Bicycle Dynamics and Control

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Introduction

This paper treats bicycles from the perspective of control. Models of different complexity are presented starting with simple ones and ending with more realistic models generated from multibody software. Models that capture essential behavior such as self-stabilization and demonstrate the difficulties with rear wheel steering are presented. Experiences of using bicycles in control education are presented with suggestions for fun and thought-provoking experiments with proven student attraction. The paper also describes design of adapted bicycles for children with disabilities and clinical experiences of their use.

The bicycle is used everywhere, for transportation, exercise, and recreation. The bicycle's evolution over time was a product of necessity, ingenuity, materials, industrialization, invention, imagination, and yet with comparatively little theoretical insight. The bicycle is efficient, highly maneuverable, and yet it represents a tantalizing enigma. Learning to ride a bicycle is an acquired skill, often acquired with some difficulty, and yet once mastered the skill becomes subconscious and second nature, literally just "as easy as riding a bike."

Bicycles have interesting dynamic behavior. They are statically unstable, like inverted pendulums, but can under certain conditions be stable in forward motion. They also exhibit non-minimum phase steering behavior. There are also nonlinearities due to geometry and tire-road interactions.

Bicycles have intrigued scientists ever since they appeared in the middle of the nineteenth century and there is a considerable literature on bicycles. The book by Sharp is a classic from 1896 that has recently been reprinted [1], the books [2], [3] give a broad engineering perspective. There are early papers from the 19th century [4] - [8]. Famous scientists like Rankine [4], Klein and Sommerfeld have analyzed bicycles [9]. It is notable that Klein and Sommerfeld were particularly interested in the effect of gyroscopic action of the front wheel. Papers on bicycles appear regularly in literature [10] - [22]. The first publications of differential equations describing the motion of an idealized bicycle appeared towards the end of the 19th century. Notable are Whipple [7], Carvallo [8], [23], who derived equations of motion, linearized around the vertical equilibrium configuration, from Lagranges equations. During the early 20th century several authors studied the problems of bicycle self stability, balancing, and steering the bicycle, with variable success. The paper [24] presented one of the first computer simulations of a nonlinear bicycle model. Neĭmark and Fufaev [25] derived a comprehensive set of linear models by approximating potential and kinetic energy by quadratic terms and applying Lagrange's equations to these expressions. They used different wheel models, ideal disks as well as pneumatic tires. Their model is elaborated in the book [26]. Modeling of bicycles became a popular topic for dissertations in the later half of the last century, [27] - [33]. Simple models of bicycle dynamics are given in the text books [34], [35] and [36]. Nonlinear models are presented in [21], [37], [38] and [39].

The orders of typical models for bicycles range from 2 to 10 depending on the simplifying assumptions. The early models were derived by hand calculations. It is natural to view the bicycle as a multibody system [40], [41], [42] [43] and use software for multibody systems for detailed modeling. Software for multibody systems was applied to bicycle modeling in the PhD theses [44] and [45] and in the paper [22]. It is useful to complement models from multibody programs with simpler models that give physical insight and understanding.

Bicycles share many properties with motorcycles. Sharp [46], [47] developed a detailed linearized motorcycle model of order 10 that also included tire slip. Weir [29] built on Sharp's work and added models of the rider to study handling characteristics [16]. A comprehensive discussion of motorcycle models is given in the book by Pacejka [48], which gives a detailed treatment of tire-road interaction with many references.

Several attempts have been made to develop autopilots for bicycles and motorcycles. The paper [49] describes a complete system for a motorcycle including results from practical tests. Autopilots are also described in [50], [37], [38], [51] and [52].

The bicycle is ideally suited to illustrate modeling, dynamics, stabilization, and feedback. It can also illustrate the fundamental limitations on control that are caused by poles and zeros in the right half plane. Simple experiments in dynamics and control can be performed using bicycles. Special bicycles with highly spectacular properties can be designed. The bicycle is well suited to activities for K-12, open houses, and demonstrations.

Geometry and Coordinate Systems

When modeling a physical object it is important to keep the purpose of the model in mind. In this paper we will present models that can be used to discuss the basic balancing and steering problems for the bicycle.

A detailed model of a bicycle is complex because the system has many degrees of freedom and the geometry is constrained. Some important aspects to consider are: what bicycle parts to include in a model; how to treat elasticity of the bicycle parts; how complicated a model of the rider that must be included; and how to treat tire-road interaction. Successful control and maneuvering of a bicycle depends critically on the forces between the wheels and the ground. Acceleration and braking require longitudinal forces; balancing and turning depend on lateral forces. A good understanding of these forces is necessary to make appropriate assumptions about valid models of the rolling conditions.

We assume that the bicycle consists of four rigid parts; two wheels, a frame, and a front fork with handlebars. The influence of other moving parts, such as pedals, chain, brakes, on the dynamics and control characteristics is thus disregarded for the purposes of this work. This simplification is intuitively justified as a bicycle's behavior remains essentially the same when the bike and rider are coasting and thus pedaling is stopped. For the cases where a rider is included in the analysis, the rider's upper body is modeled as a point mass that can move laterally with respect to the bike frame. The rider can also apply a torque on the handlebars.

When a rigid wheel rolls without sliding on a rigid surface, forces between the wheel and the ground may be transferred without losses. In reality, the tire is deformed due to the forces acting between ground and wheel. A common method to quantify the deformation, is to introduce slip as a correction normalized by the velocity. For many riding conditions, typical values of longitudinal slip are at most 5 to 10 percent. A turning wheel also tends to lag the path specified by its axis by a few degrees. Since we are not considering extreme conditions we assume that the bicycle tire rolls without longitudinal and lateral slippage.

Control of acceleration and braking is not considered explicitly but we will often assume that the forward velocity is constant. Sliding that can occur in tight turns is also neglected. We will simply assume that the bicycle moves on a horizontal plane and that the wheels always maintain contact with the ground.

Geometry

The parameters that describe the geometry of a bicycle are defined in Figure 1. The key parameters are: wheelbase b, head angle λ and trail c. The front fork is angled and shaped so that the contact point of the front wheel with the road is behind the extension of the steer axis. Trail is defined as the horizontal distance c between the contact point and the steer axis when the bicycle is in the upright reference configuration with zero steer angle. The riding properties of the bicycle are strongly affected by the trail, a large trail improves the stability but makes the steering less agile. Typical values for c range from 0.03 m to 0.08 m.

Geometrically it is convenient to consider the bicycle as composed of two hinged planes, the frame plane and the front fork plane. The frame and the rear wheel lie in the frame plane and the front wheel lies in the front fork plane. The planes are joined at the steer axis. The points P_1 and P_2 are the contact points of the wheels with the horizontal plane and the point P_3 is the intersection of the extended steer axis with the horizontal plane, see Figure 1.

Coordinates

The coordinates used to analyze the bicycle are defined in Figure 2. There is an inertial system with axes $\xi\eta\zeta$ and origin O. The coordinate system xyz has its origin at the contact point P_1 of the rear wheel and the horizontal plane. The x axis is aligned with the line of contact of the rear plane with the horizontal plane. The axis also goes through the point P_3 which is the intersection between the extended steer axis and the horizontal plane. The orientation of the rear wheel plane is defined by the angle ψ, which is the angle between the ξ-axis and the x-axis. The z axis is vertical and y is perpendicular to x and positive on the left side of the bike so that a right hand system is obtained. The roll angle φ of the rear frame is positive when leaning to the right. The roll angle of the front fork plane is φ_f. The steer angle is the angle of intersection between the rear and front planes, positive when steering left. The effective steer angle δ_f is the angle between the lines of intersection of the rear and front planes with the horizontal plane.

Simple Second Order Models

Second order models will now be derived based on several simplifying assumptions. It is assumed that the bicycle rolls on the horizontal plane, that the rider has fixed position and orientation relative the frame and that the forward velocity at the rear wheel V is assumed constant. To begin with it is also assumed that the steer axis is vertical, $\lambda = 90^\circ$ and that the trail c is zero. The steer angle δ is the control variable. The rotational degree of freedom associated with the front fork then disappears and the system only has one degree of freedom, the roll angle φ. See Figure 3. All angles are assumed to be small so that the equations can be linearized.

Top and rear views of the bicycle are shown in Figure 3. The coordinate system xyz rotates around the vertical axis with the angular velocity $\omega = V\delta/b$. An observer fixed to the coordinate system therefore experiences forces due to the acceleration of the coordinate system relative to inertial space.

Let m be the total mass of the system and b the wheel base. Consider the rigid body obtained when the wheels, the rider and the front fork assembly are fixed to the rear frame with $\delta = 0$, let J the moment inertia of this body with respect to the x-axis, and $D = -J_{xz}$ its product of inertia with respect to the xz plane. Furthermore let the x and z coordinates of the center of mass be a and h, respectively. The angular momentum of the system with respect to the x axis is

$$L_x = J\frac{d\varphi}{dt} - D\omega = J\frac{d\varphi}{dt} - \frac{VD}{b}\delta.$$

See [53]. The torques acting on the system are due to gravity and centrifugal action. The angular momentum balance now becomes

$$J\frac{d^2\varphi}{dt^2} - mgh\varphi = \frac{DV}{b}\frac{d\delta}{dt} + \frac{mV^2h}{b}\delta. \tag{1}$$

The last term of the left hand side is the torque generated by gravity. The terms of the right hand side are the torques generated by steering, the first term is due to inertial forces and the second term is due to centrifugal forces. Approximating the moment of inertia as $J \approx mh^2$ and the inertia product as $D \approx mah$ the model becomes

$$\frac{d^2\varphi}{dt^2} - \frac{g}{h}\varphi = \frac{aV}{bh}\frac{d\delta}{dt} + \frac{V^2}{bh}\delta.$$

This model has been used in [34] and [19].

The simple model (1) is a linear dynamical system of second order with two real poles and one zero

$$
\begin{aligned}
p_{1,2} &= \pm\sqrt{mgh/J} \approx \pm\sqrt{g/h} \\
s &= -\frac{mVh}{D} \approx -\frac{V}{a}.
\end{aligned} \tag{2}
$$

The gain is proportional to velocity V. It follows from (1) that the transfer function from steer angle δ to tilt angle φ is

$$
G_{\varphi\delta}(s) = \frac{V(Ds + mVh)}{b(Js^2 - mgh)} = \frac{VD}{bJ}\frac{s + \dfrac{mVh}{D}}{s^2 - \dfrac{mgh}{J}} \approx \frac{aV}{bh}\frac{s + \dfrac{V}{a}}{s^2 - \dfrac{g}{h}}. \tag{3}
$$

The model (3) is an unstable system and cannot explain why it is possible to ride “no-hands”.

