NTNU – Trondheim
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Science and Technology

TTT4120 Digital Signal Processing
Fall 2017

Design of Digital Filters: FIR

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Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 10.2.2 Design of linear-phase FIR filters using windows
  - 10.2.4 Design of optimum equiripple linear-phase FIR filters

- A compressed overview of topics treated in the lecture, see "Design av digitale filtre" on Blackboard

*Level of detail is defined by lectures and problem sets

Contents and learning outcomes

- Filter specifications
- FIR versus IIR
- Window method
- Equirippel design

Filter design

- Systems may be designed to amplify or attenuate parts of the input signal (e.g., remove noise, suppress interference)

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$$

$X(\omega),\ X(z)$ $\quad$ $H(\omega),\ H(z)$

$$Y(\omega) = H(\omega)X(\omega)$$

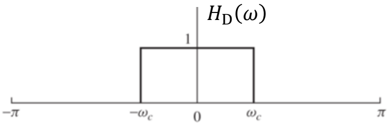
$$Y(z) = H(z)X(z)$$

- A discrete-time filter modifies the Fourier representation of $x[n]$
  - Lowpass
  - Highpass
  - Bandpass
  - Bandstop, etc.

Filter design...

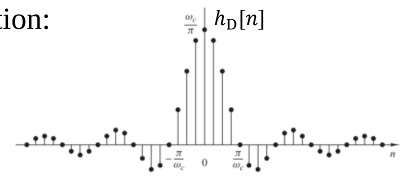
- Ideal lowpass filter:

$$H_{\mathrm{D}}(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \omega_c \leq \omega \leq \pi \end{cases}$$



- Impulse response in the sinc function:

$$h_{\mathrm{D}}[n] = \frac{\omega_c}{\pi} \frac{\sin \omega_c n}{\omega_c n}$$



- Problems:
  - Ideal filters are not causal $\Longrightarrow$ not physically realizable
  - Infinite complexity and delay, not BIBO stable, etc.

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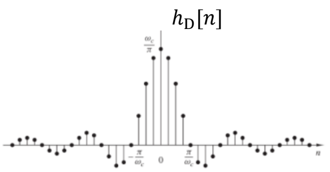
Filter design...



$\approx$?

- We want causal linear-phase filters $\Longrightarrow$ approximations needed
  - Truncate time-domain pulse (windowing)
  - Control frequency response (equirippel design)

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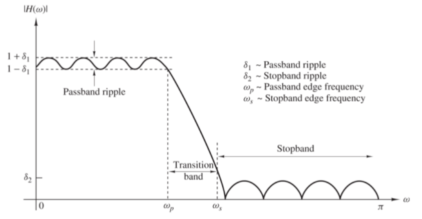
Filter design...



- In practice, ideal filter characteristics are not absolutely necessary
- Find filter of minimum complexity satisfying a given specification
  - Nonconstant magnitude in passband (small ripple)
  - Non-zero stopband (small value or small amount of ripple)
  - Allow for non-zero transition band from passband to stopband
- The more restrictions on the design, the more complex it will be

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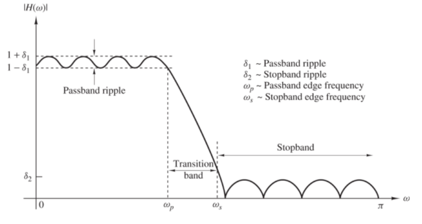
Filter design...



- Real-valued, causal filters of the form: $H(z) = \frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}}$
- FIR: $H(z) = \sum_{k=0}^{M-1} b_k z^{-k} \Rightarrow y[n] = \sum_{k=0}^{M-1} b_k x[n-k]$
- IIR: $H(z) = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{1+\sum_{k=1}^{N-1} a_k z^{-k}} \Rightarrow y[n] = -\sum_{k=1}^{N-1} a_k z^{-k} + \sum_{k=0}^{M-1} b_k x[n-k]$
- Find $\{a_k\}$ and $\{b_k\}$ that satisfy filter specification

