NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF ENGINEERING CYBERNETICS

Exam
TTK4150 Nonlinear Control Systems

Wednesday December 19, 2012

SOLUTION

Problem 1 (15%)

a The angular position and the angular velocity are chosen as state variables.

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$

This gives us the following state equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$

The Jacobian is

$$A = \frac{\partial f}{\partial x}(x) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix}$$

b For the equilibrium points x^* we have that

$$\dot{x}_1^* = 0 = x_2^* \tag{1}$$
$$\dot{x}_2^* = 0 = \frac{g}{l}\sin x_1^* - \frac{k}{m}x_2^* \tag{2}$$

From (1) we have that $x_2^* = 0$. Combining this and (2) we get that $\sin x_1^* = 0$. This gives $x_1^* = n\pi$ with $n = \pm 0, \pm 1, \pm 2, \cdots$.

To find the qualitative behavior of the system around these equilibrium points we first have to linearize it around the equilibrium points. The Jacobian was found in **a**, and inserting x_1^* gives

$$\Delta\dot{x} = A\Delta x$$

where

$$\begin{bmatrix} 0 & 1 \\ -\frac{g}{l}(-1)^n & -\frac{k}{m} \end{bmatrix}$$

and $\Delta x = x - x^*$. The eigenvalues for the linearized system are given by

$$\lambda = -\frac{k}{2m} \pm \sqrt{\left(\frac{k}{2m}\right)^2 - \frac{g}{l}(-1)^n}$$

For the equilibrium points where n is odd, we get that

$$\lambda = -\frac{k}{2m} \pm \sqrt{\left(\frac{k}{2m}\right)^2 + \frac{g}{l}} = -\alpha_1 \pm \alpha_2$$

where $0 < \alpha_1 < \alpha_2$. Hence we have two real eigenvalues, where one is positive and one is negative. This means that we have saddle points in these equilibrium points.

For the equilibrium points where n is even, we get that

$$\lambda = -\frac{k}{2m} \pm \sqrt{\left(\frac{k}{2m}\right)^2 - \frac{g}{l}} = -\alpha \pm j\beta$$

since for the given values of m, l, k and g, we know that $\frac{g}{l} > \left(\frac{k}{2m}\right)^2$. Hence, the linearized system has two complex eigenvalues with negative real parts, and the qualitative behavior of the equilibrium points with even n is stable focus.

c For even n the eigenvalues of the Jabobian has strictly negative real parts, and by Lyapunov's indirect method, these equilibrium points are stable. This corresponds to the pendulum hanging down. For odd n the Jacobian has two real eigenvalues. Since one of them is strictly positive, Lyapunov's indirect methods proves that these equilibrium points are unstable. They correspond to the pendulum balancing in the upright position.

d Here we apply the Bendixon Criterion.

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0 + \left(-\frac{k}{m}\right) = -\frac{k}{m} \tag{3}$$

Since (3) is not identically zero and does not change sign on $\mathbb{R}^2$, there are no periodic orbits in $\mathbb{R}^2$.

Problem 2 (15%)

a For the equilibrium point x^* it is given that

$$\dot{x}_1^* = 0 = -x_2^* \tag{4}$$

$$\dot{x}_2^* = 0 = x_1^* + \left(-\left(x_1^*\right)^2 - 1\right) x_2^* \tag{5}$$

From (4) it is clear that $x_2^* = 0$ and given this plus (5) it is clear that $x_1^* = 0$. Hence the only equilibrium is at the origin $x_1^* = x_2^* = 0$.

b As shown in **a**, the origin is the only equilibrium point of the system, and it is also clear that the system equations are continuously differentiable on $\mathbb{R}^2$.