The Front Fork

The design of the front fork has a major impact on bicycle dynamics. The simple model (1) does not capture this because of the assumptions of zero trail $c = 0$ and a head angle $\lambda = 90°$, see Figure 1. The torque applied to the handlebars will be considered as the control variable instead of the steer angle.

The front fork assembly will be modeled by a static torque balance. The contact forces between tire and road exert a torque on the front fork assembly when there is a tilt. Under certain conditions these forces turn the front fork towards the lean. The centrifugal force acting on the bicycle then counteracts the lean and can under certain circumstances stabilize the system.

We first observe that since the steer axis is tilted the effective front fork angle is given by

$$
\delta_f = \delta \sin\lambda. \tag{4}
$$

When the head angle differs from 90° a change of the steer angle also gives a tilt. For small angles the front fork roll angle is

$$\varphi_f = \varphi - \delta \cos \lambda. \tag{5}$$

Let N_f and F_f be the vertical and horizontal components of the forces acting on the front wheel at the ground contact,see Figure 4. Neglecting dynamics and centrifugal effects we have $N_f = amg/b$ and

$$F_f = \frac{amV^2}{b^2}\delta_f = \frac{amV^2 \sin \lambda}{b^2}\delta.$$

Also neglecting the weight of the front fork assembly the static torque balance of the front fork becomes

$$T - (F_f + N_f \varphi_f)c \sin \lambda = 0.$$

Introducing the expressions for F_f, N_f, δ_f and φ_f the torque balance can be written as

$$T - \frac{acmg \sin \lambda}{b}\varphi - \frac{acm \sin \lambda}{b^2}(V^2 \sin \lambda - bg \cos \lambda)\delta = 0. \tag{6}$$

Consider the situation when a bicycle with positive trail is moved forward with no external torque on the handlebars $T = 0$, zero tilt $\varphi = 0$, and constant velocity. It follows from (6) that there is a torque that turns the front fork assembly towards $\delta = 0$ if $V > V_{sa}$ where

$$V_{sa} = \sqrt{bg \cot \lambda} \tag{7}$$

is the self-alignment velocity. This can easily be verified experimentally by leading a bicycle at different speeds by holding it in the saddle.

The torque balance for the front fork assembly (6) can be written as

$$\delta = k_1(V)T - k_2(V)\varphi, \tag{8}$$

where the parameters are

$$\begin{aligned} k_1(V) &= \frac{b^2}{(V^2 \sin\lambda - bg\cos\lambda)mac\sin\lambda} \\ k_2(V) &= \frac{bg}{V^2\sin\lambda - bg\cos\lambda}. \end{aligned} \tag{9}$$

This model can be verified qualitatively by determining the torque applied to the front fork when biking in a straight path. Leaning the body towards the left causes the bicycle frame to tilt to the right (positive φ). A positive torque (counter clockwise) is then required to maintain a straight path. The steady-state parameters of the model can be experimentally determined from measurements of the torque and the tilt for different velocities.

Self-stabilization

The frame model (1) changes because of the geometry of the front fork. The steering angle δ has to be replaced by the effective steering angle δ_f given by (4). The center of mass of the frame is also shifted when the steering wheel is turned which gives the torque

$$-\frac{mgac\sin\lambda}{b}\delta.$$

The angular momentum balance for the frame given by (1) is then changed to

$$J\frac{d^2\varphi}{dt^2} - mgh\varphi = \frac{DV\sin\lambda}{b}\frac{d\delta}{dt} + \frac{m(V^2h - acg)\sin\lambda}{b}\delta \tag{10}$$

Inserting the expression (8) for steer angle δ in (10) gives

$$
\begin{aligned}
J\frac{d^2\varphi}{dt^2} + \frac{DVg}{V^2\sin\lambda - bg\sin\lambda}\frac{d\varphi}{dt} + \frac{mg^2(bh\cos\lambda - ac\sin\lambda)}{V^2\sin\lambda - bg\sin\lambda}\varphi \\
= \frac{DVb}{acm(V^2\sin\lambda - bg\sin\lambda)}\frac{DT}{dt} + \frac{b(V^2h - acg)}{ac(V^2\sin\lambda - bg\sin\lambda)}
\end{aligned}
\tag{11}
$$

This system is stable if

$$
\begin{aligned}
V > V_c = \sqrt{bg\cot\lambda} \\
bh > ac\tan\lambda.
\end{aligned}
\tag{12}
$$

The critical velocity V_c is thus equal to the self-aligning velocity V_{sa} given by (7).

Taking the action of the front fork into account the bicycle can be described (10) and (8). The front fork model (8) gives a negative feedback from tilt to steering angle as is illustrated in the block diagram in Figure 5. The figure shows that the bicycle can be regarded as a feedback system. The tilt angle φ influences the front wheel angle δ as described by the frame model (10) and the front wheel angle δ influences the tilt angle φ as described by front fork (8).

The model given by (10) and (8) gives a correct qualitative explanation for self-stabilization. There are however severe deficiencies in the model. The stabilizing action of the front fork is delayed because of dynamics and the critical velocity will therefore be larger than predicted by (12). The mass of the front fork assembly and gyroscopic effects also has some influence. To include these effects it is convenient to use software for multibody systems, which is done later when more detailed linearized models are presented.

Manual Control

The significant difference between the cases considered arises when the control variable is changed from steer angle to steer torque. A practical consequence, realized by skilled riders, is that the self-stabilizing action of the front fork is best achieved by lightly gripping the handlebars. When teaching children to bike it is important to remind them that they should not hold the handlebars too stiffly. This nuance is difficult for children with learning difficulties, as with fear often comes involuntary increase in the rigidity of the body. The rigidity of the child's arms and hands, in particular, blocks the communication being sent by the handlebars.

The rider controls the bicycle by leaning and by applying a torque on the handlebars. The effects of manual steering torques can be seen from (8). A simple model of rider behavior is to assume that the rider acts like a proportional controller by applying a steer torque proportional to bicycle lean, thus $T = -k\varphi$. Inserting this proportionality into (8) yields

$$\delta = -\big(kk_1(V) + k_2(V)\big)\varphi.$$

The consequence of rider action is thus to increase the value of parameter k_2 in (8). A more complex model of rider behavior is to account for the neural muscular delay of humans. A study of motorcycles by Weir [29] gave a neuromuscular delay of 0.1 s for applying steering torque and 0.3 s for upper body lean.

Gyroscopic Effects

The gyroscopic action of the rotation of the front wheel can be taken into account in the model of the front fork. The model (8) is then replaced by

$$\delta = k_1(V)T - k_2(V)\varphi - k_3(V)\frac{d\varphi}{dt},$$

which shows that gyroscopic effects give rise to derivative action. The parameter $k_3(V)$ is proportional to the angular momentum of the front wheel and thus also to the velocity.

Rear-wheel Steering

Rear-wheel steering is an interesting issue as is clear from the following quote from [2].

> Many people have seen theoretical advantages in the fact that front-drive, rear-steered recumbent bicycles would have simpler transmissions than front driven recumbents and could have the center of mass nearer the front wheel than the rear. The U.S. Department of Transportation commissioned the construction of a safe motorcycle with this configuration. It turned out to be safe in an unexpected way: No one could ride it.

The story behind this quote, which is elaborated in Side-bar A, is a nice illustration of the fact that it is possible to design systems that are useless even if their static properties are nearly ideal. The lessons embedded provide motivation to study dynamics and control.

A model for a bicycle with rear-wheel steering is obtained simply by reversing the velocity in (1). The transfer function from steer angle to tilt given by (3) then changes to

$$G_{\varphi\delta}(s) = \frac{-VDs + mV^2h}{b(Js^2 - mgh)} = \frac{VD}{bJ}\frac{-s + \dfrac{mVh}{D}}{s^2 - \dfrac{mgh}{J}} \approx \frac{aV}{bh}\cdot\frac{-s + V/a}{s^2 - g/h}, \qquad (13)$$

This transfer function has both poles and zeros in the right half plane which make the system difficult to control.

To find the effects of the rear fork we can simply reverse the sign of the velocity in (11). The system (11) is unstable for all velocities if $bh\cos\lambda > ac\sin\lambda$ because the term proportional to φ is negative for $V < V_{sa} = \sqrt{bg\cot\lambda}$ and the damping term is negative for $V > V_{sa}$.

Difficulties of Controlling a System with RHP Poles and Zeros

The calculation above was made with a specific control law. It turns out that the difficulty remains even if an arbitrarily complex control law is used. There are fundamental problems in controlling a system with poles and zeros in the right half plane [54], [55]. Robust control of such systems requires that the ratio between the RHP zero z and the RHP pole p be sufficiently large. In the particular case of the system (13) we have.

$$\frac{z}{p} = \frac{mVh}{D}\sqrt{\frac{J}{mgh}} \approx \frac{V}{a}\sqrt{\frac{h}{g}}.$$

Notice that the ratio depends on the velocity. The difficulties decrease with increasing velocity.

The zeros of a system depend on the interconnections between sensors, actuators and states. The zeros will change when sensors are changed and yet they disappear when all state variables are measured. Problems that are caused by RHP zeros can thus be eliminated, for example, by introducing more sensors. For the rear-steered bike the problem caused by the RHP zero can be eliminated by introducing tilt and yaw rate sensors. The problem cannot be alleviated by using an observer. See the comments in Side-bar 2.

Maneuvering

Having obtained some insight into stabilization of bicycles we will now turn to the problem of maneuvering. A central problem is to determine how the torque on the handlebars influences the path of the bicycle. Later we will also investigate the effects of leaning. The first step is to investigate how the steer torque T influences the steer angle δ. It follows from the block diagram in Figure 5 that

$$G_{\delta T}(s) = \frac{k_1(V)}{1 + k_2(V) G_{\varphi\delta}(s)},$$

where $G_{\varphi\delta}(s)$ is the transfer function from steer torque T to tilt angle φ given by (3). We thus find that

$$G_{\delta T}(s) = \frac{k_1(V)\Big(s^2 - \dfrac{mgh}{J}\Big)}{s^2 + \dfrac{k_2(V)DV}{bJ}s + \dfrac{k_2(V)V^2 mh}{bJ} - \dfrac{mgh}{J}}. \tag{14}$$

We note that the poles of the transfer function $G_{\varphi\delta}$ appear as zeros of the transfer function $G_{\delta T}$. The need for stabilization of the bicycle thus implies that the maneuvering dynamics has a zero in the right half plane. This imposes limitations on the maneuverability of the bicycle.

