NTNU – Trondheim
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TTT4120 Digital Signal Processing
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Wiener Filter Design

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Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 12.7.1 FIR Wiener filter
  - 12.7.3 IIR Wiener filter
  - 12.7.4 Noncausal Wiener filter

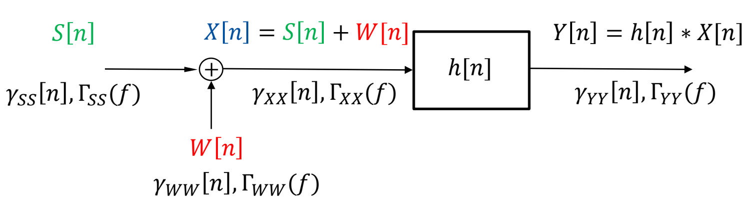
- A compressed overview of topics treated in the lecture, see "Wiener filter design" on Blackboard

*Level of detail is defined by lectures and problem sets

Contents and learning outcomes

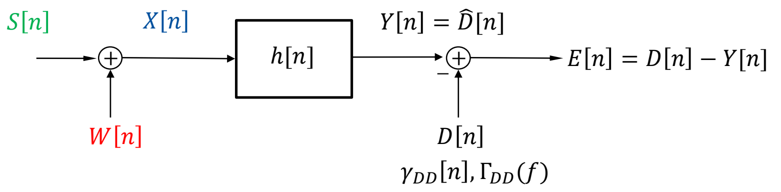
- Optimum MSE filter
  - Non-causal Wiener filter
  - Causal FIR Wiener filter
  - Causal IIR Wiener filter

Signal estimation



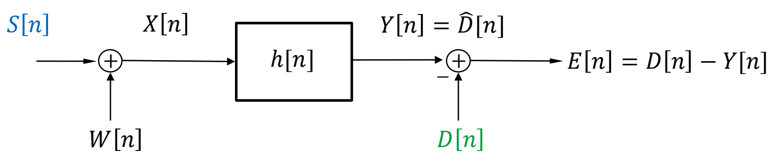
- Input signal $X[n]$ consists of a desired signal $S[n]$ and an undesired interference $W[n]$
- Design a filter $h[n]$ that suppress the undesired signal component
- Objective: Filter out the additive interference $W[n]$ while preserving the characteristics of desired signal $S[n]$
  - Interference suppression turns into the problem of signal estimation in presence of noise

Signal estimation…



- Estimator is constrained to be a linear filter whose output approximates some desired signal sequence $D[n]$
  - Input to filter: $X[n] = S[n] + W[n]$
  - Sequence $S[n]$ stationary with known $\gamma_{SS}[n], \Gamma_{SS}(f)$
  - Sequence $D[n]$ stationary with known properties $\gamma_{DD}[n], \Gamma_{DD}(f)$
  - Sequence $W[n]$ white with known (or estimated) σ_W^2
- Error between $Y[n]$ and $D[n]$ measures similarity

Choice of target sequence $D[n]$

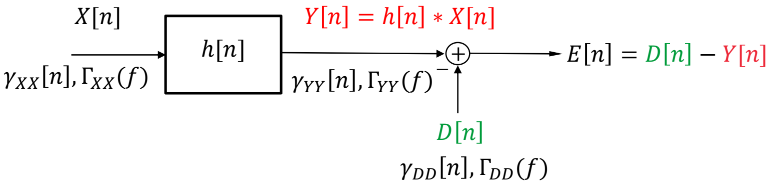


Three important choices of target sequence $D[n]$:

1. Noise reduction or filtering: $D[n] = S[n] \Longrightarrow \gamma_{DS}[l] = \gamma_{SS}[l]$
2. Smoothing: $D[n] = S[n - n_d], n_d > 0 \Longrightarrow \gamma_{DS}[l] = \gamma_{SS}[l - n_d]$
3. Prediction in noise: $D[n] = S[n + n_d], n_d > 0 \Rightarrow \gamma_{DS}[l] = \gamma_{SS}[l + n_d]$

