TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
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Fall 2016 - Solution to Assignment 5

1. Using $V(x) = a\int_0^x h(\sigma)$, we have

$$\dot{V} = ah(x)\dot{x} = h(x)\left[-x + \frac{1}{k}h(x) + u\right] = \frac{1}{k}h(x)\left[h(x) - kx\right] + h(x)u$$

By using the sector condition $h \in [0, k]$ and Definition 6.2 (in Khalil on p. 232), we know that

$$h(x)\left[h(x) - kx\right] \leq 0$$

which leads to

$$\dot{V} = \frac{1}{k}h(x)\left[h(x) - kx\right] + h(x)u \leq yu$$

Thus, by definition 6.3 the system is passive.

2. The system is given by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -h(x_1) - ax_2 + u \\
y &= kx_2 + u
\end{aligned}$$

where

$$\begin{aligned}
a &> 0 \\
k &> 0 \\
h &\in [\alpha_1, \infty] \\
&\Rightarrow zh(z) \geq \alpha_1 z^2 \\
\alpha_1 &> 0
\end{aligned}$$

A storage function is given by

$$V(x) = k\int_0^{x_1} h(z)\,dz + x^T P x$$

where $p_{11} = ap_{12}$, $p_{22} = \frac{k}{2}$ and $0 < p_{12} < \min\left\{2\alpha_1,\ \frac{ak}{2}\right\}$. The time derivative of the storage functions along the trajectories of the system is found as

$$\begin{aligned}
\dot{V}(x) &= k\frac{\partial}{\partial x_1}\left(\int_0^{x_1} h(z)\,dz\right)\dot{x}_1 + \dot{x}^T P x + x^T P \dot{x} \\
&= kh(x_1)\dot{x}_1 + 2x^T P\dot{x} \\
&= kh(x_1)x_2 - h(x_1)kx_2 + kux_2 - 2h(x_1)x_1p_{12} + 2ux_1p_{12} - akx_2^2 + 2x_2^2p_{12} \\
&= -akx_2^2 + 2p_{12}x_2^2 - 2p_{12}h(x_1)x_1 + 2p_{12}ux_1 + kux_2 \\
&= -akx_2^2 + 2p_{12}x_2^2 - 2p_{12}h(x_1)x_1 + 2p_{12}ux_1 + (kx_2 + u)u - u^2 \\
&= -akx_2^2 + 2p_{12}x_2^2 - 2p_{12}h(x_1)x_1 + 2p_{12}ux_1 - u^2 + yu
\end{aligned}$$

By rewriting this last expression it can be seen that

$$\begin{aligned}
yu &= \dot{V}(x) + akx_2^2 - 2p_{12}x_2^2 + 2p_{12}h(x_1)x_1 - 2p_{12}ux_1 + u^2 \\
&= \dot{V}(x) + (ak - 2p_{12})x_2^2 + 2p_{12}h(x_1)x_1 + (u - p_{12}x_1)^2 - p_{12}^2x_1^2 \\
&\geq \dot{V}(x) + (ak - 2p_{12})x_2^2 + 2p_{12}\alpha_1x_1^2 - p_{12}^2x_1^2 + (u - p_{12}x_1)^2 \\
&= \dot{V}(x) + (ak - 2p_{12})x_2^2 + \left(2p_{12}\alpha_1 - p_{12}^2\right)x_1^2 + (u - p_{12}x_1)^2 \\
&\geq \dot{V}(x) + (ak - 2p_{12})x_2^2 + \left(2p_{12}\alpha_1 - p_{12}^2\right)x_1^2 \\
&= \dot{V}(x) + \psi(x)
\end{aligned}$$

where

$$\psi(x) = (ak - 2p_{12})x_2^2 + p_{12}(2\alpha_1 - p_{12})x_1^2$$

Since $0 < p_{12} < \min\left\{2\alpha_1,\ \frac{ak}{2}\right\}$ we have that $\psi(x)$ is positive definite. Hence, by definition 6.3 the system is strictly passive.

3. (Duckmaze)

   (a) We have

$$\begin{aligned}
\dot{V} &= \tilde{x}_1\dot{\tilde{x}}_1 + m\tilde{x}_2\dot{\tilde{x}}_2 \\
&= \tilde{x}_1\tilde{x}_2 + \tilde{x}_2\left[-f_3\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] - f_1\tilde{x}_1 - d\tilde{x}_2 + \tilde{u}\right]
\end{aligned}$$

   Selecting $\tilde{u}$ as

$$\tilde{u} = f_3\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] + f_1\tilde{x}_1 - \tilde{x}_1 + v$$

   yields

$$\dot{V} = -d\tilde{x}_2^2 + \tilde{x}_2 v \tag{1}$$

   This means that the system is passive from the input v to the output $y = \tilde{x}_2$.