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \tag{6}$$

The eigenvalues of A are $\lambda = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2}$, and hence they both have negative real parts. By Corollary 4.3 in Khalil, the equilibrium point is exponentially stable.

c $V(x)$ is continuously differentiable, radially unbounded and positive definite. We also have that

$$\begin{aligned}
\dot{V}(x) &= -x_1 x_2 + x_2 \left(x_1 + \left(-x_1^2 - 1\right) x_2 \right) \\
&= -x_1 x_2 + x_1 x_2 + \left(-x_1^2 - 1\right) x_2^2 \\
&= \left(-x_1^2 - 1\right) x_2^2 = -x_1^2 x_2^2 - x_2^2 \\
&\leq 0 \quad \forall\, x \in \mathbb{R}^2
\end{aligned}$$

Hence $\dot{V}(x)$ is negative semidefinite. We continue by applying Corollary 4.3. First we find $S = \left\{ x \in \mathbb{R}^2 | \dot{V}(x) = 0 \right\} = \left\{ x \in \mathbb{R}^2 | x_2 = 0 \right\}$. For $x_2 = 0$ we have that $\dot{x}_2 = 0$ $\to x_1 = 0$. This means that no solution can stay identically in S other than the trivial solution $x_1 = x_2 = 0$. In addition $V(x)$ is radially unbounded, which means that we can conclude that the origin of the nonlinear system is globally asymptotically stable.

d Again we turn to Lyapunov's indirect method.

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \tag{7}$$

The eigenvalues of A are now $\lambda = \frac{1}{2} \pm \frac{\sqrt{-3}}{2}$, which both have positive real parts. By Theorem 4.7 in Khalil, the equilibrium point is unstable.

OBS!!! It is not possible to show instability of an equilibrium by showing that a Lyapunov function *candidate* has a positive definite time derivative. (By definition a Lyapunov function has a negative semidefinite or negative definite time derivative.)

Showing that the candidate you have chosen is not cut out to be a Lyapunov function for the given system (pos.def derivative) does not mean that there does not exists another Lyapunov function candidate with negative semidefinite or negative definite time derivative. The conditions of Lyapunov's direct stability theorem(s) are only *sufficient*!

By Lyapunov theory there are two ways to prove instability covered by the curriculum of this course. The first is through Lyapunov's indirect method (Theorem 4.7, page 139 in Khalil), which was used here, or through Chetaev's theorem (Theorem 4.3, page 125 in Khalil).

Problem 3 (10%)

a

$$\frac{df}{dx} = -1 + a\cos x \tag{8}$$

Both $f(x)$ and $\frac{df}{dx}$ are continuous on $\mathbb{R}$. In addition $\frac{df}{dx}$ is uniformly bounded on $\mathbb{R}$ since $|\frac{df}{dx}| \leq 1 + a$. This means that by Lemma 3.3 in Khalil, $f(x)$ is globally Lipschitz.

b

$$\frac{df}{dx} = \frac{1}{\cos^2 x} \tag{9}$$

Both $f(x)$ and $\frac{df(x)}{dx}$ are continuous on any set $D = \left\{x \in \mathbb{R} \middle| x \neq \frac{n\pi}{2}\right\}$ where $n = \pm 1, \pm 3, \cdots$. Hence it is locally Lipschitz on any such set D. The function is not globally Lipschitz since neither $\tan(x)$ or $\frac{df(x)}{dx}$ is not continuous on the entire state space $\mathbb{R}$.

c A function is locally Lipschitz if the Lipschitz condition holds for a domain D, but not necessarily with the same constant L for all points in D. It is Lipschitz in the domain D if the Lipschitz condition holds in D with the same constant L, and the function is globally Lipschitz if it is Lipschitz on $\mathbb{R}^n$ (same, constant L for $\mathbb{R}^n$).