- Remember definition: $\gamma_{DS}[l] = E\{D[n]S[n-l]\} = E\{D[n+l]S[n]\}$
  $= \gamma_{SD}[-l]$

Optimal MSE filtering



- Find filter $h[n]$ that minimizes mean-square error (MSE)

$$h_{\text{opt}}[n] = \arg\min_{h} E\left\{(D[n] - Y[n])^2\right\}$$

- Possible solutions depend on conditions set on filter $h[n]$
  - IIR and noncausal
  - IIR and causal, or FIR and causal

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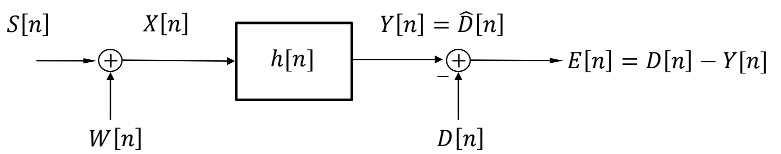
Optimum MSE noncausal IIR filter



- Filter $h[n]$ allowed to include both infinite past and infinite future of sequence $X[n]$ in forming output $Y[n]$

$$Y[n] = \sum_{k=-\infty}^{\infty} h[k] X[n-k]$$

- Filter $h[n]$ is unrealizable but serves as a best-case scenario
- Design filter to minimize $\sigma_E^2 = E\{(D[n] - Y[n])^2\}$

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Optimum MSE noncausal IIR filter…

- Mean-square error (MSE);

$$\sigma_E^2 = E\{(D[n] - Y[n])^2\}$$

$$= E\{(D[n] - \sum_{k=-\infty}^{\infty} h[k]X[n-k])^2\}$$

$$= \gamma_{DD}[0] - 2\sum_{k=-\infty}^{\infty} h[k]\gamma_{DX}[k] +$$

$$+ \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h[k]h[l]\gamma_{XX}[k-l]$$

- Minimum MSE (MMSE) when

$$\frac{d\sigma_E^2}{dh[k]} = 0, -\infty < k < \infty$$

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Optimum MSE noncausal IIR filter…

- Minimum MSE (MMSE) attained for $h[n]$ satisfying equation

$$\sum_{k=-\infty}^{\infty} h[k]\gamma_{XX}[l-k] = \gamma_{DX}[l], |l| \geq 0$$

- Minimum achievable MSE obtained by above filter

$$\sigma_E^2 = \gamma_{DD}[0] - \sum_{k=-\infty}^{\infty} h[k]\gamma_{DX}[k]$$

- Equation system for $h[k]$ not solvable in time domain
- Take z-transform (or DTFT):

$$\Gamma_{DX}(z) = H(z)\Gamma_{XX}(z)$$

$$\boxed{\Longrightarrow H(z) = \frac{\Gamma_{DX}(z)}{\Gamma_{XX}(z)}}$$

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Optimum MSE noncausal IIR filter…

- White noise $W[n]$ is uncorrelated with all other signals, i.e.,

$$\gamma_{XX}[l] = \gamma_{SS}[l] + \sigma_W^2 \delta[l], |l| \geq 0$$

$$\gamma_{DX}[l] = \gamma_{DS}[l], |l| \geq 0$$

- Optimal filter given by:

$$H(z) = \frac{\Gamma_{DS}(z)}{\Gamma_{SS}(z) + \sigma_W^2}$$

- Time-domain impulse response:

$$h[n] = \mathcal{Z}^{-1}\{H(z)\}$$

Optimum MSE noncausal IIR filter…

- Example: $X[n] = S[n] + W[n]$, and $W[n] \sim N(0, \sigma_W^2 = 1)$

  $S[n] = 0.6S[n-1] + N[n]$, and $N[n] \sim N(0, \sigma_N^2 = 0.64)$

  Design a noncausal IIR Wiener filter to estimate $S[n]$
- From earlier lectures:

$$\gamma_{SS}[l] = \frac{0.64}{1-0.6^2} 0.6^{|l|} = 0.6^{|l|} = \gamma_{DS}[l]$$

$$\Gamma_{SS}(z) = \frac{0.64}{(1-0.6z^{-1})(1-0.6z)} = \Gamma_{DS}(z)$$

$$\Gamma_{XX}(z) = \frac{0.64}{(1-0.6z^{-1})(1-0.6z)} + 1 = \frac{1.8\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{3}z\right)}{(1-0.6z^{-1})(1-0.6z)}$$

