NTNU – Trondheim
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TTT4120 Digital Signal Processing
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Lecture: Z-Transform - Introduction

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Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 3.1 The z-transform
  - 3.2 Properties of the z-transform
  - 3.3 Rational z-transforms

*Level of detail is defined by lectures and problem sets

2

Contents and learning outcomes

- Definition of z-transform and its existence
- Some properties of the z-transform
- Rational z-transforms: poles and zeros

Motivation

- Linear time-invariant system:

$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$

$e^{j\omega n} \longrightarrow y[n] = e^{j\omega n} H(\omega)$

$X(\omega) \longrightarrow Y(\omega) = H(\omega)X(\omega)$

- What if $h[n] = 2^n u[n]$?
  - System is unstable $\sum|h[n]|$ not finite
  - DTFT of $h[n]$ does not exist
- Can we analyze such systems using a transform method while retaining the good properties of the DTFT?

Basic idea

- Capture the source of instability or inapplicability of the DTFT
- Apply the DTFT to the modified (captured) signal

Basic idea

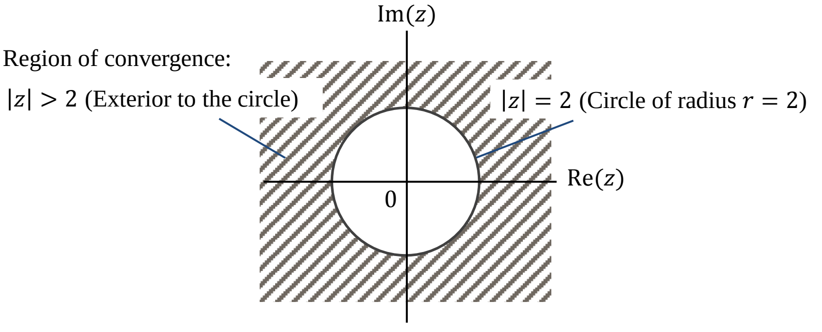
- Example: Suppose we have signal $x[n] = 2^n u[n]$
  - Problem is due to the exponential growth
  - Capture the signal by multiplying it by a decaying exponential stronger than the growing one, i.e., $r^{-n}x[n], r > 0$
  - What values of r allow for a DTFT for $r^{-n}x[n]$?

$$\sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n} = \sum_{n=0}^{\infty} 2^n r^{-n} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty}\left(2r^{-1}e^{-j\omega}\right)^n = \frac{1}{1-2r^{-1}e^{-j\omega}}$$

Convergence if $\left|2r^{-1}e^{-j\omega}\right| < 1$ or $r > 2$

Basic idea…



- Define complex number $z = re^{j\omega}$ in previous expression

$$\sum_{n=0}^{\infty} 2^n r^{-n} e^{-j\omega n} = \sum_{n=0}^{\infty} 2^n z^{-n} = \frac{1}{1-2z^{-1}}, \forall |z| > 2$$

- Convergence has only to do with $r = |z|$ and not ω
- We have a more general transform of the sequence $x[n]$

7

Definition of z-transform

- The z-transform of a discrete-time signal $x[n]$ is

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- Notation: $x[n] \overset{\mathcal{Z}}{\leftrightarrow} X(z)$ $\quad x[n] = \mathcal{Z}^{-1}\{X(z)\}$
- Transforms $x[n]$ into its complex-plane representation $X(z)$
- Transform only exists whenever power series converges
- Region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value

8

Definition of z-transform…

- Example: Z-transforms of finite-length sequences

$$x_1[n] = \{\underline{1}, 2, 5, 0, 1\}$$
$$= \delta[n] + 2\delta[n-1] + 5\delta[n-2] + \delta[n-4]$$
$$x_2[n] = \{1, \underline{2}, 5, 0, 1\}$$
$$x_3[n] = 2\delta[n]$$

- ROC for finite-length signals is entire z-plane, except possibly when $z \to 0$ or $z \to \infty$
  - either z^k or z^{-k} grow unbounded

