2.3 Linear systems

Parseval's theorem is very useful in the study of passive linear systems, as shown next. It is now recalled for the sake of completeness.

Theorem 2.2 (Parseval's theorem) Provided that the integrals exist, the following relation holds

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega)y^*(j\omega)d\omega \tag{2.15}$$

where y^* denotes the complex conjugate of y and $x(j\omega)$ is the Fourier transform of $x(t)$. $x(t)$ is a complex function of t, Lebesgue integrable. ♠♠

Proof The result is established as follows: the Fourier transform of the time function $x(t)$ is

$$x(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{2.16}$$

while the inverse Fourier transform is

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega)e^{j\omega t}d\omega \tag{2.17}$$

Insertion of (2.17) in (2.15) gives

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega)e^{j\omega t}d\omega\right]y^*(t)dt \tag{2.18}$$

By changing the order of integration this becomes

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega)\left[\int_{-\infty}^{\infty} y^*(t)e^{j\omega t}dt\right]d\omega \tag{2.19}$$

Here

$$\int_{-\infty}^{\infty} y^*(t)e^{j\omega t}dt = \left[\int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt\right]^* = y^*(j\omega) \tag{2.20}$$

and the result follows. ♠♠

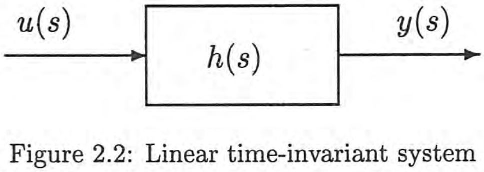

Figure 2.2: Linear time-invariant system

We will now present an important property of linear time-invariant passive systems. **Re**[·] denotes the real part and **Im**[·] denotes the imaginary part

Theorem 2.3 Given a linear time-invariant linear system

$$y(s) = h(s)u(s) \tag{2.21}$$

with a rational transfer function $h(s)$. Assume that all the poles of $h(s)$ have real parts less than zero. Then the following assertions hold:

1. The system is passive $\Leftrightarrow$ $\mathbf{Re}[h(j\omega)] \geq 0$ for all ω.
2. The system is input strictly passive $\Leftrightarrow$ There is a $\delta > 0$ such that $\mathbf{Re}[h(j\omega)] \geq \delta > 0$ for all ω.
3. The system is output strictly passive $\Leftrightarrow$ There is an $\epsilon > 0$ such that

$$\begin{array}{rcl} \mathbf{Re}[h(j\omega)] & \geq & \epsilon\,|h(j\omega)|^2 \\ & \Updownarrow & \\ \left(\mathbf{Re}[h(j\omega)] - \frac{1}{2\epsilon}\right)^2 + (\mathbf{Im}[h(j\omega)])^2 & \leq & \left(\frac{1}{2\epsilon}\right)^2 \end{array}$$

♠♠

Remark 2.3 The notation *for all* ω means *for all* $\omega \in I\!R \cup \{-\infty, \infty\}$, which is the extended real line.

Proof

The proof is based on the use of Parseval's theorem. In this theorem the time integration is over $t \in [0, \infty)$. In the definition of passivity there is an integration over $t \in [0, T]$. To be able to use Parseval's theorem in this proof we introduce the truncated function

$$u_T(t) = \begin{cases} u(t) & \text{when} \quad t \leq T \\ 0 & \text{when} \quad t > T \end{cases} \tag{2.22}$$

which is equal to $u(t)$ for all t less than or equal to T, and zero for all t greater than T. The Fourier transform of $u_T(t)$, which is denoted $u_T(j\omega)$, will be used in Parseval's theorem.