- Optimum filter:

$$H(z) = \frac{\Gamma_{SS}(z)}{\Gamma_{XX}(z)} = \frac{0.64}{1.8} \frac{1}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{3}z\right)} = \frac{0.4}{\left(1-\frac{1}{3}z^{-1}\right)} + \frac{\frac{0.4}{3}z}{\left(1-\frac{1}{3}z\right)}$$

Optimum MSE noncausal IIR filter…

- Impulse response $h[n]$:

$$h[n] = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{0.4}{\left(1-\frac{1}{3}z^{-1}\right)} + \frac{\frac{0.4}{3}z}{\left(1-\frac{1}{3}z\right)}\right\}$$

$$= \ 0.4\left(\frac{1}{3}\right)^{n} u[n] - 0.4 \cdot 3^{n} u[-n-1]$$

$$= 0.4\left(\frac{1}{3}\right)^{|n|}$$

- Minimum MSE

$$\sigma_E^2 = 1 - \sum_{k=-\infty}^{\infty} 0.4\left(\frac{1}{3}\right)^{|k|} 0.6^{|k|} = 0.4$$

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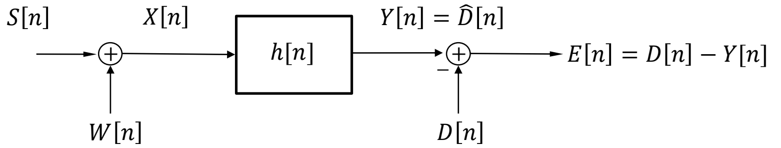
Optimum MSE causal FIR filter



- Filter $h[n]$ constrained to be causal and length M
- Output $Y[n]$ depends on $X[n], X[n-1], \dots X[n-M+1]$

$$Y[n] = \sum_{k=0}^{M-1} h[k]X[n-k]$$

- Design filter to minimize $\sigma_E^2 = E\{(D[n] - Y[n])^2\}$

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Optimum MSE causal FIR filter…

- Mean-square error (MSE);

$$\sigma_E^2 = E\{(D[n] - Y[n])^2\}$$

$$= E\left\{\left(D[n] - \sum_{k=0}^{M-1} h[k]X[n-k]\right)^2\right\}$$

$$= \gamma_{DD}[0] - 2\sum_{k=0}^{M-1} h[k]\gamma_{DX}[k] +$$

$$+ \sum_{k=0}^{M-1}\sum_{l=0}^{M-1} h[k]h[l]\gamma_{XX}[k-l]$$

- Minimum MSE (MMSE) when

$$\frac{d\sigma_E^2}{dh[k]} = 0, 0 < k < M-1$$

Optimum MSE FIR filter

- Minimum MSE (MMSE) attained for $h[n]$ satisfying equation

$$\sum_{k=0}^{M-1} h[k]\gamma_{XX}[l-k] = \gamma_{DX}[l], l = 0, 1, \ldots, M-1$$

- In matrix notation:

$$\underbrace{\begin{bmatrix} \gamma_{XX}[0] & \cdots & \gamma_{XX}[M-1] \\ \vdots & \ddots & \vdots \\ \gamma_{XX}[M-1] & \cdots & \gamma_{XX}[0] \end{bmatrix}}_{\mathbf{\Gamma}_{XX}} \underbrace{\begin{bmatrix} h[0] \\ \vdots \\ h[M-1] \end{bmatrix}}_{\mathbf{h}} = \underbrace{\begin{bmatrix} \gamma_{DX}[0] \\ \vdots \\ \gamma_{DX}[M-1] \end{bmatrix}}_{\boldsymbol{\gamma}_{DX}}$$

where $M \times M$ autocorrelation matrix $(\mathbf{\Gamma}_{XX})_{lk} = \gamma_{XX}[l-k]$ and $M \times 1$ cross-correlation vector $(\boldsymbol{\gamma}_{DX})_l = \gamma_{DX}[l]$

