TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2016 - Solution to Assignment 3

1. (a) Consider the following Lyapunov function candidate

$$V = \frac{1}{2}\left[\tilde{x}_1^2 + m\tilde{x}_2^2\right] \tag{1}$$

Differentiation of this function along the trajectories of the system yields

$$\begin{aligned} \dot{V} &= \tilde{x}_1\dot{\tilde{x}}_1 + m\tilde{x}_2\dot{\tilde{x}}_2 & (2)\\ &= \tilde{x}_1\tilde{x}_2 + \tilde{x}_2\left\{-f_3\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] - f_1\tilde{x}_1 - d\tilde{x}_2 + \tilde{u}\right\} & (3) \end{aligned}$$

The input $\bar{u}$ is selected as

$$\tilde{u} = f_3\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] + f_1\tilde{x}_1 - \tilde{x}_1 - k_2\tilde{x}_2$$

which gives that

$$\dot{V} = -\left(d + k_2\right)\tilde{x}_2^2 \leq 0 \quad \forall \quad x, \quad (d + k_2) > 0 \tag{4}$$

(b) The closed-loop system is found by inserting $\tilde{u}$ into the system equations

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ m\dot{\tilde{x}}_2 &= -\tilde{x}_1 - (d + k_2)\tilde{x}_2 \end{aligned} \tag{5}$$

$$\downarrow$$

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{m} & -\frac{1}{m}(d + k_2) \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \tag{6}$$

1. $\dot{V}$ from the previous question is negative semidefinite. V is a continuously differentiable, positive definite, radially unbounded function such that $\dot{V} \leq 0$ for all $\tilde{x} \in \mathcal{R}^2$ and $V(0) = 0$. Let $S \in \left\{\tilde{x} \in \mathcal{R}^2 | \dot{V} = 0\right\}$, i.e. $S \in \{\tilde{x} \in \mathcal{R}^2 | \tilde{x}_2 = 0\}$. Inserting $\tilde{x}_2 = 0$ gives
$$\tilde{x}_2 = 0 \Longrightarrow \tilde{x}_1 = 0 \tag{7}$$
i.e. the only solution that can stay in S is the trivial solution $\tilde{x} = 0$. LaSalle's theorem states that the origin is a globally asymptotically stable equilibrium point of the closed-loop system.
2. The eigenvalues of the closed loop system can be calculated from
$$\begin{vmatrix} \lambda & -1 \\ \frac{1}{m} & \lambda + \frac{1}{m}(d + k_2) \end{vmatrix} = 0 \tag{8}$$
which gives
$$\lambda^2 + \lambda\frac{1}{m}(d + k_2) + \frac{1}{m} = 0 \tag{9}$$
$$\lambda = \frac{1}{2}\left(-\left(\frac{d + k_2}{m}\right) \pm \sqrt{\left(\frac{d + k_2}{m}\right)^2 - \frac{4}{m}}\right) \tag{10}$$

Since d, k_2, $m > 0$

$$\frac{d+k_2}{m} > \sqrt{\left(\frac{d+k_2}{m}\right)^2 - \frac{4}{m}} \tag{11}$$

and the eigenvalues will always lie in the left half plane which means that A is Hurwitz. Any linear system on the form

$$\dot{x} = Ax \tag{12}$$

is exponentially stable as long as A is Hurwtitz, and the system (6) is hence globally exponentially stable.

(c) From (5) it is easy to see that increasing k_2 will introduce more damping into the system. Therefore the system dynamics gets slower as the controller gain k_2 increases.

(d) From (10) it is seen that no controller gain is part of the last term. This means that the real and imaginary part of each eigenvalue are dependent on each other, and the poles of the system cannot be placed arbitrarily. (They can only be placed arbitrarily on the real axis, but then the imaginary part would be given, or vice versa.)

2. Since P is positive definite we should have $p_{11} > 0$ and $p_{11}p_{22} - p_{12}^2 > 0$ (i.e. all leading principal minors are positive, see Appendix from Assignment 2). The eigenvalue λ of P is such that

$$\begin{gathered}
\det(\lambda I - P) = 0 \\
\iff \det\left(\begin{bmatrix} \lambda - p_{11} & -p_{12} \\ -p_{12} & \lambda - p_{22} \end{bmatrix}\right) = 0 \\
\iff (\lambda - p_{11})(\lambda - p_{22}) - p_{12}^2 = 0 \\
\iff \lambda^2 - (p_{11} + p_{22})\lambda + p_{11}p_{22} - p_{12}^2 = 0
\end{gathered}$$

Then we have

$$\begin{aligned}
\lambda_{\min} &= 0.5\left(p_{11} + p_{22} - \sqrt{(p_{11} - p_{22})^2 + 4p_{12}^2}\right) \\
\lambda_{\max} &= 0.5\left(p_{11} + p_{22} + \sqrt{(p_{11} - p_{22})^2 + 4p_{12}^2}\right).
\end{aligned}$$