Without loss of generality we will assume that $y(t)$ and $u(t)$ are equal to zero for all $t \leq 0$. Then according to Parseval's theorem

$$\int_0^T y(t)u(t)dt = \int_{-\infty}^{\infty} y(t)u_T(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} y(j\omega)u_T^*(j\omega)d\omega \tag{2.23}$$

Insertion of $y(j\omega) = h(j\omega)u_T(j\omega)$ gives

$$\int_0^T y(t)u(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(j\omega)u_T(j\omega)u_T^*(j\omega)d\omega \tag{2.24}$$

where

$$h(j\omega)u_T(j\omega)u_T^*(j\omega) = \{\mathbf{Re}[h(j\omega)] + j\mathbf{Im}[h(j\omega)]\}|u_T(j\omega)|^2 \tag{2.25}$$

The left side of (2.24) is real, and it follows that the imaginary part on the right hand side is zero. This implies that

$$\int_0^T u(t)y(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{Re}[h(j\omega)]|u_T(j\omega)|^2 d\omega \tag{2.26}$$

First, assume that $\mathbf{Re}[h(j\omega)] \geq \delta \geq 0$ for all ω. Then

$$\int_0^T u(t)y(t)dt \geq \frac{\delta}{2\pi}\int_{-\infty}^{\infty} |u_T(j\omega)|^2 d\omega = \delta \int_0^T u^2(t)dt \tag{2.27}$$

the equality is implied by Parseval's theorem. It follows that the system is passive, and in addition input strictly passive if $\delta > 0$.
Then, assume that the system is passive. Thus there exists a $\delta \geq 0$ so that

$$\int_0^T y(t)u(t)dt \geq \delta \int_0^T u^2(t)dt = \frac{\delta}{2\pi}\int_{-\infty}^{\infty} |u_T(j\omega)|^2 d\omega \tag{2.28}$$

for all u, where the initial conditions have been selected so that $\beta = 0$. Here $\delta = 0$ for a passive system, while $\delta > 0$ for a strictly passive system. Then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{Re}[h(j\omega)]|u_T(j\omega)|^2 d\omega \geq \frac{\delta}{2\pi}\int_{-\infty}^{\infty} |u_T(j\omega)|^2 d\omega \tag{2.29}$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} (\mathbf{Re}[h(j\omega)] - \delta)|u_T(j\omega)|^2 d\omega \geq 0 \tag{2.30}$$

If there exists a ω_0 so that $\mathbf{Re}[h(j\omega_0)] < \delta$, then inequality will not hold for all u because the integral on the left hand side can be made arbitrarily small if the control signal is selected to be $u(t) = U\cos\omega_0 t$. The results 1 and 2 follow.
To show result 3 we first assume that the system is output strictly passive, that is, there is an $\epsilon > 0$ such that

$$\int_0^T y(t)u(t)dt \geq \epsilon \int_0^T y^2(t)dt = \frac{\epsilon}{2\pi}\int_{-\infty}^{\infty} |h(j\omega)|^2|u_T(j\omega)|^2 d\omega \tag{2.31}$$

This gives the inequality (see (2.26))

$$\mathbf{Re}[h(j\omega)] \geq \epsilon\,|h(j\omega)|^2 \tag{2.32}$$

which is equivalent to

$$\epsilon\left[(\mathbf{Re}[h(j\omega)])^2 + (\mathbf{Im}[h(j\omega)])^2\right] - \mathbf{Re}[h(j\omega)] \leq 0 \tag{2.33}$$

and the second inequality follows by straighforward algebra. The converse result is shown similarly as the result for input strict passivity. ♠♠
Note that according to the theorem a passive system will have a transfer function which satisfies

$$|\angle h(j\omega)| \leq 90^\circ \quad \text{for all} \quad \omega \tag{2.34}$$

In a Nyquist diagram the theorem states that $h(j\omega)$ is in the closed half plane $\mathbf{Re}\,[s] \geq 0$ for passive systems, $h(j\omega)$ is in $\mathbf{Re}\,[s] \geq \delta > 0$ for input strictly passive systems, and for output strictly passive systems $h(j\omega)$ is inside the circle with center in $s = 1/(2\epsilon)$ and radius $1/(2\epsilon)$. This is a circle that crosses the real axis in $s = 0$ and $s = 1/\epsilon$.

Remark 2.4 A transfer function $h(s)$ is rational if it is the fraction of two polynomials in the complex variable s, that is if it can be written in the form

$$h(s) = \frac{Q(s)}{R(s)} \tag{2.35}$$

where $Q(s)$ and $R(s)$ are polynomials in s. An example of a transfer function that is not rational is $h(s) = \tanh s$ which appears in connection with systems described by partial differential equations.

