TTK4150 Nonlinear Control Systems
Lecture 5

Stability analysis for autonomous system

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continued

NTNU
Innovation and Creativity

Previous lecture

Previous lecture:

Lyapunov's direct method:

- Lyapunov functions - a generalization of energy functions
- Lyapunov's theorems for
  - stability
  - local and global asymptotic stability
  - local and global exponential stability
- How to apply Lyapunov's direct method

Outline I

1. Introduction
   - Previous lecture
   - Today's goals
   - Literature
2. The Invariance Principle
   - Invariant sets
   - LaSalle's theorem
   - Prove asymptotic stability when $\dot{V} \leq 0$
   - Estimate Region of attraction
   - Convergence to other invariant sets
3. Methods for choosing Lyapunov function candidates
   - Variable gradient method
   - Lyapunov functions for linear systems
4. How to handle terms with indeterminate sign
   - Tools for dominating cross-terms

Outline II

5. Next lecture

Today's goals

After this lecture you should...

- Know La Salle's theorem, and how to use this
  - $\dot{V} \leq 0$ asymptotic stability of equilibrium points
  - Regions of attraction - find an estimate
  - Convergence to other invariant sets than equilibrium points
- Know some methods for finding Lyapunov function candidates (LFCs)

Literature

Today's lecture is based on

Khalil	Section 4.1 p. 120-122
	Sections 4.2-4.3
	Section 8.2

Part I

La Salle's theorem

Invariant sets

Let $x(t)$ be a solution of $\dot{x} = f(x) \quad f : \mathbb{D} \to \mathbb{R}^n$

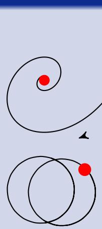
Positive limit point

A point p is a <u>positive limit point</u> of $x(t)$ iff

$\exists$ sequence $\{t_n\}$ in $\mathbb{R}_+$ with $t_n \xrightarrow{n\to\infty} \infty$

such that

$$x(t_n) \xrightarrow{n\to\infty} p$$

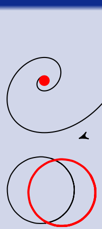


Invariant sets cont.

Positive limit set

The positive limit set of $x(t)$ is:

The set of all positive limit points of $x(t)$



Lemma 4.1

If a solution $x(t)$ is bounded and belongs to $\mathbb{D}$ for $t \geq 0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches L^+ as $t \to \infty$.

Invariant sets cont.

Definition (Invariant set)

A set M is an invariant set with respect to $\dot{x} = f(x)$ iff

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \in \mathbb{R}$$

Definition (Positively invariant set)

A set M is a positively invariant set with respect to $\dot{x} = f(x)$ iff

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \geq 0$$

The invariance principle: LaSalle's theorem

$\dot{x} = f(x) \quad f : \mathbb{D} \to \mathbb{R}^n$ locally Lipschitz

Theorem 4.4 (LaSalle's theorem)

If $\exists\ V : \mathbb{D} \to \mathbb{R}$ such that

i) V is C^1

ii) $\exists\ c > 0$ such that $\Omega_c = \{x \in \mathbb{R}^n | V(x) \leq c\} \subset \mathbb{D}$ is bounded

iii) $\dot{V}(x) \leq 0 \quad \forall\ x \in \Omega_c$

Let $E = \left\{x \in \Omega_c | \dot{V}(x) = 0\right\}$
Let M be the largest invariant set contained in E
Then

$$x(0) \in \Omega_c \Rightarrow x(t) \xrightarrow{t \to \infty} M$$

La Salle's theorem ii)

Note: V does not have to be positive definite

- V positive definite $\Rightarrow \Omega_c$ bounded for small c
- V radially unbounded $\Rightarrow \Omega_c$ bounded for $\forall\ c$

Special cases:

- Cor. 4.1 ($M = \{0\}$)
- Cor. 4.2 (Global version)

Application

Applications of La Salle's theorem:

- $\dot{V} \leq 0$ Prove asymptotic stability of equilibrium points
- Regions of attraction - find an estimate
- Convergence to other invariant sets than equilibrium points