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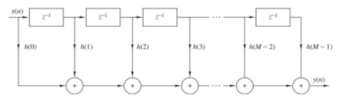
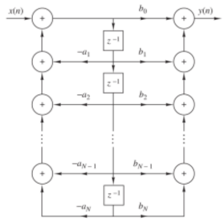
FIR versus IIR

- FIR filters:
  - Always stable
  - Can achieve exactly linear phase
  - Easily designed with linear methods
  - Easy to implement
- IIR filters:
  - Fewer parameters (low filter order)
  - Less memory
  - Low delay
  - Lower computational complexity
  - Typically designed by transforming an analog filter design





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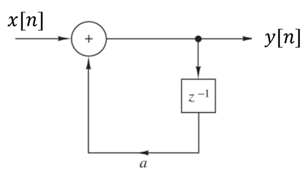
FIR versus IIR...

- Example: $H(z) = \frac{1}{1-az^{-1}}$, $|a| < 1$

- IIR implementation: $y[n] = ay[n-1] + x[n]$



- FIR approximation: $y[n] = \sum_{k=0}^{M} a^k \, x[n-k]$, M large

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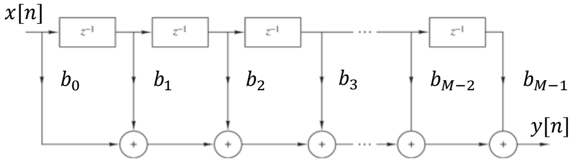
Linear-phase FIR filters

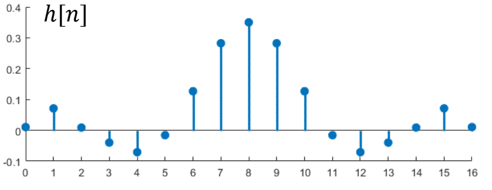


- Moving average filter, or an all-zero filter, of order M

$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k} = b_0 z^{-(M-1)} \prod_{k=1}^{M-1} (z - z_k)$$

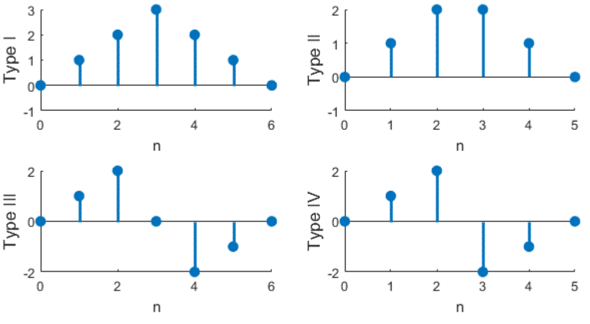
- Design of $\{b_k\} \Leftrightarrow$ moving zeros in the z-plane
  – Can be designed using some optimality criterion
- Impulse response $h[n]$ of an FIR filter given by the filter weights
  – Easily verified by setting $x[n] = \delta[n]$

Linear-phase FIR filters...



- FIR filters can be causal and have linear phase
  – Implies a linear shift in time domain (no distortion)
  – Exact linear phase not possible in IIR filters
- Linear phase filters must have symmetric impulse response
  – Four possibilities: M even/odd, $h[n]$ symmetric/antisymmetric

Linear-phase FIR filters...



- Let us review the case of M odd and $h[n]$ symmetric (Lecture 7)
  $\Rightarrow M-1$ even and $h[n] = h[M-1-n]$

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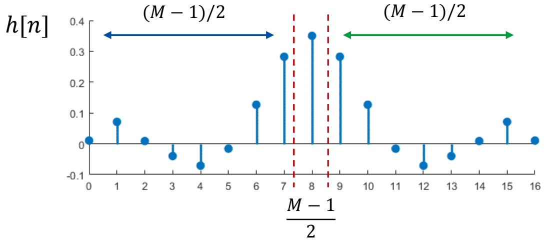
Linear-phase FIR filters...