To show $\lambda_{\min} I \leq P$ we need to show that

$$\begin{aligned}
\Lambda_{\min} &= P - \lambda_{\min} I \\
&= \begin{bmatrix} -\lambda_{\min} + p_{11} & p_{12} \\ p_{12} & -\lambda_{\min} + p_{22} \end{bmatrix}
\end{aligned}$$

is positive semidefinite. The leading principal minors of $\Lambda_{\min}$ are

$$\begin{aligned}
\mu_{\min,1} &= -\lambda_{\min} + p_{11} \\
\mu_{\min,2} &= \det(\Lambda_{\min})
\end{aligned}$$

and we have $\mu_{\min,1} = 0.5\left(p_{11} - p_{22} + \sqrt{(p_{11} - p_{22})^2 + 4p_{12}^2}\right) > 0$ (since $(p_{11} - p_{22}) < \sqrt{(p_{11} - p_{22})^2 + 4p_{12}^2}$)) and $\mu_{\min,2} = \det(P - \lambda_{\min} I) = 0$. Thus $\Lambda_{\min}$ is positive semidefinite. This implies that

$$\begin{aligned}&x^T\Lambda_{\min}x \geq 0\\ \Longleftrightarrow\ &x^TPx - \lambda_{\min}x^Tx \geq 0\\ \Longleftrightarrow\ &x^TPx - \lambda_{\min}\|x\|_2^2 \geq 0\\ \Longleftrightarrow\ &x^TPx \geq \lambda_{\min}\|x\|_2^2\end{aligned}$$

for all x. To show $P \leq \lambda_{\max} I$ we need to show that

$$\begin{aligned}\Lambda_{\max} &= \lambda_{\max} I - P\\ &= \begin{bmatrix} \lambda_{\max} - p_{11} & -p_{12}\\ -p_{12} & \lambda_{\max} - p_{22}\end{bmatrix}\end{aligned}$$

is positive semidefinite. The leading principal minors of $\Lambda_{\max}$ are

$$\begin{aligned}\mu_{\max,1} &= \lambda_{\max} - p_{11}\\ \mu_{\max,2} &= \det(\Lambda_{\max})\end{aligned}$$

and we have $\mu_{\max,1} = 0.5\left(p_{22} - p_{11} + \sqrt{(p_{11}-p_{22})^2 + 4p_{12}^2}\right) > 0$ and $\mu_{\min,2} = \det(\lambda_{\max} I - P) = 0$. Thus $\Lambda_{\max}$ is positive semidefinite. This implies that

$$\begin{aligned}&x^T\Lambda_{\max}x \geq 0\\ \Longleftrightarrow\ &\lambda_{\max}x^Tx - x^TPx \geq 0\\ \Longleftrightarrow\ &\lambda_{\max}\|x\|_2^2 - x^TPx \geq 0\\ \Longleftrightarrow\ &x^TPx \leq \lambda_{\max}\|x\|_2^2\end{aligned}$$

for all x.

3. The function is given by

$$V(x) = \frac{(x_1+x_2)^2}{1+(x_1+x_2)^2} + (x_1-x_2)^2$$

(a) Let $x_1 = 0$, then $V(x)$ is given by

$$V(x) = \frac{x_2^2}{1+x_2^2} + x_2^2$$

and it can be seen that $V(x) = \frac{x_2^2}{1+x_2^2} + x_2^2 \to \infty$ as $|x_2| \to \infty$.
Let $x_2 = 0$, then $V(x)$ is given by

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_1^2$$

and it can be seen that $V(x) = \frac{x_1^2}{1+x_1^2} + x_1^2 \to \infty$ as $|x_1| \to \infty$.

(b) On the set $x_1 = x_2$ the function is given by

$$V(x) = \frac{4x_1^2}{1+4x_1^2}$$

and it can be seen that $V(x) = \frac{4x_1^2}{1+4x_1^2} \to 1$ as $|x_1| \to \infty$, and $V(x)$ is therefore not radially unbounded.

4. The system is given by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -h_1(x_1) - x_2 - h_2(h_3)\\ \dot{x}_3 &= x_2 - x_3\end{aligned}$$

(a) From the system equations it can be seen that the equilibrium point is given by

$$\begin{aligned}0 &= x_2\\ 0 &= -h_1(x_1) - x_2 - h_2(x_3)\\ 0 &= x_2 - x_3\end{aligned}$$

which is equivalent to

$$\begin{aligned}x_2 &= 0\\ -h_1(x_1) - h_2(0) &= 0\\ x_3 &= 0\end{aligned}$$

since $h_1(x_1) = 0$ only when $x_1 = 0$, origin is a unique equilibrium point.

