TTK4150 Nonlinear Control Systems
Lecture 7

Input-to-State Stability (ISS)

and

Input-Output Stability (IOS)

NTNU
Innovation and Creativity

Previous lecture

Previous lecture:

Lyapunov's direct method for nonautonomous systems

- Time-varying Lyapunov functions candidates
- Lyapunov's theorems for
  - stability
  - uniform stability (US)
  - uniform asymptotic stability (UAS)
  - global uniform asymptotic stability (GUAS)
  - local and global exponential stability (GES $\Rightarrow$ GUAS)
- Barbalat's lemma

Outline I

Outline II

- Introduction
- $\mathcal{L}_p$ norms and spaces
- Definition
- Causal operators
- Examples

Today's goals

After today you should...

- Know that there exists other stability concepts than Lyapunov stability

In particular

- Understand the motivation and the definition of Input-to-State stability (ISS)
- Be able to analyze ISS using ISS-Lyapunov functions
- Know some relations between ISS and Lyapunov stability

- Know the definition of Input-Output Stability (IOS)
- Be able to analyze IOS using the definition
- Know the small-gain theorem

Literature

Today's lecture is based on

Khalil Section 4.9

Background material:

- Paper and talk by E.D. Sontag:
  The ISS Philosophy as a Unifying Framework for Stability-Like Behavior
- Mini-course by A. Loria:
  Cascaded nonlinear time-varying systems: analysis and design

Sections 5.1 and 5.4

(5.2 - 5.3 and Ex. 5.14 are additional material)

Part I

Input-to-state stability (ISS)

Systems with inputs

System

We want to analyse systems on the form

$$\dot{x} = f(t, x, u) \qquad (\Sigma)$$

$f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

Input

$u(t)$ pieciewise continuous, bounded function

- disturbance
- modelling error

When $u(t) = 0$

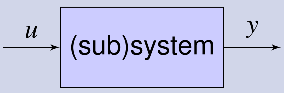
$$\dot{x} = f(t, x, 0)$$

$x = 0$ is GUAS (0-GUAS)

What if $u(t) \neq 0$?

Motivation

Motivation



- Adding to control system theorist's "toolkit" for studying systems via decomposition
- Quantify response to external signals
- Unify state-space and i/o stability theory

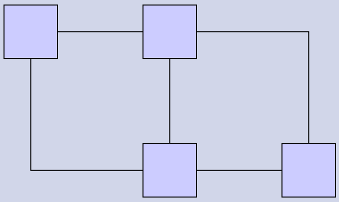
Motivation: Decomposition

Motivation: Decomposition (Cascades)

Even if the original system is autonomous

$$\dot{x} = f(x)$$

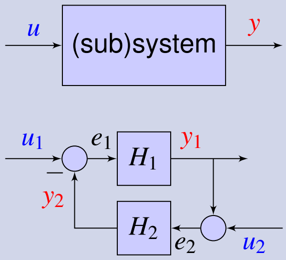
we may study "systems with i/o signal"



(Otherwise, how do we interconnect them?)

Motivation: Response to external signals

Motivation: Response to external signals

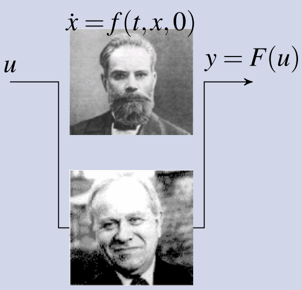


$u = (u_1, u_2)$ = noise, disturbance, modelling error, ...
$y = (y_1, y_2)$ = distance to desired states, tracking error, ...