$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k} =$$

$$= \sum_{k=0}^{(M-3)/2} h[k]z^{-k} + h\left[\frac{M-1}{2}\right] z^{-(M-1)/2} + \sum_{k=(M+1)/2}^{M-1} h[k]z^{-k}$$

$$= \sum_{k=0}^{(M-3)/2} h[k]z^{-k} + h\left[\frac{M-1}{2}\right] z^{-(M-1)/2} + \sum_{k=(M+1)/2}^{M-1} h[M-1-k]z^{-k}$$

$$= \sum_{k=0}^{(M-3)/2} h[k]z^{-k} + h\left[\frac{M-1}{2}\right] z^{-(M-1)/2} + \sum_{l=0}^{(M-3)/2} h[l]z^{l-(M-1)}$$

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Linear-phase FIR filters...

$$H(z) = \sum_{k=0}^{(M-3)/2} h[k]z^{-k} + h\left[\frac{M-1}{2}\right] z^{-(M-1)/2} + \sum_{l=0}^{(M-3)/2} h[l]z^{l-(M-1)}$$

$$= h\left[\frac{M-1}{2}\right] z^{-(M-1)/2} + \sum_{k=0}^{(M-3)/2} h[k]\left(z^{-k} + z^{k-(M-1)}\right)$$

$$= h\left[\frac{M-1}{2}\right] z^{-(M-1)/2} + \sum_{k=0}^{(M-3)/2} h[k]z^{-(M-1)/2}\left(z^{-(k-(M-1)/2)} + z^{k-(M-1)/2}\right)$$

$$= \left(h\left[\frac{M-1}{2}\right] + \sum_{k=0}^{(M-3)/2} h[k]\left(z^{-(k-(M-1)/2)} + z^{k-(M-1)/2}\right)\right) z^{-(M-1)/2}$$

- Frequency response obtained by substituting $z = e^{j\omega}$

$$H(\omega) = \left(h\left[\frac{M-1}{2}\right] + 2\sum_{k=0}^{(M-3)/2} h[k]\cos[\omega((M-1)/2 - k)]\right) e^{-j\omega(M-1)/2}$$

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Linear-phase FIR filters...

- Frequency response M odd and $h[n]$ symmetric

$$H(\omega) = \left(h\left[\frac{M-1}{2}\right] + 2\sum_{k=0}^{\frac{M-3}{2}} h[k]\cos\left[\omega\left(\frac{M-1}{2} - k\right)\right]\right) e^{-\frac{j\omega(M-1)}{2}}$$

$$= H_{\mathrm{R}}(\omega)\, e^{-\frac{j\omega(M-1)}{2}}$$

- Amplitude of filter $H_{\mathrm{R}}(\omega) \in \mathbb{R}$ similar to $|H(\omega)|$ since $|H_{\mathrm{R}}(\omega)| = |H(\omega)|$
- However, note that $H_{\mathrm{R}}(\omega)$ can be less than 0
- Linear shift $e^{-\frac{j\omega(M-1)}{2}}$: $H(\omega)$ has piecewise linear phase
  - When $H_{\mathrm{R}}(\omega)$ changes sign, phase jumps π radians (usually in stopband)

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Linear-phase FIR filters...

- All possibilities (similar derivations):
  - Type I. Symmetric, $h[n] = h[M-1-n]$, M odd:

$$H(\omega) = \left(h\left[\frac{M-1}{2}\right] + 2\sum_{n=0}^{(M-3)/2} h[n]\cos\left[\frac{M-1}{2} - n\right]\right) e^{-j\omega(M-1)/2}$$

  - Type II. Symmetric, $h[n] = h[M-1-n]$, M even:

$$H(\omega) = \left(2\sum_{n=0}^{(M-2)/2} h[n]\cos\left[\frac{M-1}{2} - n\right]\right) e^{-j\omega(M-1)/2}$$

  - Type III. Antisymmetric, $h[n] = -h[M-1-n]$, M odd:

$$H(\omega) = \left(2\sum_{n=0}^{(M-3)/2} h[n]\cos\left[\frac{M-1}{2} - n\right]\right) e^{-j\left[\frac{\omega(M-1)}{2}+\frac{\pi}{2}\right]}$$

