NTNU – Trondheim
Norwegian University of Science and Technology

Department of Engineering Cybernetics

Examination paper for TTK4150 Nonlinear Systems

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Examination date: December 5, 2013

Examination time (from-to): 15.00-19.00

Permitted examination support material:

D – No printed or written materials allowed.

NTNU type approved calculator with empty memory allowed.

Other information:

This exam counts for 100% of the final grade.

Language: English

Number of pages: 6

Number of pages enclosed: 1

Checked by:

Date Signature

Problem 1 (10%)

Consider the system

$$\dot{x}_1 = x_2 \tag{1}$$
$$\dot{x}_2 = -\sin(x_1) - \left(5 + x_2^2 + x_2^4\right) x_2 \tag{2}$$

a [7%] Find all equilibrium points and classify the qualitative behavior of each of them. Sketch the phase portraits.

b [3%] Prove that there are no periodic orbits in $\mathbb{R}^2$.

Problem 2 (15%)

Consider the following system

$$\dot{x}_1 = -x_1 x_2^2 - x_2^3 \tag{3}$$
$$\dot{x}_2 = -4x_2 - x_1^2 x_2 + x_1^3 \tag{4}$$

a [3%] Show that the origin is the only equilibrium point.

b [5%] Which conclusion can be drawn about the stability of the origin using the indirect Lyapunov method?

c [7%] Use the Lyapunov function candidate $V(x) = \frac{1}{4}x_1^4 + \frac{1}{4}x_2^4$ to prove that $(0,0)$ is an asymptotically stable equilibrium point. Is it globally asymptotically stable? Justify your answer.

Problem 3 (17%)

Consider the system

$$\dot{x}_1 = f_1(t, x_1, u) = -\left(1 + g(t)\right) x_1 + u \tag{5}$$

where $0 \leq g(t) \leq g_0 \ \forall t \geq 0$, and g_0 is a positive constant.

a [3%] Writing (5) as

$$\dot{x}_1 = f_1(t, z) \tag{6}$$

where $z = [x_1, u]^T$. Show that there exists a constant $L > 0$ such that

$$\left\| \frac{\partial f_1(t, z)}{\partial z} \right\|_1 \leq L \tag{7}$$

on $[0, \infty) \times \mathbb{R}^2$. Is the function locally/globally Lipschitz?

b [5%] Show that $x_1 = 0$ is a globally exponentially stable (GES) equilibrium point for the unforced system ($u = 0$).

c [4%] Show that the system is input-to-state stable (ISS).

d [5%] With the following state added to the system

$$\dot{x}_2 = f_2(t, x_2) = -2x_2 - x_2 \sin(t) \tag{8}$$

and with $u = x_2$ in (5), show that $(x_1, x_2) = (0, 0)$ is a globally uniformly asymptotically stable (GUAS) equilibrium point for the overall system.

Problem 4 (20%)

Consider the system

$$\dot{x}_1 = -x_1^2 x_2^2 + x_2 \tag{9}$$
$$\dot{x}_2 = -x_1^3 + x_1^5 x_2 - x_2 + u \tag{10}$$
$$y = x_2 \tag{11}$$

a [5%] Show that the system is output strictly passive.
Hint: Use the storage function $V_1(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

b [5%] Show that the system is zero-state observable.

c [4%] State the criteria for the function $\phi(y)$, so that the controller $u = -\phi(y)$ globally stabilizes the origin of the system. How can $\phi(y)$ be chosen to satisfy possible constraints on u (upper and lower bounds)?

d [6%] Show that the dynamic controller

$$\dot{x}_3 = -4x_3 + x_3 e^{-|x_3|} + y \tag{12}$$
$$-u = x_3 \tag{13}$$

will result in a system where the origin of the closed loop system is globally asymptotically stable.

Problem 5 (28%)

Consider the following system equation

$$\dddot{x} + 4\ddot{x} + 4\dot{x} + 12x|x| = 0 \tag{14}$$

a [4%] The system can be written as a feedback connection of a linear system $y = G(s)u$ with the nonlinear feedback $u = -\psi(y)$, where $\psi(y) = y|y|$. See Fig. 1. Given initial conditions equal to zero, show that the linear system $G(s)$ is

$$G(s) = \frac{12}{s^3 + 4s^2 + 4s} \tag{15}$$

What is the output from the linear system y?

b [4%] What do we have to assume about the transfer function $G(s)$ in order to justify using the describing function method? Why?

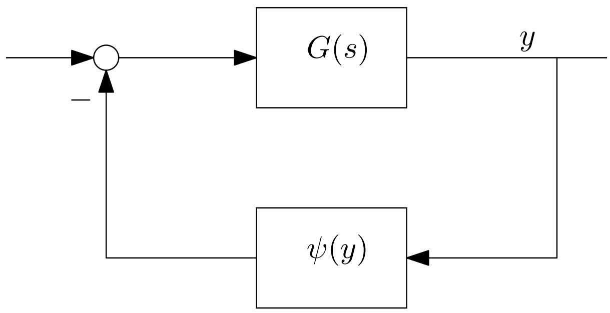

Figure 1: Feedback connection of linear system with nonlinear feedback.