Example 2.1 Note the difference between the condition $\mathbf{Re}[h(j\omega)] > 0$ and the condition for input strict passivity that there exists a $\delta > 0$ so that $\mathbf{Re}[h(j\omega_0)] \geq \delta > 0$ for all ω. An example of this is

$$h_1(s) = \frac{1}{1 + Ts} \tag{2.36}$$

We find that $\mathbf{Re}[h_1(j\omega)] > 0$ for all ω because

$$h_1(j\omega) = \frac{1}{1 + j\omega T} = \frac{1}{1 + (\omega T)^2} - j\frac{\omega T}{1 + (\omega T)^2} \tag{2.37}$$

However there is no $\delta > 0$ that ensures $\mathbf{Re}[h(j\omega_0)] \geq \delta > 0$ for all ω. This is seen from the fact that for any $\delta > 0$ we have

$$\mathbf{Re}[h_1(j\omega)] = \frac{1}{1 + (\omega T)^2} < \delta \quad \text{for all} \quad \omega > \sqrt{\frac{1-\delta}{\delta}} \tag{2.38}$$

This implies that $h_1(s)$ is not input strictly passive.
We note that for this system

$$|h_1(j\omega)|^2 = \frac{1}{1 + (\omega T)^2} = \mathbf{Re}[h_1(j\omega)] \tag{2.39}$$

which means that the system is output strictly passive with $\epsilon = 1$.

Example 2.2 Consider a system with the transfer function

$$h_2(s) = \frac{s+c}{(s+a)(s+b)} \tag{2.40}$$

where a, b and c are positive constants. We find that

$$\begin{aligned} h_2(j\omega) &= \frac{j\omega+c}{(j\omega+a)(j\omega+b)} \\ &= \frac{(c+j\omega)(a-j\omega)(b-j\omega)}{(a^2+\omega^2)(b^2+\omega^2)} \\ &= \frac{abc+\omega^2(a+b-c)+j[\omega(ab-ac-bc)-\omega^3]}{(a^2+\omega^2)(b^2+\omega^2)} \end{aligned}$$

From this it is clear that

1. If $c \leq a+b$, then $\mathbf{Re}[h_2(j\omega)] > 0$ for all ω. As $\mathbf{Re}[h_2(j\omega)] \to 0$ when $\omega \to \infty$, *the* system is not input strictly passive.

2. If $c > a+b$, then $h_2(s)$ is not passive because $\mathbf{Re}[h_2(j\omega)] < 0$ for $\omega > \sqrt{abc/(c-a-b)}$.

Example 2.3 The systems with the transfer functions

$$h_3(s) = 1 + Ts \tag{2.41}$$

$$h_4(s) = \frac{1+T_1 s}{1+T_2 s}, \quad T_1 < T_2 \tag{2.42}$$

are input strictly passive because

$$\mathbf{Re}[h_3(j\omega)] = 1 \tag{2.43}$$

and

$$\mathbf{Re}[h_4(j\omega)] = \frac{1+\omega^2 T_1 T_2}{1+(\omega T_2)^2} \in \left(\frac{T_1}{T_2}, 1\right] \tag{2.44}$$

Moreover $|h_4(j\omega)|^2 \leq 1$, so that

$$\mathbf{Re}[h_4(j\omega)] \geq \frac{T_1}{T_2} \geq \frac{T_1}{T_2} |h_4(j\omega)|^2 \tag{2.45}$$

which shows that the system is output strictly passive with $\epsilon = T_1/T_2$. The reader may verify from a direct calculation of $|h_4(j\omega)|^2$ and some algebra that it is possible to have $\mathbf{Re}[h_4(j\omega)] \geq |h_4(j\omega)|^2$, that is, $\epsilon = 1$. This agrees with the Nyquist plot of $h_4(j\omega)$.