  - Type IV. Antisymmetric, $h[n] = -h[M-1-n]$, M even:

$$H(\omega) = \left(2\sum_{n=0}^{(M-2)/2} h[n]\sin\left[\frac{M-1}{2} - n\right]\right) e^{-j\left[\frac{\omega(M-1)}{2}+\frac{\pi}{2}\right]}$$

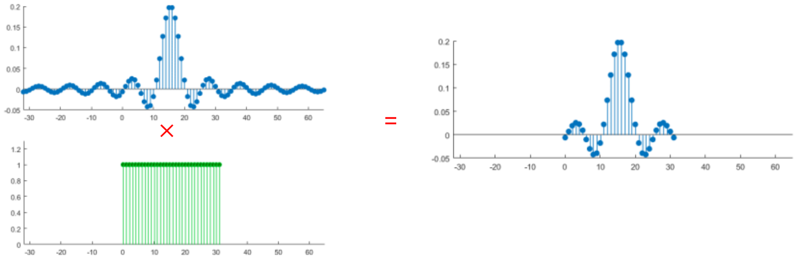
Linear-phase design using windowing

- Basic design principle: Start with a desired frequency specification $H_{\mathrm{D}}(\omega)$ and determine impulse response

$$h_{\mathrm{D}}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{D}}(\omega)d\omega$$

- In general $h_{\mathrm{D}}[n]$ is of infinite length and need to be truncated
- To obtain causal FIR filter of length M we can multiply $h_{\mathrm{D}}[n]$ with a rectangular window

Linear-phase design using windowing...



- Truncation of $h_{\mathrm{D}}[n]$ $\Leftrightarrow$ multiplying $h_{\mathrm{D}}[n]$ by window $w[n]$

$$h[n] = h_{\mathrm{D}}[n]w_{\mathrm{R}}[n]$$

where

$$w_{\mathrm{R}}[n] = \begin{cases} 1 & \text{for } 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

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Linear-phase design using windowing...

- Multiplication in time-domain corresponds to

$$H(\omega) = H_{\mathrm{D}}(\omega) * W(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\lambda)W(\omega-\lambda)d\lambda$$

with $W(\omega) = e^{-j\omega(M-1)/2}\frac{\sin\frac{\omega M}{2}}{\sin\frac{\omega}{2}}$

- Rectangular window has a mainlobe and sidelobes
  - Mainlobe smoothens desired frequency response
  - Sidelobes introduce ringing effects

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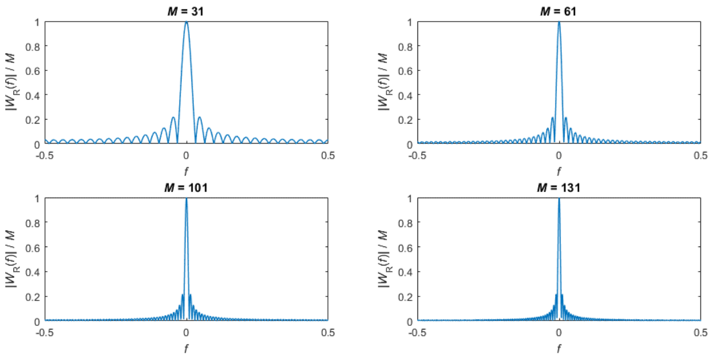
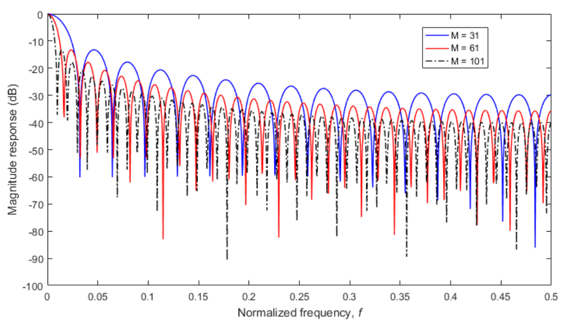
Linear-phase design using windowing...