Problem 4 (26%)

a A storage function must be continuously differentiable and positive semidefinite. Since $\psi(\theta)\theta > 0 \,\forall\, \theta \neq 0$ and $\psi(0) = 0$, the graph of $\psi(\theta)$ lies strictly in the first and third quadrants. Therefore, the integral $\int_0^{x_1} \psi(\theta)d\theta$ is a positive definite function of x_1. The function $\frac{1}{2}x_2^2$ is also a positive definite function of x_2, and they are both continuously differentiable. The sum

$$V(x) = \int_0^{x_1} \psi(\theta)d\theta + \frac{1}{2}x_2^2 \tag{10}$$

is therefore both continuously differentiable and positive definite, hence it is a storage function.

b We find the time derivative of the storage function.

$$\begin{aligned}
\dot{V}(x) &= (2x_1 + \sin x_1)\,\dot{x}_1 + x_2\dot{x}_2 \\
&= (2x_1 + \sin x_1)\,x_2 + x_2\Big(-\left(1 + x_1^4\right)x_2^3 - (2x_1 + \sin x_1) + u\Big) \\
&= (2x_1 + \sin x_1)\,x_2 - x_2^4\left(1 + x_1^4\right) - x_2\,(2x_1 + \sin x_1) + x_2 u \\
&= -x_2^4\left(1 + x_1^4\right) + x_2 u \\
&\leq -x_2^4 + x_2 u = -y^4 + yu
\end{aligned}$$

Hence, by Definition 6.3 in Khalil, the system is output strictly passive.

c Defining $S = \left\{x \in \mathbb{R}^2 \middle| y = 0\right\} = \left\{x \in \mathbb{R}^2 \middle| x_2 = 0\right\}$. For the unforced system ($u = 0$), we have that $x_2 = 0$ gives $2x_1 + \sin x_1 = 0$. Since the function $\psi(x_1) = 2x_1 + \sin x_1$ lies strictly in the first and third quadrants, this results in $x_1 = 0$. Hence, the only solution of the unforced system that can stay identically in S is the trivial solution $x = 0$, which implies that the system is zero-state observable.

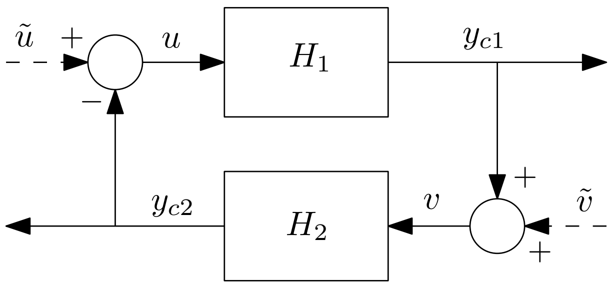

Figure 1: Negative feedback interconnection.

d By now we know that the open loop system is

- Output strictly passive, and
- zero state observable.

In addition to this the storage function $V(x)$ is radially unbounded. From Theorem 14.4 in Khalil, we then know that the origin can be globally stabilized by a $u = -\phi(y)$, where ϕ is any locally Lipschitz function such that $\phi(0) = 0$ and $y\phi(y) > 0$ for all $y \neq 0$.

$u = -2x_2 - \sin x_2 = -\psi(y)$, where ψ fulfills the requirements of Theorem 14.4. Hence the origin of the closed loop system is globally asymptotically stable.

e

$$\begin{aligned}
\dot{V}(z) &= (2z_1 + \sin z_1)\,\dot{z}_1 + z_2\dot{z}_2 \\
&= (2z_1 + \sin z_1)\,z_2 + z_2\left(\left(-z_1^2 + v\right)z_2^3 - (2z_1 + \sin z_1) + v\right) \\
&= -z_1^2 z_2^4 + z_2^4 v + z_2 v \\
&= -z_1^2 z_2^4 + \left(z_2 + z_2^4\right)v \leq \left(z_2 + z_2^4\right)v
\end{aligned}$$

Hence, if we choose $\bar{y} = h(z) = z_2 + z_2^4$, we obtain $\dot{V} \leq \bar{y}v$, which means that the system is passive with input v and output $\bar{y} = z_2 + z_2^4$.

f We will use the fact that a negative feedback interconnection of two passive systems (as in Figure 1) is a passive system with input $u_c = (\tilde{u}, \tilde{v})$ and output $y_c = (y, \bar{y})$ (Theorem 6.1 in Khalil).