To determine how the bicycle's path is influenced by the steer angle we use the coordinate system in Figure 2. Assume that the angle ψ is small and linearizing around a straight line path along the ξ axis we get

$$\begin{aligned}\frac{d\eta}{dt} &= V\psi \\ \frac{d\psi}{dt} &= \frac{V}{b}\delta.\end{aligned}$$

The transfer function from steer angle δ to path deviation y is

$$G_{\eta\delta}(s) = \frac{V^2}{bs^2}.$$

Combining this with (14) gives the following transfer function from steer torque T to y,

$$G_{\eta T}(s) = \frac{k_1(V)V^2}{b} \frac{s^2 - mgh/J}{s^2\bigg(s^2 + \dfrac{k_2(V)VD}{bJ}s + \dfrac{mgh}{J}(\dfrac{V^2}{V_c^2} - 1)\bigg)}.$$

Figure 6 shows the path of a bicycle when a step input of torque is applied to the handlebars.

The response of the path is typical for a system with a right half plane zero. This behavior can be explained physically as follows. When a positive steer torque is applied, the bicycle's contact points with the ground initially track in the direction of the applied torque. This generates a reaction force that tilts the bicycle around the positive x axis. This tilt in turn generates a torque on the front fork, due in large to the upward action of the ground supporting the front wheel, which turns the front fork in the negative direction. Thus, both lean angle and steer angle tend to be, as the bicycle's transient decays, in the direction opposite to the torque torque applied to the handlebars. Recall again that the roll angle φ of the rear frame is positive when leaning to the right. In Figure 6 the lean angle settles to $\varphi = 0.14$ rad and the steer angle settles to $\delta = -0.14$ rad. Given the curves in Figure 6 it is straight forward to give a physical explanation. It is difficult to apply causal reasoning directly because of the closed loop nature of the system. The physical argument also indicates a remedy because the centrifugal force can be opposed by gravitational forces if the biker leans his upper body appropriately relative to the frame. This is precisely what experienced bikers do.

The inverse response nature of the response to steer torque has has contributed to a number of motorcycle accidents. To explain this, visualize a motorcycle driven along a straight path. Assume that a car might suddenly appear from the right. The intuitive reaction is then to try to steer left and thus away from the car. The motorcycle will initially do so when the steer torque is applied to the left, but because of the nature of the maneuvering dynamics the motorcycle will steer into the car as indicated in Figure 6. Again the remedy is to simultaneously lean and counter-steer. Counter-steering consists of a momentary turn of the handlebars in the opposite direction of the intended travel. This establishes a lean, which aids in turning the motorcycle away from the danger (or the obstacle that has appeared in front of the rider). Failure to counter-steer often occurs with novice riders and is responsible for a significant number of motorcycle accidents and resulting deaths. Leaning without counter-steering suffices in many normal traveling conditions, like routine lane changes. Counter-steer is, however, necessary for fast lane changes because the dynamics of leaning alone has a long response time.

Effects of Rider Lean

In terms of model development it has been assumed that the rider sits rigidly on the bicycle without leaning. A simple way of describing the effect of leaning is to consider the rider as composed of two rigid pieces, the lower body and the upper body, where the upper body can be rotated as is illustrated in Figure 7.

Let the upper body lean angle be ϕ relative to a plane through the bicycle. The linearized momentum balance given by (1) is then replaced by

$$J\frac{d^2\varphi}{dt^2} + J_r\frac{d^2\phi}{dt^2} = mgh\varphi + m_r g h_r \phi + \frac{DV}{b}\frac{d\delta}{dt} + \frac{mV^2h}{b}\delta.$$

where J_r is the moment of inertia of the upper body with respect to the x axis, m_r is the mass of the upper body, h_r is the distance from the center of mass of the upper body and its turning axis. Combining this equation with the static model of the front fork (8) gives the following model of the bicycle with a leaning rider

$$\begin{aligned} J\frac{d^2\varphi}{dt^2} &+ \frac{DVk_2(V)}{b}\frac{d\varphi}{dt} + \Big(\frac{mV^2hk_2(V)}{b} - mgh\Big)\varphi \\ &= \frac{DVk_1(V)}{b}\frac{dT}{dt} + \frac{mV^2k_1(V)}{b}T - J_r\frac{d^2\phi}{dt^2} + m_r g h_r \phi. \end{aligned} \tag{15}$$

The dynamics relating steer angle to steer torque and lean is a second order dynamical system with two independent inputs. The difficulties associated with the right half plane zero in (14) can be avoided because the system has two inputs. By proper coordination of handlebar torque and upper body lean it is possible to alleviate the problems occurring when using only handlebar torque for steering. This is what we normally do intuitively when biking. The system theoretic interpretation is that the right half plane zero is eliminated by introducing an extra control variable.

Bicycles in Education

Bicycles are simple, inexpensive and highly attractive to use in education. One advantage is that many students use bikes and have some feel for their behavior. The bicycle can be used to illustrate a wide variety of aspects on control such as modeling, dynamics of nonlinear nonholonomic systems, stabilization, fundamental limitations, the role of right half plane poles and zeros, the role of sensing and actuation, control design, adaptation, integrated process, and control design. The bicycle is also well suited in activities for K-12 and the general public. Self-stabilization of conventional bikes and the counter-intuitive properties of rear-steered bicycles are useful in this context. A few experiments will be presented in this section.

Pioneering work was done at Cornell University, [56] and at the University of Illinois at Urbana-Champaign (UIUC), [57]- [59]. An adaptive controller for a bicycle was designed at the Department of Theoretical Cybernetics at the St. Petersburg State University [50]. A number of universities have incorporated bicycles into the teaching of dynamics and controls, for example, Lund University, University of California Santa Barbara, and University of Michigan. Bicycles are also used in mechanics and physics.

Instrumentation

There are many simple experiments that can be conducted with only modest instrumentation. A basic on-board instrumentation system can provide measurements of wheel speed, lean angle, relative lean angle of the rider, front fork steer angle, and steer torque. Gyros and accelerometers give additional information. Video cameras on the ground, on the frame, and on the handlebars are useful alternatives. Examples of on-board instrumentation are given in Figures 8 and 9.

The Front Fork

The front fork is essential for the behavior of the bicycle. The front fork is influenced by many factors such as rider applied steer torque, gravitational forces, gyroscopic reaction torques, and ground reaction forces due to castor and camber. A simple way to develop insight is to ride a bike in a straight path on a flat surface. The rider can lean gently to one side, and apply a steer torque to maintain a straight-line path. The torque can be sensed by holding the handlebars with a light fingered grip. The experiment can be repeated for different velocities.

Most students are astonished to realize that in many "no-hands" riding conditions one initiates a turn by leaning into the turn, but the matter of maintaining the steady state turn requires a reverse lean - in other words - leaning relatively speaking opposite to the direction of turn. This is a good illustration of the bicycle's counter-intuitive nature associated with its non-minimum phase properties.

Quantitative data can be obtained by measuring four variables: steer torque, rider lean angle relative to the frame, steer angle, and forward velocity. In the UIUC experiments steer torque was measured using a torque wrench. Rider lean angle was measured using a potentiometer pivoted to the rear of the seat attached to a vertical rod and then attached to a vest worn by the rider. A DC voltage calibrated to rider lean angle was indicated by a voltmeter mounted on the handlebars in view of the rider. Steer angle was also measured in a similar way. Velocity was measured using a common bicycle computer.

A nice experiment that can be conducted with basic instrumentation is to ride in a circle marked on the ground, a radius of 10 m is a good value. The objective of the experiment is to use lean angle as the primary input, while steer torque is nulled to be maintained close to zero. Small adjustments of steer torque are required to maintain stability and the desired circular path. Another experiment is to keep the relative rider lean angle at zero degrees and applying the steer torque required to maintain the steady state turning. The experiment should be conducted at a constant forward speed. The behavior of the system is different for speeds below and above the critical speed.

Stabilization

Many experiments can be done to illustrate instability and stabilization of riderless bikes in various configurations. Start by holding the bicycle still and upright, and then releasing it to get a feel for the time constant of the basic instability of the stationary bicycle as an inverted pendulum. Then walk forward holding the bicycle in the saddle, observe the behavior of the front fork as the velocity is changed, and determine the self alignment velocity (12). The next sequence of experiments can be done on an open surface with a gentle down-slope. One person starts the bicycle rolling with a firm initial push. Another person positioned downhill hits the bike sideways with a hand as the riderless bike passes. The impact causes a brief transient and then the self-stabilizing action restores the bicycle to a straight upright position, although the path of travel will be at a different angle. Give the bicycle a push downhill and observe its self-stabilizing behavior. The downhill slope and initial speed should be sufficient so that the critical velocity is exceeded. Tests at different speeds combined with side impulse jabs will give a feel for critical velocity. When velocity is subcritical a bike will oscillate with growing amplitude when jabbed. Conversely, the oscillation is damped when the velocity is above the critical speed. Next, it is particularly useful to see how critical velocity varies with trail, and then with the addition of mass to the front wheel. A simple way to increase the mass of the front wheel is to wrap a lead cable or chains around the rim and secure in place with tape around the circumference of the rim and tire. This increased mass increases the gyroscopic action on the front wheel substantially and lowers the critical velocity.

Yet another experiment is to arrange a destabilizing spring torque to the front fork by attaching a rubber cord between the steering neck stem and the seat post, see Figure 10. The rubber cord acts as a negative spring. It is destabilizing at zero velocity but surprisingly it decreases the critical velocity. Stability is improved because the front wheel, when moving, reacts faster to the lean.

The Rocket Bike

The idea is to provide a riderless bicycle with a model rocket attached to the handlebars to one side so that it can create a torque on the front fork. In the experiment the bike is pushed off in a straight direction. The rocket is ignited remotely. The startling counter-intuitive result is that the path of travel resembles the one in Figure 6. Spectators are frequently astonished to see that the bicycle ends up turning in the opposite direction of the applied torque. This presents convincing empirical evidence that the bicycle is counter-intuitive, aptly demonstrating the non-minimum phase character of the bicycle. Moreover, the bicycle when pushed off will either remain stable in a turn when velocity is above its critical value, or it will fall inwards in an increasingly tighter turn if velocity is below the critical value. These rocket-push experiments can be combined with stability augmentation such as weighted wheels and differing trail configurations described above.

Rear-Wheel Steering

A bicycle with rear-wheel steering has many pedagogical uses. Almost anyone that tries it is confounded by its properties, particularly that a device that is seemingly similar to what they use regularly exhibits such a strange behavior. Our experience is that the rear-steered bike always stirs up interest from K-12 to senior engineers in industry. It is also an ideal tool to motivate students to study dynamics and control. A rear-steered bike can be used to illustrate the importance of integrated process and control design, effects of poles and zeros in the right half plane, and the fundamental limitations caused by bad system dynamics. It can also lead into discussions about the role of sensors and actuators. An example is provided by the scenario given in Side-bar 3.