- Illustration: $\frac{1}{M}|W(\omega)| = \frac{1}{M}\frac{|\sin\frac{\omega M}{2}|}{|\sin\frac{\omega}{2}|}$



Linear-phase design using windowing

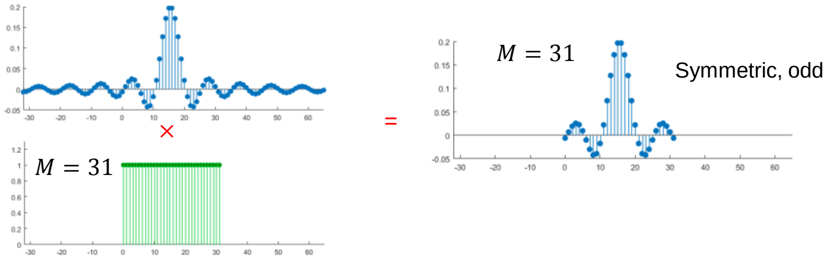


- Large dynamic range $\Longrightarrow$ plot magnitude response in dB

Matlab
```
M   = 31;
wR  = window(@rectwin,M);
[WR,w]=freqz(wR,1,1024);
plot(w/2/pi,20*log10(abs(WR)/M))
```

Linear-phase design using windowing...

- Design example: $H_{\mathrm{D}}(\omega) = \begin{cases} 1 \cdot e^{-j\omega(M-1)/2}, & |\omega| \leq \omega_c \\ 0, & \omega_c \leq \omega \leq \pi \end{cases}$
- Corresponding impulse response $h_{\mathrm{D}}[n] = \frac{\omega_c}{\pi} \frac{\sin \omega_c[n-(M-1)/2]}{\omega_c[n-(M-1)/2]}$
- Truncated response: $h[n] = w_{\mathrm{R}}[n] h_{\mathrm{D}}[n]$



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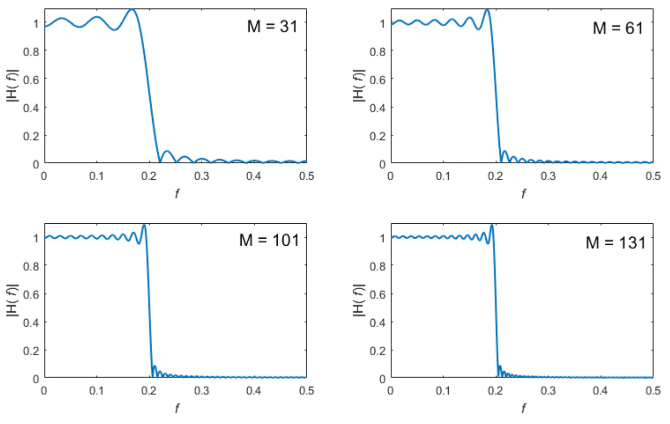
Linear-phase design using windowing...



- Oscillations do not disappear as M increases (Gibbs)
- Use other windows to reduce ripples in passband and stopband

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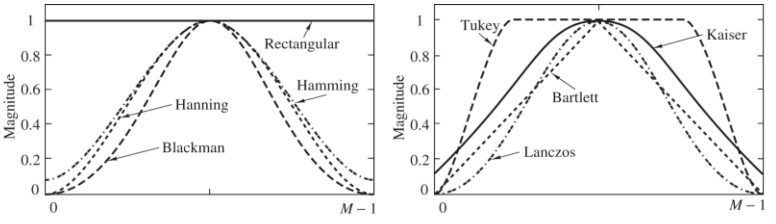
Different windows in time domain



- Type ‘window’ at Matlab command prompt
- Transition bandwidth depends on window length and type
- Passband attenuation
  - Depends on window chosen
- Rectangular window narrowest mainlobe
  - Smallest transition region but worst attenuation in stopband

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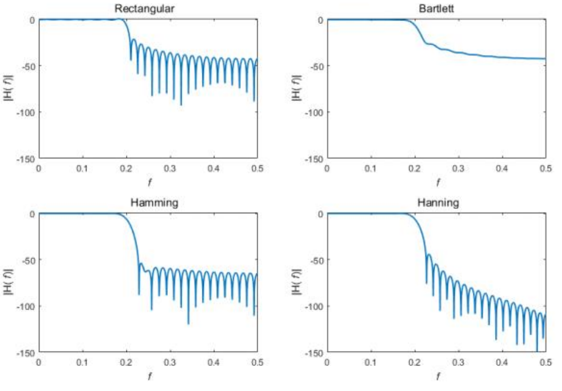
Different windows in time domain...



