TTK4150 Nonlinear Control Systems

Lecture Summary

NTNU
Innovation and Creativity

Outline I

Lecture's goals

What did we learn in Lecture 1?

- Know some important mathematical definitions and results necessary to be able to follow the course TTK4150 Nonlinear Control Systems.
- Know the basic differences between linear and nonlinear systems
- Recognize the need for new analysis and control design methods
- Know when to use nonlinear methods for analysis and design
- Know how to calculate equilibrium points
- Be able to describe the common nonlinearities saturation, dead-zone, sign and backlash

Nonlinear dynamic systems

We consider nonlinear dynamic systems in the form

$$\dot{x} = f_p(t, x, u) \tag{1}$$

$$\dot{x}_1 = f_{p_1}(t, x_1, \ldots, x_n, u_1, \ldots, u_m) \tag{2}$$

$$\vdots$$

$$\dot{x}_n = f_{p_n}(t, x_1, \ldots, x_n, u_1, \ldots, u_m) \tag{3}$$

$$\downarrow u = \gamma(t, x) \tag{4}$$

$$\dot{x} = f(t, x) \qquad \text{Nonautonomous} \tag{5}$$

Special case

$$\dot{x} = f(x) \qquad \text{Autonomous} \tag{6}$$

Nonlinear dynamic systems

Linear systems (LTI)

$$\dot{x} = ax + bu$$

$$u = -K_p(x + \frac{1}{T_i}\int x d\tau + T_d\dot{x})$$

$$z \triangleq [x,\ \dot{x}]^T$$

$$\dot{z} = Az$$

Solution:
$z(t) = e^{At}z(0)$

Analysis:
Laplace transform $\rightarrow$
Transfer function

Nonlinear systems

$$\dot{x} = f_p(t, x, u)$$

$$u = \gamma(t, x)$$

$$\dot{x} = f(t, x)$$

Solution:
Generally no analytical solution

Analysis:
No Laplace transform
$\Rightarrow$ need new methods

Lecture's goals

What did we learn in Lecture 2?

- Fundamental properties
  - Be able to validate a mathematical model by ensuring the existence and uniqueness of solutions of the initial value problem
- The first analysis tool
  - Be able to use the comparison principle to find an upper bound for the solution $x(t)$ without computing the solution itself

Lecture's goals cont.

What did we learn in Lecture 2?

- Phase plane analysis: Analysis of 2D systems
  - Phase portraits: graphical analysis tools
  - Know how to construct phase portraits and interpret them
    - Analytical method
    - Computer simulations
    - Vector field diagrams
  - Be able to describe a periodic orbit and a limit cycle
  - Be able to tell whether a periodic orbit may or may not exist for a 2D system

How to do phase plane analysis

Summary: How to do phase plane analysis

1. Find the equilibrium points of the system
2. Perform a local analysis (about each equilibrium point)
    1. Linearize about the equilibrium point
    2. Find the eigenvalues $\lambda(A)$
    3. Classify the (isolated) equilibria

    In order to obtain qualitative knowledge about the system behavior locally around the equilibria. (This will guide you in the next step)
3. Construct a phase portrait using
    a) the analytical method
    b) a vector field diagram
    c) computer simulations
4. Try to find possible periodic orbits and limit cycles

Periodic orbits and Limit cycles: Existence criteria

Existence criteria for periodic orbits

1. Poincaré-Bendixson
2. The Bendixson (negative) criterion
3. The index method

Lecture's goals

What did we learn in Lecture 3?

- Understand how the need for stabilization of equilibrium points arise in control problems
- **Lyapunov stability properties**
  Know and understand the following stability definitions for autonomous systems
  - Stability
  - Asymptotic stability
  - Exponential stability
  - Global versus local
- **Lyapunov stability analysis**
  - Lyapunov's indirect method

Stability analysis

How do we analyze the Lyapunov stability properties?

- Definitions
  - If we have solution $x(t) = \ldots$ OK
- Phase plane analysis (dim $x = 2$)
  - Phase portrait
  - Local phase plane analysis

    Phase portrait of linearized system $\sim$ local phase portrait of nonlinear system
- New method: Lyapunov's indirect method

Lyapunov's indirect method/Linearization method

Theorem 4.7 (Lyapunov's indirect method)

Let $x = 0$ be an equilibrium point for

$$\dot{x} = f(x) \quad f : \mathbb{D} \to \mathbb{R}^n \quad \text{is} \quad C^1$$

1) Linearize the system about $x = 0$, $\dot{x} = Ax$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{x=0}$$

2) Find the eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$

Lyapunov's indirect method cont.

