TTK4150 Nonlinear Control Systems
Lecture 3

Stability

and

Stability analysis of equilibrium points

NTNU
Innovation and Creativity

Previous lecture

Previous lecture:

- Fundamental properties
  - Existence and uniqueness of solutions
- Phase plane analysis
  - How to construct phase portraits and interpret these
    - Analytical method
    - Vector field diagrams
    - Computer simulations
  - Local phase plane analysis (nodes, foci, saddle and center points)
  - Periodic orbits and Limit cycles (general, not only in the plane)
  - Existence criteria for periodic orbits in the plane

Outline I

Outline II

- Introduction
- Lyapunov's indirect method

5 Summary

6 Next lecture

Today's goals

After this lecture you should...

- Understand how the need for stabilization of equilibrium points arise in control problems
- **Lyapunov stability properties**
  Know and understand the following stability definitions for autonomous systems
  - Stability
  - Asymptotic stability
  - Exponential stability
  - Global versus local
- **Lyapunov stability analysis**
  - Lyapunov's indirect method

Literature

Today's lecture is based on

Khalil	Section 4.1
	Theorem 4.7, Section 4.3
	Corollary 4.3, Section 4.7

Part I

The control problem

The control problem

The control problem:

- Given a physical process with
  - inputs (actuators)
  - outputs (sensors)

- Given a set of specifications of the desired system behavior

- Model the physical plant by a set of differential equations

- **Design** a control law $\rightleftarrows$ - **Analysis** of the closed-loop system

- Implement the control law

The Regulation problem: x_{ref} = constant

Process

$$\dot{x} = f_p(t, x, u)$$

Control law design

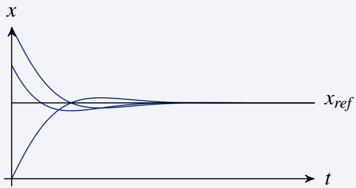
Find

$$u = \gamma(t, x)$$

such that the closed-loop (CL) system

$$\dot{x} = f_p(t, x, \gamma(t, x)) =: f(t, x)$$

has desired behavior.



Desired CL system behavior?

- x_{ref} an equilibrium point
- convergence
- start close ⇒ stay close

Asymptotic stabilization problem

Find $\gamma(t, x)$ such that x_{ref} is an asymptotically stable equilibrium point of $\dot{x} = f(t, x)$.

The Regulation problem: x_{ref} = constant

Process

$$\dot{x} = f_p(t,x,u)$$

Control law design

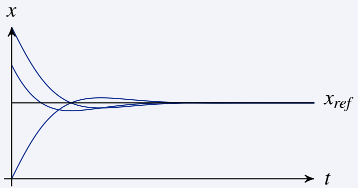
Find

$$u = \gamma(t,x)$$

such that the closed-loop (CL) system

$$\dot{x} = f_p(t,x,\gamma(t,x)) =: f(t,x)$$

has desired behavior.



Desired CL system behavior?

- x_{ref} an equilibrium point
- convergence
- start close $\Rightarrow$ stay close

Asymptotic stabilization problem

Find $\gamma(t,e)$, $e = x - x_{ref}$, s.t. $e = 0$ is an asymptotically stable equilibrium point of $\dot{e} = f(t, e + x_{ref})$.

The Tracking/Servo problem: $x_{ref}(t)$

Process

$$\dot{x} = f_p(t,x,u)$$

Control law design

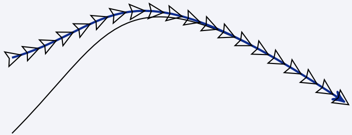
Find

$$u = \gamma(t,x)$$

such that the closed-loop (CL) system

$$\dot{x} = f_p(t,x,\gamma(t,x)) =: f(t,x)$$

has desired behavior.