9

Definition of z-transform…

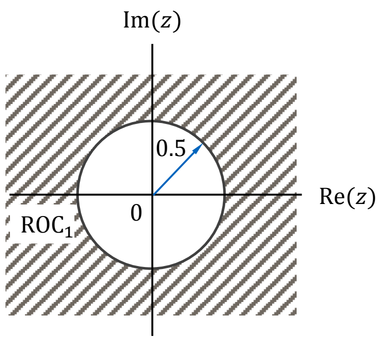
- Example: Compute z-transforms of infinite-length sequences

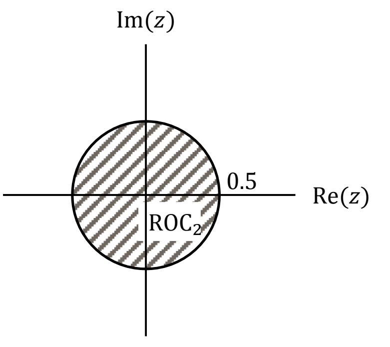
$$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$x_2[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$$

10

Definition of z-transform…





$$X_1(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \qquad X_2(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

Definition of z-transform…

- Observations for infinite-duration sequences:
  - z-transform expression alone does not uniquely specify the time-domain signal. ROC resolves ambiguity
  - ROC causal sequence is the exterior of a circle
  - ROC anti-causal sequence is the interior of a circle

ROC of z-transform

- In ROC of $X(z)$, we have $|X(z)| < \infty$
- Using polar form of z, i.e., $z = re^{j\theta}$, we get

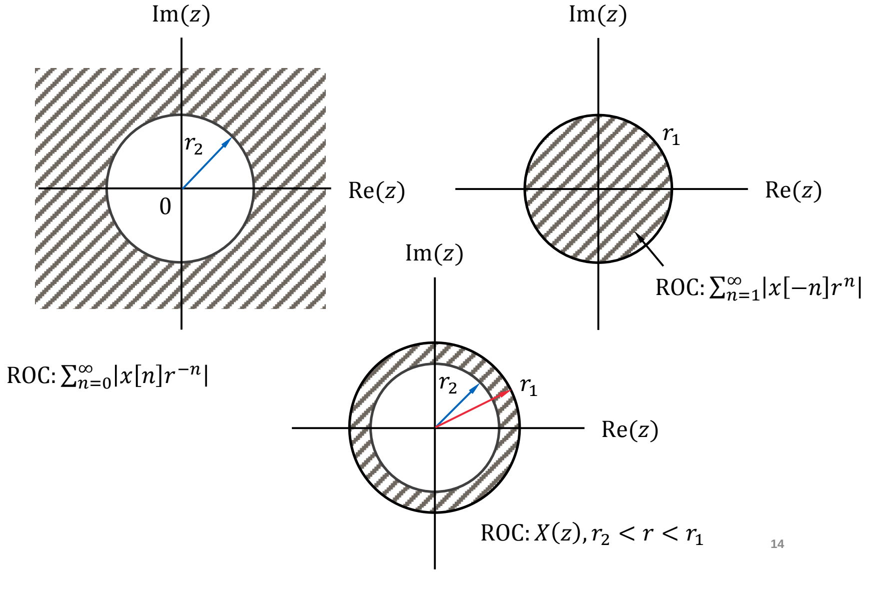
$$|X(z)| = \left|\sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\theta n}\right| \leq \sum_{n=-\infty}^{\infty}\left|x[n]r^{-n}e^{-j\theta n}\right|$$

$$\boxed{\leq \sum_{n=1}^{\infty}|x[-n]r^{n}| + \sum_{n=0}^{\infty}|x[n]r^{-n}|}$$

- Observations:
  - both series should converge, $\mathrm{ROC} = \mathrm{ROC}_1 \cap \mathrm{ROC}_2$
  - for r sufficiently small, $r \leq r_1 < \infty$, first sum may converge
  - for r sufficiently large, $r \geq r_2$, second sum may converge