Theorem 4.7 (Lyapunov's indirect method) cont.

3)
a) $\forall i \quad \mathrm{Re}(\lambda_i) < 0 \quad \Rightarrow \quad x = 0$ is locally asymptotically stable

b) $\exists i \quad \mathrm{Re}(\lambda_i) > 0 \quad \Rightarrow \quad x = 0$ is unstable

c) $\begin{array}{ll} \forall i & \mathrm{Re}(\lambda_i) \leq 0 \\ \exists i & \mathrm{Re}(\lambda_i) = 0 \end{array}$ $\Rightarrow$ No conclusion

Comments

+ Simple to use
÷ Not always conclusive
÷ Only <u>local</u> results

Corollary 4.3

Corollary 4.3, Sec. 4.7

Let $x = 0$ be an equilibrium point for

$$\dot{x} = f(x) \quad f : \mathbb{D} \to \mathbb{R}^n \text{ is } C^1$$

$$\forall i \quad \mathrm{Re}(\lambda_i) < 0 \quad \Leftrightarrow \quad x = 0 \text{ is (locally) exponentially stable}$$

Lecture's goals

Lyapunov's direct method

- Lyapunov functions - a generalization of energy functions
- Lyapunov's theorems for
  - stability
  - local and global asymptotic stability
  - local and global exponential stability
- How to apply Lyapunov's direct method

Advantages and disadvantages

+ General
÷ No general way to find $V(x)$
+ Can give global results

- Some methods for choosing Lyapunov Function candidates

Lyapunov functions

The system

Consider the autonomous system

$$\dot{x} = f(x)$$

where $f : \mathbb{D} \to \mathbb{R}^n$ is locally Lipschitz.
$x = 0 \in \mathbb{D}$ is an equilibrium point of the system.

Lyapunov function candidate

Let $V : \mathbb{D} \to \mathbb{R}$ be a continuously differentiable (C^1) function

The derivative of V along the system trajectories is:

$$\dot{V} = \frac{dV(x)}{dt} = \frac{dV}{dx} f(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

Lyapunov functions

Definition (Lyapunov function)

V is a Lyapunov function for $x = 0$ iff

i) V is C^1

ii) $V(0) = 0$
$V(x) > 0 \quad \text{in} \quad \mathbb{D} \backslash \{0\}$

iii) $\dot{V}(0) = 0$
$\dot{V}(x) \leq 0 \quad \text{in} \quad \mathbb{D} \backslash \{0\}$

If, moreover,

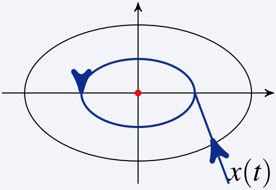
$$\dot{V}(x) < 0 \quad \text{in} \quad \mathbb{D} \backslash \{0\}$$

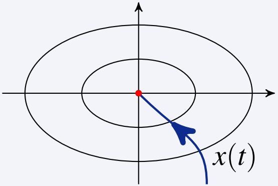
then V is a strict Lyapunov function.

Lyapunov's direct method

Theorem 4.1

- If $\exists$ Lyapunov function for $x = 0$, then $x = 0$ is stable
- If $\exists$ strict Lyapunov function for $x = 0$, then $x = 0$ is asymptotically stable





Global asymptotic stability

Theorem 4.2: Global asymptotic stability

If $\exists$ strict Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ for $x = 0$

and

V is radially unbounded

then $x = 0$ is globally asymptotically stable.

Exponential stability

Theorem 4.10: Exponential stability

If there exists constants $a, k_1, k_2, k_3 > 0$ such that

i) V is C^1

ii) $k_1 \|x\|^a \leq V(x) \leq k_2 \|x\|^a \quad \forall x \in \mathbb{D}$

iii) $\dot{V}(x) \leq -k_3 \|x\|^a \quad \forall x \in \mathbb{D}$

then $x = 0$ is exponentially stable.