(b) Since $V(x)$ is a sum of nonnegative functions ($h_i(y) \geq 0\ \forall y \geq 0$) it is a positive semi definite function. To show that it is positive definite, we need to show that

$$V(x) = 0 \Rightarrow x = 0$$

Since $y h_i(y) > 0\ \forall y \neq 0$, the integral $\int_0^{x_i} h_i(y)\, dy$ vanish if and only if $x_i = 0$, and it follows that $V(x)$ *is* positive definite.

(c) The time derivative of the function

$$V(x) = \int_0^{x_1} h_1(y)\, dy + \frac{1}{2}x_2^2 + \int_0^{x_3} h_2(y)\, dy$$

along the trajectories of the system is found as

$$\begin{aligned}\dot{V}(x) &= h_1(x_1)\dot{x}_1 + x_2\dot{x}_2 + h_2(x_3)\dot{x}_3\\ &= h_1(x_1)x_2 + x_2(-h_1(x_1) - x_2 - h_2(x_3)) + h_2(x_3)(x_2 - x_3)\\ &= -x_2^2 - h_2(x_3)x_3\\ &= -\left(x_2^2 + h_2(x_3)x_3\right)\end{aligned}$$

since $h_2(x_3)x_3 > 0\ \forall x_3 \neq 0$ we have that $\dot{V}(x)$ is negative semi definite. In order to prove asymptotic stability, we apply Corollary 4.1. From $\dot{V}(x)$ it can be seen that the set $\mathcal{S}$ is given by

$$\mathcal{S} = \left\{x \in \mathcal{R}^3 | x_2^2 + h_2(x_3)x_3 = 0\right\} = \left\{x \in \mathcal{R}^3 | x_2 = 0, x_3 = 0\right\}$$

and it can be seen from the system equation that no solution can stay identical in $\mathcal{S}$ other than the trivial solution $x = 0$, and asymptotic stability of the origin follows.

(d) To show global asymptotically stability the function $V(x)$ need to be radially unbounded. This is the case if the functions h_i satisfies $\int_0^z h_i(y)\, dy \to \infty$ as $|z| \to \infty$.

5. (a) The Jacobian of the vector field is given by

$$\begin{aligned}\dot{x}_1 &= -(x_1+2x_2)(x_1+2)\\ \dot{x}_2 &= -8x_2(2+2x_1+x_2)\end{aligned}$$

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} -(x_1+2)-(x_1+2x_2) & -2(x_1+2) \\ -16x_2 & -8(2+2x_1+x_2)-8x_2 \end{bmatrix}$$

Computing at the origin we have

$$A = \frac{\partial f(0)}{\partial x} = \begin{bmatrix} -2 & -4 \\ 0 & -16 \end{bmatrix}$$

and the eigenvalues are $-2, -6$. Thus the origin is asymptotically stable.

(b) The derivative of Lyapunov function along the trajectory of the system is

$$\begin{aligned}\dot{V} &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2\\ &= -2x_1(x_1+2x_2)(x_1+2) - 16x_2^2(2+2x_1+x_2)\\ &= -2x_1\left(x_1^2+2x_1+2x_1x_2+4x_2\right) - 16x_2^2(2+2x_1+x_2)\\ &= -2x_1^2(x_1+2+2x_2) - 8x_1x_2 - 16x_2^2(2+2x_1+x_2)\end{aligned}$$

In the region $\mathcal{D} = \{x \in \mathcal{R}^2 | x_1+2x_2+1 \geq 0 \text{ and } 2x_1+x_2+1 \geq 0\}$ we have

$$\begin{aligned}x_1+2x_2+2 &\geq 1\\ 2x_1+x_2+2 &\geq 1\end{aligned}$$

and it follows that

$$\dot{V} \leq -2x_1^2 - 8x_1x_2 - 16x_2^2 = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 4 \\ 4 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -x^TQx$$

The matrix Q is positive definite. Thus $\dot{V}$ is negative definite which implies that the origin is asymptotically stable.

(c) Since $\Omega_{\frac{1}{9}}$ is bounded and contained in $\mathcal{D}$, the set $\Omega_{\frac{1}{9}} \cap \mathcal{D}$ is an estimate of the region of attraction (see page 122 in Khalil). With $x(0) = \left(0, \frac{1}{3}\right)$, the trajectory converges to the origin because $x(0) = \left(0, \frac{1}{3}\right)$ is in the set $\Omega_{\frac{1}{9}} \cap \mathcal{D}$ (region of attraction). For the case of $x(0) = \left(-\frac{4}{3}, 2\right)$ there is no $c > 0$ which will give $x(0) \in \Omega_c \cap \mathcal{D}$, where Ω_c is bounded and contained in $\mathcal{D}$. This implies that it is possible for the trajectory not to converge to the origin when we start from $x(0) = \left(-\frac{4}{3}, 2\right)$. See Figure 1.