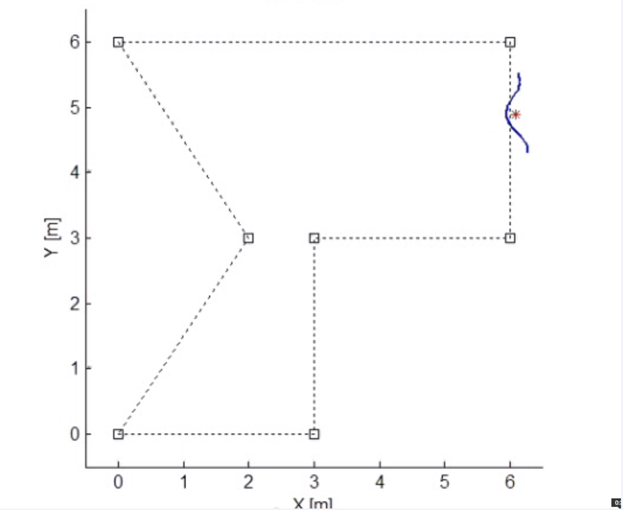
Desired CL system behavior?

- on trajectory $\Rightarrow$ stay on trajectory
- convergence to trajectory
- start close $\Rightarrow$ stay close

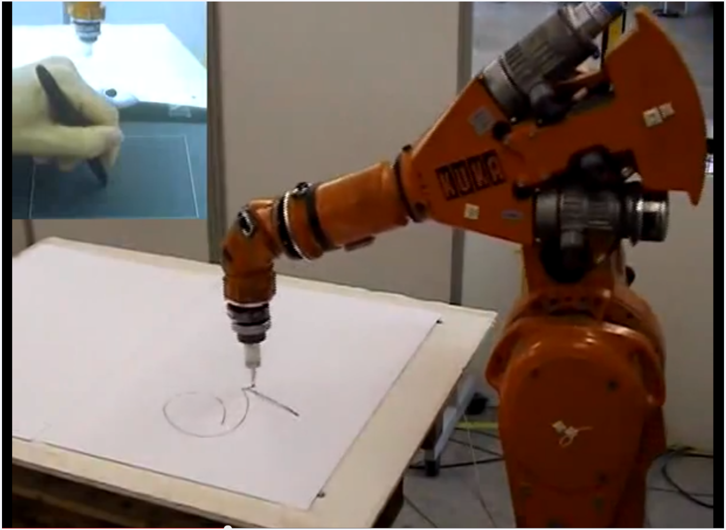
Asymptotic stabilization problem

Find $\gamma(t,e)$ such that $e = x - x_{ref}(t) = 0$ is an asymptotically stable equilibrium point of $\dot{e} = \bar{f}_p(t,e,\gamma(t,e)) = \bar{f}(t,e)$.

Simulated straight line path following control of a snake robot

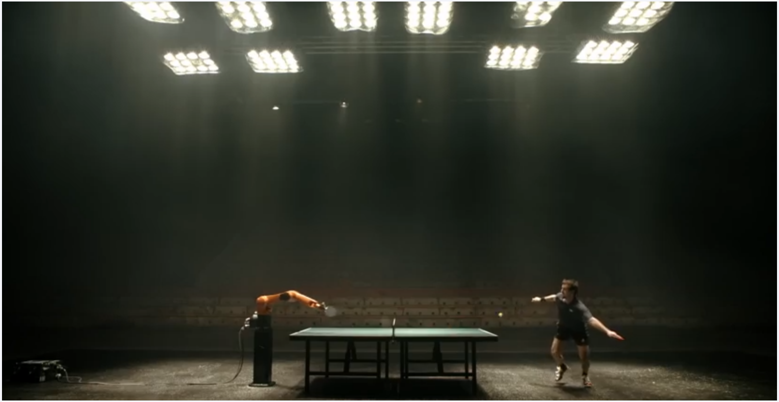


Optimal trajectory tracking by robot



KUKA KR210 Robot Arm Plays Ball

The Duel: Timo Boll vs. KUKA Robot



Part II

Lyapunov stability

Lyapunov stability properties

Autonomous systems

$$\dot{x} = f(x), \qquad f: \quad D \subset \mathbb{R}^n \to \mathbb{R}^n$$

f locally Lipschitz

$x = 0$ is the equilibrium point of interest

We will define the following stability properties:

- Stability (Local property)
- Asymptotic stability = Stability + Local convergence
- Exponential stability
    - Region of attraction
- Global asymptotic stability
- Global exponential stability