Motivation: Unify state-space and i/o stability theory

Motivation: Merge Lyapunov/Zames

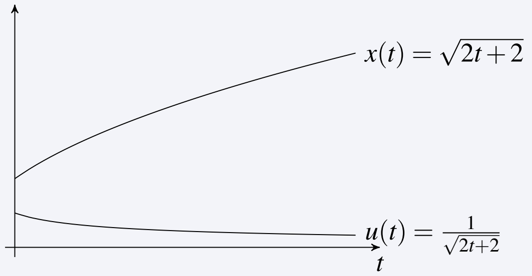
- We have Lyapunov theory for systems without inputs and outputs
- We have a rich theory for stability of input/output operators developed by George Zames, and many others
- ISS allows us to combine features of both



Motivation: $\dot{x} = f(x,0)$ Stable is not enough

For linear $\dot{x} = Ax + Bu$, A Hurwitz $\Rightarrow$ $(u \to 0 \Rightarrow x \to 0)$
i.e. Bounded Input Bounded State (BIBS)

This is NOT true for nonlinear systems. Ex: $\dot{x} = -x + (x^2+1)u$



even though $\dot{x} = f(x,0)$ is GES: $\dot{x} = -x$.

Motivation: Require I/O boundedness

We must bound the solution $\|x(t, x_0, u)\|$ in a "nonlinear gain" sense

$\|x(t)\|$ ("ultimately") $\leq \gamma(\|u(\cdot)\|_\infty)$

$\gamma \in \mathcal{K}_\infty$:
$\gamma(0) = 0$
$C^0, \nearrow +\infty$

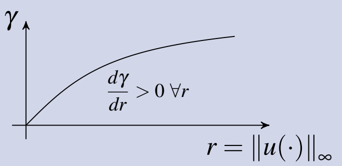


Figure: Example class $\mathcal{K}_\infty$ function γ

Motivation: $\dot{x} = f(x,0)$ GAS

Repetition (from last lecture):

Global asymptotic stability (GAS) of the origin means

$\exists$ class $\mathcal{KL}$ function β s.t. $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \quad \forall t \geq t_0 \geq 0 \quad \forall \|x(t_0)\|$

$\|x(t)\| \leq \beta(\|x(t_0)\|, 0) \rightsquigarrow$ stability (small overshoot)
$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \xrightarrow{(t-t_0)\to\infty} 0 \rightsquigarrow$ convergence

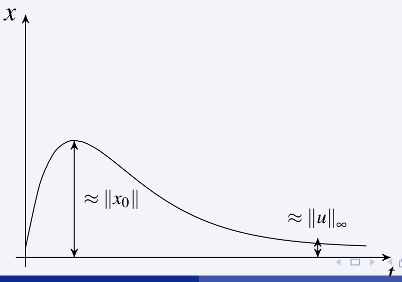
Definition of ISS

Original definition

$\exists \beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ s.t.

$$\|x(t, x_0, u)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(\|u\|_\infty)\}$$

Transient (overshoot) depends on x_0
When $(t - t_0)$ is large $x(t)$ bounded by $\gamma(\|u\|_\infty)$ independent of x_0



Definition of ISS: Khalil

An alternative definition is found in Khalil

Definition

Consider

$$\Sigma : \dot{x} = f(t, x, u)$$

The system Σ is ISS if $\exists \beta \in \mathcal{KL}$ and $\exists \gamma \in \mathcal{K}$ such that
$\forall\ u \in \mathcal{L}_p$ and $x_0 = x(0) \in \mathbb{R}^n$ (the solution $x(t)$ exists $\forall t \geq t_0$ and)

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|)$$

Linear case, for comparison

Example: Linear case

Given a stable linear system:
(i.e. the matrix A is Hurwitz: Re$(\lambda_i(A)) < 0 \quad \forall i = 1, \ldots, n$)

$$\dot{x} = Ax + Bu$$

Is this an input-to-state stable system?

Well-known that the system solution is:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$\|x(t)\| \leq \left\|e^{A(t-t_0)}\right\| \|x(t_0)\| + \int_{t_0}^{t} \left\|e^{A(t-\tau)}\right\| \|B\| \|u(\tau)\| d\tau$$

Theorem 4.11: A Hurwitz $\Leftrightarrow$ $\left\|e^{A(t-t_0)}\right\| \leq ke^{-\lambda(t-t_0)} \quad k, \lambda > 0$