In our case, system (1) is passive with input u and output $y = x_2$, and system (2) is passive with input v and output $\bar{y} = z_2 + z_2^4$. Therefore, if we choose the input as

$$\begin{aligned}
u &= -\bar{y} + \tilde{u} = -z_2 - z_2^4 + \tilde{u} \\
v &= y + \tilde{v} = x_2 + \tilde{v}
\end{aligned}$$

and output

$$y_c = \begin{bmatrix} y \\ \bar{y} \end{bmatrix} = \begin{bmatrix} x_2 \\ z_2 + z_2^4 \end{bmatrix}$$

then the closed-loop system will be passive. This corresponds to

$$\begin{aligned}
\alpha(z) &= -z_2 - z_2^4 \\
\beta(x) &= x_2
\end{aligned}$$

Problem 5 (26%)

a We differentiate the output $y = x_1$ to find the relative degree:

$$\dot{y} = \dot{x}_1 = x_3 - x_2$$
$$\ddot{y} = \dot{x}_3 - \dot{x}_2 = x_1^2 - x_3 + u + x_2 + u = x_1^2 + x_2 - x_3 + 2u$$

The relative degree of the system is $\rho = 2$. It exists and is well defined, hence the system is input-output linearizable.

b The system can be written as

$$\dot{x} = f(x) + g(x)u$$

where

$$f(x) = \begin{bmatrix} x_3 - x_2 \\ -x_2 \\ x_1^2 - x_3 \end{bmatrix} \qquad g(x) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

- First, the external dynamics ξ are found. Since $\rho = 2$, ξ is of dimension 2.

$$\xi_1 = y = x_1$$
$$\xi_2 = L_f y = \frac{\partial y}{\partial x} f(x) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} f(x) = x_3 - x_2$$

- Then the internal dynamics η are found. Since ξ is of dimension 2, η is of dimension $3 - 2 = 1$. With $\eta = \phi(x)$ we have that

$$\frac{\partial \phi(x)}{\partial x} g(x) = 0$$

Inserting for $g(x)$ we have that

$$-\frac{\partial \phi(x)}{\partial x_2} + \frac{\partial \phi(x)}{\partial x_3} = 0$$

A $\phi(x)$ which satisfies this is

$$\phi(x) = x_2 + x_3$$

- This gives the following diffeomorphism candidate

$$z = T(x) = \begin{bmatrix} \eta \\ \cdots \\ \xi \end{bmatrix} = \begin{bmatrix} x_2 + x_3 \\ \cdots\cdots \\ x_1 \\ x_3 - x_2 \end{bmatrix}$$

Clearly $T(x)$ is continuously differentiable, but we also need to check that its inverse exists and is continuously differentiable.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \frac{1}{2}(\eta - \xi_2) \\ \frac{1}{2}(\eta + \xi_2) \end{bmatrix} = T^{-1}(z)$$

$T^{-1}(z)$ exists, and it is also continuously differentiable, hence $T(x)$ is a diffeomorphism.

- Next, $\gamma(x)$ and $\alpha(x)$ are found

$$
\begin{aligned}
\gamma(x) &= L_g L_f y = 2 \\
\alpha(x) &= -\frac{L_f^2 y}{\gamma(x)} = -\frac{1}{2}\left(x_1^2 + x_2 - x_3\right)
\end{aligned}
$$

- And finally, the system is written in normal form

$$
\begin{aligned}
\dot{\eta} &= \xi_1^2 - \eta = f_0(\eta, \xi) \\
\dot{\xi} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 2 \left[u + \frac{1}{2}\left(x_1^2 + x_2 - x_3\right) \right] \\
&= A_c \xi + B_c \gamma(x) \left[u - \alpha(x) \right]
\end{aligned}
$$

