TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2016 - Assignment 4
Due date: Friday 11 November at 16.00.

1. For each of the following scalar systems, investigate input-to-state stability:

   (1) $\dot{x} = -(1+u)x^3$

   (2) $\dot{x} = -(1+u)x^3 - x^5$

   (3) $\dot{x} = -x + x^2 u$

   (4) $\dot{x} = x - x^3 + u$

   Hint: If a system is ISS, then:

   (a) for $u(t) \equiv 0$ the origin is globally asymptotically stable.

   (b) for a bounded input $u(t)$, every solution $x(t)$ is bounded.

   If one of these is not satisfied, the system can **not** be ISS.

2. For each of the following scalar systems, investigate input-to-state stability:

   (1) $\dot{x}_1 = -x_1 + x_1^2 x_2 \qquad \dot{x}_2 = -x_1^3 - x_2 + u$

   (2) $\dot{x}_1 = -x_1 + x_2 \qquad \dot{x}_2 = -x_1^3 - x_2 + u$

   (3) $\dot{x}_1 = (x_1 - x_2 + u)(x_1^2 - 1) \qquad \dot{x}_2 = (x_1 + x_2 + u)(x_1^2 - 1)$

   (4) $\dot{x}_1 = -x_1 + x_1^2 x_2 \qquad \dot{x}_2 = -x_2 + x_1 + u$

   Hint for part (2): Read example 4.27 before doing this exercise.
   Hint for part (4): For $u(t) \equiv 0$ an ISS system needs to have a globally asymptotically stable origin. This requires the absence of other equilibria.

3. Using Lemma 4.7 in Khalil, show that the origin of the system

$$\dot{x}_1 = -x_1^3 + x_2 \qquad \dot{x}_2 = -x_2^3$$

   is globally asymptotically stable.

4. Consider a system defined by the memoryless function $y = u^{1/3}$.

   (a) Show that the system is $\mathcal{L}_\infty$ stable with zero bias.

   (b) For any positive constant a, show that the system is finite-gain $\mathcal{L}_\infty$ stable with $\gamma = a$ and $\beta = (1/a)^{1/2}$.

   (c) Compare the two statements.

5. Consider a system defined by the memoryless function by $y = h(u)$ where $h : R^m \to R^q$ is globally Lipschitz. Investigate $\mathcal{L}_p$ stability for each $p \in [1, \infty]$ when

   (1) $h(0) = 0$.

   (2) $h(0) \neq 0$.

6. Consider the feedback connection of Figure 5.1 in Khalil, where H_1 and H_2 are linear time-invariant systems represented by the transfer function $H_1(s) = (s-1)/(s+1)$ and $H_2(s) = 1/(s-1)$. Find the closed-loop transfer function from (u_1, u_2) to (y_1, y_2) and from (u_1, u_2) to (e_1, e_2). Use these transfer functions to discuss why we have to consider both inputs (u_1, u_2) and both outputs (e_1, e_2) (or (y_1, y_2)) in studying the stability of the feedback connection.

7. Consider the system

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + (x_1 + a)x_2 \\
\dot{x}_2 &= -x_1(x_1 + a) + bx_2 \\
a &\neq 0
\end{aligned}
$$

   (a) Let $b = 0$. Show that the origin is globally asymptotically stable. Is it exponentially stable?

   (b) Let $b > 0$. Show that the origin is exponentially stable for $b < \min\{1, a^2\}$.

   (c) Show that the origin is not globally asymptotically stable for any $b > 0$.

   (d) Discuss the results of part (a) through (c) in view of the robustness results of Section 9.1 in Khalil, and show that when $b = 0$ the origin is not globally exponentially stable.

8. Consider the scalar system $\dot{x} = -x/(1 + x^2)$ and $V(x) = x^4$.

   (a) Show that inequalities (9.11) through (9.13) in Khalil are satisfied globally with

$$
\begin{aligned}
\alpha_1(r) = \alpha_2(r) &= r^4 \\
\alpha_3(r) &= \frac{4r^4}{1 + r^2} \\
\alpha_4(r) &= 4r^3
\end{aligned}
$$

   (b) Verify that these functions belong to class $\mathcal{K}_\infty$.

(c) Show that the right-hand side of (9.14) in Khalil approaches zero as $r \to \infty$.

(d) Consider that perturbed system $\dot{x} = -x/(1+x^2) + \delta$, where δ is a positive constant. Show that whenever $\delta > 1/2$, the solution $x(t)$ escapes to ∞ for any initial state $x(0)$.

9. Study, using the averaging method, each of the following scalar systems.

(1) $\dot{x} = \epsilon(x - x^2)\sin^2(t)$.

(2) $\dot{x} = \epsilon(x\cos^2(t) - \frac{1}{2}x^2)$.

10. For each of the following systems, show that, for sufficient small $\epsilon > 0$, the origin is exponentially stable:

(1) $\dot{x}_1 = \epsilon x_2$
$\dot{x}_2 = -\epsilon(1 + 2\sin(t))x_2 - \epsilon(1 + \cos(t))\sin(x_1)$.

(2) $\dot{x}_1 = \epsilon[(-1 + 1.5\cos^2(t))x_1 + (1 - 1.5\sin(t)\cos(t))x_2]$
$\dot{x}_2 = \epsilon[(-1 - 1.5\sin(t)\cos(t))x_1 + (-1 + 1.5\sin^2(t))x_2]$.