A regular bike can be converted to rear-wheel steering with a moderate effort. Such a bicycle is shown in Figure 11. Trying to ride a rear-steered bicycle is a direct way to get a feel for the difficulties caused by dynamics with poles and zeros in the right half plane. The bicycle built by Klein is un-ridable, as is a similar bicycle built in Lund. The simple analysis based on the second order model (13) gives insight into what is required to ride a rear-steered bicycle. The RHP pole is largely fixed, $p \approx \sqrt{g/h}$, but the RHP zero $z \approx V/a$ depends on the velocity. The zero is thus at the origin at zero velocity and it moves to the right with increasing velocity. To have a good separation between the pole and the zero it is desirable to have a large velocity V, a small distance from the contact point of the front wheel and the projection of the center of mass on the ground a, and a large height h. Such a bicycle has been built by Klein, see Figure 12 and it can indeed be ridden successfully. The UCSB bicycle in Figure 13, is designed so that it can be ridden under certain conditions. One way is to start by leaning forward (making a small) and standing up on the pedals (making h large), and then pedaling rapidly to get up to high speed fast. It takes fortitude but several students and one faculty member, Roy Smith, have mastered the technique. Bicycles with rear wheel steering are clumsy to handle. Lunze [60] has made a clever design of a bicycle with rear-wheel steering that also can be used as a regular bike, see Figure 14.

Adapted Bicycles for Teaching Children with Disabilities

Even if it is an intellectual challenge to make un-ridable bikes it is perhaps more useful to make bicycles that are easy to ride. Klein and colleagues have developed a program and a methodology to permit children to ride bikes more easily [61] [62]. The program has been used successfully to teach children with disabilities to ride bicycles. It is based on knowledge of bicycle dynamics, rider perceptions and limitations. Children who are fearful of riding, or children who have yet to learn to ride, typically fear speed and thus timidly pedal bikes in anticipation of falling. The low speed, the rigidity of the child's tense body, and previous exposure to training wheels, tend to make the experience uncomfortable and erratic for the child. In particular, the children cannot benefit from the self-stabilizing property which requires that the critical speed is exceeded.

When placing intrepid children on a bicycle it is desirable to provide stability augmentation and to slow down the dynamics. This makes the bicycle more forgiving and easier to ride. Experience from behavior studies indicate that humans, including children with disabilities, can learn, refine, and encode motor tasks as long as they they have some success [63]. By adapting the bicycles it is possible to create conditions so that children, even those with developmental challenges, can often develop riding skills.

By making a bicycle more stable and thus easier to ride, we necessarily diminish maneuverability and thus restrict controllability in a systems theoretic sense. Getting a fearful first time rider to be successful in staying upright on a bike takes precedence over issues of maneuverability. Thus, the program uses special bicycles with properties of enhanced stability and slower dynamics. Moreover, pedagogical methods have also been developed to use these bikes to teach children with an array of disabilities and handicaps. The key is that bikes should behave like ordinary bicycles but with more benign dynamics.

The simple model given by (1) shows that the bicycle has an unstable mode. The pole associated with this mode is approximately $\sqrt{g/h}$. The unstable mode can be made slower by replacing the ordinary wheels with crowned rollers as shown in Figure 16. Let the lateral contour radius of the roller be R, see Figure 15.

If $R < h$ the gravity term $mgh\varphi$ in (1) is replaced by $mg(h-R)\varphi$ and the unstable pole becomes

$$p = \sqrt{\frac{mg(h-R)}{J}} \approx \sqrt{\frac{(1-R/h)g}{h}}. \tag{16}$$

The unstable pole can thus be made arbitrarily slow by making R large. The pole is zero for $R = h$. For $R > h$ the system (1) has poles on the imaginary axis for zero velocity and close to the imaginary axis for low velocities. The critical velocity also decreases. By using crowned wheels and changing the gearing it is possible to design bicycles that behave much like an ordinary bicycle, but which can be pedaled at a slower forward speed while maintaining the self-stabilizing property.

In essence the use of crowned rollers slows down the time scale of the dynamics by reducing gravity's action, as the effective moment arm for gravity is reduced. By varying the lateral contour radius of the crown, the two dominant open poles can be moved along the positive real axis or to the imaginary axis. Bicycles with crowned rollers function dynamically as if we had taken a conventional bicycle and placed it in a fractional gravity environment, such as a lunar environment.

For fearful children, we can start with nearly flat rollers which make the bicycle statically stable, see Figure 16. We can then proceed by decreasing the radii of the rollers first making the unstable pole slow and gradually moving towards faster unstable poles as the children acquire balance and riding skills. When trained instructors observe that the children are "participating in the steering" by turning into the direction of the lean, we advance them onto bikes that are more like conventional bikes, but still have modifications, see Figure 17. Once children are ready for more challenge they are placed on a more conventional adapted bike. Even these bikes have a number of modified parameters: increased front wheel mass using a turf style tire; then later a weighted wheel; positioning of the seat to be forward and yet comfortable: pedals repositioned to be lower and more forward; shortened the pedal crank length; and raised handlebars so as to promote better forward vision. See Figure 18. In this way it is easier for children to learn how to ride conventional two-wheelers. The approach is similar to the "wind-surfer" approach to adaptive control proposed in [64].

Klein also uses adapted trainer bikes incorporating "dither" inputs on the handlebars, analogous to signal stabilization inputs pioneered by Oldenburger. A modestly high frequency cyclical torque variation can be superimposed on the handlebars as the learning child experiences riding. The presence of the dither signal causes the rigidity of the rider's arms to relax, thus causing the rider to better sense what the bicycle dynamics are trying to do - and to go along as opposed to fighting against the self-stabilizing character of the bicycle's front fork. Signal stabilization when injected into nonlinear feedback systems, usually just ahead of a nonlinearity, will cause the nonlinearity to soften in a describing function sense. This is exactly what happens when the children experience the handlebar dither.

The adapted bicycle designs used so far have incorporated mechanical adaptations. We opted for this instead of electronic stability augmentation which is not conducive for many fearful children. Also, instructors who lack technical expertise in servomechanism technology do best with mechanically adapted bicycle trainers.

The therapy has been effective in over a dozen U.S. based clinics with children, and adults, with a wide array of disabilities, including Down syndrome, autism, mild cerebral palsy, Asperger's syndrome, and more. The age ranges of children typically run from six to twenty years. The therapy has been applied to about 600 children. To date, favorable results are achieved in about 70 to 80 percent of the children. The children typically enjoy the camp/clinic environments, and smiles abound. As the children learn, the progression of adapted bikes allows them to take over more and more of the control, and thus enjoy increased maneuverability. Astonishing conclusions, based on clinical trials to date, are that the bicycle, once made stable, can provide a kinesthetic learning environment, and that even children with cognitive and developmental issues can become successful riders in relatively brief periods when immersed in the therapy.

More Complicated Models

The simple models used so far give insight but they are not detailed enough to give accurate answers to some interesting questions, such as how the critical velocity depends on the geometry and mass distribution of the bicycle. There are also many physical phenomena that could be considered in greater depth, such as gyroscopic effects and tire-road interaction. A nonlinear model can be used to determine continuously turning solutions, analyze stability and bifurcations of these solutions as well as the upright forward moving reference solution.

Many models of bicycles of different complexity are available in the literature. Most models are linear because they were developed before adequate computer tools were available. Today it is feasible to develop complex models using general object oriented software like Modelica [65]. A system is then viewed as an interconnection of subsystems. Both subsystems and interconnections are described in an object oriented manner which permits effective use of inheritance. Equations for each subsystem are given as balances of mass, momentum, and energy together with constitutive equations. Component libraries can be built and modeling is then done simply by graphically combining components. A Modelica library for multibody systems is described in [66].The software gathers all equations for subsystems and interconnections and uses extensive symbolic computations to eliminate redundant equations. The resulting equations are then transformed into ordinary differential equations or differential algebraic equations which are then integrated numerically. Linearized equations are obtained as a side product. Special purpose software for multibody systems [40], [41], [42], [43] can also be used. The nonlinear model analyzed in this chapter is derived with a multibody software described in [67]. A similar approach is used in [45].

A Linear Fourth Order Model

The simple second order model consists of a dynamic torque balance for the frame (1) and a static momentum balance for the front fork assembly (8). A natural extension is to replace the static front fork model with a dynamic model, which results a model of fourth order. Such a model was derived by Neĭmark and Fufaev [25], [26] who presented a model of the form

$$M \begin{pmatrix} \ddot{\varphi} \\ \ddot{\delta} \end{pmatrix} + CV \begin{pmatrix} \dot{\varphi} \\ \dot{\delta} \end{pmatrix} + (K_0 + K_2 V^2) \begin{pmatrix} \varphi \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ T \end{pmatrix}, \tag{17}$$

where V is the forward velocity. Expressions for the elements of the matrices are given as functions of the geometry and the mass distribution of the bicycle. A similar approach is used in [56] and [28] where minor errors in [25] are corrected. The results are summarized in [22] where the model is compared with numeric linearization of nonlinear models obtained from two multibody programs. The bicycle is described by 23 parameters: wheel base b =1.00 m, trail c =0.08 m, head angle $\lambda = 70°$, wheel radii $R_{rw} = R_{fw}$ =0.35 m and the parameters that characterize the mass distribution. The parameters in Table 1, with $g = 9.81$ m/s^2

Table 1 Masses and inertia tensors for standard bicycle with and (without) rider.

	Rear Frame	*Front Frame*	*Rear Wheel*	*Front Wheel*
Mass m [kg]	87 (12)	2	1.5	1.5
Center of Mass				
x [m]	0.492 (0.439)	0.866	0	b
z [m]	1.028 (0.579)	0.676	R_{rw}	R_{fw}
Inertia Tensor				
J_{xx} $[kg/m^2]$	3.28 (0.476)	0.08	0.07	0.07
J_{xz} $[kg/m^2]$	-0.603 (-0.274)	0.02	0	0
J_{yy} $[kg/m^2]$	3.880 (1.033)	0.07	0.14	0.14
J_{zz} $[kg/m^2]$	0.566 (0.527)	0.02	J_{xx}	J_{xx}

give the following numerical values for the matrices in (17)

$$
\begin{aligned}
M &= \begin{pmatrix} 96.8\ (6.00) & -3.57(-0.472) \\ -3.57\ (-0.472) & 0.258\ (0.152) \end{pmatrix} \\
C &= \begin{pmatrix} 0 & -50.8\ (-5.84) \\ 0.436\ (0.436) & 2.20\ (0.666) \end{pmatrix} \\
K_0 &= \begin{pmatrix} -901.0\ (-91.72) & 35.17\ (7.51) \\ 35.17\ (7.51) & -12.03\ (-2.57) \end{pmatrix} \\
K_2 &= \begin{pmatrix} 0 & -87.06\ (-9.54) \\ 0 & 3.50\ (0.848) \end{pmatrix}.
\end{aligned}
\tag{18}
$$

The values in parentheses are for a bicycle without rider. This model is well suited for design of autopilots for bicycles.