- Minimum MSE: $\sigma_E^2 = \gamma_{DD}[0] - \sum_{k=0}^{M-1} h[k]\gamma_{DX}[k]$

Optimum MSE FIR filter

- Can be solved directly in time-domain

$$\mathbf{h} = \mathbf{\Gamma}_{XX}^{-1}\boldsymbol{\gamma}_{DX}$$

- Matrix $\mathbf{\Gamma}_{XX}$ symmetric and Toeplitz $\Longrightarrow$ Efficient algorithms exist
- Minimum achievable MSE obtained by above filter

$$\sigma_E^2 = \gamma_{DD}[0] - \sum_{k=0}^{M-1} h[k]\gamma_{DX}[k]$$
$$= \gamma_{DD}[0] - \mathbf{h}^{\mathrm{T}}\boldsymbol{\gamma}_{DX}$$

- FIR filters are popular for signal estimation as they can be adapted continuously in dynamic environments (adaptive filters)

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Optimum MSE FIR filter…

- Example: $X[n] = S[n] + W[n]$, and $W[n] \sim N(0, \sigma_W^2 = 1)$

  $S[n] = 0.6S[n-1] + N[n]$, and $N[n] \sim N(0, \sigma_N^2 = 0.64)$

  Design FIR filter with $M = 2$ coefficients

- From before $\gamma_{SS}[l] = 0.6^{|l|} = \gamma_{DX}[l], \gamma_{XX}[l] = \gamma_{SS}[l] + \delta[l]$

$$\begin{bmatrix} \gamma_{XX}[0] & \gamma_{XX}[1] \\ \gamma_{XX}[1] & \gamma_{XX}[0] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \end{bmatrix} = \begin{bmatrix} \gamma_{DX}[0] \\ \gamma_{DX}[1] \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0.6 \\ 0.6 & 2 \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \end{bmatrix} = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}$$

- We get $h[0] = 0.451$ and $h[1] = 0.165$
- Minimum MSE: $\sigma_E^2 = 1 - \sum_{k=0}^{1} h[k]\gamma_{XX}[k] = 0.45$

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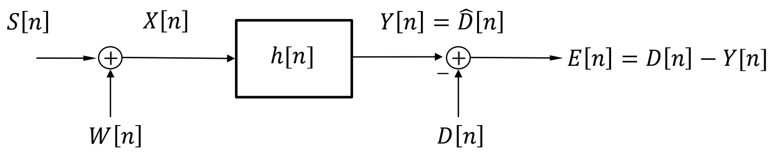
Optimum MSE causal IIR filter



- Filter $h[n]$ constrained to be causal but can be of infinite duration
- Output $Y[n]$ depends on $X[n], X[n-1], \ldots$

$$Y[n] = \sum_{k=0}^{\infty} h[k]X[n-k]$$

- Design filter to minimize $\sigma_E^2 = E\{(D[n] - Y[n])^2\}$

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Optimum MSE IIR causal filter…

- Mean-square error (MSE):

$$\sigma_E^2 = E\{(D[n] - Y[n])^2\}$$

$$= E\{(D[n] - \sum_{k=0}^{\infty} h[k]X[n-k])^2\}$$

$$= \gamma_{DD}[0] - 2\sum_{k=0}^{\infty} h[k]\gamma_{DX}[k] +$$

$$+ \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} h[k]h[l]\gamma_{XX}[k-l]$$

- Minimum MSE when

$$\frac{d\sigma_E^2}{dh[k]} = 0, k = 0, 1, \ldots$$

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Optimum MSE IIR causal filter…