Examples

Example: $\dot V \le 0$ Prove asymptotic stability of eq.point

$$\ddot x + b(\dot x) + c(x) = 0$$

$$\dot x_1 = x_2$$
$$\dot x_2 = -b(x_2) - c(x_1)$$

$b, c \quad C^0$

$b(0) = c(0) = 0$

$x_1 c(x_1) > 0 \quad x_1 \neq 0 \quad x_1 \in (-a_1, a_1)$

$x_2 b(x_2) > 0 \quad x_2 \neq 0 \quad x_2 \in (-a_2, a_2)$

Analyze the stability properties of $x = 0$ using Lyapunov theory.

Pendulum with friction, $x =$ angle, $\dot x =$ angle velocity

$c(x_1) = \frac{g}{l}\sin x_1$
$b(x_2) = \frac{k}{m}x_2$

$a_1 = \pi$
$a_2 \to \infty$

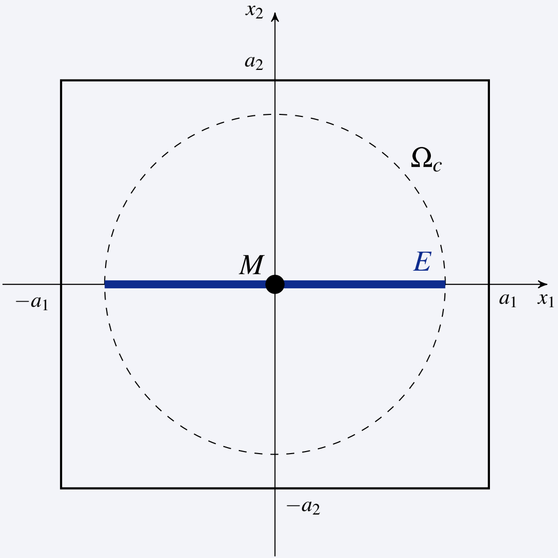
Mass-spring-damper, $x =$ position, $\dot x =$ velocity

$c(x_1)$ = spring force (kx_1)
$b(x_2)$ = damping force (dx_2)

RLC circuit, $x =$ charge, $\dot x =$ current

$c(x_1)$ = Capacitor voltage ($\frac{1}{C}x_1$)
$b(x_2)$ = Resistor voltage (Rx_2)

Example cont.



Region of attraction

Definition (The Region of attraction)

Let $\phi(t, x_0)$ be the solution of $\dot{x} = f(x)$ that starts at initial state x_0 at time $t = 0$. The region of attraction of the origin, denoted by R_A, is defined by

$$R_A = \{x_0 \in \mathbb{D} | \phi(t, x_0) \text{ is defined } \forall\, t \geq 0 \text{ and } \phi(t, x_0) \to 0 \text{ as } t \to \infty\}$$

Is $\mathbb{D}$ an estimate of R_A?

Given a strict Lyapunov function

$$\left.\begin{array}{ll} V & \text{is } C^1 \\ V & \text{pos.def.} \\ \dot{V} & \text{neg.def.} \end{array}\right\} \quad \forall\, x \in \mathbb{D}$$

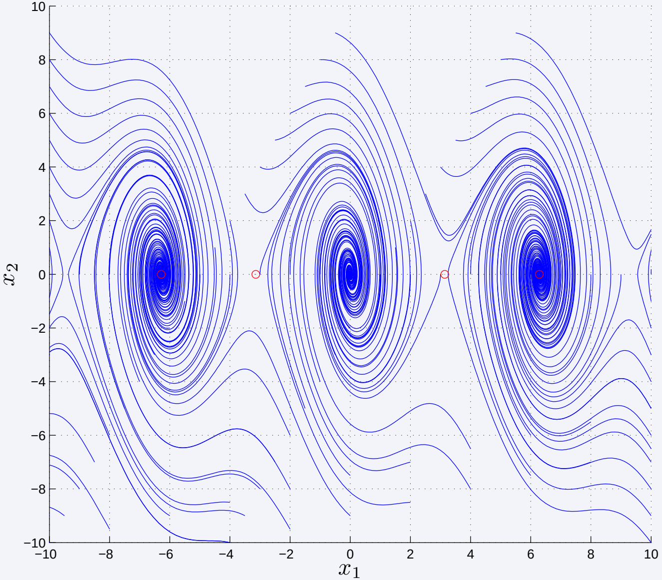
Is $\mathbb{D}$ a region attraction?