Global exponential stability

If the conditions in the theorem are satisfied with

$$\mathbb{D} = \mathbb{R}^n$$

then $x = 0$ is globally exponentially stable.

How to apply Lyapunov's direct method

How to apply Lyapunov's direct method - revisited

1) Choose a Lyapunov function **candidate** $V(x)$
   - Electrical/mechanical systems
     - $V(x)$ = total energy
   - Others
     - $V(x) = \frac{1}{2}x^T P x$
     - $V(x) = \frac{1}{2}(x_1^2 + a_2 x_2^2 + \ldots + a_n x_n^2)$
     - *some* methods exist for choosing $V(x)$

2) Determine whether $V(x)$ satisfies the conditions of any of the Lyapunov theorems.

3) If the answer is yes:

   The equilibrium point is Stable/Asymptotically stable/Exponentially stable

   If the answer is no:

Lecture's goals

What did we learn in Lecture 5?

- Know La Salle's theorem, and how to use this
  - $\dot{V} \leq 0$ asymptotic stability of equilibrium points
  - Regions of attraction - find an estimate
  - Convergence to other invariant sets than equilibrium points
- Know some methods for finding Lyapunov function candidates (LFCs)

The invariance principle: LaSalle's theorem

$\dot{x} = f(x) \quad f : \mathbb{D} \to \mathbb{R}^n$ locally Lipschitz

Theorem 4.4 (LaSalle's theorem)

If $\exists\, V : \mathbb{D} \to \mathbb{R}$ such that

i) V is C^1

ii) $\exists\, c > 0$ such that $\Omega_c = \{x \in \mathbb{R}^n | V(x) \leq c\} \subset \mathbb{D}$ is bounded

iii) $\dot{V}(x) \leq 0 \quad \forall\, x \in \Omega_c$

Let $E = \left\{x \in \Omega_c | \dot{V}(x) = 0\right\}$
Let M be the largest invariant set contained in E
Then

$$x(0) \in \Omega_c \Rightarrow x(t) \xrightarrow{t\to\infty} M$$

La Salle's theorem ii)

Note: V does not have to be positive definite

- V positive definite $\Rightarrow \Omega_c$ bounded for small c
- V radially unbounded $\Rightarrow \Omega_c$ bounded for $\forall\ c$

Special cases:

- Cor. 4.1 $(M = \{0\})$
- Cor. 4.2 (Global version)

Application

Applications of La Salle's theorem:

- $\dot{V} \leq 0$ Prove asymptotic stability of equilibrium points
- Regions of attraction - find an estimate
- Convergence to other invariant sets than equilibrium points

Methods for choosing Lyapunov function candidates

Methods for choosing LFCs

- Total energy
- LFCs with quadratic terms $\frac{1}{2}x^T P x$
  - $V(x) = \frac{1}{2}(x_1^2 + x_2^2 + \cdots + x_n^2)$
  - $V(x) = \frac{1}{2}(x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2)$
  - $V(x) = \frac{1}{2}x^T P x$
- $V(x) = \frac{1}{2}\ln(1 + x_1^2 + \cdots + x_n^2)$
- The variable gradient method
- LFCs for linear time-invariant systems
- Krasovskii's method (Assignment)
- $\vdots$

Lecture's goals

What did we learn in Lecture 6?

Know **Lyapunov's direct method for nonautonomous systems.** In particular,

- Know comparison functions of class $\mathscr{K}$, $\mathscr{K}_\infty$ and $\mathscr{K}\mathscr{L}$
- Know the stability definitions of nonautonomous systems (and how they deviate from the stability definitions of autonomous systems)
- Be able to use Lyapunov's direct method to analyze the stability properties of an equilibrium point of a nonautonomous system.
- Be able to use Barbalat's lemma to analyze the convergence properties when $\dot{V}(t,x) \leq 0$

Nonautonomous systems

Nonautonomous systems

$$\dot{x} = f(t,x) \quad f : [0,\infty) \times \mathbb{D} \to \mathbb{R}^n$$

- $f(t,x)$ Piecewise continuous in t
  locally Lipschitz in x on $[0,\infty) \times \mathbb{D}$
- $x = 0 \in \mathbb{D}$