- Minimum MSE (MMSE) attained for $h[n]$ satisfying equation

$$\sum_{k=0}^{\infty} h[k]\gamma_{XX}[l-k] = \gamma_{DX}[l], l \geq 0$$

- Minimum achievable MSE obtained by above filter

$$\sigma_E^2 = \gamma_{DD}[0] - \sum_{k=0}^{\infty} h[k]\gamma_{DX}[k]$$

- We cannot directly solve for $h[k]$ using z-transform, since equations only consider $l \geq 0$
- Instead we consider an alternative solution via the innovations representation of $X[n]$

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Optimum MSE IIR causal filter…

- Definition: Let $[A(z)]_+$ denote the causal part of $A(z)$, i.e.,

$$A(z) = \sum_{k=-\infty}^{\infty} a[k]z^{-k} \Rightarrow [A(z)]_+ = \sum_{k=0}^{\infty} a[k]z^{-k}$$

- Example: $A(z) = \frac{1}{1-0.5z^{-1}} + \frac{0.5z}{1-0.5z}$, ROC: $0.5 < |z| < 2$

$$\Rightarrow [A(z)]_+ = \sum_{k=0}^{\infty} a[k]z^{-k} = \frac{1}{1-0.5z^{-1}}, \quad \text{ROC: } |z| > 0.5$$

$$a[n] = \left(\frac{1}{2}\right)^n u[n]$$

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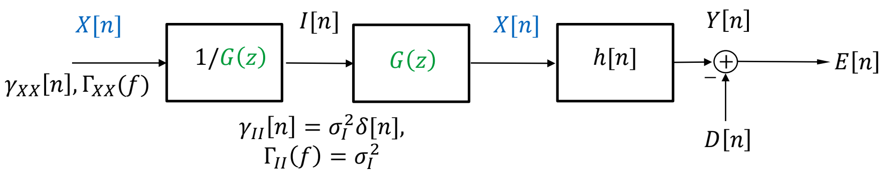
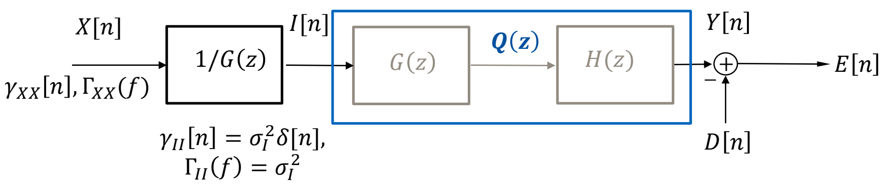
Optimum MSE IIR causal filter…



- Express $\Gamma_{XX}(z) = \sigma_I^2 G(z)G(z^{-1})$ with $G(z)$ being minimum-phase
  - Remember definition that $G(z)$ *causal and stable with causal and stable inverse* $1/G(z) \Rightarrow G(z)$ must be minimum-phase
- Use the innovations representation of $X[n]$ to simplify the design

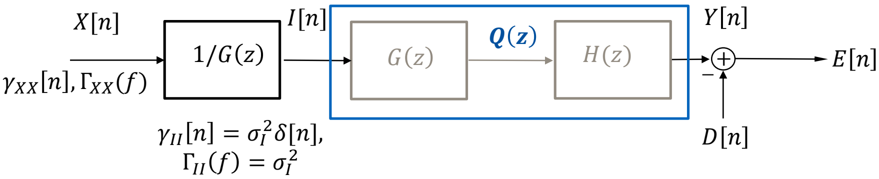
Optimum MSE IIR causal filter…



- Study system $Q(z) = G(z)H(z)$ with input $I[n]$ and derive optimal $Q(z)$ using the MSE formulation
- Once $Q(z)$ obtained we can get $H(z)$ from relation

$$H(z) = Q(z)/G(z)$$

Optimum MSE IIR causal filter…



- Filter $q[n]$ constrained to be causal but can be of infinite duration
- Output $Y[n]$ depends on $I[n], I[n-1], \dots$

$$Y[n] = \sum_{k=0}^{\infty} q[k] I[n-k]$$

- Design filter to minimize $\sigma_E^2 = E\{(D[n] - Y[n])^2\}$

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Optimum MSE IIR causal filter…