13

ROC of z-transform…



14

ROC of z-transform

- Example: Two-sided infinite-length sequences

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[-n-1]$$

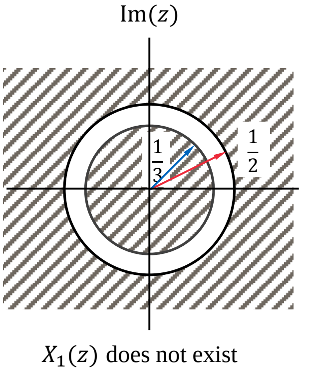
$$x_2[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[-n-1]$$

$$X_1(z) = ?, X_2(z) = ?$$

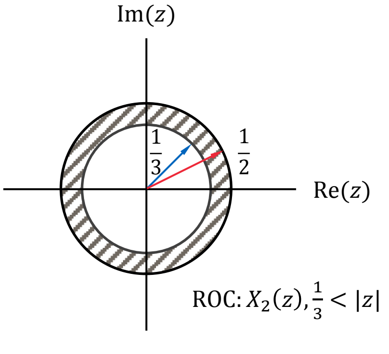
From earlier: $\alpha^n u[n] \overset{\mathcal{Z}}{\leftrightarrow} \frac{1}{(1-\alpha z^{-1})}$, ROC: $|z| > \alpha$

$$-\alpha^n u[-n-1] \overset{\mathcal{Z}}{\leftrightarrow} \frac{1}{(1-\alpha z^{-1})}, \text{ROC: } |z| < \alpha$$

ROC of z-transform…



$X_1(z)$ does not exist



ROC: $X_2(z), \frac{1}{3} < |z| < \frac{1}{2}$

$$X_2(z) = \frac{1}{1-\frac{1}{3}z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}$$

Properties of the z-transform

- Linearity
- Time-shift
- Scaling
- Time-reversal
- Convolution
- Differentiation
- Initial value theorem

Properties of the z-transform…

- Linearity:

$$x_3[n] = a_1 x_1[n] + a_2 x_2[n] \overset{z}{\leftrightarrow} X_3(z) = a_1 X_1(z) + a_2 X_2(z)$$

  for <u>any</u> constants a_1 and a_2

- ROC of $X_3(z)$ at least $\mathcal{R}_{X_1} \cap \mathcal{R}_{X_2}$ but can extend beyond intersection
- Example: $x_1[n] = (3 \cdot 2^n - 4 \cdot 3^n)u[n]$

$$x_2[n] = (3 \cdot 2^n + 4 \cdot 3^n)u[n]$$

$$x_3[n] = x_1[n] + x_2[n]$$

Properties of the z-transform…

- Time-shift: $x[n-k] \overset{z}{\leftrightarrow} z^{-k}X(z)$
- ROC of $z^{-k}X(z)$ same as $X(z)$ except at $z = 0$ and $z \to \infty$
- Coefficient of z^{-n} becomes $z^{-(n+k)}$
- Example: $x[n] = \{1, \underline{2}, -1, 0, 3\}$
  $x[n+2]$
  $x[n-2]$

Properties of the z-transform…

- Scaling: $a^n x[n] \overset{z}{\leftrightarrow} X(a^{-1}z)$
- If ROC of $X(z)$ is $r_1 < |z| < r_2$, then ROC of $X(a^{-1}z)$ is
  $$|a|r_1 < |z| < |a|r_2$$
- Example: $x[n] = 2^n u[n]$

Properties of the z-transform…

- Time reversal: $x[-n] \overset{z}{\leftrightarrow} X(z^{-1})$
- If ROC of $X(z)$ is $r_1 < |z| < r_2$, then ROC of $X(z^{-1})$ is

$$1/r_2 < |z| < 1/r_1$$

- Example: $x[n] = u[-n]$

21

Properties of the z-transform…

- Convolution: $x[n] = x_1[n] * x_2[n] \overset{z}{\leftrightarrow} X_1(z)X_2(z)$
- ROC at least the intersection of that of $X_1(z)$ and $X_2(z)$
- Many cases much easier to carry out in z-domain
- Example: $x_1[n] = \{\underline{1}, -1\}$
  $x_2[n] = \{\underline{1}, 1\}$