Example: Pendulum with friction

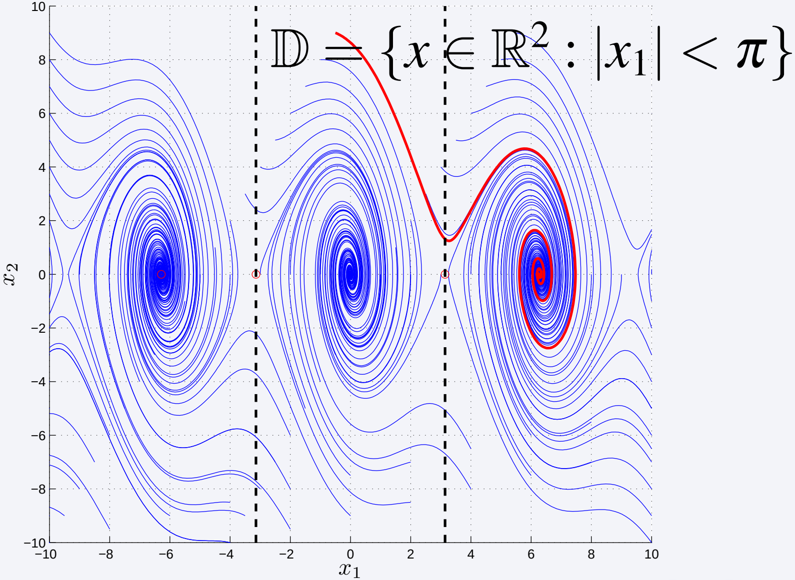
$$V(x) = \frac{g}{l}(1 - \cos x_1) + \frac{1}{2} x^T P x$$

$$\mathbb{D} = \{x \in \mathbb{R}^2 : |x_1| < \pi\}$$

Estimate the region of attraction



Estimate the region of attraction



An estimate of the region of attraction

Starting point:
You have proved asymptotic stability of the origin by either finding a strict Lyapunov function or by using LaSalle's theorem

Estimate R_A using Ω_c

1) Choose the largest set

$$\Omega_c = \{x \in \mathbb{R}^n : \; V(x) \leq c\}$$

   that is contained in $\mathbb{D}$ (where $V > 0$ and $\dot{V} < 0$) or in which $\dot{V} \leq 0$ (LaSalle) and which is bounded

2) Choose the connected component in this set that contains the origin.

Then this is a subset of the region of attraction of the origin, and can hence be used as an estimate.

Example: An estimate of the region of attraction

(Do not always trust your intuition)

Example

$$\dot{z}_1 = -z_1 + z_1^2 z_2$$
$$\dot{z}_2 = -z_2$$

Equilibrium point (0,0)
Lyapunov linearization method: Locally asymptotically stable
Corollary 4.3: Locally exponentially stable

Q: Is it globally asymptotically/exponentially stable?
Intuition may suggest yes...

Example cont.

Example cont.

For this particular system it is possible to find an analytical solution:

$$z_1(t) = \frac{2z_1(t_0)}{z_1(t_0)z_2(t_0)e^{-t} + [2 - z_1(t_0)z_2(t_0)]e^{t}} \tag{1}$$