6. If $r_1 \geq r_2$ we have that $r_1 + r_2 \leq 2r_1$ which implies that

$$\alpha(r_1+r_2) \leq \alpha(2r_1) \leq \alpha(2r_1) + \alpha(2r_2)$$

and if $r_2 \geq r_1$ we have that $r_1 + r_2 \leq 2r_2$ which implies that

$$\alpha(r_1+r_2) \leq \alpha(2r_2) \leq \alpha(2r_1) + \alpha(2r_2)$$

where it has been used that a class K function is strictly increasing in its argument. Using the two different cases, we can conclude that the inequality $\alpha(r_1+r_2) \leq \alpha(2r_1) + \alpha(2r_2)$ is always satisfied.

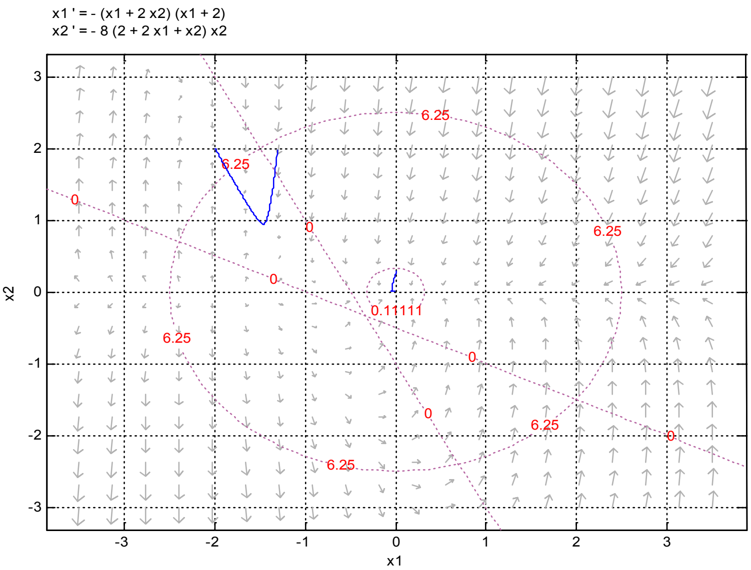

Figure 1: Phase portrait

7. (a) For class $\mathcal{KL}$ functions we know that

$$\beta(r, t) \leq \beta(r, 0)$$

Hence

$$\|x(t)\| \leq \beta(\|x(0)\|, 0)$$

We want to find $\delta > 0$ such that

$$\|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq \epsilon \quad \forall t \geq 0$$

To guarantee this, given $\epsilon > 0$ we choose $\delta > 0$ s.t. $\beta(\delta, 0) < \epsilon$. This is possible because $\lim\limits_{r \to 0} \beta(r, 0) = 0$. By properties of $\mathcal{KL}$ functions, if $\|x(0)\| < \delta$ then $\beta(\|x(0)\|, 0) < \beta(\delta, 0) < \epsilon$. See Figure 2.
Hence, $x = 0$ is stable.

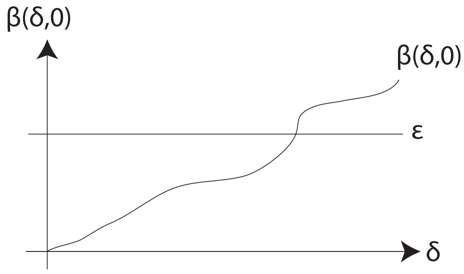

Figure 2: Plot of $\beta(\delta, 0)$ vs. ϵ

(b) Since β is of class $\mathcal{KL}$ then $\lim\limits_{t \to \infty} \beta(\|x(0)\|, t) = 0$. Thus $\lim\limits_{t \to \infty} \|x(t)\| = 0$. By definition of norm it follows that $\lim\limits_{t \to \infty} x(t) = 0$. We conclude that the origin of the system is globally asymptotically stable.

8. From $V = 0.5\left(bx_1^2 + ax_2^2\right)$ we have

$$\begin{aligned}
\dot{V} &= bx_1\dot{x}_1 + ax_2\dot{x}_2 && (13)\\
&= bx_1\left(-\phi(t)x_1 + a\phi(t)x_2\right) + ax_2\left(b\phi(t)x_1 - ab\phi(t)x_2 - c\psi(t)x_2^3\right) && (14)\\
&= -\phi(t)b\left(x_1^2 - 2ax_1x_2 + a^2x_2^2\right) - ac\psi(t)x_2^4 && (15)
\end{aligned}$$

$$\begin{aligned}
\dot{V} &= -b\phi(t)(x_1 - ax_2)^2 - ac\psi(t)x_2^4\\
&\le -b\phi_0(x_1 - ax_2)^2 - ac\psi_0x_2^4 \triangleq -W_3(x)
\end{aligned}$$

It can be verified that $W_3(x)$ is positive definite for all x. Hence, by Theorem 4.9, the origin is globally uniformly asymptotically stable.