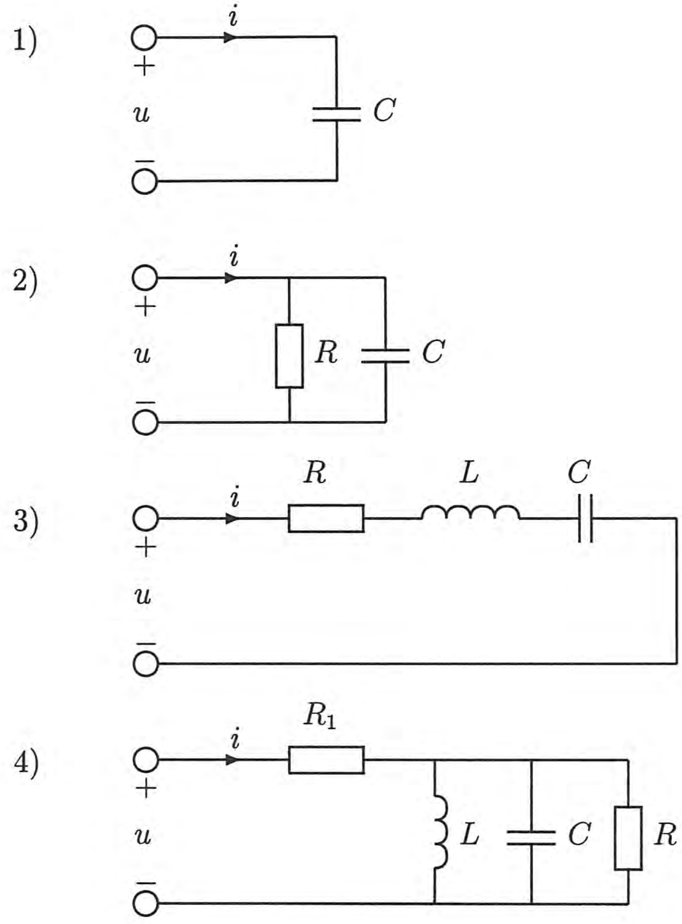

Figure 2.3: Passive electrical one-ports

Example 2.4 A dynamic system describing an electrical one-port with resistors, inductors and capacitors is passive if the voltage over the port is input and the current into the port is output, or *vice versa.* In figure 2.3 different passive one-port are shown. We consider the voltage over the port to be the input and the current into the port as the output. The resulting transfer functions are admittances, which are the inverses of the impedances. Circuit 1 is a capacitor, circuit 2 is a resistor in parallel with a capacitor, circuit 3 is a resistor in series with a inductor and a capacitor, while circuit 4 is a resistor in series with a parallel connection of an inductor, a capacitor and a resistor. The transfer functions are:

$$h_1(s) = Cs \tag{2.46}$$

$$h_2(s) = \frac{1}{R}(1 + RCs) \tag{2.47}$$

$$h_3(s) = \frac{Cs}{1 + RCs + LCs^2} \tag{2.48}$$

$$h_4(s) = \frac{1}{R_1} \frac{1 + \frac{L}{R}s + LCs^2}{1 + (\frac{L}{R_1} + \frac{L}{R})s + LCs^2} \tag{2.49}$$

Systems 1, 2, 3 and 4 are all passive as the poles have real parts that are less than zero, and in addition $\mathbf{Re}[h_i(j\omega)] \geq 0$ for all ω and $i \in \{1, 2, 3, 4\}$. It follows that the transfer functions have phases that satisfy $|\angle h_i(j\omega)| \leq 90°$. In addition system 2 is input strictly passive as $\mathbf{Re}[h_2(j\omega)] = 1/R > 0$ for all ω. For system 4 we find that

$$\mathbf{Re}[h_4(j\omega)] = \frac{1}{R_1} \frac{(1 - \omega^2 LC)^2 + \omega^2 \frac{L^2}{R_1(R_1+R)}}{(1 - \omega^2 LC)^2 + \omega^2 \frac{L^2}{(R_1+R)^2}} \geq \frac{1}{R_1 + R} \tag{2.50}$$

which means that system 4 is input strictly passive. ♠♠

So far we have only considered systems where the transfer functions $h(s)$ have poles with negative real parts. There are however passive systems that have transfer functions with poles on the imaginary axis. This is demonstrated in the following example:

Example 2.5 Consider the system $\dot{y}(t) = u(t)$ which is represented in transfer function description by $y(s) = h(s)u(s)$ where $h(s) = \frac{1}{s}$. This means that the transfer function has a pole at the origin, which is on the imaginary axis. For this system $\mathbf{Re}[h(j\omega)] = 0$ for all ω. However, we cannot establish passivity using theorem 2.3 as this theorem only applies to systems where all the poles have negative real parts. Instead, consider

$$\int_0^T y(t)u(t)dt = \int_0^T y(t)\dot{y}(t)dt \tag{2.51}$$

A change of variables $\dot{y}dt = dy$ gives

$$\int_0^T y(t)u(t)dt = \int_{y(0)}^{y(T)} y(t)dy = \frac{1}{2}[y(T)^2 - y(0)^2] \geq -\frac{1}{2}y(0)^2 \tag{2.52}$$

and passivity is shown with $\beta = -\frac{1}{2}y(0)^2$. ♠♠

It turns out to be relatively involved to find necessary and sufficient conditions on $h\,(j\omega)$ for the system to be passive when we allow for poles on the imaginary axis. The conditions are relatively simple and are given in the following theorem.

Theorem 2.4 Consider a linear time-invariant system with a rational transfer function $h(s)$. The system is passive if and only if

1. $h(s)$ has no poles in $\mathbf{Re}[s] > 0$.
2. $\mathbf{Re}[h(j\omega)] \geq 0$ for all ω such that $j\omega$ is not a pole of $h(s)$.
3. If $j\omega_0$ is a pole of $h(s)$, then it is a simple pole, and the residual in $s = j\omega_0$ is real and greater than zero, that is, $Res_{s=j\omega_0} h(s) = \lim_{s \to j\omega_0}(s - j\omega_0)h(j\omega) > 0$.

♠♠

The above result is established in section 2.11.

Corollary 2.2 If a system with transfer function $h(s)$ is passive, then $h(s)$ has no poles in $\mathbf{Re}[s] > 0$. ♠♠

Proposition 2.1 Consider a rational transfer function

$$h(s) = \frac{(s+z_1)(s+z_2)\dots}{s(s+p_1)(s+p_2)\dots} \tag{2.53}$$

where $\mathbf{Re}[p_i] > 0$ and $\mathbf{Re}[z_i] > 0$ which means that $h(s)$ has one pole at the origin and the remaining poles in $\mathbf{Re}[s] < 0$, while all the zeros are in $\mathbf{Re}[s] < 0$. Then the system with transfer function $h(s)$ is passive if and only if $\mathbf{Re}[h(j\omega)] \geq 0$ for all ω. ♠♠

Proof
The residual of the pole on the imaginary axis is

$$\mathrm{Res}_{s=0} h(s) = \frac{z_1 z_2 \dots}{p_1 p_2 \dots} \tag{2.54}$$

Here the constants z_i and p_i are either real and positive, or they appear in complex conjugated pairs where the products $z_i z_i^* = |z_i|^2$ and $p_i p_i^* = |p_i|^2$ are real and positive. It is seen that the residual at the imaginary axis is real and positive. As $h(s)$ has no poles in $\mathbf{Re}[s] < 0$ by assumption, it follows that the system is passive if and only if $\mathbf{Re}[h(j\omega)] \geq 0$ for all ω. ♠♠

Example 2.6 Consider two systems with transfer functions

$$h_1(s) = \frac{s^2 + a^2}{s(s^2 + \omega_0^2)}, \quad a \neq 0, \omega_0 \neq 0 \tag{2.55}$$

$$h_2(s) = \frac{s}{s^2 + \omega_0^2}, \quad \omega_0 \neq 0 \tag{2.56}$$

where all the poles are on the imaginary axis. Thus condition 1 in theorem 2.4 is satisfied. Moreover,

$$h_1(j\omega) = -j\frac{a^2-\omega^2}{\omega(\omega_0^2-\omega^2)} \tag{2.57}$$

$$h_2(j\omega) = j\frac{\omega}{\omega_0^2-\omega^2} \tag{2.58}$$

so that also condition 2 holds in view of $\mathbf{Re}[h_1(j\omega)] = \mathbf{Re}[h_2(j\omega)] = 0$ for all ω so that $j\omega$ is not a pole in $h(s)$. We now calculate the residual, and find that

$$\text{Res}_{s=0}h_1(s) = \frac{a^2}{\omega_0^2} \tag{2.59}$$

$$\text{Res}_{s=\pm j\omega_0}h_1(s) = \frac{\omega_0^2-a^2}{2\omega_0^2} \tag{2.60}$$

$$\text{Res}_{s=\pm j\omega_0}h_2(s) = \frac{1}{2} \tag{2.61}$$

We see that according to theorem 2.4 the system with transfer function $h_2(s)$ is passive, while $h_1(s)$ is passive whenever $a < \omega_0$.