The momentum balance for the roll axis is similar to the simple model (1), but there is an additional inertia term $m_{12}d^2\delta/dt^2$ due to the shift of the mass when the steering wheel is turned. The model (17) gives the self-alignment velocity of the front fork as $V_{sa} = \sqrt{7.51/2.57} = 1.74$ which can be compared with the value 1.89 given by (7).

A Fourth Order Nonlinear Model

A complete nonlinear model was used for more detailed investigations. The model in [44] was generalized so that toroidal wheels could be investigated. The model was developed using Sophia [67], which is a package of Maple procedures for symbolic representation of mechanics that is suitable for analysis of multibody systems. The model has the form

$$\begin{aligned} \mathcal{M}\dot{u} &= f_u(u, q) \\ \dot{q} &= f_q(u), \end{aligned} \tag{19}$$

where q is a vector of lean and steer angles, u is a vector of their velocities and the angular velocity of the front wheel projected onto its axis. The system is of fourth order even if five coordinates are used because energy is conserved. The matrix $\mathcal{M}$, and the functions f_u and f_q are generated from the multibody program. They are much to too long to publish in text form. Linearization of the nonlinear equations by symbolic methods gives the linear model (17).

The bifurcation diagram in Figure 19 gives fixed points of a bicycle with rider that is rigidly attached to the rear frame. It illustrates the complex behavior of the nonlinear model. Apart from the solution $\varphi = 0$, and $\delta = 0$ there are continuously turning solutions and several saddle-node bifurcations. These solutions are all unstable.

The fourth order model captures many aspects of the bicycle but there are still effects that are neglected. Several extensions of the model are of interest.

Frame elasticities are noticeable even in conventional bicycles with diamond-shaped frames and even more so in ladies bikes, see Figure 14, and mountain bikes with suspension springs. Pneumatic tires also have elasticities that have to be considered as in [26] and [46]. Effects of the interaction between tire and road can also be considered. It is also interesting to have a more realistic model of the rider.

Stability of an Uncontrolled Bicycle

The simple second order model given by (10) and (8) indicates that a bicycle is self-stabilizing provided that the velocity is sufficiently large. This phenomena will now be explored further using the fourth order linear model (17). Figure 20 shows the root locus of the model (17) with matrices given by (18) for the bicycle with a rider. When the velocity is zero the system has four real poles at $p_{1,2} = \pm 3.05$ and $p_{3,4} = \pm 9.18$, marked with circles in Figure 20. The first pair of poles correspond to the pendulum poles which are approximately given by $\sqrt{mgh/J} \approx \sqrt{g/h}$. The other pole-pair depends on interaction of frame and front fork. As velocity increases the poles p_1 and p_3 meet and combine to a complex pole pair at a velocity that is close to the self-alignment velocity. The real part of the complex pole-pair decreases as velocity increases. Following [46] this mode is called the weave mode. This mode becomes stable at the critical velocity V_c =5.95 m/s. The pendulum pole p_2, which correspond to the capsize mode, remains real and moves towards the right with increasing velocity. It becomes unstable at the velocity V =10.36 m/s. The determinant of the matrix $K_0 + V^2 K_2$ vanishes at this velocity. The instability of the capsize mode at higher velocities has little influence on practical ridability, because the real part is small and easy to stabilize by manual feedback. The pole p_4 moves to the left as velocity increases. Asymptotically for large V it follows from (17) that one pole is zero and three poles are proportional to the velocity, $p = (-0.65 \pm 1.33i)V$, and -1.40V.

The velocity range where the the bicycle is stable is more clearly visible in Figure 21 which shows the real part of the poles as functions of velocity. The values for negative velocities correspond to a rear-steered bicycle. Figure 21 shows that the poles for a rear-steered bicycle have positive real parts for all velocities.

The nonlinear model reveals even more complex behavior. There is a stable limit cycle for velocities below the critical velocity as illustrated in Figure 22, which shows trajectories for a bicycle without rider for initial velocities above and below the critical velocity. The stable periodic orbit is clearly visible in the figure. There is period doubling when velocity is reduced further. The periodic orbit can be observed in carefully executed experiments.

Gyroscopic Effects

An often posed question is to what extent gyroscopic effects contribute to stabilization of bicycles. This question was raised by Klein and Sommerfeld [9]. Sommerfeld summarizes the situation as follows [68]:

> " That the gyroscopic effects of the wheels are very small compared with these (centrifugal effects) can be seen from the construction of the wheel; if one wanted to strengthen the gyroscopic effects, one should provide the wheels with heavy rims and tires instead of making them as light as possible. It can nevertheless be shown that these weak effects contribute their share to the stability of the system."

This is in contrast with the conclusion in [69] who claims that gyroscopic action plays very little part in the riding of a bicycle at normal speeds. Jones wanted to design an unridable bicycle and he investigated the effects of gyroscopic action experimentally by mounting a second wheel on the front fork to cancel or augment the angular momentum of the front wheel. We quote from [12]:

> "This creation, "Unridable Bicycle MK1" unaccountably failed; it could easily be ridden, both with the extra wheel spinning at high speed in either direction and with it stationary. Its "feel" was a bit strange, a fact I attribute to the increased moment of inertia about the front forks, but it did not tax my (average) riding skill even at low speed."

Klein and students at UIUC conducted a number of precession canceling and altering experiments in the 1980's. Two experimental UIUC bikes are shown respectively in Figures 25 and 26. These two bikes share similarities, except for the front fork geometries. The first bike has an unaltered steering head and a positive trail. Similar to Jones' results, this precession-canceling bike is easily ridable. The second bike, see Figure 26, has an altered frame thereby accommodating a vertical steering head and front fork, as well as zero trail. This bike, called the "naïve bicycle," has no predisposition for the handlebars to turn when the bike leans. While the naïve bike is modestly wobbly when ridden, it is indeed ridable.

To get further insight into the problem we will investigate the effects of precession by calculating the poles for different moments of inertia of the wheels for the riderless bicycle where the effects are more pronounced. The moment of inertias were varied from 0 to 0.184 kgm^2, which corresponds to the case when all mass is at the perimeter of the wheel. The results are given in Figure 23 which shows that the velocity interval where the bicycle is stable is shifted towards lower velocities when the front fork inertia is increased. This is also well documented with experiments with bicycles like the one shown in Figure 10.

Notice in the figure that the velocity where the capsize mode becomes unstable is inversely proportional to the square of the moments of inertia for the wheels. This suggests that there is a simple formula for the velocity limit, but it remains to be found. Asymptotically for low moments of inertia it appears that the ratio of the velocities is constant.

The difficulties in manual control of unstable systems can be expressed in terms of the product $p\tau$, where p is the unstable pole, and τ is the neural delay of the human controller expressed in seconds. A system with $p\tau > 2$ cannot be stabilized at all. In [55] it is shown that comfortable stabilization requires that $p\tau < 0.2$. A crude estimate of the neural delay for bicycle steering is $\tau = 0.2$ ss [29]. A bicycle can thus be ridden comfortably so long as the unstable pole has a real part that is less than 1 rad/s. This means that the effect of gyroscopic action has little practical effect because a skilled rider can easily stabilize a slow unstable pole.

Jones' work and the collective UIUC precession experiments suggest that precession has a minor effect for conventional bicycles.

Wheels

Many bicycles have thin tires where a typical tube diameter is an order of magnitude smaller than the wheel radius. The crowned rollers used in bicycles for children with disabilities change the dynamics significantly. This is illustrated in Figure 24 which shows the poles for an adapted bicycle with rider. The bicycle has a rear wheel with a crowned roller with lateral radius 0.4 m and the height of the center of mass is 0.4 m. The weave mode becomes stable at 0.54 m/s and the capsize mode remains stable for all velocities. The dynamics of the system is also slower as predicted by (16).

Conclusions

Control is becoming ubiquitous with an increasing number of applications in widely different areas. One reason is the huge potential benefits of feedback: to reduce effects of disturbances and parameter variations, to change dynamic behavior of systems, the ability to stabilize an unstable system. Another reason is the ease of implementation of control. An understanding of control is also essential for understanding the behavior of natural and man-made systems. The wider range of potential uses of feedback gives challenges to control. A recent panel for assessment of the field [70], [71] expressed it as follows:

> "The Panel believes that control principles are now a required part of any educated scientist's or engineer's background, and we recommend that the community and funding agencies *Invest in new approaches to education and outreach for the dissemination of control concepts and tools to non-traditional audiences.*

When teaching control to students from a broader spectrum of disciplines it is essential to strive for a balance between theory and application. One way to do this is to relate the abstract concepts to something concrete and familiar. The bicycle allows us to show students how to understand dynamic behavior, and to model and design control systems.

In this paper we have approached the bicycle from the point of view of control. By using models of different complexity we have explored some interesting properties, for example that the front fork creates a feedback that under certain circumstances stabilizes the bicycles and that rear-driven bicycles are difficult to ride. These properties can be explained qualitatively using simple models. More elaborate models which give quantitative results are also given. A number of experiments that can be performed with modest equipment have been described. Insights into dynamics and control have been used to develop adapted bicycles for children with cognitive and motor developmental disabilities. Clinical experiences from using adapted bicycles are also presented.

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Side-bar 1 - The NHSA Rear-Steered Motorcycle

The National Highway Safety Administration funded a project aimed at developing a safe motorcycle in the late 1970s. The key ideas were to have a low center of mass , a long wheel base, and separation of steering and braking. The last requirement leads naturally to a design with rear-wheel steering because the front wheel gives the major contribution to the braking force. Rear-wheel steering combined with a long wheel base also makes it possible to have a low center of mass. A contract to analyze a motorcycle with rear-wheel steering and to build a test vehicle was given to South Coast Technology in Santa Barbara, California, with Robert Schwarz as principal investigator. A theoretical study was performed by taking the mathematical model developed by Sharp [46] and simply reversing the sign of the velocity. A derivation based on first principles showed that this does indeed give the correct model. The model was linearized and the eigenvalues were investigated. Two complex pole pairs and a real pole representing weave, wobble, and capsize dominate the dynamics. A range of geometrical configurations was investigated. The eigenvalues were plotted as functions of velocity for each configuration. The real part of the unstable poles typically covered a range from 4 to 12 rad/s for speeds ranging from 3 to 50 m/s. It was concluded the instability was too fast to be stabilized by a human rider. The result was reported to NHSA with a recommendation that it was pointless to build a prototype because the motorcycle could apparently not be ridden. NHSA was of a different opinion based on an in-house study and they insisted that a prototype should be tested. South Coast Technology built the test buck shown in Figure 27. Notice the outrigger with the support wheels. The tests showed conclusively that the motorcycle was unridable even by the most skilled riders. We quote from [72].