Definition: Equilibrium point

x^* is an equilibrium point for $\dot{x} = f(t,x)$ at $t = 0$ iff

$$f(t,x^*) = 0 \quad \forall t \geq 0$$

Stability theorems

$\dot{x} = f(t,x)$ $\quad f : [0,\infty) \times \mathbb{D} \to \mathbb{R}^n$ piecewise continuous in t locally Lipschitz

Stability theorem (Theorem 4.8 - 4.9)

Let $V : [0,\infty) \times \mathbb{D} \to \mathbb{R}$ $\quad C^1$

The equilibrium point $x^* = 0$ is

	Stable	Uniformly stable	Uniformly as. st.	GUAS
V	Pos.def.	Pos.def. Decrescent	Pos.def. Decrescent	Pos.def. Decrescent Rad. unb.
$\dot{V}$	Neg.semidef	Neg.semidef.	Neg.def.	Neg.def.
	$\forall x \in \mathbb{D}$	$\forall x \in \mathbb{D}$	$\forall x \in \mathbb{D}$	$\forall x \in \mathbb{D} = \mathbb{R}^n$

Exponential stability

Exponential stability (Theorem 4.10)

Let $V : [0, \infty) \times \mathbb{D} \to \mathbb{R} \quad C^1$

If there exists constants $a, k_1, k_2, k_3 > 0$ such that

- $k_1 \|x\|^a \leq V(t,x) \leq k_2 \|x\|^a, \quad \forall t \geq 0, \quad \forall\, x \in \mathbb{D}$
- $\dot{V}(t,x) \leq -k_3 \|x\|^a, \qquad \forall t \geq 0, \quad \forall\, x \in \mathbb{D}$

then $x^* = 0$ is exponentially stable.

Global exponential stability

If the conditions in the theorem are satisfied with

$$\mathbb{D} = \mathbb{R}^n$$

then $x^* = 0$ is globally exponentially stable.

Summary

Lyapunov's direct method for nonautonomous systems

- Time-varying Lyapunov functions candidates
- Lyapunov's theorems for
  - stability
  - uniform stability (US)
  - uniform asymptotic stability (UAS)
  - global uniform asymptotic stability (GUAS)
  - local and global exponential stability (GES $\Rightarrow$ GUAS)
- Barbalat's lemma

Lecture's goals

What did we learn in Lecture 7?

- Know that there exists other stability concepts than Lyapunov stability

In particular

- Understand the motivation and the definition of Input-to-State stability (ISS)
- Be able to analyze ISS using ISS-Lyapunov functions
- Know some relations between ISS and Lyapunov stability

- Know the definition of Input-Output Stability (IOS)
- Be able to analyze IOS using the definition
- Know the small-gain theorem

Systems with inputs

System

We want to analyse systems on the form

$$\dot{x} = f(t, x, u) \qquad (\Sigma)$$

$f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

Input

$u(t)$ pieciewise continuous, bounded function

- disturbance
- modelling error

When $u(t) = 0$

$$\dot{x} = f(t, x, 0)$$

$x = 0$ is GUAS (0-GUAS)

What if $u(t) \neq 0$?

How to check ISS?

Definition: ISS Lyapunov function (ISS-LF)

$V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is an ISS-LF for Σ iff

i) V is C^1

$\exists\ \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{K}$ s.t.

ii) $\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|)$

iii) $\dot{V}(t,x) = \dfrac{\partial V}{\partial x} f + \dfrac{\partial V}{\partial t} \leq -W_3(x) \quad \|x\| \geq \rho(\|u\|) > 0$

where $W_3(x)$ is a C^0 positive definite function on $\mathbb{R}^n$.

A Lyapunov-like theorem for ISS

Theorem 4.19

$$\exists \text{ ISS-LF for } \Sigma \Rightarrow \Sigma \text{ is ISS}$$

Sontag & Wang 1995

For autonomous systems: Σ is ISS $\Leftrightarrow \exists$ ISS-LF for Σ

$$\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$$

ISS vs. Lyapunov stability properties

ISS vs. 0-GUAS

$$\Sigma \text{ is ISS} \Rightarrow \Sigma \text{ is 0-GUAS}$$

$$\Updownarrow$$

$$\neg(\Sigma \text{ is 0-GUAS}) \Rightarrow \neg(\Sigma \text{ is ISS})$$

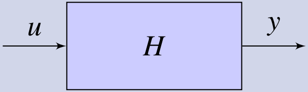
ISS vs. 0-GES (Lemma 4.6)