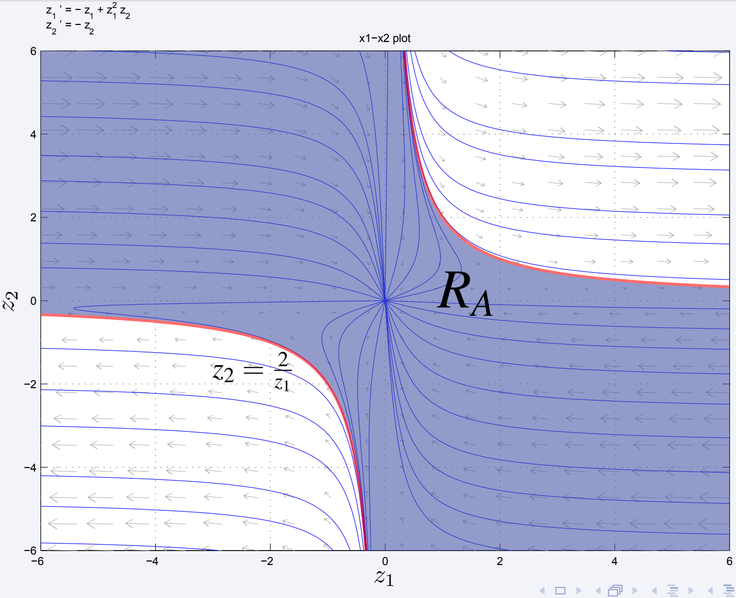
$$z_2(t) = z_2(t_0)e^{-t} \tag{2}$$

If $z_1(t_0)z_2(t_0) > 2$, the denominator in Eq. (1) becomes zero at the time

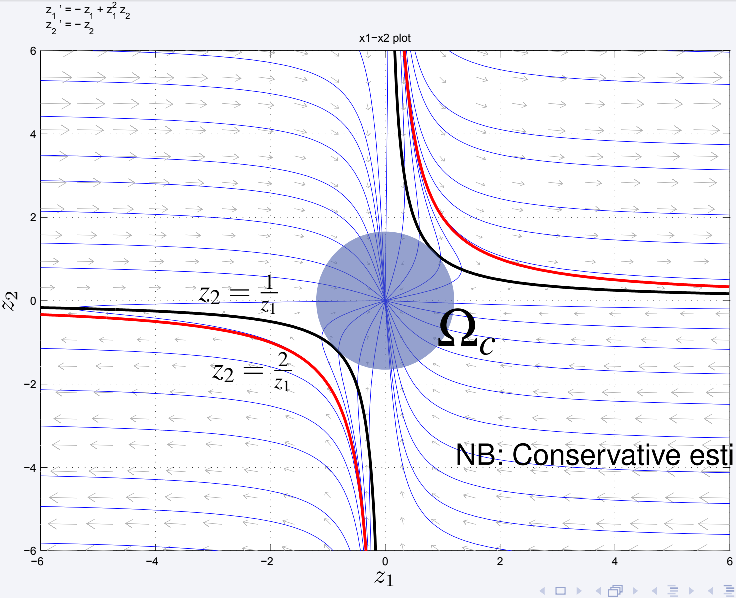
$$t_{esc} = \frac{1}{2}\ln\left(\frac{z_1(t_0)z_2(t_0)}{z_1(t_0)z_2(t_0) - 2}\right)$$

The equilibrium point is clearly not globally asymptotically stable. It is locally exponentially stable and the region of attraction is given by $z_1(t_0)z_2(t_0) < 2$.

Example: Region of attraction



Example: Estimate of region of attraction



NB: Conservative estimate

Convergence to other invariant sets

Example

Consider the system

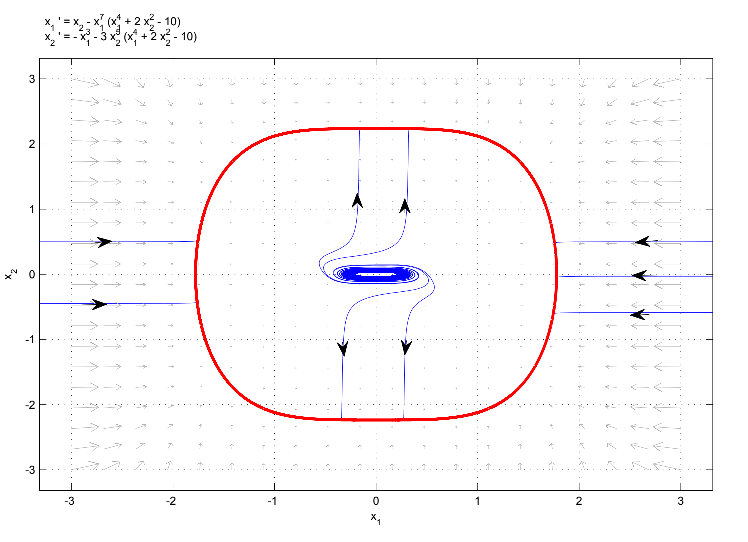
$$\dot{x}_1 = x_2 - x_1^7\left(x_1^4 + 2x_2^2 - 10\right)$$
$$\dot{x}_2 = -x_1^3 - 3x_2^5\left(x_1^4 + 2x_2^2 - 10\right)$$