The outriggers were essential; in fact, the only way to keep the machine upright for any measurable period of time was to start out down on one outrigger, apply a steer input to generate enough yaw velocity to pick up the outrigger an then attempt to catch it as the machine approached vertical. Analysis of film data indicated that the longest stretch on two wheels was about 2.5 s.

Side-bar 2 - The Role of Sensors and Actuators

The problem with right half plane zeros can to be eliminated by introducing extra sensors and actuators. Consider the standard linear system

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx.$$

The zeros of the system are the values of s where the determinant of the matrix

$$\begin{pmatrix} sI - A & B \\ C & 0 \end{pmatrix}$$

vanishes. A system has no zeros if either of the matrices B or C has full rank. Difficulties with the rear- steered bicycle can thus be eliminated, by introducing a feedback system with a rate gyro and a tilt sensor. However, the system then becomes critically dependent on sensing and actuation for its proper function which means that control is mission critical [73].

It also follows that a system has no zeros if the matrix B is of full rank. The problem with right half plane zeros can thus also be eliminated by using additional actuators. An example of this is maneuvering of bicycles using both steer torque and rider lean. In fact, skilled riders at times use variations in forward speed as an additional control variable, as centrifugal forces can be altered to some extent.

Side-bar 3 - Control is Important for Design

The design of engineering systems is traditionally made based on static reasoning which does not account for stability and controllability. A strong reason for studying control is that dynamics of a system may impose fundamental limitations on design options. Control theory helps to quantify these constraints. Here is a scenario that has been used successfully in many introductory courses.

Start a lecture by discussing the design of a recumbent bicycle. Lead the discussion into a configuration with a front wheel drive and rear wheel steering. Have students elaborate the design, then take a break and say: “I have a device with this configuration. Let us go outside and try it!”. Bring the students to the yard for experiments with the rear-steered bike and observe their reactions. The rear-steered bike riding challenge invariably will bring forth a willing and overly courageous test rider; a rider who is destined to fail in spite of repeated attempts. After a sufficient number of failed attempts, bring them back into the class for a discussion. Emphasize that the design was beautiful from a static point of view but useless because of dynamics. Start a discussion about what knowledge is required to avoid this trap emphasizing the role of dynamics and control. You can spice the presentation with the true story about the NHSA motorbike in Side-bar 1. You can also mention briefly that the concepts required to understand what goes on are poles and zeros in the right half plane. Later in the course there are many natural ways to return to the rear-steered bike when more material has been presented. Tell students how important it is to recognize systems that are difficult to control because of inherently bad dynamics. Make sure that everyone knows that the presence of poles and zeros in the right half plane indicates that there are severe difficulties in controlling a system.

This approach has been used by one of the authors in many introductory classes on control. It shows why a basic knowledge of control is essential for all engineers. It also illustrates that it is a good practice to formulate a simple dynamic model at an early stage in a design project to quickly find potential problems caused by unsuitable system dynamics.

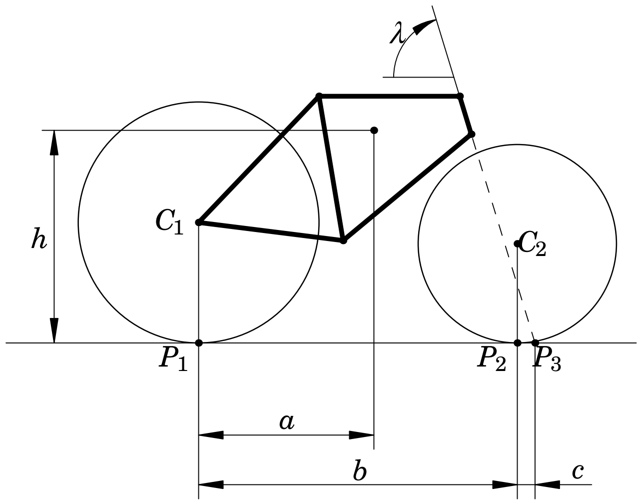

Figure 1 Parameters defining bicycle geometry. The points P_1 and P_2 are the contact points of the wheels and the point P_3 is the intersection of the extended steer axis with the horizontal plane.

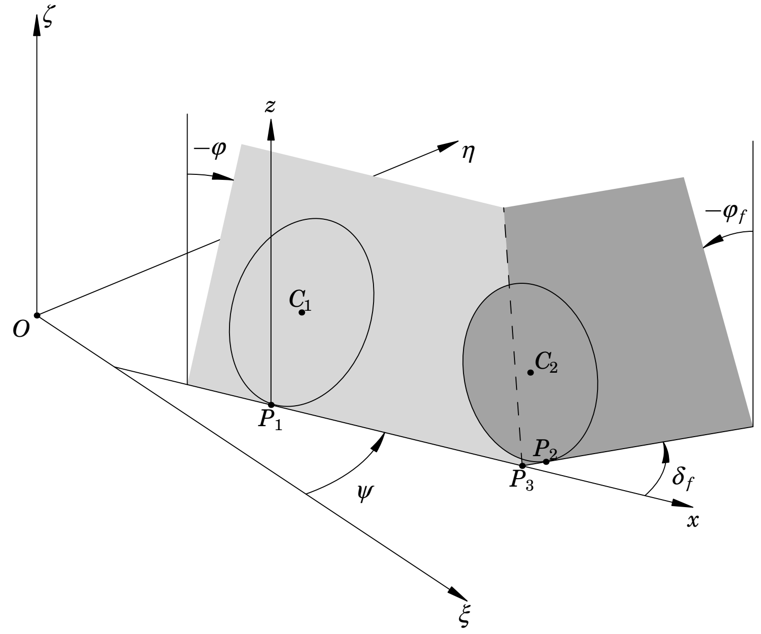

Figure 2 Coordinate systems. The system $\xi\eta\zeta$ is fixed to inertial space. The system xyz has its origin at the contact point of the rear wheel with the xy plane. The x axis is in the direction of the contact line of the rear wheel with the xy plane. The point P_3 lies on this line. The z axis is vertical.

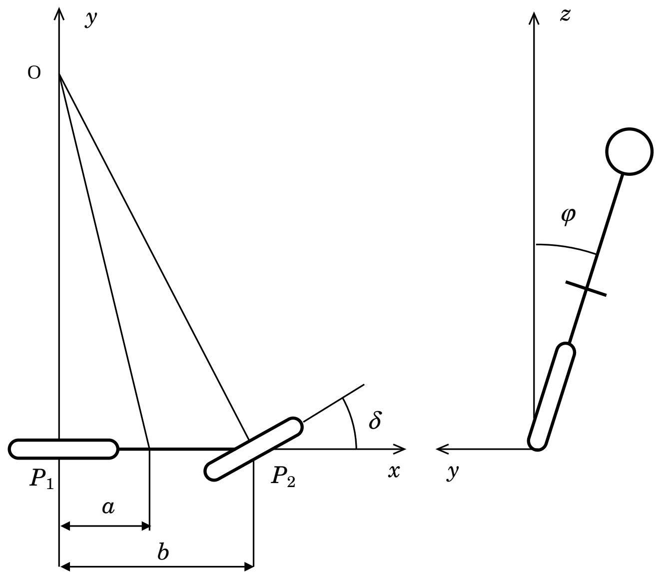

Figure 3 Schematic top (left) and rear views (right) bicycle. The steer angle is δ, the roll angle is φ, and the heading angle is $\lambda = 0$.

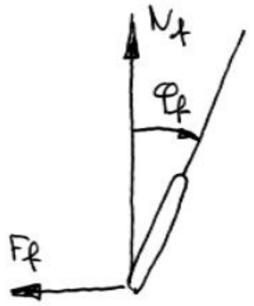

Figure 4 Contact forces acting on the front fork. The normal component $N_f = mga/b$ is due to gravity and horizontal component $F_f = amV^2\delta_f/b^2$ is due to the centrifugal force.

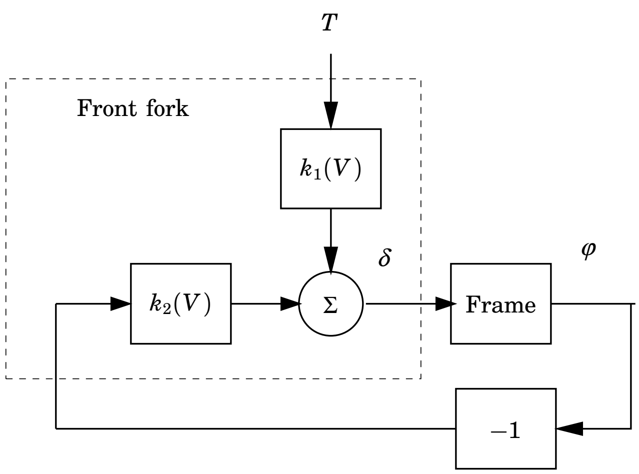

Figure 5 Block diagram of bicycle with a front fork. The steer torque applied to the handlebars is T, φ is the roll angle and δ is the steer angle. Notice that the front fork creates a feedback system.

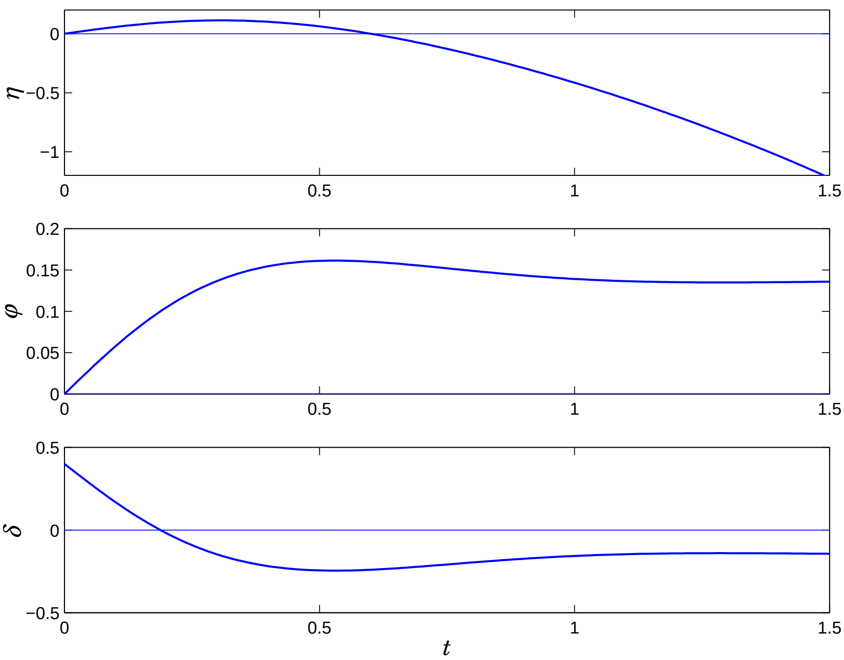

Figure 6 Simulation of bicycle with constant positive steer torque. From top to bottom the plots show the time histories of path deviation η, tilt angle φ, and steer angle δ.