Matlab
```
M  = 61;
wc = 2*pi*0.2;
hB = fir1(M-1,wc/pi,bartlett(M));
[HB,w]=freqz(B,1,1024);
plot(w/2/pi,20*log10(abs(HB)))
```

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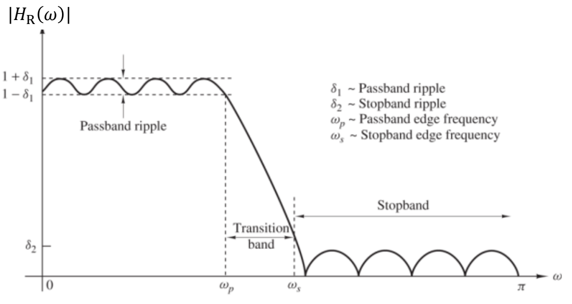
Equiripple design of linear-phase filters

- Major disadvantage of window method is the lack of precise control of the critical frequencies at band edges, i.e., ω_p and ω_s
- Instead, find filter coefficients $h_{\mathrm{R}}[n]$ to minimize the maximal deviation from a desired response $H_{\mathrm{D}}(\omega)$



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Equiripple design of linear-phase filters...

- Define an error function $E(\omega) = W(\omega)[H_{\mathrm{D}}(\omega) - H_{\mathrm{R}}(\omega)]$
  - $H_{\mathrm{D}}(\omega)$ is the desired frequency response
  - $H_{\mathrm{R}}(\omega)$ is the frequency response with filter coefficients $h[n] = b_n$
  - $W(\omega)$ is a weight function given by the filter specs

$$W(\omega) = \begin{cases} \frac{\delta_2}{\delta_1} & \omega \le \omega_p \\ 0 & \omega \ge \omega_s \end{cases}$$

- Find filter coefficients $h[n]$ that minimizes maximal deviation

$$\min_{h[n]} \max_{\omega} |W(\omega)[H_{\mathrm{D}}(\omega) - H_{\mathrm{R}}(\omega)]|$$

- Result of minimization is a filter with equirippel characteristic

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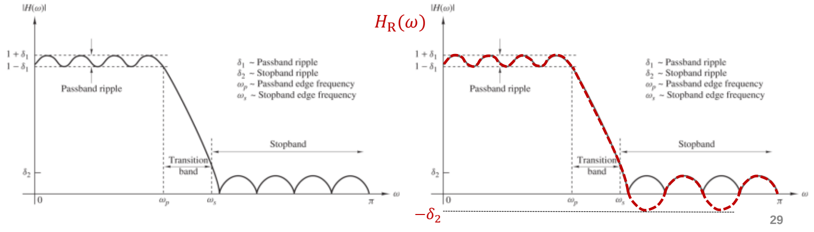
Equiripple design of linear-phase filters...

- Let us consider linear-phase filter of Type 1 (symmetric, M odd):

$$H(\omega) = H_{\mathrm{R}}(\omega)\, e^{-\frac{j\omega(M-1)}{2}},$$

$$\text{with } H_{\mathrm{R}}(\omega) = h\left[\frac{M-1}{2}\right] + 2\sum_{k=0}^{(M-3)/2} h[k]\cos\left[\omega\left(\frac{M-1}{2} - k\right)\right]$$

- Design $H(\omega)$ equivalent to design $H_{\mathrm{R}}(\omega)$: slightly different specs



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Equiripple design of linear-phase filters