Investigate the stability properties of the invariant set

$$Q = \{x \in \mathbb{R}^2 | x_1^4 + 2x_2^2 - 10 = 0\}$$

using

$$V(x) = \left(x_1^4 + 2x_2^2 - 10\right)^2$$


x1 ' = x2 - x1^7 (x1^4 + 2 x2^2 - 10)
x2 ' = - x1^3 - 3 x2^5 (x1^4 + 2 x2^2 - 10)
x1
x2
-3
-2
-1
0
1
2
3

Part II

Methods for choosing Lyapunov function candidates

Methods for choosing Lyapunov function candidates

Methods for choosing LFCs

- Total energy
- LFCs with quadratic terms $\frac{1}{2}x^T Px$
  - $V(x) = \frac{1}{2}(x_1^2 + x_2^2 + \cdots + x_n^2)$
  - $V(x) = \frac{1}{2}(x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2)$
  - $V(x) = \frac{1}{2}x^T Px$
- $V(x) = \frac{1}{2}\ln(1 + x_1^2 + \cdots + x_n^2)$
- The variable gradient method
- LFCs for linear time-invariant systems
- Krasovskii's method (Assignment)
- $\vdots$

Variable gradient method

Variable gradient method

$\dot{V} = \frac{dV}{dx} f(x) = g^T(x) f(x)$ Choose $g(x)$ such that

$$\begin{cases} g(x) \text{ is the gradient of a scalar function} \\ V(x) = \int_0^x g^T(y)dy \text{ is positive definite} \\ \dot{V}(x) = g^T(x)f(x) \text{ is negative definite} \end{cases}$$

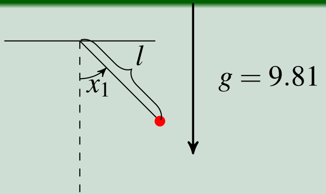
$$\Leftrightarrow \frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i} \quad \forall\, i,j = 1,\ldots,n$$

$$V(x) = \int_0^x \sum_{i=1}^n g_i(y)dy_i = \int_0^{x_1} g_1(y_1,0,0,\ldots,0)dy_1$$
$$+ \int_0^{x_2} g_2(x_1,y_2,0,\ldots,0)dy_2 + \ldots + \int_0^{x_n} g_n(x_1,x_2,\ldots,y_n)dy_n > 0$$

Example

Pendulum with friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$



Find a LFC for this system using the variable gradient method.

$$\frac{\partial g_1(x)}{\partial x_2} = \frac{\partial g_2(x)}{\partial x_1}$$

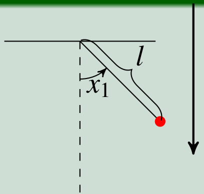
$$V(x) = \int_0^{x_1} g_1(y_1, 0)dy_1 + \int_0^{x_2} g_2(x_1, y_2)dy_2$$

$$\dot{V} = \begin{bmatrix} g_1(x) & g_2(x) \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$

Example

Pendulum with friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$



$g = 9.81$

Choose a structure for $g(x)$ (Trial and error)

$$g_1(x) = a_{11}(x)x_1 + a_{12}(x)x_2$$
$$g_2(x) = a_{21}(x)x_1 + a_{22}(x)x_2$$

i.e. $g(x) = P(x)x$

Linear time-invariant systems

LTI systems

The linear time-invariant system

$$\dot{x} = Ax \qquad (\det A \neq 0)$$

has one equilibrium point $x = 0$

Hurwitz

A is Hurwitz iff

$$\text{Re}(\lambda_i) < 0 \quad \forall\, i = 1, \ldots, n$$

LFC

Which Lyapunov function candidate do we choose?