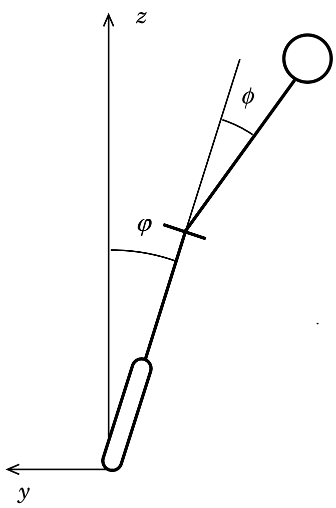

Figure 7 Schematic rear view of bicycle with leaning rider. Bicycle lean is φ, and rider lean relative the bicycle is ϕ.

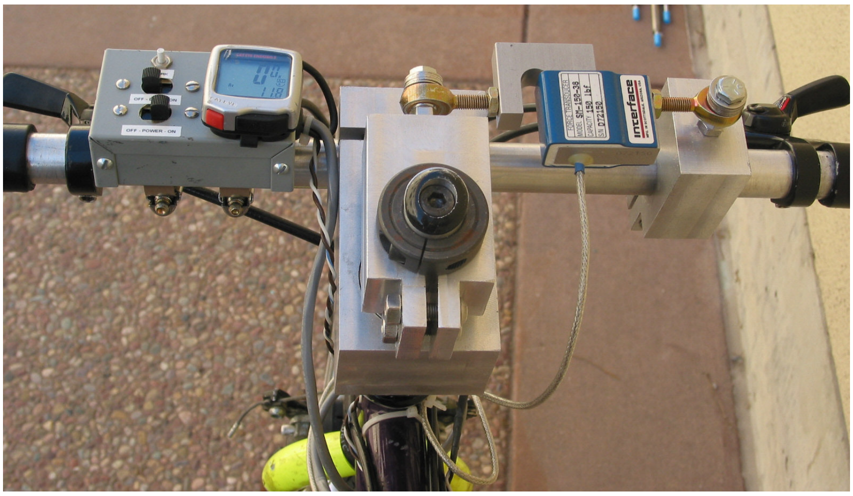

Figure 8 A UCSB bicycle with sensors for velocity and handle bar torque.

Figure 9 A UCSB bicycle with sensor for rider lean, data logger and power pack.

Figure 10 Bicycle from Lund University with a weighted front wheel and an elastic cord attached to a neck on the front fork.

Figure 11 Klein's original un-ridable rear-steered bicycle. There is a US 1000$ prize for riding this bike under specified conditions.

Figure 12 Klein's ridable rear-steered bike. This bicycle is ridable because it has a high center of gravity and a small distance from vertical projection of center of mass to driving wheel contact point.

Figure 13 The UCSB ridable rear-steered bike. This bicycle is ridable provided that you dare to get up to speed quickly as demonstrated by Dave Bothman who supervised the construction of the bike.

Figure 14 Lunze's bicycle. This bike can be used both for front and rear-wheel steering.

The trail by moving the wheel to different positions on the front.

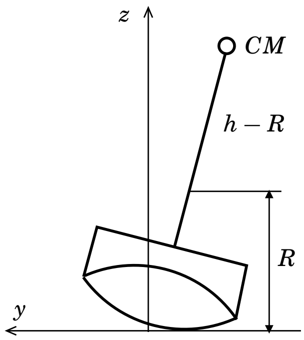

Figure 15 Schematic rear view of bicycle with crowned rollers. The tilting torque due to gravity decreases by increasing R.

Figure 16 Child on an entry level adapted bike. The wheels are replaced with crowned rollers. Virtually all children love the idea of being independent bike riders. This child rides comfortably because the bike is stable.

Figure 17 Child on adapted bike with crowned roller on the rear wheel and a regular front wheel. The child is participating in steering by gracefully turning the handlebars into the direction of tilt.

Figure 18 Teenage boy on an adapted bike with weighted front wheel.

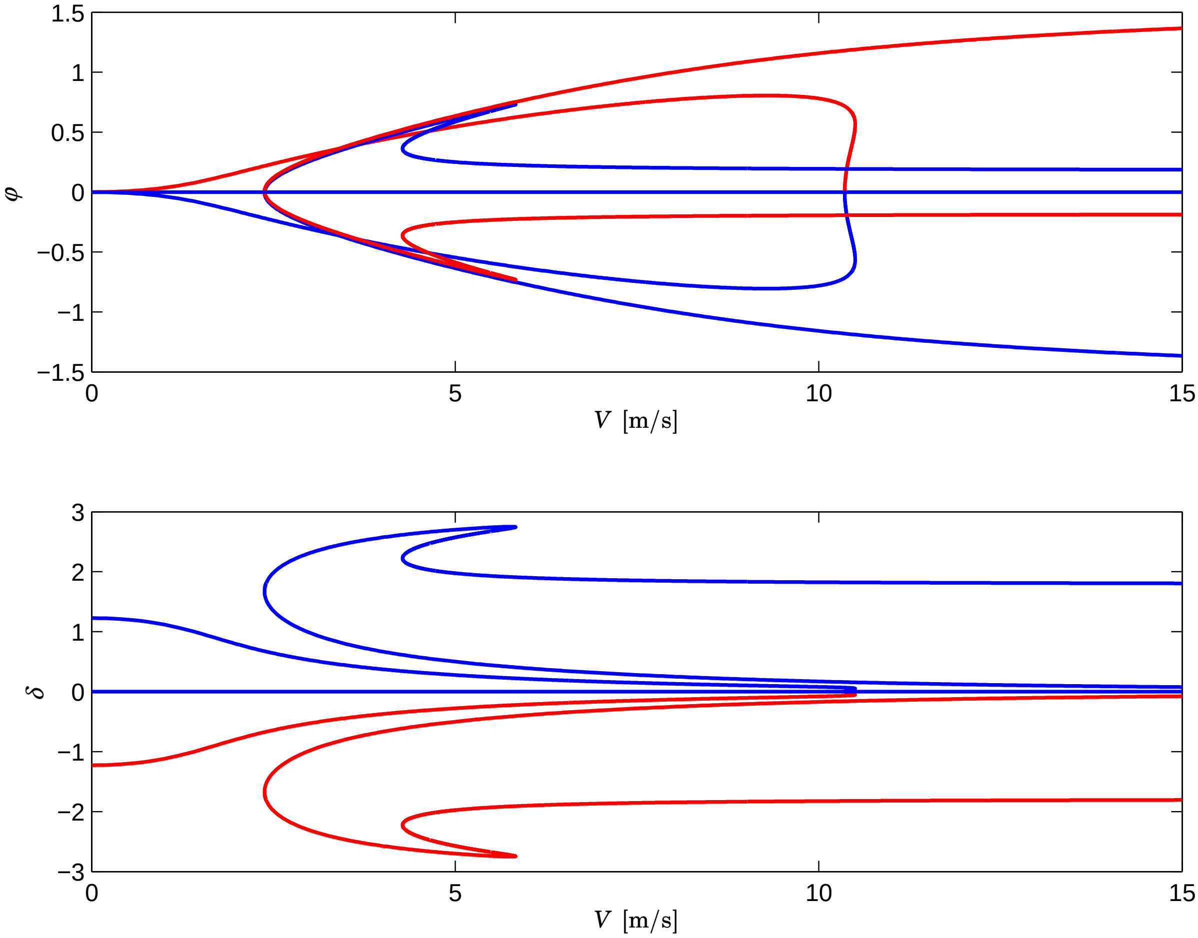

Figure 19 Bifurcation diagram for nonlinear model of bicycle with rider.

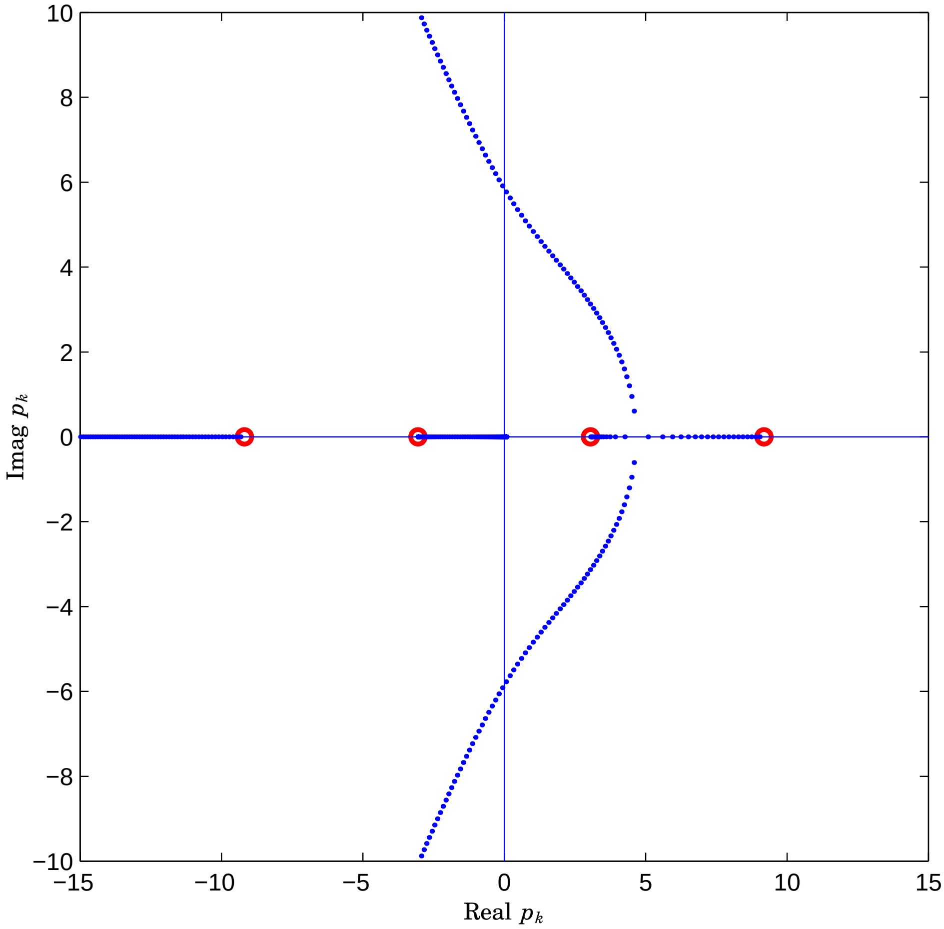

Figure 20 Root locus for bicycle with respect to velocity. The model is given by (17) with parameters (18). The poles at zero velocity are marked with red o.