9. The system is given by

$$\begin{aligned}
\dot{x}_1 &= \frac{1}{L(t)}x_2\\
\dot{x}_2 &= -\frac{1}{C(t)}x_1 - \frac{R(t)}{L(t)}x_2
\end{aligned}$$

where $L(t)$, $C(t)$ and $R(t)$ continuously differentiable and bounded from below and above. The Lyapunov function candidate is given by

$$V(t,x) = \left(R(t) + \frac{2L(t)}{R(t)C(t)}\right)x_1^2 + 2x_1x_2 + \frac{2}{R(t)}x_2^2$$

(a) The function can be upper bounded by

$$V(t,x) \le \left(k_6 + \frac{2k_2}{k_3k_5}\right)x_1^2 + 2x_1x_2 + \frac{2}{k_5}x_2^2$$

and lower bounded by

$$V(t,x) \ge \left(k_5 + \frac{2k_1}{k_4k_6}\right)x_1^2 + 2x_1x_2 + \frac{2}{k_6}x_2^2$$

Using the upper bounds it is clear that $V(t,x)$ is decresent. If we try to use the lower bounds to show that $V(t,x)$ is positive definite, we will have to restrict the constants to

$$\frac{2k_5}{k_6} + \frac{4k_1}{k_6^2k_4} - 1 > 0$$

Instead of making this restriction, we work directly with $V(t,x)$ and rewrite it as

$$\begin{aligned}
V(t,x) &= x^T\begin{bmatrix} \left(R(t) + \frac{2L(t)}{R(t)C(t)}\right) & 1\\ 1 & \frac{2}{R(t)}\end{bmatrix}x\\
&\ge x^T\begin{bmatrix} R(t) & 1\\ 1 & \frac{2}{R(t)}\end{bmatrix}x\\
&= x^T\tilde{P}x
\end{aligned}$$

The eigenvalues of $\tilde{P}$ are calculated as

$$\lambda_{1,2} = \frac{1}{2}\left(\left(R + \frac{2}{R}\right) \pm \sqrt{\left(R + \frac{2}{R}\right)^2 - 4}\right)$$

The smallest eigenvalue is given by

$$\lambda_{\min} = \frac{1}{2}\left(\left(R + \frac{2}{R}\right) - \sqrt{\left(R + \frac{2}{R}\right)^2 - 4}\right)$$

where it is easily seen that

$$\left(R + \frac{2}{R}\right) > \sqrt{\left(R + \frac{2}{R}\right)^2 - 4} \tag{16}$$

Since $R(t) > k_5$ where k_5 is positive, it is clear that there is a positive constant c such that $\lambda_{\min} \geq c$ for all t, which shows that $V(t, x)$ is positive definite.

(b) The time derivative of $V(t, x)$ is found as

$$\begin{aligned}\dot{V}(t, x) &= -\frac{2}{C(t)}\left(1 + \dot{R}(t)\left(\frac{L(t)}{R^2(t)} - \frac{C(t)}{2}\right) + \frac{L(t)\dot{C}(t)}{R(t)C(t)} - \frac{\dot{L}(t)}{R(t)}\right)x_1^2 \\ &\quad -\frac{2}{L(t)}\left(1 + \frac{L(t)\dot{R}(t)}{R^2(t)}\right)x_2^2\end{aligned}$$

Suppose $\dot{L}(t)$, $\dot{C}(t)$ and $\dot{R}(t)$ satisfy

$$\begin{aligned}1 + \dot{R}(t)\left(\frac{L(t)}{R^2(t)} - \frac{C(t)}{2}\right) + \frac{L(t)\dot{C}(t)}{R(t)C(t)} - \frac{\dot{L}(t)}{R(t)} &> c_3 \\ 1 + \frac{L(t)\dot{R}(t)}{R^2(t)} &> c_4\end{aligned}$$

Then

$$\dot{V}(t, x) < -\frac{2c_3}{k_3}x_1^2 - \frac{2c_4}{k_1}x_2^2$$

and $\dot{V}(t, x)$ is negative definite. This implies that the origin is uniformly asymptotically stable. Using Theorem 4.10 it is concluded that the origin is exponentially stable.

10. The system is given by

$$\begin{aligned}\dot{x}_1 &= h(t)x_2 - g(t)x_1^3 \\ \dot{x}_2 &= -h(t)x_1 - g(t)x_2^3\end{aligned}$$

where $h(t)$ and $g(t)$ are bounded, continuously differentiable functions and $g(t) \geq k > 0$ $\forall t \geq 0$.