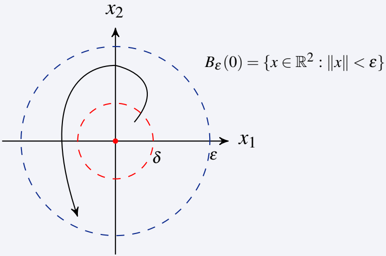
Stability

Definition (Stability)

$x = 0$ is stable iff

$$\forall\ \varepsilon > 0 \quad \exists\ \delta(\varepsilon) > 0 \quad s.t. \quad \|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall t \geq 0$$

Visual interpretation ($n = 2$):

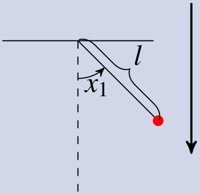


Note: $\forall\ \varepsilon$
Else: Unstable

Stability: Example

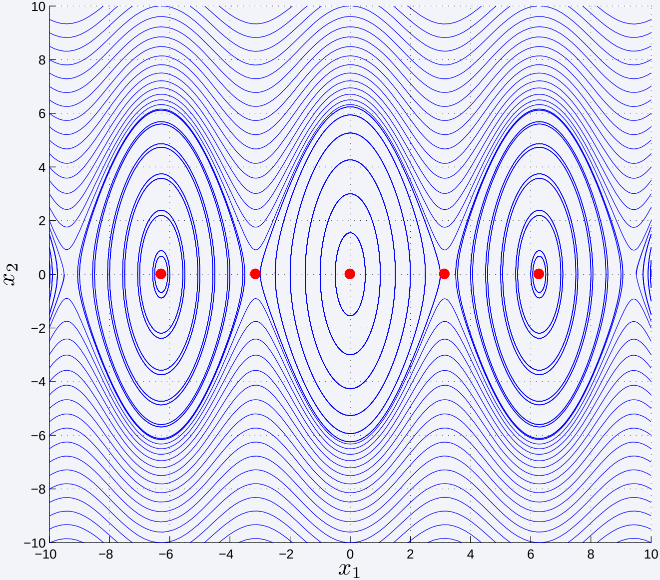
Pendulum without friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1$$

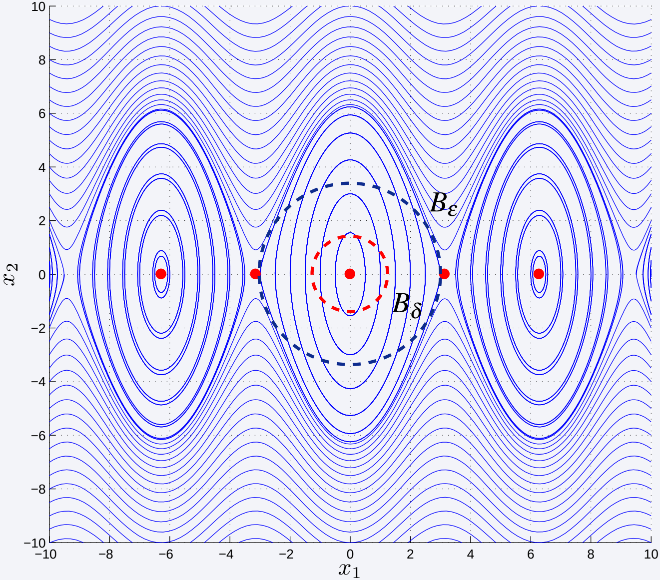


$g = 9.81$

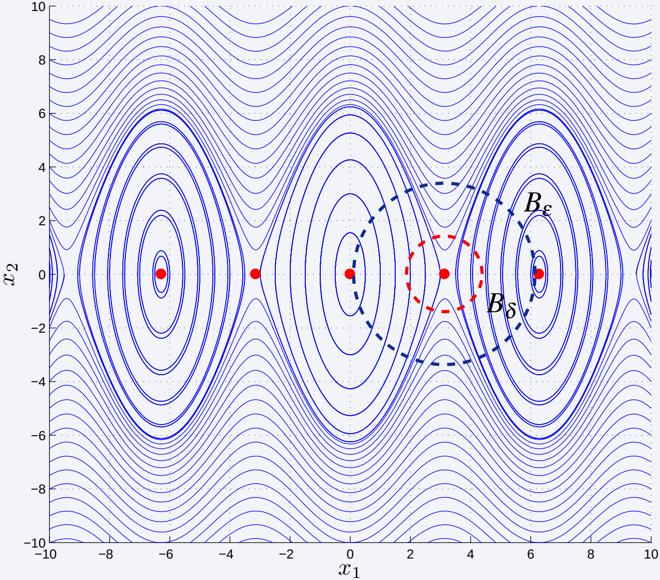
Pendulum without friction, cont.