Linear case, for comparison

$$\|x(t)\| \leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + \frac{k\|B\|}{\lambda} \sup_{t_0 \leq \tau \leq t} \|u(\tau)\|$$

$$\|x(t)\| \leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + \frac{k\|B\|}{\lambda} \|u(\tau)\|_\infty$$

$$\rightsquigarrow \boxed{\|x(t)\| \leq \beta(t)\|x(t_0)\| + \gamma\|u\|_\infty}$$

$$\beta(t) = ke^{-\lambda(t-t_0)} \xrightarrow{(t-t_0)\to\infty} 0$$

$$\gamma = \frac{k\|B\|}{\lambda}$$

This is a particular case of the ISS estimate

$$\boxed{\|x(t,x_0,u)\| \leq \beta(\|x(t_0)\|, t-t_0) + \gamma(\|u\|_\infty)}$$

How to check ISS?

Definition: ISS Lyapunov function (ISS-LF)

$V : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is an ISS-LF for Σ iff

i) V is C^1

$\exists\ \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{K}$ s.t.

ii) $\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|)$

iii) $\dot{V}(t,x) = \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial t} \leq -W_3(x) \quad \|x\| \geq \rho(\|u\|) > 0$

where $W_3(x)$ is a C^0 positive definite function on $\mathbb{R}^n$.

A Lyapunov-like theorem for ISS

Theorem 4.19

$\exists$ ISS-LF for $\Sigma \Rightarrow \Sigma$ is ISS

Sontag & Wang 1995

For autonomous systems: Σ is ISS $\Leftrightarrow \exists$ ISS-LF for Σ

$$\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$$

Example

Example

$$\dot{x} = -x^3 + x^2 u$$

The system is 0-GUAS ($\dot{x} = -x^3$)

Determine the system's ISS properties using the ISS-LFC $V(x) = \frac{1}{2}x^2$

Read

Read Examples 4.25 - 4.27

ISS vs. Lyapunov stability properties

ISS vs. 0-GUAS

Σ is ISS $\Rightarrow$ Σ is 0-GUAS

$$\Updownarrow$$

$\neg$(Σ is 0-GUAS) $\Rightarrow$ $\neg$(Σ is ISS)

ISS vs. 0-GES (Lemma 4.6)

$\Sigma : \dot{x} = f(t,x,u)$ $\quad$ f is C^1 and <u>globally</u> Lipschitz in (x,u)

Σ is 0-GES $\Rightarrow$ Σ is ISS

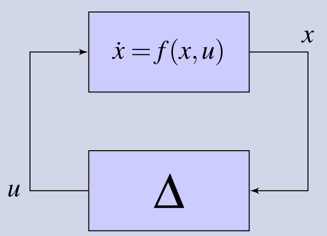
How do we use this?

ISS $\equiv$ Robust Stability

Y. Wang & E.D. Sontag, Systems and Control Letters 1995

ISS $\Leftrightarrow$ $\exists$ "margin of stability" $\rho \in \mathcal{K}_\infty$

$\boxed{\dot{x} = f(t, \Delta(t,x))}$



has GUAS origin $\forall$ time-varying feedback laws Δ s.t. $|\Delta(t,x)| \leq \rho(\|x\|)$

How do we use ISS: Cascades

Stability of cascades

$$\boxed{\Sigma_2} \longrightarrow \boxed{\Sigma_1}$$

$$\begin{aligned}\Sigma_1: \quad \dot{x}_1 &= f_1(t, x_1, x_2)\\ \Sigma_2: \quad \dot{x}_2 &= f_2(t, x_2)\end{aligned}$$

$f_1 : [0,\infty) \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ and $f_2 : [0,\infty) \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ are piecewise continuous in t and locally Lipschitz in x

Lemma 4.7

$$\underbrace{\boxed{\text{GUAS}} \longrightarrow \boxed{\text{ISS}}}_{\text{GUAS}}$$

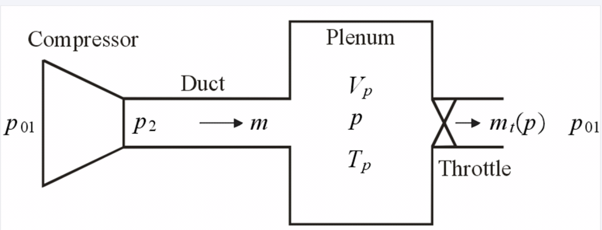
Example

Example

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + x_1^2 x_2\\ \dot{x}_2 &= -kx_2 \quad k > 0\end{aligned}$$