(a) It can be recognized from the model that $x = 0$ is an equilibrium point. The stability properties are analyzed using the Lyapunov function candidate

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$$

The time derivative along the trajectories of the system is found as

$$\begin{aligned}
\dot{V}(x) &= x_1\left(h(t)\,x_2 - g(t)\,x_1^3\right) + x_2\left(-h(t)\,x_1 - g(t)\,x_2^3\right) \\
&= -g(t)\,x_1^4 + h(t)\,x_1x_2 - h(t)\,x_1x_2 - g(t)\,x_2^4 \\
&= -g(t)\,x_1^4 - g(t)\,x_2^4 \\
&= -g(t)\left(x_1^4 + x_2^4\right) \\
&\leq -k\left(x_1^4 + x_2^4\right)
\end{aligned}$$

Setting $W_1 = W_2 = V(x)$ and $W_3 = -k\left(x_1^4 + x_2^4\right)$, Theorem 4.9 states that the origin is uniformly asymptotically stable.

(b) The Lyapunov function does not satisfy Theorem 4.10. The next step is to use Corollary 4.3, where

$$\begin{aligned}
A(t) &= \frac{\partial f(t,0)}{\partial x} \\
&= \begin{bmatrix} 0 & h(t) \\ -h(t) & 0 \end{bmatrix}
\end{aligned}$$

The eigenvalues of A are

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -h(t) \\ h(t) & \lambda \end{vmatrix} = \lambda^2 + h^2(t) = 0 \tag{17}$$

$$\downarrow$$

$$\lambda = \pm hi \tag{18}$$

which gives $\mathcal{R}\rceil(\lambda) = 0$, and A is not Hurwitz. Hence, the system is not exponentially stable.

(c) Since $V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$ is a radially unbounded Lyapunov function for the system with a time derivative satisfying $\dot{V}(x) \leq -k\left(x_1^4 + x_2^4\right)$ globally, we conclude by Theorem 4.9 that the origin is globally uniformly asymptotically stable.

(d) Since the system is not exponentially stable, it can not be globally exponentially stable.

11. The system is given as

$$\begin{aligned}
\dot{x}_1 &= -6x_1 \\
\dot{x}_2 &= 2x_1 - 6x_2 - 2x_2^3
\end{aligned}$$

The Jacobian matrix of the considered system is

$$A = \left[\frac{\partial f}{\partial X}\right] = \begin{bmatrix} -6 & 0 \\ 2 & -6 - 6x_2^2 \end{bmatrix}$$

Applying Krasovskiis Theorem we get

$$F = A^T + A = \begin{bmatrix} -12 & 2 \\ 2 & -12 - 12x_2^2 \end{bmatrix}$$

and the Eigenvalues of F are

$$\begin{aligned} \begin{vmatrix} \lambda + 12 & -2 \\ -2 & \lambda + 12 + 12x_2^2 \end{vmatrix} &= 0 \\ (\lambda + 12)^2 + 12x_2^2(\lambda + 12) - 4 &= 0 \\ \lambda^2 + 24\lambda + 144 + 12x_2^2\lambda + 144x_2^2 - 4 &= 0 \\ \lambda^2 + (24 + 12x_2^2)\lambda + (140 + 144x_2^2) &= 0 \end{aligned}$$

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2}\left[-(24 + 12x_2^2) \pm \sqrt{(24 + 12x_2^2)^2 - 4(140 + 144x_2^2)}\right] \\ &= -(12 + 6x_2^2) \pm \sqrt{(12 + 6x_2^2)^2 - (140 + 144x_2^2)} \end{aligned}$$

where

$$0 < \sqrt{(12 + 6x_2^2)^2 - (140 + 144x_2^2)} < (12 + 6x_2^2)$$

which means that

$$\lambda_{1,2} = -(12 + 6x_2^2) \pm \sqrt{(12 + 6x_2^2)^2 - (140 + 144x_2^2)} < 0 \quad \forall\ x_2 \in R$$

and A is negative define in R^2. Moreover,

$$\begin{aligned} V(x) &= f^T(x)f(x) \\ &= (-6x_1)^2 + (2x_1 - 6x_2 - 2x_2^3)^2 \to \infty \text{ as } ||x|| \to \infty \end{aligned}$$

and it can be concluded that the origin of x is globally asymptotically stable.

12. (a) The function $V_1(x_1, x_2, t)$ is given by

$$V_1(x_1, x_2, t) = x_1^2 + \left(1 + e^t\right) x_2^2$$

Since $e^t \to \infty$ when $t \to \infty$, the term $(1 + e^t)$ may not be upper bounded uniformly in t. Hence, the function is not decrescent. The function may however be lower bounded by

$$\begin{aligned} V_1(x_1, x_2, t) &= x_1^2 + \left(1 + e^t\right) x_2^2 \\ &\geq x_1^2 + x_2^2 \\ &= W_1(x) \end{aligned}$$

where $W_1(x)$ is positive definite. This implies that the function $V_1(x_1, x_2, t)$ is positive definite.