   (b) The zero state observability is checked:

$$y = 0 \implies \tilde{x}_2 = 0 \implies \dot{\tilde{x}}_2 = 0 \implies -f_3\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] - f_1\tilde{x}_1 = 0 \tag{2}$$

   This means that $\tilde{x}_1 = 0$, so the system is zero state observable.

   (c) We have shown that the system is passive and zero-state observable, and it is clear that the storage function $V = \frac{1}{2}\left(\tilde{x}_1^2 + m\tilde{x}_2^2\right)$ is radially unbounded and positive definite. Hence, according to Theorem 14.4 in Khalil, the origin can be globally stabilized by $v = -\phi(y)$ where ϕ is any locally Lipschitz function such that $\phi(0) = 0$ and $y\phi(y) > 0$ for all $y \neq 0$.
   The function ϕ is selected as $\phi = k_2 y = k_2\tilde{x}_2$ which gives the controller

$$v = -k_2\tilde{x}_2 \tag{3}$$

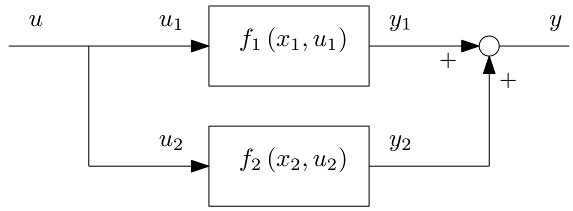

Figure 1: Parallel connected systems.

This controller makes the origin globally asymptotically stable. The coordinates are transformed back to the original coordinates x_1, x_2 by using the relationships $\tilde{x}_1 = x_1 - x_{1d}, \tilde{x}_2 = x_2$. Recall from assignment 2 that $u = u_0 + \tilde{u}$ where u_0 was found to be

$$u_0 = f_3 x_{1d}^3 + f_1 x_{1d} + mg \tag{4}$$

The result is

$$\begin{aligned} u &= u_0 + \tilde{u} \\ &= f_3 x_{1d}^3 + f_1 x_{1d} + mg + f_3 \left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3 \right] + f_1 \tilde{x}_1 - \tilde{x}_1 + v \\ &= f_3 x_{1d}^3 + f_1 x_{1d} + mg + f_3 (\tilde{x}_1 + x_{1d})^3 - f_3 x_{1d}^3 + f_1 \tilde{x}_1 - \tilde{x}_1 + v \\ &= f_3 (\tilde{x}_1 + x_{1d})^3 + f_1 (\tilde{x}_1 + x_{1d}) - \tilde{x}_1 + v \\ &= f_x x_1^3 + f_1 x_1 - (x_1 - x_{1d}) - k_2 x_2 \end{aligned} \tag{5}$$

(d) In Assignment 2 the input u was biased to move the equilibrium point to a desired equilibrium point. The disturbance w can be seen on as a contribution to this bias. This means that the input u is now described by $u = (u_0 + w) + \tilde{u}$, not $u = u_0 + \tilde{u}$ as before and the equilibrium point will not be moved to $(x_{1d}, 0)$ but to another equilibrium point which will be called $(x_{1d,w}, 0)$. A similar change of coordinate as in Assignment 2 (Exercise 1b) will show that the controller $u = (u_0 + w) + \tilde{u}$ makes the equilibrium point $(x_{1d,w}, 0)$ globally asymptotically stable. The conclusion is therefore that in the presence of the constant disturbance $w \neq 0$ the controller (5) will give a stationary deviation in the position x_1 from the reference value x_{1d} since $x_{1d,w} \neq x_{1d}$ when $w \neq 0$.

4. A parallel connection, as seen in Figure 1, is characterized by

$$\begin{aligned} u &= u_1 = u_2 \\ y &= y_1 + y_2 \end{aligned}$$

where the two systems is given by

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, u_1) \\ \dot{x}_2 &= f_2(x_2, u_2) \end{aligned}$$

with the storage functions $V_1(x_1)$ and $V_2(x_2)$.
Suppose the overall storage function $V(x) = V_1(x) + V_2(x)$ where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$. Then

$$
\begin{aligned}
\dot{V} &= \frac{\partial}{\partial x} V(x) f(x,u) = \begin{bmatrix} \frac{\partial V_1(x_1)}{x_1} & \frac{\partial V_2(x_2)}{x_2} \end{bmatrix} \begin{bmatrix} f_1(x_1,u_1) \\ f_2(x_2,u_2) \end{bmatrix} \\
&= \frac{\partial V_1(x_1)}{x_1} f_1(x_1,u_1) + \frac{\partial V_2(x_2)}{x_2} f_2(x_2,u_2) \qquad (6)
\end{aligned}
$$