Pendulum without friction, cont.



Pendulum without friction, cont.



Asymptotic stability

Asymptotic stability

The equilibrium point $x = 0$ is (locally) asymptotically stable iff

i) it is stable

ii) $\exists r > 0 \quad \text{s.t.} \quad \|x(0)\| < r \Rightarrow \lim_{t\to\infty} x(t) = 0$ (Convergence)

Region of attraction

$$B_r = \{r \in \mathbb{R}^n : \|x\| < r\}$$

Global asymptotic stability

The equilibrium point $x = 0$ is globally asymptotically stable iff

i) it is stable

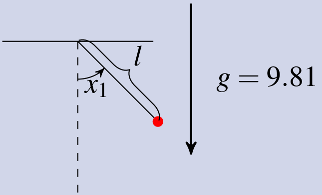
ii) $\forall x(0) \qquad \lim_{t\to\infty} x(t) = 0$

NB: This implies that $x = 0$ is the only equilibrium point.

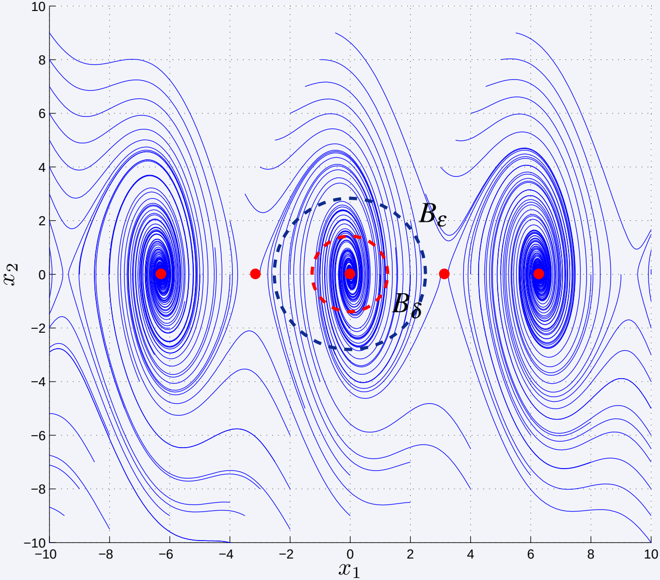
As. stability: Pendulum with friction

Pendulum with friction

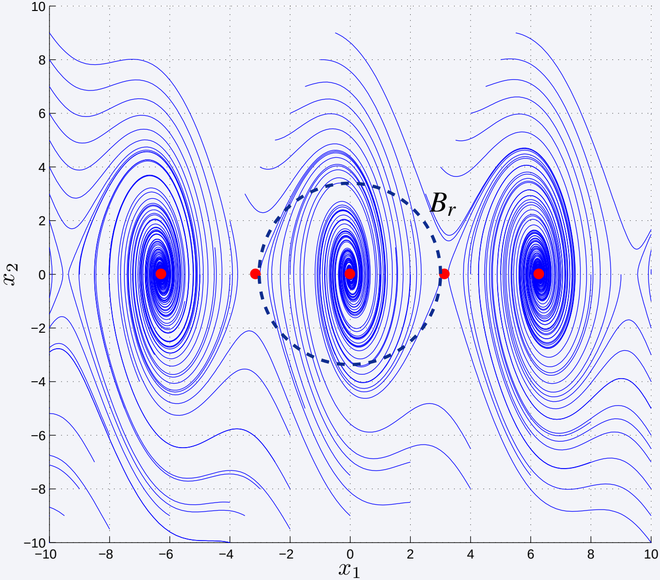
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$



Pendulum with friction, cont.