Since $\alpha(x) = -\frac{L_f^2 y}{\gamma(x)}$, the transformation is valid when $\gamma(x) \neq 0$. Since $\gamma(x) = 2$, this transformation is valid for the entire $\mathbb{R}^3$ space.

c An input-output linearizing controller is given by

$$
u = \alpha(x) + \beta(x) v
$$

where $\beta(x) = \frac{1}{\gamma(x)} = \frac{1}{2}$ and $\alpha(x) = -\frac{1}{2}\left(x_1^2 + x_2 - x_3\right)$. This gives

$$
u = -\frac{1}{2}\left(x_1^2 + x_2 - x_3\right) + 0.5v \tag{11}
$$

d The external dynamics ξ with the input-output linearizing controller inserted is given by:

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= v
\end{aligned}
$$

$\xi = [\xi_1 \quad \xi_2]^T$ can be stabilized by $v = -k_1\xi_1 - k_2\xi_2$, which gives the closed loop dynamics

$$
\dot{\xi} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \xi = A\xi
$$

The eigenvalues of the closed loop system are

$$
\lambda = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1}}{2}
$$

which have negative real parts as long as $k_1, k_2 > 0$. This means that A is Hurwitz and $\dot{\xi} = A\xi$ is asymptotically stable at the origin with $v = -k_1\xi_1 - k_2\xi_2$, $k_1, k_2 > 0$.

e The system is minimum phase if $T(0) = 0$ (the origin is kept as an equilibrium point for $[\eta \ \xi]^T$) and $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable at the origin. The zero dynamics is given by

$$
\dot{\eta} = f_0(\eta, 0) = -\eta
$$

Differentiating the Lyapunov function

$$V(\eta) = \frac{1}{2}\eta^2$$

along the trajectories of the zero dynamics gives

$$\dot{V}(\eta) = -\eta^2$$

Since $V(\eta)$ is continuously differentiable and positive definite, and $\dot{V}(\eta)$ is negative definite, $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable at the origin. The system is therefore minimum phase.

f Since the system is minimum phase, and the closed-loop external dynamics is asymptotically stable at the origin, the closed-loop system $[\eta \quad \xi]^T$ is asymptotically stable at the origin.

<u>Problem 6 (8%)</u>

Using the Lyapunov function $V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$, we have that

$$\alpha_1\left(\|x\|\right) = \frac{1}{2}\|x\|_2^2 \leq V(x) \leq \frac{1}{2}\|x\|_2^2 = \alpha_2\left(\|x\|\right) \tag{12}$$

where $\alpha\left(\|x\|\right) = \alpha_1\left(\|x\|\right) = \alpha_2\left(\|x\|\right)$ is a $\mathcal{K}_\infty$-function.

$$\begin{aligned}
\dot{V}(x) &= x_1\left(-x_1 + x_2\right) + x_2\left(-x_1 - x_2 + u\right) \\
&= -\|x\|_2^2 + x_2 u \leq -\|x\|_2^2 + |x_2||u| \\
&\leq -\|x\|_2^2 + \|x\|_2 |u| \\
&= -\|x\|_2^2 + \|x\|_2 |u| + \theta\|x\|_2^2 - \theta\|x\|_2^2 \\
&= -(1-\theta)\|x\|_2^2 + |u|\|x\|_2 - \theta\|x\|_2^2 \\
&= -(1-\theta)\|x\|_2^2 - \left(\theta\|x\|_2 - |u|\right)\|x\|_2 \\
&\leq -(1-\theta)\|x\|_2^2 \ \forall \ \|x\|_2 \geq \frac{|u|}{\theta}
\end{aligned}$$

where $\theta \in (0, 1)$. Hence, by Theorem 4.19 in Khalil, the system is input-to-state stable.