Use cascaded systems theory to prove that the origin $(x_1, x_2) = (0, 0)$ of this system is globally uniformly asymptotically stable (GUAS)

Application example: Compressor



$$\dot{m} = \frac{A_1}{L_c}(p_2(m,\omega) - p)$$

$$\dot{p} = \frac{a_{01}^2}{V_p}(m - m_t(p))$$

$$\dot{\omega} = \frac{1}{J}(\tau_d - \sigma r_2^2 |m| \omega)$$

- Objective: Active surge control
  - High efficiency
  - Avoid surging: pressure and mass flow oscillations
- Need mass flow observer
  - Bøhagen & Gravdahl (2004) - reduced order observer

Compressor application cont.

- Suggested observer

$$\dot{z} = \frac{A_1}{L_c}(p_2 - p - u) + k_{\tilde{m}}(m_t(p) - \hat{m})$$
$$\hat{m} = z + k_{\tilde{m}}\frac{V_p}{a_{01}^2}p$$

- Observer error is GES

$$\dot{\tilde{m}} = -k_{\tilde{m}}\tilde{m}$$

- CE control yields the cascade

$$\Sigma_1 : \quad \dot{x}_1 = f_1(x_1) + g(x_1, x_2)$$
$$\Sigma_2 : \quad \dot{x}_2 = f_2(x_2)$$

- Interconnection

$$|g(x_1, x_2)| \leq g_1|x_2|$$

- Hence, Σ_1 is ISS wrt x_2

$\Rightarrow$ The cascade is GUAS

- Moreover, Σ_1 is 0-GES
  - The cascade is GES

Background material

Autonomous systems

$$\Sigma_1 : \quad \dot{x}_1 = f_1(x_1) + g(x_1, x_2)$$
$$\Sigma_2 : \quad \dot{x}_2 = f_2(x_2)$$

$$\underbrace{\boxed{\text{GAS + LES}} \rightarrow \boxed{\text{iISS}}}_{\text{GAS}}$$

$$\underbrace{\boxed{\text{GAS + LES}} \rightarrow \boxed{\text{GAS} + |g| \leq}}_{\text{GAS}}$$

For more information see

`http://www.math.rutgers.edu/~sontag`

Background material

Nonautonomous systems

$$\Sigma_1 : \quad \dot{x}_1 = f_1(t, x_1) + g(t, x)x_2$$
$$\Sigma_2 : \quad \dot{x}_2 = f_2(t, x_2)$$

Panteley & Loria (Automatica 2001)

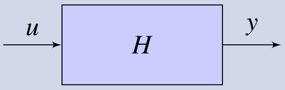
Loria (Tutorial)

Part II

Input-output stability (IOS)

Introduction

Input-output models



We consider systems on the form

$$y = Hu$$

$u : [0,\infty) \to \mathbb{R}^m$ piecewise continuous
$y : [0,\infty) \to \mathbb{R}^q$ piecewise continuous

Input-output stability

How do we analyze stability of such systems?

$\mathscr{L}_p$ Norms and spaces

We need a measure of the size of a signal ($u(t)$ and $y(t)$)

Recall from Lecture 1: Norm

Norms on $C[0,\infty)$

$$\left.\begin{aligned} \|f\|_p &= \left(\int_0^\infty |f(t)|^p \, dt\right)^{\frac{1}{p}} \\ \|f\|_\infty &= \sup_{0 \le t \le \infty} |f(t)| \end{aligned}\right\} \mathscr{L}_p - \text{norms}$$

$\mathscr{L}_p$-space

$$(C[0,\infty), \mathscr{L}_p - \text{norm})$$

- $f \in \mathscr{L}_p \Leftrightarrow \|f\|_p$ is bounded $\quad (\exists\, c : \|f\|_p \le c)$

$\mathscr{L}_p^m$ space

Extension to multivariable, piecewise continuous functions $u : [0,\infty) \to \mathbb{R}^m$