$\Sigma : \dot{x} = f(t,x,u)$ $\quad$ f is C^1 and <u>globally</u> Lipschitz in (x,u)

$$\Sigma \text{ is 0-GES} \Rightarrow \Sigma \text{ is ISS}$$

Introduction

Input-output models



We consider systems on the form

$$y = Hu$$

$u : [0, \infty) \to \mathbb{R}^m$ piecewise continuous
$y : [0, \infty) \to \mathbb{R}^q$ piecewise continuous

Input-output stability

How do we analyze stability of such systems?

Input-output stability

Consider the mapping

$$H : \mathscr{L}_{pe}^{m} \to \mathscr{L}_{pe}^{q}$$

$\mathscr{L}_p$ stable

$H : \mathscr{L}_{pe}^{m} \to \mathscr{L}_{pe}^{q}$ is $\mathscr{L}_p$ stable iff

i) $\exists \alpha$ class $\mathscr{K}$ $\alpha : [0,\infty) \to [0,\infty)$

ii) $\exists$ constant $\beta \geq 0$

s.t.

$$\|(Hu)_\tau\|_{\mathscr{L}_p} \leq \alpha(\|u_\tau\|_{\mathscr{L}_p}) + \beta \qquad \forall u \in \mathscr{L}_{pe}^{m} \text{ and } \tau \in [0,\infty)$$

Input-output stability cont.

Finite-gain $\mathscr{L}_p$ stable

$H : \mathscr{L}_{pe}^m \to \mathscr{L}_{pe}^q$ is finite-gain $\mathscr{L}_p$ stable iff

$$\exists \text{ constants } \gamma, \beta \geq 0$$

s.t.

$$\|(Hu)_\tau\|_{\mathscr{L}_p} \leq \gamma \|u_\tau\|_{\mathscr{L}_p} + \beta$$

$\mathscr{L}_p$ gain

Bias term

BIBO stability $\equiv \mathscr{L}_\infty$ stability

Lecture's goals

What did we learn in Lecture 8?

- Be able to analyze the stability properties of a system under the influence of disturbances
- Know the difference between
  - Vanishing perturbations
  - Nonvanishing perturbations
- Learn useful tools in order to study the stability of a stable system $\dot{x} = f(t,x)$ which is perturbed by another vanishing or nonvanishing vector field $g(t,x)$

Perturbed Systems

We want to analyse systems on the form

$$\dot{x} = f(t,x) + g(t,x) \tag{7}$$

- $D \subset \mathbb{R}^n$ is a domain that contains the origin $x^* = 0$
- f and $g : [0,\infty) \times D \to \mathbb{R}^n$, piecewise continuous in t and locally Lipschitz in x on $[0,\infty) \times D$
- Nominal system
$$\dot{x} = f(t,x). \tag{8}$$

The Perturbation term $g(t,x)$

often unknown, but with a known upper bound on $||g(t,x)||$

- modeling errors, uncertainties, disturbances etc.

Exponentially Stable Equilibrium Point

Lemma 9.1

- Let $x^* = 0$ be an exponentially stable equilibrium point of the nominal system $\dot{x} = f(t,x)$
- ...

Then, the origin is an exponentially stable equilibrium point of the perturbed system $\dot{x} = f(t,x) + g(t,x)$.

Uniformly Asymptotically Stable $x^* = 0$

Uniformly Asymptotic Stability

- Suppose $x^* = 0$ is a uniformly asymptotically stable equilibrium point of the nominal system $\dot{x} = f(t,x)$, and let
- ...

the perturbed system

$$\dot{x} = f(t,x) + g(t,x)$$

is asymptotically stable.

Nonvanishing Additive Perturbations

Nominal system $\dot{x} = f(t,x)$

Perturbed system $\dot{x} = f(t,x) + g(t,x),\ g(t,0) \neq 0$

- In this case, $x^* = 0$ may not be an equilibrium point of the perturbed system
- It can no longer be study the stability of the origin or expect that the solution of the perturbed system approaches the origin as $t \to \infty$.