Pendulum with friction, cont.



Convergence

Convergence

$$\|x(0)\| < r \Rightarrow \lim_{t\to\infty} x(t) = 0$$

NB!

Convergence $\not\Rightarrow$ Stability

Vinograd's counter example:

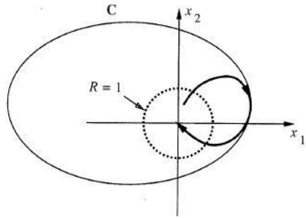


Figure 3.5 : State convergence does not imply stability

Exponential stability

Exponential stability

The equilibrium point $x = 0$ is (locally) exponentially stable iff $\exists r, k, \lambda > 0$ such that

$$\|x(t_0)\| < r \Rightarrow \|x(t)\| \leq k \|x(t_0)\| e^{-\lambda(t-t_0)} \quad \forall t \geq t_0$$

(Local exponential convergence)

Remark

local exponential stability $\Rightarrow$ local asymptotic stability

Global exponential stability

The equilibrium point $x = 0$ is globally exponentially stable iff

$$\forall x(t_0) \qquad \|x(t)\| \leq k \|x(t_0)\| e^{-\lambda(t-t_0)} \quad \forall t \geq t_0$$

(Global exponential convergence)

Part III

Lyapunov stability analysis

Stability analysis

How do we analyze the Lyapunov stability properties?

- Definitions
  - If we have solution $x(t) = \dots$ OK
- Phase plane analysis (dim $x = 2$)
  - Phase portrait
  - Local phase plane analysis

    Phase portrait of linearized system $\sim$ local phase portrait of nonlinear system
- New method: Lyapunov's indirect method

Lyapunov's indirect method/Linearization method

Theorem 4.7 (Lyapunov's indirect method)

Let $x = 0$ be an equilibrium point for

$$\dot{x} = f(x) \quad f : \mathbb{D} \to \mathbb{R}^n \quad \text{is} \quad C^1$$

1) Linearize the system about $x = 0$, $\dot{x} = Ax$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{x=0}$$

2) Find the eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$

Lyapunov's indirect method cont.

Theorem 4.7 (Lyapunov's indirect method) cont.

3)
- a) $\forall i \quad \mathrm{Re}(\lambda_i) < 0 \quad \Rightarrow \quad x = 0$ is locally asymptotically stable
- b) $\exists i \quad \mathrm{Re}(\lambda_i) > 0 \quad \Rightarrow \quad x = 0$ is unstable
- c) $\begin{array}{ll} \forall i & \mathrm{Re}(\lambda_i) \leq 0 \\ \exists i & \mathrm{Re}(\lambda_i) = 0 \end{array} \quad \Rightarrow$ No conclusion

Comments

- $+$ Simple to use
- $\div$ Not always conclusive
- $\div$ Only <u>local</u> results

Lyapunov's indirect method: Example

Example

Given

$$\dot{x} = ax - x^3.$$

Analyze the stability properties of the equilibrium point $x = 0$ using Lyapunov's indirect method.

Corollary 4.3

Corollary 4.3, Sec. 4.7

Let $x = 0$ be an equilibrium point for

$$\dot{x} = f(x) \quad f : \mathbb{D} \to \mathbb{R}^n \text{ is } C^1$$

$$\forall i \quad \mathrm{Re}(\lambda_i) < 0 \quad \Leftrightarrow \quad x = 0 \text{ is (locally) exponentially stable}$$

Summary

Summary:

- The need for stabilization of equilibrium points arises in regulation and servo/trajectory tracking control problems
- **Lyapunov stability properties**
  For autonomous systems
  - Stability
  - Asymptotic stability
  - Exponential stability
  - Global versus local
- **Lyapunov stability analysis**
  - Lyapunov's indirect method

Next lecture

- How to use Lyapunov's direct method to analyze the stability properties of an equilibrium point.
- Lyapunov's theorems for
  - stability
  - local and global asymptotic stability
  - local and global exponential stability
- Preparation

Khalil	Section 4.1
	Theorem 4.10, Section 4.5