- Minimum MSE (MMSE) attained for $q[n]$ satisfying equation

$$\sum_{k=0}^{\infty} q[k]\gamma_{II}[l-k] = \gamma_{DI}[l], l \geq 0$$

- We know that $I[n]$ is white noise with $\gamma_{II}[l] = \sigma_I^2\delta[l]$

$$q[l]\gamma_{II}[0] = \gamma_{DI}[l], l \geq 0$$

$$\Rightarrow q[l] = \frac{\gamma_{DI}[l]}{\gamma_{II}[0]} = \frac{\gamma_{DI}[l]}{\sigma_I^2}, l \geq 0$$

- Coefficients of filter $q[n]$ is related to $\Gamma_{DI}(z)$ as

$$Q(z) = \sum_{k=0}^{\infty} q[k] z^{-k} = \frac{1}{\sigma_I^2}\sum_{k=0}^{\infty} \gamma_{DI}[k] z^{-k} = \frac{1}{\sigma_I^2}[\Gamma_{DI}(z)]_+$$

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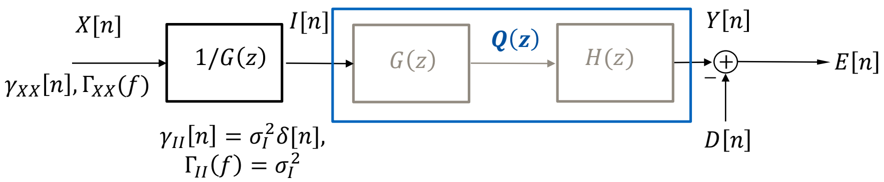
Optimum MSE IIR causal filter…



- To find $[\Gamma_{DI}(z)]_+$ we express $I[n]$ in terms of $X[n]$
- Let $v[n]$ denote the impulse response of $1/G(z)$

$$I[n] = \sum_{k=0}^{\infty} v[k]X[n-k]$$

$$\gamma_{DI}[l] = E\{D[n]I[n-l]\} = \sum_{k=0}^{\infty} v[k]E\{D[n]X[n-k-l]\}$$

$$= \sum_{k=0}^{\infty} v[k]\gamma_{DX}[k+l]$$

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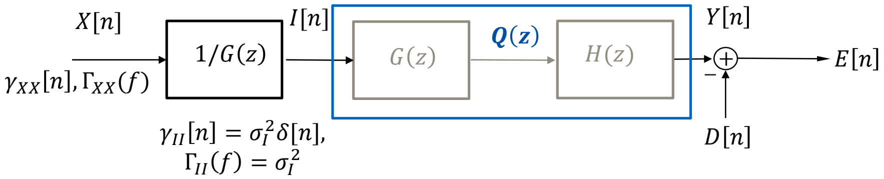
Optimum MSE IIR causal filter…



- $\Gamma_{DI}(z)$ in terms of $X[n]$

$$\Gamma_{DI}(z) = \sum_{l=-\infty}^{\infty} \gamma_{DI}[l]z^{-l} = \sum_{l=-\infty}^{\infty} \left(\sum_{k=0}^{\infty} v[k]\gamma_{DX}[k+l]\right)z^{-l}$$

$$= V(z^{-1})\Gamma_{DX}(z) = \Gamma_{DX}(z)/G(z^{-1})$$

- Consequently

$$\boxed{H_{opt}(z) = \frac{Q(z)}{G(z)} = \frac{\frac{1}{\sigma_I^2}[\Gamma_{DI}(z)]_+}{G(z)} = \frac{1}{\sigma_I^2 G(z)}\left[\frac{\Gamma_{DX}(z)}{G(z^{-1})}\right]_+}$$