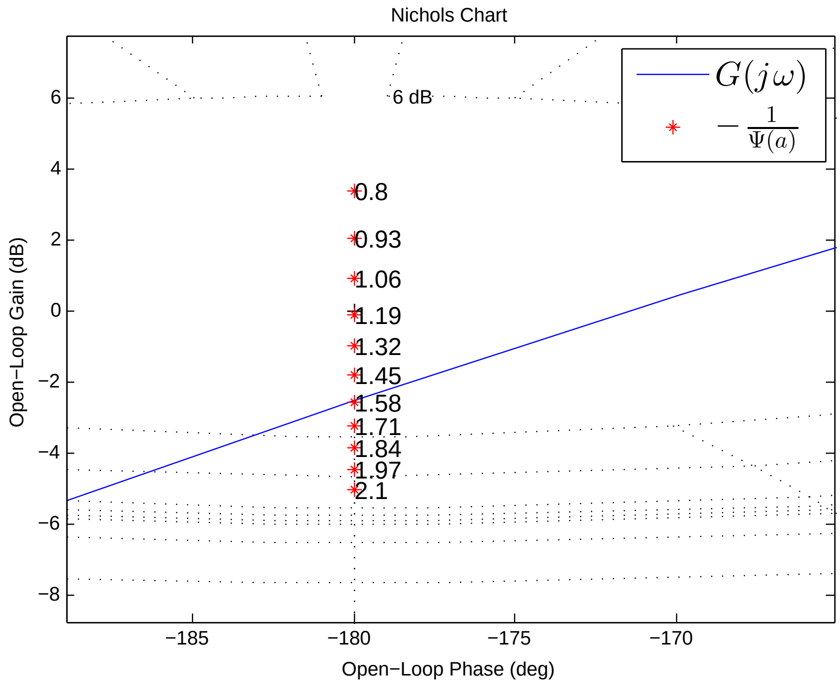

Figure 2: Nichols plot for $G(j\omega)$ and $-\frac{1}{\Psi(a)}$.

c **[6%]** Find the describing function of the odd, time-invariant, memoryless nonlinearity $\psi(y) = y|y|$.

Hint:

$$\int_0^{\pi/2} \sin^u du = \begin{cases} \frac{1\cdot3\cdot5\cdots(n-1)}{2\cdot4\cdot6\cdots n} \cdot \frac{\pi}{2} & \text{if } n \geq 2 \text{ and is an even integer} \\ \frac{2\cdot4\cdot6\cdots(n-1)}{3\cdot5\cdot7\cdots n} & \text{if } n \geq 3 \text{ and is an odd integer} \end{cases} \tag{16}$$

d **[8%]** Use an analytic approach to estimate the frequency and amplitude of possible periodic solutions in the system.

e **[6%]** The Nichols plot for $|G(j\omega)|$ and $-\frac{1}{\Psi(a)}$ is given in Fig. 2. Use this plot to investigate the stability properties of possible periodic soultions. (The numbers next to $-\frac{1}{\Psi(a)}$ is a.)

Problem 6 (10%)

Consider the following system

$$\dot{x}_1 = (x_1 - x_2)^2 + (x_1 - x_2)\, x_2 + u \tag{17}$$

$$\dot{x}_2 = -x_2 + u \tag{18}$$

a [3%] Explain why backstepping cannot be applied directly to the given system.

b [7%] Introduce the change of variables

$$\bar{x}_1 = x_1 - x_2 \tag{19}$$

$$\bar{x}_2 = x_2 \tag{20}$$

and explain why backstepping now can be applied? Use backstepping to design a nonlinear controller that makes the origin globally asymptotically stable.

Appendix: Formulae

$$xy \leq \frac{\varepsilon^p}{p}|x|^p + \frac{1}{q\varepsilon^q}|y|^q$$

$$y(t) \approx a\sin\theta$$
$$\theta = \omega t$$

$$\psi(y(t)) \approx b + c_c\cos\theta + c_s\sin\theta = b + c\sin(\theta+\phi)$$

$$b = \frac{1}{2\pi}\int_0^{2\pi}\psi(a\sin\theta)\mathrm{d}\theta$$
$$c_c = \frac{1}{\pi}\int_0^{2\pi}\psi(a\sin\theta)\cos\theta\mathrm{d}\theta$$
$$c_s = \frac{1}{\pi}\int_0^{2\pi}\psi(a\sin\theta)\sin\theta\mathrm{d}\theta$$

$$c = \sqrt{{c_s}^2 + {c_c}^2}$$
$$\phi = \arctan\left(\frac{c_c}{c_s}\right)$$

$$\Psi(a,\omega) = \frac{c\,\mathrm{e}^{j(\theta+\varphi)}}{a\,\mathrm{e}^{j\theta}} = \frac{c\,\mathrm{e}^{j\varphi}}{a} = \frac{c_s + jc_c}{a}$$
$$|\Psi(a,\omega)| = \frac{c}{a}$$
$$\angle\Psi(a,\omega)| = \phi$$

$$G(j\omega)\Psi(a,\omega) + 1 = 0$$

$$\mathrm{e}^{\alpha+j\beta} = \mathrm{e}^{\alpha}(\cos\beta + j\sin\beta)$$
$$\cos^2\alpha + \sin^2\alpha = 1$$
$$\sin\alpha = \pm\sqrt{1-\cos^2\alpha}$$
$$\cos\alpha = \pm\sqrt{1-\sin^2\alpha}$$