(b) The function $V_2(x_1, x_2, t)$ is given by

$$V_2(x_1, x_2, t) = \frac{x_1^2 + x_2^2}{1 + t}$$

Since $\frac{1}{1+t} \to 0$ when $t \to \infty$ the function $V_2(x_1, x_2, t)$ may not be lower bounded uniformly in t. Hence, the function is not positive definite. The function may however be upper bounded by

$$\begin{aligned} V_2(x_1, x_2, t) &= \frac{x_1^2 + x_2^2}{1 + t} \\ &\leq x_1^2 + x_2^2 \\ &= W_2(x) \end{aligned}$$

where $W_2(x)$ is positive definite. This implies that the function $V_2(x_1, x_2, t)$ is decrescent.

(c) The function $V_3(x_1, x_2, t)$ is given by

$$V_3(x_1, x_2, t) = \left(1 + \cos^4 t\right)\left(x_1^2 + x_2^2\right)$$

Since $1 \leq (1 + \cos^4 t) \leq 2$, the function may be lower and upper bounded according to

$$\begin{aligned} V_3(x_1, x_2, t) &= \left(1 + \cos^4 t\right)\left(x_1^2 + x_2^2\right) \\ &\geq x_1^2 + x_2^2 \\ &= W_1(x) \end{aligned}$$

and

$$\begin{aligned} V_3(x_1, x_2, t) &= \left(1 + \cos^4 t\right)\left(x_1^2 + x_2^2\right) \\ &\leq 2\left(x_1^2 + x_2^2\right) \\ &= W_2(x) \end{aligned}$$

Since $W_1(x)$ and $W_2(x)$ are both positive definite, we conclude that the function $V_3(x_1, x_2, t)$ is positive definite and decrescent.

13. The system is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - c(t)\, x_2 \end{aligned}$$

A Lyapunov function candidate is taken as

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right) \tag{19}$$

The time derivative of $V(x)$ along the trajectories of the system is

$$\begin{aligned} \dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1x_2 + x_2\left(-x_1 - c(t)\, x_2\right) \\ &= x_1x_2 - x_1x_2 - c(t)\, x_2^2 \\ &= -c(t)\, x_2^2 \\ &\leq -k_1x_2^2 \end{aligned} \tag{20}$$

and it can be seen that $\dot{V}(x)$ is negative semidefinite. By Theorem 4.8 we conclude that the origin is uniformly stable ($V(x)$ is positive definite and decrescent). In order to prove that $x_2 \to 0$ as $t \to \infty$ we apply Barbalat's lemma. Since $\dot{V}(x) = -c(t)\, x_2^2$ where $c(t)$ is some bounded value greater than zero, $\dot{V}(x) = 0 \Leftrightarrow x_2 = 0$. Following the notation of Lemma 8.2, let $\phi(t) = \dot{V}(t)$. $\dot{V}(t)$ is uniformly continuous in t if $\ddot{V}(t)$ is bounded

$$\begin{aligned}
\ddot{V}(t) &= -\dot{c}(t)\, x_2^2 - 2c(t)\, x_2 \dot{x}_2 \\
&= -\dot{c}(t)\, x_2^2 - 2c(t)\, x_2 (-x_1 - c(t)\, x_2) \\
&= -\dot{c}(t)\, x_2^2 + 2c(t)\, x_1 x_2 + 2c^2(t)\, x_2^2
\end{aligned}$$

Since the time derivative of $\dot{V}(t)$ is negative semidefinite, it follows that $V(t) \leq V(t_0)$, and since $V(x_1, x_2, t)$ is radially unbounded in x this again implies that x_1 and x_2 are bounded. Since x_1 and x_2 are bounded and it is given that $c(t)$ and $\dot{c}(t)$ are bounded, it follows that $\ddot{V}(t)$ is bounded. The bound on $\ddot{V}(t)$ guarantees that $\dot{V}(t)$ is uniformly continuous. In order to conclude by Barbalat's lemma we also need to prove that $\lim_{t\to\infty} \int_0^t \dot{V}(\tau)\, d\tau$ exists and is finite. This is proven according to

$$\begin{aligned}
\lim_{t\to\infty} \int_0^t \dot{V}(\tau)\, d\tau &= \lim_{t\to\infty} (V(t) - V(0)) \\
&= \lim_{t\to\infty} V(t) - V(0)
\end{aligned}$$

where we know that $\lim_{t\to\infty} V(t) = V_\infty$ is a finite number since $V(t) \geq 0 \forall t$ and $\dot{V}(t) \leq 0 \forall t$. By Lemma 8.2 it is shown that $\dot{V}(t) \to 0$ as $t \to \infty$, and hence that $x_2 \to 0$ as $t \to \infty$.