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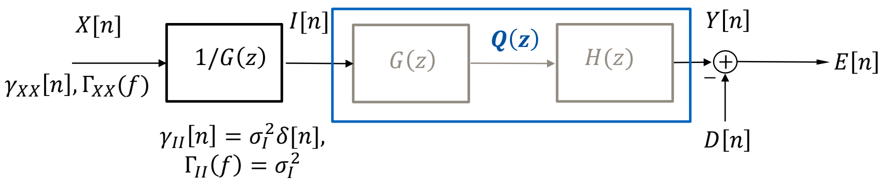
Optimum MSE IIR causal filter…



- Summary of steps:
  1. Express $\Gamma_{XX}(f)$ as $\Gamma_{XX}(f) = \sigma_I^2 G(z) G(z^{-1})$
  2. Compute $H_{opt}(z) = \frac{1}{\sigma_I^2 G(z)} \left[\frac{\Gamma_{DX}(z)}{G(z^{-1})} \right]_+$

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Optimum MSE IIR causal filter…

- Example: $X[n] = S[n] + W[n]$, and $W[n] \sim N(0, \sigma_W^2 = 1)$

  $S[n] = 0.6S[n-1] + N[n]$, and $N[n] \sim N(0, \sigma_N^2 = 0.64)$

  Design a causal IIR Wiener filter to estimate $S[n]$
- From before:

$$\gamma_{SS}[l] = 0.6^{|l|} = \gamma_{DX}[l]$$

$$\Gamma_{SS}(z) = \frac{0.64}{(1-0.6z^{-1})(1-0.6z)} = \Gamma_{DX}(z) = \Gamma_{DS}(z)$$

$$\Gamma_{XX}(z) = \Gamma_{SS}(z) + \Gamma_{WW}(z)$$

$$= \frac{1.8\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{3}z\right)}{(1-0.6z^{-1})(1-0.6z)} = \sigma_I^2 G(z) G(z^{-1})$$

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Optimum MSE IIR causal filter…

- System function of optimal IIR filter:

$$H_{opt}(z) = \frac{1}{\sigma_I^2 G(z)}\left[\frac{\Gamma_{DX}(z)}{G(z^{-1})}\right]_+ = \frac{(1-0.6z^{-1})}{1.8\left(1-\frac{1}{3}z^{-1}\right)}\left[\frac{0.64(1-0.6z)}{(1-0.6z^{-1})(1-0.6z)\left(1-\frac{1}{3}z\right)}\right]_+$$

$$= \frac{(1-0.6z^{-1})}{1.8\left(1-\frac{1}{3}z^{-1}\right)}\left[\frac{0.8}{(1-0.6z^{-1})} + \frac{0.266z}{\left(1-\frac{1}{3}z\right)}\right]_+ = \frac{(1-0.6z^{-1})}{1.8\left(1-\frac{1}{3}z^{-1}\right)}\frac{0.8}{(1-0.6z^{-1})}$$

$$= \frac{4}{9}\frac{1}{1-\frac{1}{3}z^{-1}}$$

- Impulse response:

$$h[n] = \frac{4}{9}\left(\frac{1}{3}\right)^n u[n]$$

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Optimum MSE IIR causal filter…

- Minimum MSE:

$$\sigma_E^2 = \gamma_{DD}[0] - \sum_{k=0}^{\infty} h[k]\gamma_{DX}[k]$$

with

$$\gamma_{SS}[l] = 0.6^{|l|} = \gamma_{DX}[l]$$

$$h[n] = \frac{4}{9}\left(\frac{1}{3}\right)^n u[n]$$

we finally obtain

$$\sigma_E^2 = 1 - \frac{4}{9}\sum_{k=0}^{\infty} 0.6^k \left(\frac{1}{3}\right)^k = 1 - \frac{4}{9}\sum_{k=0}^{\infty}\left(\frac{1}{5}\right)^k = \frac{4}{9} \approx 0.44$$

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Summary

- Today we discussed:
  - Wiener filters (noncausal and causal design)
- Next:
  - Filter implementation