. **234** (2023), no. 3, 931–935.

[4] Joshua Evan Greene and Andrew Lobb, *Floer homology and square pegs*, 2024.

[5] Cole Hugelmeyer, *A solution to the periodic square peg problem*, 2024.

[6] Dusa McDuff and Dietmar Salamon, *Introduction to symplectic topology*, third ed., Oxford Graduate Texts in Mathematics, 2017.

[7] Yong-Geun Oh, *Floer homology and its continuity for non-compact Lagrangian submanifolds*, Turkish J. Math. **25** (2001), no. 1, 103–124.

[8] Andrés Pedroza, *A quick view of Lagrangian Floer homology*, Geometrical themes inspired by the N-body problem, Springer, Cham, 2018, pp. 91–125.

[9] Marcin Poźniak, *Floer homology, Novikov rings and clean intersections*, Northern California Symplectic Geometry Seminar, Amer. Math. Soc., Providence, RI, 1999, pp. 119–181.

[10] Terence Tao, *An integration approach to the Toeplitz square peg problem*, Forum Math. Sigma **5** (2017), Paper No. e30, 63.

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