- Goal is to find optimal $H_{\mathrm{R}}(\omega)$ that complies with specifications

$$H_{\mathrm{R}}(\omega) = h\left[\frac{M-1}{2}\right] + 2\sum_{k=0}^{(M-3)/2} h[k]\cos\left[\omega\left(\frac{M-1}{2} - k\right)\right]$$

- Optimization over the filter taps $h[n],\ n = 0, \dots, (M-1)/2 + 1$
- The weighted error function is

$$E(\omega) = W(\omega)[H_{\mathrm{D}}(\omega) - H_{\mathrm{R}}(\omega)]$$
$$= W(\omega)\left[H_{\mathrm{D}}(\omega) - \left(h\left[\frac{M-1}{2}\right] + 2\sum_{k=0}^{(M-3)/2} h[k]\cos\left[\omega\left(\frac{M-1}{2} - k\right)\right]\right)\right]$$

- Alternation theorem: The optimal $H_{\mathrm{R}}(\omega)$ will touch the error bounds at $(M-1)/2 + 2$ frequencies in interval $[0, \pi]$

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Equiripple design of linear-phase filters...

- Alternation theorem: The optimal $H_{\mathrm{R}}(\omega)$ will touch the error bounds at $(M-1)/2+2$ frequencies in interval $[0,\pi]$
- Remez Exchange algorithm finds coefficients $h[k]$ such that $H_{\mathrm{R}}(\omega)$ satisfies the alternation theorem
  - Always converges to an equiripple solution
  - May not have the passband/stopband characteristics needed for a given M $\Rightarrow$ increase M

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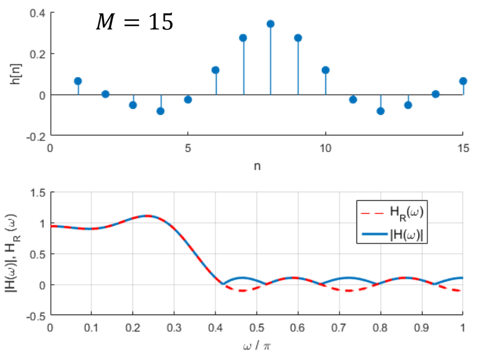
Equiripple design of linear-phase filters...



Matlab

```
E = [0 0.3 0.4 1];
A = [1 1 0 0];
M = 15;
B = firpm(M-1, E, A)
w = linspace(0,pi,500);
H = freqz(B,1,w);
figure
subplot(2,1,1),
stem(B);
subplot(2,1,2),
plot(w/pi,abs(H));
```

- $(M-1)/2+2=(15-1)/2+2=9$ alterations
- Change width of transition band (comment on result)

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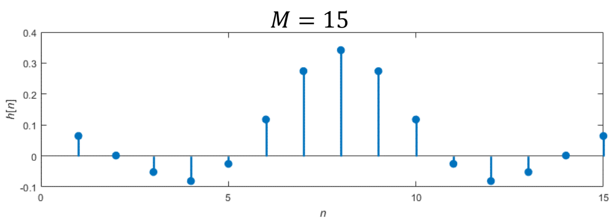
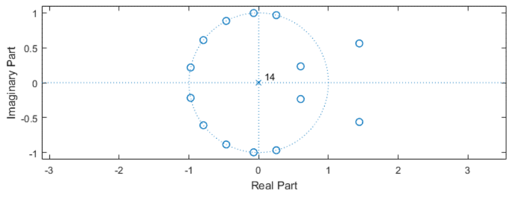
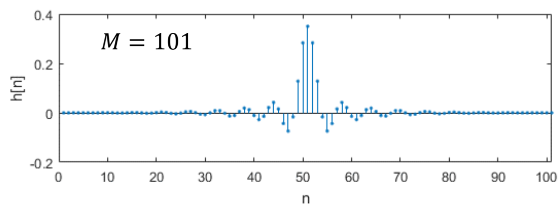
Equiripple design of linear-phase filters...