$\mathscr{L}_p^m$ space

$$u \in \mathscr{L}_p^m \quad 1 \le p < \infty \quad \Leftrightarrow \quad \|u\|_{\mathscr{L}_p} = \left(\int_0^\infty \|u(t)\|_{\bar{p}}^p \, dt\right)^{\frac{1}{p}} < \infty$$

$\mathscr{L}_2^m$ space (with $\bar{p} = 2$)

$$u \in \mathscr{L}_2^m \quad \Leftrightarrow \quad \|u\|_{\mathscr{L}_2} = \sqrt{\int_0^\infty u^T(t)u(t)\, dt} < \infty$$

$\mathscr{L}_\infty^m$ space

$$u \in \mathscr{L}_\infty^m \quad \Leftrightarrow \quad \|u\|_{\mathscr{L}_\infty} = \sup_{t\ge 0} \|u(t)\|_{\bar{p}} < \infty$$

Input-output stability | $\mathscr{L}_p$ norms and spaces

$\mathscr{L}_p^m$ space

Extension to multivariable, piecewise continuous functions $u : [0,\infty) \to \mathbb{R}^m$

$\mathscr{L}_p^m$ space

$$u \in \mathscr{L}_p^m \quad 1 \le p < \infty \quad \Leftrightarrow \quad \|u\|_{\mathscr{L}_p} = \left(\int_0^\infty \|u(t)\|_{\bar{p}}^p \, dt\right)^{\frac{1}{p}} < \infty$$

Arbitrary $\bar{p}$-norm on $\mathbb{R}^m$

$\mathscr{L}_2^m$ space (with $\bar{p} = 2$)

$$u \in \mathscr{L}_2^m \quad \Leftrightarrow \quad \|u\|_{\mathscr{L}_2} = \sqrt{\int_0^\infty u^T(t)u(t)\, dt} < \infty$$

$\mathscr{L}_\infty^m$ space

$$u \in \mathscr{L}_\infty^m \quad \Leftrightarrow \quad \|u\|_{\mathscr{L}_\infty} = \sup_{t\ge 0} \|u(t)\|_{\bar{p}} < \infty$$

34 Lecture 7: ISS and IOS

Input-output stability | $\mathscr{L}_p$ norms and spaces

$\mathscr{L}_p^m$ space

Extension to multivariable, piecewise continuous functions $u : [0,\infty) \to \mathbb{R}^m$

$\mathscr{L}_p^m$ space

$$u \in \mathscr{L}_p^m \quad 1 \le p < \infty \quad \Leftrightarrow \quad \|u\|_{\mathscr{L}_p} = \left(\int_0^\infty \|u(t)\|_{\bar{p}}^p \, dt\right)^{\frac{1}{p}} < \infty$$

$\mathscr{L}_2$: "Space of piecewise continuous, square-integrable functions"
$\mathscr{L}_\infty$: "Space of piecewise continuous, bounded functions"

Notation

$u \in \mathscr{L}_p^m \qquad u \in \mathscr{L}^m \qquad u \in \mathscr{L}$

$\mathscr{L}_{pe}^m$ - space

To be able to handle unbounded signals we introduce an extended space:

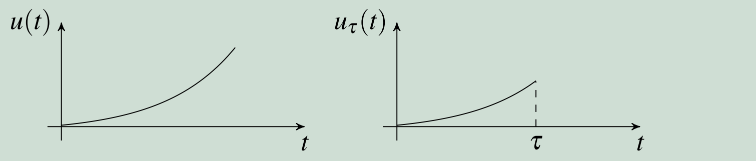
$\mathscr{L}_{pe}^m$ - space

$$u \in \mathscr{L}_{pe}^m \Leftrightarrow u_\tau \in \mathscr{L}_p^m \quad \forall\, \tau \in [0, \infty)$$

where

$$u_\tau(t) = \begin{cases} u(t), & t \in [0, \tau] \\ 0, & t > \tau \end{cases} \quad \text{truncation}$$