14. Optional exercise:

(a) Given $f(x) = \int_0^1 \frac{\partial}{\partial x} f(\sigma x)\, x d\sigma$

$$\begin{aligned}
x^T P f(x) + f^T(x) P x &= x^T P \int_0^1 \frac{\partial}{\partial x} f(\sigma x)\, x d\sigma + \left(\int_0^1 \frac{\partial}{\partial x} f(\sigma x)\, x d\sigma \right)^T P x \\
&= x^T P \int_0^1 \frac{\partial}{\partial x} f(\sigma x)\, x d\sigma + \int_0^1 x^T \left(\frac{\partial}{\partial x} f(\sigma x) \right)^T d\sigma P x \\
&= x^T \left(P \int_0^1 \frac{\partial}{\partial x} f(\sigma x)\, d\sigma + \int_0^1 \left(\frac{\partial}{\partial x} f(\sigma x) \right)^T d\sigma P \right) x \\
&= x^T \int_0^1 \left(P \frac{\partial}{\partial x} f(\sigma x) + \left(\frac{\partial}{\partial x} f(\sigma x) \right)^T P \right) d\sigma x
\end{aligned}$$

and by using $P \frac{\partial}{\partial x} f(\sigma x) + \left(\frac{\partial}{\partial x} f(\sigma x) \right)^T P \leq -I$ the expression may be upper bounded by

$$x^T P f(x) + f^T(x) P x \leq x^T (-I)\, x = -x^T x = -\|x\|_2^2$$

(b) Given the function $V(x) = f^T(x) P f(x)$ where P is symmetric and positive definite. To show that $V(x)$ is positive definite, we need to show that $f(x) = 0$ if and only if $x = 0$. In other words we need to show that the origin is a unique equilibrium point. Suppose, to the contrary that there is a $p \neq 0$ such that $f(p) = 0$. Then

$$p^T p \leq -\left(p^T P f(p) + f^T(p) P p \right) = 0$$

which is a contradiction since $p \neq 0$ (in order to satisfy the above inequality p needs to equal zero). Hence the origin is a unique equilibrium point. To see that the function is radially unbounded, we first assume that $V(x)$ bounden as

$$V(x) = |V(x)| \leq a, \quad \forall x \in \mathbb{R}^n$$

which means that

$$|V(x)| = |f(x)^\top P f(x)| \leq \lambda_{max}(P)||f(x)||_2^2 \leq a.$$

We can conclude if $V(x)$ is bounded then $||f(x)||_2$ needs to be bounded as followed

$$||f(x)||_2 \leq \sqrt{\frac{a}{\lambda_{max}(P)}} = c$$

where $c > 0$. Now we see if there exist a $c > 0$ such that $||f(x)||_2 \leq c$. Here,

$$x^\top P f(x) + f(x)^\top P x \leq -x^\top x$$

is used. First we see that

$$x^\top P f(x) + f(x)^\top P x = 2x^\top P f(x) \leq -x^\top x. \tag{21}$$

Next, we show that

$$2x^\top P f(x) \leq 2||x||_2 \lambda_{max}(P)||f(x)||_2 \leq ||x||_2 b, \tag{22}$$

where

$$b = 2\lambda_{max}(P)c.$$

It can be seen that (21) is equivalent to

$$2x^\top P f(x) \leq -||x||_2^2. \tag{23}$$

By summing 23 and (22) together we get

$$\begin{aligned}
4x^\top P f(x) &\leq ||x||_2(b - ||x||_2) \\
\frac{4x^\top P f(x)}{||x||_2} &\leq b - ||x||_2 \\
\lim_{x\to\infty} \frac{4x^\top P f(x)}{||x||_2} &\leq b - \lim_{x\to\infty} ||x||_2 \\
0 &\leq -\infty,
\end{aligned}$$

which is clear contradiction. Thus as $||x||_2 \to \infty$, the magnitude of $f(x)$ must approach ∞, which shows that $V(x) \to \infty$ as $||x||_2 \to \infty$.

(c) We have sown that $V(x)$ is positive definite and radially unbounded. The time derivative of the function is found as

$$\begin{aligned}
\dot{V}(x) &= \dot{f}^T(x) P f(x) + f^T(x) P \dot{f}(x) \\
&= \left(\frac{\partial}{\partial x} f(x) \dot{x}\right)^T P f(x) + f^T(x) P \left(\frac{\partial}{\partial x} f(x) \dot{x}\right) \\
&= \left(\frac{\partial f(x)}{\partial x} f(x)\right)^T P f(x) + f^T(x) P \left(\frac{\partial f(x)}{\partial x} f(x)\right) \\
&= f^T(x) \left(\frac{\partial f(x)}{\partial x}\right)^T P f(x) + f^T(x) P \left(\frac{\partial f(x)}{\partial x} f(x)\right) \\
&= f^T(x) \left(P \frac{\partial f(x)}{\partial x} + \left(\frac{\partial f(x)}{\partial x}\right)^T P\right) f(x) \\
&\leq -f^T(x) f(x) \\
&= -\left\| f(x) \right\|_2^2
\end{aligned}$$

Since origin is a unique equilibrium point and all of the conditions are globally, the origin is a globally asymptotically stable equilibrium point.