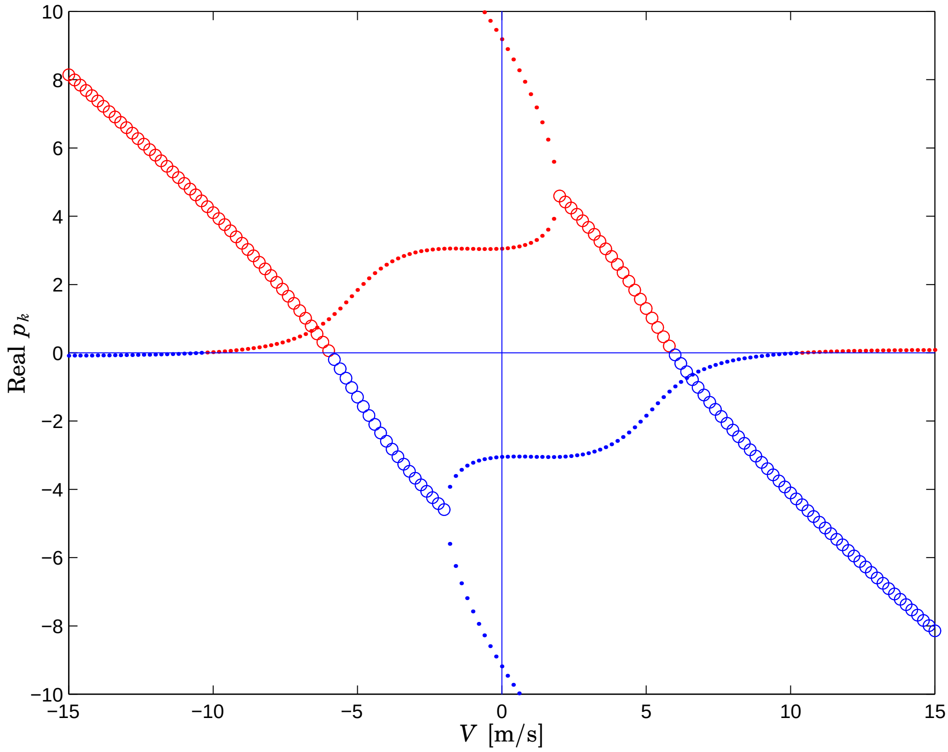

Figure 21 Real parts of poles for a bicycle for a bicycle with rider as function of velocity. Real poles are indicated by a dot, complex poles by o.

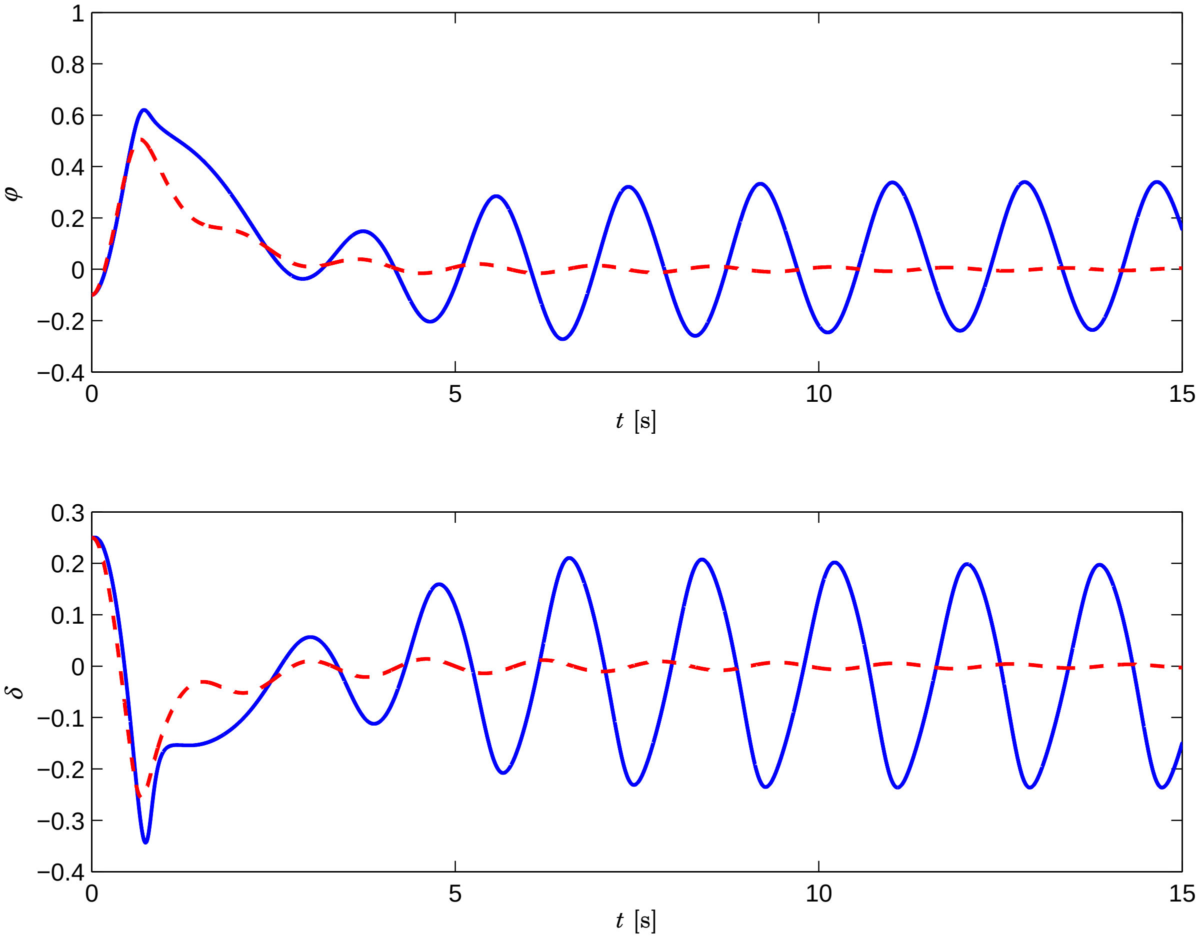

Figure 22 Simulation of nonlinear model (19) of bicycle with rider. The full lines correspond to initial conditions below the critical velocity, the dashed lines correspond to initial conditions slightly above the critical velocity.

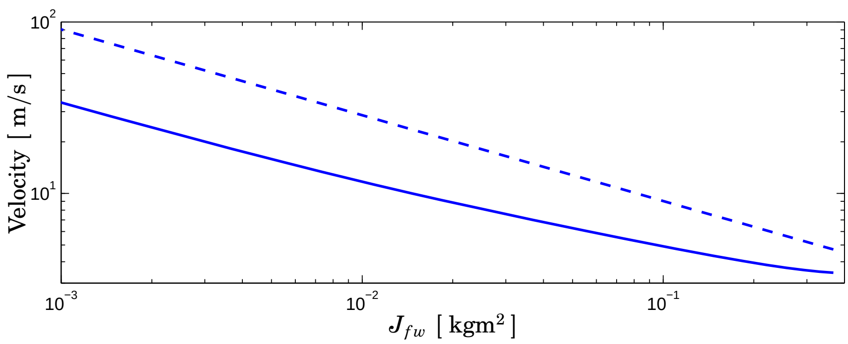

Figure 23 Critical velocities as functions of the moment of inertial of the front wheel. The full line shows the velocity where the bicycle becomes stable and the dashed line shows the velocity where the capsize mode becomes unstable. Notice that scales are logarithmic.

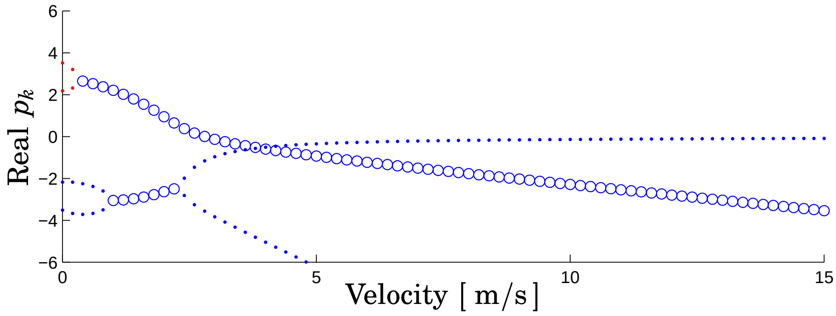

Figure 24 Real parts of poles for an adapted bicycle with rider. The bicycle has a crowned roller with lateral radius 0.3 m at the rear and a heavy front wheel.

Figure 25 A UIUC bike with precession canceling. The center of mass of the front fork assembly could also be adjusted.

Figure 26 The UIUC naïve bike. This bike has vertical front fork ($\lambda = 90^{\circ}$), zero trail, and extra slot provision to accommodate a conventional front wheel.

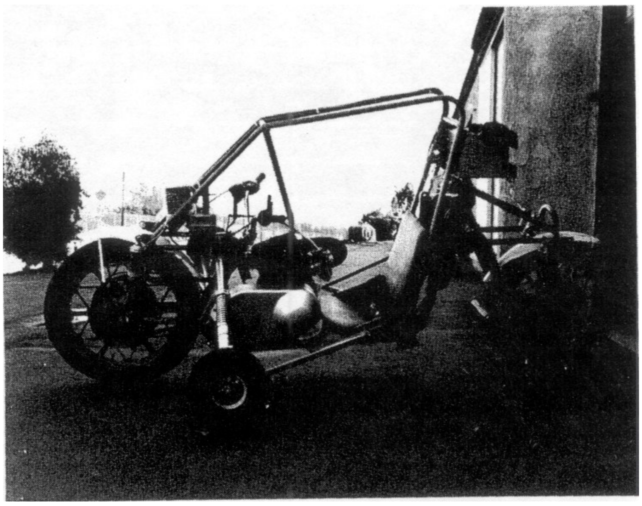

Figure 27 Schwarz prototype for a rear-steered motorcycle. This motorcycle was impossible to ride even by skilled experts. (Courtesy of R. Schwarz)