$u(t) = e^t$



Input-output stability

Consider the mapping

$$H : \mathscr{L}_{pe}^m \to \mathscr{L}_{pe}^q$$

$\mathscr{L}_p$ stable

$H : \mathscr{L}_{pe}^m \to \mathscr{L}_{pe}^q$ is $\underline{\mathscr{L}_p \text{ stable}}$ iff

i) $\exists \alpha$ class $\mathscr{K}$ $\alpha : [0, \infty) \to [0, \infty)$
ii) $\exists$ constant $\beta \geq 0$

s.t.

$$\|(Hu)_\tau\|_{\mathscr{L}_p} \leq \alpha(\|u_\tau\|_{\mathscr{L}_p}) + \beta \qquad \forall u \in \mathscr{L}_{pe}^m \text{ and } \tau \in [0, \infty)$$

Input-output stability cont.

Finite-gain $\mathscr{L}_p$ stable

$H : \mathscr{L}_{pe}^m \to \mathscr{L}_{pe}^q$ is $\underline{\text{finite-gain } \mathscr{L}_p \text{ stable}}$ iff

$$\exists \text{ constants } \gamma, \beta \geq 0$$

s.t.

$$\|(Hu)_\tau\|_{\mathscr{L}_p} \leq \gamma \, \|u_\tau\|_{\mathscr{L}_p} + \beta$$

$\mathscr{L}_p$ gain

Bias term

BIBO stability $\equiv \mathscr{L}_\infty$ stability

Causal

Definition (causal)

$H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is causal iff

$$(Hu)_\tau = (Hu_\tau)_\tau$$

If H is causal and $\mathcal{L}_p$ stable, then

$$u \in \mathcal{L}_p^m \Rightarrow Hu \in \mathcal{L}_p^q$$

and

$$\|(Hu)\|_{\mathcal{L}_p} \leq \alpha(\|u\|_{\mathcal{L}_p}) + \beta$$

If H is causal and finite-gain $\mathcal{L}_p$ stable, then

$$u \in \mathcal{L}_p^m \Rightarrow Hu \in \mathcal{L}_p^q$$

and

$$\|(Hu)\|_{\mathcal{L}_p} \leq \gamma\|u\|_{\mathcal{L}_p} + \beta$$

Examples

Example

Given

$$y = u^{\frac{1}{3}},$$

is it BIBO stable? Finite-gain $\mathcal{L}_\infty$ stable?

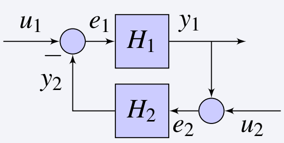
Read

Read Examples 5.1 and 5.3

Read Definition 5.2 page 201

Small gain theorem

Feedback interconnection



$$H_1 : \mathcal{L}_e^m \to \mathcal{L}_e^q \qquad H_2 : \mathcal{L}_e^q \to \mathcal{L}_e^m$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Stability of feedback interconnection

The feedback interconnection where H_1 and H_2 are finite-gain $\mathcal{L}$-stable, i.e.

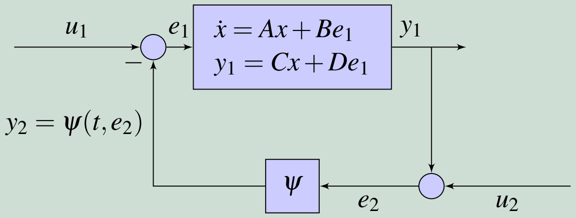
$$\|y_{1\tau}\|_{\mathcal{L}} \leq \gamma_1 \|e_{1\tau}\|_{\mathcal{L}} + \beta_1$$
$$\|y_{2\tau}\|_{\mathcal{L}} \leq \gamma_2 \|e_{2\tau}\|_{\mathcal{L}} + \beta_2$$

is finite-gain $\mathcal{L}$-stable if

$$\gamma_1\gamma_2 < 1$$

Example

Example



A Hurwitz
$G(s) = C(sI - A)^{-1}B + D$

Analyse the Input-Output stability properties of the interconnection.

Next lecture

- How to analyze the stability of perturbed systems
  - Vanishing perturbation
  - Nonvanishing perturbation
- Recommended reading
  Khalil **Chapter 9**
  Sections 9.1 and 9.2