22

Properties of the z-transform…

- Differentiation: $nx[n] \overset{Z}{\leftrightarrow} -z\frac{dX(z)}{dz}$
- ROC convergence stays the same
- Example: $x[n] = na^n u[n]$
- Initial value theorem: $x[0] = \lim_{z\to\infty} X(z)$, $x[n]$ causal

Rational z-transforms

- Family of transforms where $X(z)$ can be represented as the ratio of two polynomials in z^{-1} (or z)

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Mz^{-M}}{a_0+a_1z^{-1}+a_2z^{-2}+\cdots+b_Nz^{-N}}$$

$$= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$= \frac{b_0}{a_0}\frac{\sum_{k=0}^{M}(b_k/b_0)z^{-k}}{\sum_{k=0}^{N}(a_k/a_0)z^{-k}} \qquad (a_0, b_0 \neq 0)$$

$$= \frac{b_0}{a_0}\frac{\prod_{k=0}^{M}(1-z_kz^{-1})}{\prod_{k=0}^{N}(1-p_kz^{-1})}$$

Rational z-transforms…

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 \prod_{k=0}^{M}(1-z_k z^{-1})}{a_0 \prod_{k=0}^{N}(1-p_k z^{-1})}$$

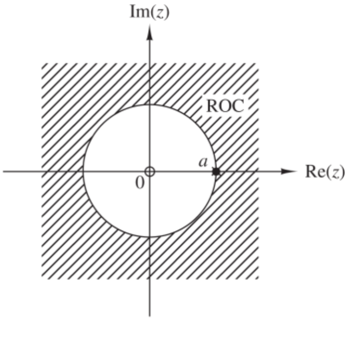
- The zeros of $X(z)$: values of z for which $X(z) = 0$, $B(z) = 0$
- The poles of $X(z)$: values of z for which $X(z) \to \infty$, $A(z) = 0$
- If a_k and b_k real-valued$\Longrightarrow$ poles (zeros) are either real-valued or must occur in conjugate pairs

25

Rational z-transforms…

- Example (pole-zero plot):

$$x[n] = a^n u[n], a > 0 \overset{z}{\leftrightarrow} X(z) = \frac{1}{1 - az^{-1}}$$

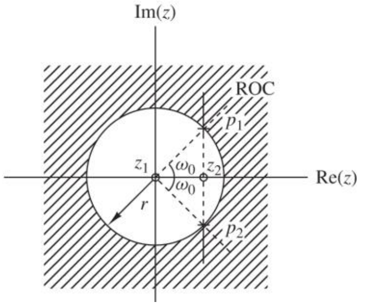


26

Rational z-transforms…

- Example (pole-zero plot):

$$x[n] = \left(\frac{5}{6}\right)^n \sin\left(\frac{\pi}{3}n\right) u[n] \overset{Z}{\leftrightarrow} X(z) = \frac{\frac{5}{6}\sin\left(\frac{\pi}{3}\right)z^{-1}}{\left(1-\frac{5}{6}e^{\frac{j\pi}{3}}z^{-1}\right)\left(1-\frac{5}{6}e^{-\frac{j\pi}{3}}z^{-1}\right)}$$

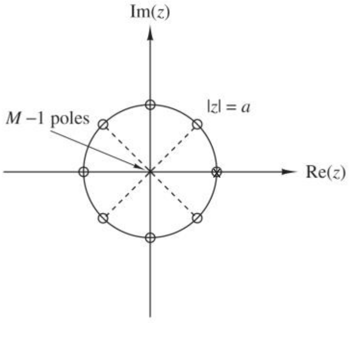


27

Rational z-transforms…

- Example (pole-zero plot):

$$x[n] = \begin{cases} a^n, 0 \le n \le M-1 \\ 0, \text{elsewhere} \end{cases} \overset{Z}{\leftrightarrow} X(z) = \frac{1-(az^{-1})^M}{1-az^{-1}}$$



28

Summary

Today:

- Z-transform and its existence (ROC)
- Properties of the z-transform
- Rational z-transforms: poles and zeros

Next:

- LTI systems: The system function, stability and causality
- Computation and sketching of frequency response function