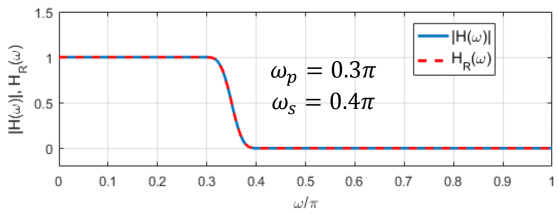




- Check pole-zero plot with `zplane(B,1)`

Equiripple design of linear-phase filters...





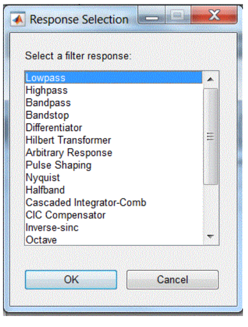
- $(M-1)/2 + 2 = (101-1)/2 + 2 = 52$ alterations

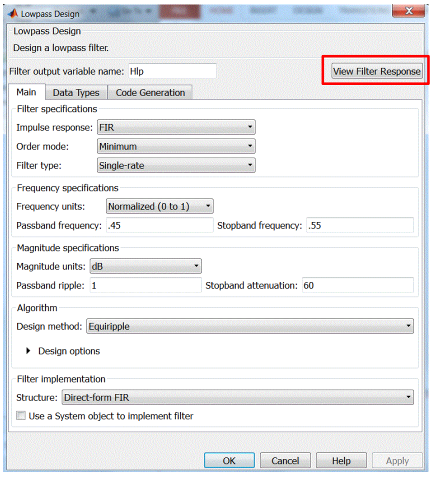
Summary

- Today we discussed:
  - Basics of filter design
  - Linear phase filters using windowing and equiripple designs
- Next:
  - IIR filter design

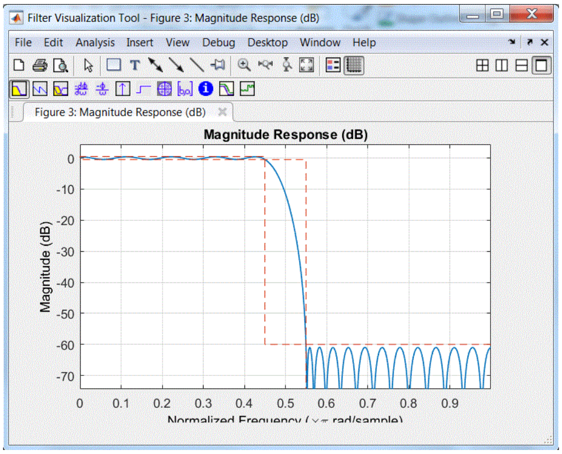
Matlab: filterbuilder…

- Type `filterbuilder` at Matlab command prompt:

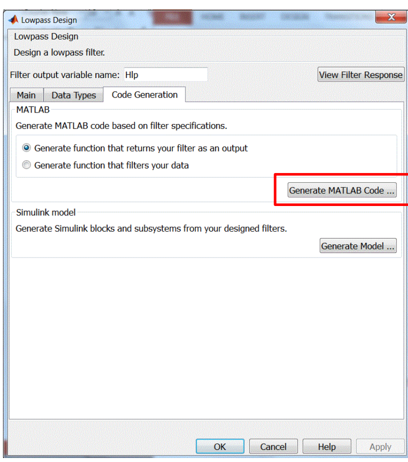




Matlab: filterbuilder



Matlab: filterbuilder…




```
function Hd = test
%TEST Returns a discrete-time filter object.

% MATLAB Code
% Generated by MATLAB(R) 8.6 and the Signal Processing Toolbox 7.1.
% Generated on: 07-Oct-2016 14:29:33

Fpass = 0.45;  % Passband Frequency
Fstop = 0.55;  % Stopband Frequency
Apass = 1;     % Passband Ripple (dB)
Astop = 60;    % Stopband Attenuation (dB)

h = fdesign.lowpass('fp,fst,ap,ast', Fpass, Fstop, Apass, Astop);

Hd = design(h, 'equiripple', ...
    'MinOrder', 'any', ...
    'StopbandShape', 'flat');
```