The best we can do is find a bound on the size of $g(t,x)$ that ensures $x(t)$ remains close to the origin.

Nonvanishing Perturbations

- Exponential stable origin of $\dot{x} = f(t,x) \Longrightarrow$ Lemma 9.2
- Uniformly asymptotically stable origin of $\dot{x} = f(t,x) \Longrightarrow$ Lemma 9.3

Lecture's goals

What did we learn in Lecture 9?

- Analyze an autonomous system under the influence of a weak periodic perturbation
- Use Averaging method-Periodic averaging

Using the Theorem 10.3

Perturbed system

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon)$$

- If $g(t, 0, \varepsilon) = 0$, the origin will be an equilibrium point of the perturbed system.
- By uniqueness of the periodic solution $\bar{x}(t, \varepsilon)$, it follows that $\bar{x}(t, \varepsilon)$ is the trivial solution $x = 0$.

NB

The Theorem 10.3 ensures that the origin is an exponentially stable equilibrium point of the perturbed system.

Introduction to Averaging Theory

The basic idea of averaging theory–deterministic or stochastic

is to approximate the **original system**

- time-varying and periodic
- almost periodic, or randomly perturbed

by **a simpler (average) system**

- time-invariant, deterministic

or some approximating **diffusion system**

- a stochastic system simpler than the original one

The averaging method has been developed as:

- **a practical tool** in mechanics/dynamics
- **a theoretical tool** in mathematics both for deterministic dynamics and for stochastic dynamics.

The basic problem in averaging

Nonautonomous System

$$\dot{x} = \varepsilon f(t, x, \varepsilon)$$

Autonomous System

$$\dot{x} = \varepsilon f_{av}(x)$$

NB

Determine in what sense the behavior of the

- **autonomous system**

approximates the behavior of the

- **nonautonomous system.**

Lecture's goals

What did we learn in Lecture 10?

- Be able to analyze the passivity properties of a system by using the definition of passivity for
  - Memoryless functions
  - Dynamical systems
- Understand the relations between passivity and
  - Lyapunov stability
  - $\mathscr{L}_2$ stability (IOS)
- Know the passivity theorems (for feedback connections)

Passivity

What is passivity?

- A tool (not a stability concept) for design and analysis of control systems
- Based on an Input-Output description of systems
- Has an interesting energy interpretation
  (allows the control engineer to relate a set of efficient mathematical tools to well known physical phenomena)

Main use:

- Relates nicely to
  - Lyapunov stability
  - $\mathscr{L}_2$ stability
- Can provide a somewhat systematic way to build Lyapunov functions
- Can give conclusions about properties of feedback connections (based on the properties of each subsystem)

Passivity definitions for dynamical systems

Definition

The dynamical system is

- passive if
  $\exists C^1$ positive semidefinite function $V(x) : \mathbb{R}^n \to \mathbb{R}$
  (Storage function)
  such that

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x,u) \qquad \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^p$$

Moreover, it is

- lossless if

$$u^T y = \dot{V}$$

Input strictly passive

Definition continued

- Input strictly passive if

$$u^T y \geq \dot{V} + u^T \varphi(u), \quad u^T \varphi(u) > 0 \quad \forall u \neq 0$$

- Output strictly passive if

$$u^T y \geq \dot{V} + y^T \rho(y), \quad y^T \rho(y) > 0 \quad \forall y \neq 0$$

- (State) Strictly passive if

$$u^T y \geq \dot{V} + \psi(x), \quad \psi(x) \quad \text{positive definite function}$$

Relations between Passivity properties and (Lyapunov) stability

Dynamical systems

$$\Sigma \qquad \begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

$f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ locally Lipschitz
$h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ continuous

$f(0,0) = 0$ and $h(0,0) = 0$

Lemma 6.6 (Lyapunov stable (0-stable))

If Σ is passive with a *positive definite* storage function $V(x)$, then

the origin of $\dot{x} = f(x,0)$ is stable

Relations between Passivity properties and $\mathscr{L}_2$ stability

Lemma 6.5 (Finite-gain $\mathscr{L}_2$ stable)

If Σ is output strictly passive with $\rho(y) = \delta y$, $\delta > 0$ then

Σ is finite-gain $\mathscr{L}_2$ stable
with $\mathscr{L}_2$-gain $\gamma \leq \frac{1}{\delta}$

Asymptotic stability of passive systems

Definition: Zero-state observability

Σ is zero-state observable iff
no solution of $\dot{x} = f(x,0)$ can stay identically in
$S = \{x \in \mathbb{R}^n | h(x,0) = 0\}$ other than the trivial solution $x(t) = 0$.