Lyapunov functions for linear systems

Theorem 4.6

Given the system $\dot{x} = Ax$
Let $V(x) = x^T P x$ and choose $Q = Q^T$ positive definite.
Seek to find a solution $P = P^T$ of Lyapunov's matrix equation

$$A^T P + PA = -Q \tag{3}$$

- If (3) does not have a solution $P = P^T$, or the solution is not unique: $x = 0$ is not asymptotically stable
- If (3) has a unique solution $P = P^T$, but P is not positive definite: $x = 0$ is not asymptotically stable
- If (3) has a unique solution $P = P^T$, and P is positive definite: $x = 0$ is asymptotically stable

Example

Example

Consider the system

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = 3x_1 - x_2$$

Analyze the stability properties of $x = 0$ using Lyapunov's direct method

Part III

How to handle terms with indeterminate sign

Handling terms with indeterminate sign

Terms with indeterminate sign

We have seen examples on how to

- Cancel
  - Adjust a_i in $V(x) = \frac{1}{2}(x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2)$ such that cross-terms $x_i x_j$ in $\dot{V}$ cancel each other
  - Adjust all the parameters in P such that $V(x) = x^T P x > 0$ $(P = P^T > 0)$ and $\dot{V} < 0$
- Dominate
  - Completion of squares
  - Write as $-x^T Q x$
  - Young's inequality
  - Cauchy-Schwarz inequality

Tools for dominating cross-terms

Completion of squares

$$(x \pm y)^2 \geq 0, \quad x, y \in \mathbb{R}$$

$$\Updownarrow$$

$$x^2 \pm 2xy + y^2 \geq 0$$

$$\Updownarrow$$

$$x^2 + y^2 \geq \pm 2xy$$

$$\Rightarrow xy \leq |xy| \leq \tfrac{1}{2}(x^2 + y^2) \qquad \Rightarrow x_1 x_2 \leq \tfrac{1}{2}(x_1^2 + x_2^2) = \tfrac{1}{2}\,\|x\|_2^2$$

Tools for dominating cross-terms cont.

Young's inequality $(x, y \in \mathbb{R})$

$$xy \leq \varepsilon x^2 + \frac{1}{4\varepsilon} y^2, \quad \forall\, \varepsilon > 0$$

Proof:

$$\varepsilon(x - \tfrac{1}{2\varepsilon} y)^2 \geq 0$$

$$\Updownarrow$$

$$\varepsilon(x^2 - \tfrac{1}{\varepsilon} xy + \tfrac{1}{4\varepsilon^2} y^2) \geq 0$$

$$\Updownarrow$$

$$\varepsilon x^2 - xy + \tfrac{1}{4\varepsilon} y^2 \geq 0$$

Tools for dominating cross-terms

Alternatively

Write $\dot{V}$ as $-x^T Qx$, where $Q = Q^T$ is positive definite

NB This is similar to completing the squares

Tools for dominating cross-terms

Completion of squares

$$\dot{V} = -x_1^2 + 6x_1x_2 - 20x_2^2$$

Handling terms with indeterminate sign

Cauchy-Schwarz inequality

$$|a_1 x_1 + a_2 x_2 + \cdots a_n x_n| \leq \sqrt{(a_1^2 + a_2^2 + \cdots a_n^2)}\, \|x\|_2$$

Example (See page 319)

$$x_1 - 2x_2 \leq |x_1 - 2x_2| \leq \sqrt{1^2 + (-2)^2}\, \|x\|_2 = \sqrt{5}\, \|x\|_2$$

Next lecture

Next lecture

- Lyapunov stability analysis for nonautonomous systems
- Recommended reading

Khalil	Sections 4.4-4.5
	Section 8.3