Example 2.7 Consider a system with transfer function

$$h(s) = -\frac{1}{s} \tag{2.62}$$

The transfer function has no poles in $\mathbf{Re}[s] > 0$, and $\mathbf{Re}[h(j\omega)] \geq 0$ for all $\omega \neq 0$. However, $Res_{s=0}h(s) = -1$, and theorem 2.4 shows that the system is not passive This result agrees with the observation

$$\int_0^T y(t)u(t)dt = -\int_{y(0)}^{y(T)} y(t)dy = \frac{1}{2}[y(0)^2 - y(T)^2] \tag{2.63}$$

where the right hand side has no lower bound as $y(T)$ can be arbitrarily large.

2.4 Passivity of the PID controllers

Proposition 2.2 Assume that $0 \leq T_d < T_i$ and $0 \leq \alpha \leq 1$. Then the PID controller

$$h_r(s) = K_p\frac{1+T_is}{T_is}\frac{1+T_ds}{1+\alpha T_ds} \tag{2.64}$$

is passive. This follows from proposition 2.1.

♠♠

Proposition 2.3 Consider a PID controller with transfer function

$$h_r(s) = K_p\beta\frac{1+T_is}{1+\beta T_is}\frac{1+T_ds}{1+\alpha T_ds} \tag{2.65}$$

where $0 \leq T_d < T_i$, $1 \leq \beta < \infty$ and $0 < \alpha \leq 1$. Then the controller is passive, and in addition, the transfer function gain has an upper bound $|h_r(j\omega)| \leq \frac{K_p\beta}{\alpha}$ and the real part of the transfer function is bounded away from zero according to $\mathbf{Re}\,[h_r(j\omega)] \geq K_p$ for all ω.

♠♠

It follows from Bode diagram techniques that

$$|h_r(j\omega)| \leq K_p\beta \cdot 1 \cdot \frac{1}{\alpha} = \frac{K_p\beta}{\alpha} \tag{2.66}$$

The result on the $\mathbf{Re}\,[h_r(j\omega)]$ can be established using Nyquist diagram, or by direct calculation of $\mathbf{Re}\,[h_r(j\omega)]$. ♠♠

2.5 Stability of a passive feedback interconnection

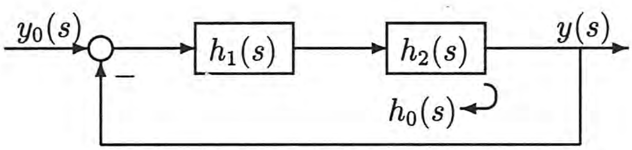

Figure 2.4: Interconnection of a passive system $h_1(s)$ and a strictly system $h_2(s)$

Consider a feedback loop with loop transfer function $h_0(s) = h_1(s)h_2(s)$ as shown in figure 2.4. If h_1 is passive and h_2 is strictly passive, then the phases of the transfer functions satisfy

$$|\angle h_1(j\omega)| \leq 90^\circ \quad \text{and} \quad |\angle h_2(j\omega)| < 90^\circ \tag{2.67}$$

It follows that the phase of the loop transfer function $h_0(s)$ is bounded by

$$|\angle h_0(j\omega)| < 180^\circ \tag{2.68}$$

As h_1 and h_2 are passive, it is clear that $h_0(s)$ has no poles in $\mathbf{Re}\,[s] > 0$. Then according to standard Bode-Nyquist stability theory the system is asymptotically stable and BIBO stable. The same result is obtained if instead h_1 is strictly passive an
We note that in view o
action is strictly stable.

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