Lemma 6.7 (Asymptotically stable (0-AS))

The origin of $\dot{x} = f(x,0)$ is asymptotically stable if Σ is either

- state strictly passive

or

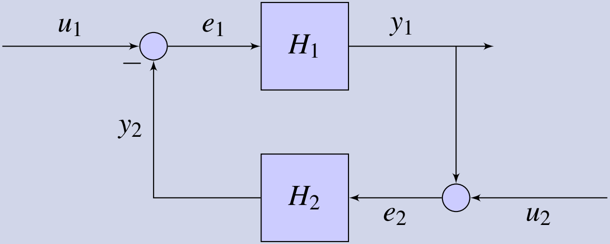
- output strictly passive
  zero-state observable

If furthermore $V(x)$ is radially unbounded, then the origin is globally asymptotically stable

Passivity theorems
Feedback systems

Feedback systems

Σ



$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Passivity theorems

Asymptotic stability of feedback connection

Theorem 6.3: Asymptotic stability of feedback connection

If

- H_1 and H_2 state strictly passive

or

- H_1 and H_2 output strictly passive and zero-state observable

or

- H_1 state strictly passive
  H_2 output strictly passive and zero-state observable
  or opposite

then Σ is 0-AS

If furthermore V_1 and V_2 are radially unbounded then Σ is 0-GAS.

Lecture's goals

What did we learn in Lecture 11?

Be able to design a passivity-based feedback control law

Energy-based Lyapunov Control Design

Energy-based Lyapunov Control Design

- Alternative B)

  Propose a Lyapunov function candidate
  = *Desired energy of the closed-loop system*

  Find a control law $u = g(t,x)$ that makes this LFC a (strict) Lyapunov function

Summary

Passivity

- Relates nicely to Lyapunov stability and $\mathscr{L}_2$ stability
- Can provide a somewhat systematic way to build Lyapunov functions
  Choosing the LFC thinking in terms of energy of the controlled system. Typically potential energy shaping, possibly kinetic energy shaping (acceleration feedback). This is also denoted Energy-based control.
- Can give conclusions about properties of feedback connections based on the properties of each subsystem
  This allows for modular analysis and design, something which simplifies the design process. (Resembling the cascade results for ISS systems, page 179-180)

Summary

Passivity

- **Robustness:** If the model possesses the same passivity properties regardless of the numerical values of the physical parameters, and a controller is designed so that stability relies on the passivity properties only, the closed-loop system will be stable regardless of the values of the physical parameters
- **A tool for choosing where to place sensors:** Passivity considerations are helpful as a guide for the choice of a suitable variable y for output feedback. This is helpful for selecting where to place sensors for feedback control.
- **A tool for choosing where to place actuators:** A guide for choice of location of actuators

Lecture's goals

What did we learn in Lecture 12?

- Know the concepts of relative degree, normal form, external dynamics, internal dynamics and zero dynamics.
- Be able to design a stabilizing control law using the input-output linearization method, including
  1) Finding the relative degree
  2) Writing the system in normal form
  3) Creating a linear input-output relation by feedback control
  4) Analyzing the zero dynamics
  5) Choosing the transformed input variable v to stabilize the origin of the system, locally or globally
- Be able to design a control law that solves the local tracking control problem, using the input-output linearization method
- Be able to discuss the advantages and the disadvantages of the input-output linearization method

Advantages/shortcomings

Advantages/shortcomings

- Cancels all dynamics $L_f h$
  - ÷ Does not take advantage of stabilizing terms
  - ÷ Robustness to modelling errors is questionable
- Requires well-defined relative degree
- Requires minimum phase system
- \+ Exponential convergence
- \+ We can use linear control design methods
- \+ Easy tuning

Lecture's goals

What did we learn in Lecture 13?

- Be able to design a stabilizing control law using the integrator backstepping method
- Be able to discuss the advantages and disadvantages of this method