Rings and Fields

As explained by "Arturo Magidin" Math StackExchange[1]

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A ring is an ordered triple, $(R, +, \times)$, where R is a set, $+\colon R \times R \to R$ and $\times\colon R \times R \to R$ are binary operations (usually written in in-fix notation) such that:

- $+$ is associative.
- There exists $0 \in R$ such that $0 + a = a + 0 = a$ for all $a \in R$.
- For every $a \in R$ there exists $b \in R$ such that $a + b = b + a = 0$.
- $+$ is commutative.
- $\times$ is associative.
- $\times$ distributes over $+$ on the left: for all $a, b, c \in R$, $a \times (b + c) = (a \times b) + (a \times c)$.
- $\times$ distributes over $+$ on the right: for all $a, b, c \in R$, $(b + c) \times a = (b \times a) + (c \times a)$.

1-4 tell us that $(R, +)$ is an abelian group. 5 tells us that $(R, \times)$ is a semigroup. 6 and 7 are the two distributive laws that you mention.

We also have the following items:

- There exists $1 \in R$ such that $1 \times a = a \times 1 = a$ for all $a \in R$.
- $1 \neq 0$.
- For every $a \in R$, $a \neq 0$, there exists $b \in R$ such that $a \times b = b \times a = 1$.
- $\times$ is commutative.

A ring that satisfies (1)-(7)+(a) is said to be a **ring with unity.** Clearly, every ring with unity is also a ring; it takes "more" to be a ring with unity than to be a ring.

A ring that satisfies (1)-(7)+(a,b,c) is said to be a **division ring**. Again, every division ring is a ring, and it takes "more" to be a division ring than to be a ring. (5)+(a)+(b)+(c) tell us that $(R - \{0\}, \times)$ is a group (note that we need to remove 0 because (c) specifies nonzero, and we need (b) to ensure we are left with *something*).

A ring that satisfies (1)-(7)+(a,b,c,d) is a **field**. Again, every field is a ring.

We do indeed have that $(R, +)$ is an abelian group, that $(R - \{0\}, \times)$ is an abelian group, and that these structures "mesh together" via (6) and (7). In a ring, we have that $(R, +)$ is an abelian group, that $(R, \times)$ is a semigroup (or better yet, a semigroup with 0), and that the two structures "mesh well".

We have that every field is a division ring, but there are division rings that are not fields (e.g., the quaternions); every division ring is a ring with unity, but there are rings with unity that are not division rings (e.g., the integers if you want commutativity, the $n \times n$ matrices with coefficients in, say, $\mathbb{R}$, $n > 1$, if you want noncommutativity); every ring with unity is a ring, but there are rings that are not rings with unity (strictly upper triangular 3×3 matrices with coefficients in $\mathbb{R}$, for instance). So

$$\text{Fields} \subsetneq \text{Division rings} \subsetneq \text{Rings with unity} \subsetneq \text{Rings}$$

and

$$\text{Fields} \subsetneq \text{Commutative rings with unity} \subsetneq \text{Commutative rings} \subsetneq \text{Rings}.$$

References

[1] Arturo Magidin. *Question: "what is difference between a ring and a field"*. May 2012. URL: `http://math.stackexchange.com/a/141255/249083`.