We know that the passivity properties of the interconnected systems may be expressed as

$$
\dot{V}_1 = \frac{\partial V_1(x_1)}{x_1} f_1(x_1,u_1) \leq u_1^T y_1 - u_1^T \varphi_1(u_1) - y_1^T \rho_1(y_1) - \psi_1(x_1) \qquad (7)
$$

$$
\dot{V}_2 = \frac{\partial V_2(x_2)}{x_2} f_2(x_2,u_2) \leq u_2^T y_2 - u_2^T \varphi_2(u_2) - y_2^T \rho_2(y_2) - \psi_2(x_2) \qquad (8)
$$

Inserting (7) and (8) into (6) leads to

$$
\begin{aligned}
\dot{V} = \frac{\partial}{\partial x} V(x) f(x,u) &\leq u_1^T y_1 - u_1^T \varphi_1(u_1) - y_1^T \rho_1(y_1) - \psi_1(x_1) \\
&+ u_2^T y_2 - u_2^T \varphi_2(u_2) - y_2^T \rho_2(y_2) - \psi_2(x_2) \\
&= u^T y - u^T \varphi(u) + y_1^T \rho_1(y_1) + y_2^T \rho_2(y_2) + \psi(x) \qquad (9)
\end{aligned}
$$

where

$$
\varphi(u) = \varphi_1(u_1) + \varphi_2(u_2) = \varphi_1(u) + \varphi_2(u) \qquad (10)
$$

$$
\psi(x) = \psi_1(x_1) + \psi_2(x_2) \qquad (11)
$$

From (9)–(11) it can be seen that the parallel connection keeps the passivity properties of passive, input strictly passive and stricly passive from the interconnected systems.
For the output strictly passive property, we assume

$$
y_i^T \rho_i(y_i) \geq \delta_i y_i^T y_i \qquad (12)
$$

for some positive δ_i. Using (12) and $\delta = \min\{\delta_1,\ \delta_2\}$ we may rewrite $y_1^T \rho_1(y_1) + y_2^T \rho_2(y_2)$ according to

$$
\begin{aligned}
y_1^T \rho_1(y_1) + y_2^T \rho_2(y_2) &\geq \delta_1 y_1^T y_1 + \delta_2 y_2^T y_2 \\
&\geq \delta y_1^T y_1 + \delta y_2^T y_2 \\
&= \delta \left(y_1^T y_1 + y_2^T y_2 \right) \\
&\geq \delta \left(\frac{1}{2} (y_1 + y_2)^T (y_1 + y_2) \right) \\
&= \frac{1}{2} \delta y^T y
\end{aligned}
$$

where we used the fact that

$$(y_1 + y_2)^T (y_1 + y_2) \leq 2 \left(y_1^T y_1 + y_2^T y_2 \right)$$

Then (9) will be expressed as

$$\dot{V} = \frac{\partial}{\partial x} V(x) f(x, u) \quad \leq \quad u^T y - u^T \varphi(u) + \frac{1}{2} \delta y^T y + \psi(x) \tag{13}$$

We see that the parallel connection also keeps the property of output strictly passive from the interconnected systems.

5. (a) The Lyapunov function candidate is continuously differentiable. Also, using that $\int \psi(z)dz \geq \int k_1 z dz = 1/2 k_1 x_1^2$, we see that $V(x) \geq 1/2 k x_1^2 + x_2^2$, and is thus both positive definite and radially unbounded. Furthermore, when $\delta = 0$ we have

$$\begin{aligned} \dot{V}(x) &= \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2 = \psi(x_1)(-3x_1 + 2x_2) + x_2(-2\psi(x_1) - x_2) \\ &= -3x_1 \psi(x_1) - x_2^2 \leq -3k_1 x_1^2 - x_2^2 \end{aligned}$$

for all x, i.e. $\dot{V}$ is negative definite in x. Since $V(x)$ satisfies all conditions of Theorem 4.2 in Khalil, it follows that the origin is GAS. (Furthermore, V(x) satisfies all conditions of Theorem 4.10, and the origin is thus GES).

(b) Using $V(x) = \int_0^{x_1} \psi(z)\, dz + \frac{1}{2} x_2^2$ we have

$$\begin{aligned} \dot{V}(x) &= \psi(x_1)(-3x_1 + 2x_2) + x_2(-2\psi(x_1) - x_2 + \delta) = -3x_1 \psi(x_1) - x_2^2 + x_2 \delta \\ &\leq -3k_1 x_1^2 - x_2^2 + x_2 \delta = -\phi(x) + \delta y \end{aligned}$$

where $\phi(x) = 3k_1 x_1^2 + x_2^2$ is positive definite. Hence it is (state) strictly passive.

(c) Using $V(x) = \int_0^{x_1} \psi(z)\, dz + \frac{1}{2} x_2^2$ we have

$$\begin{aligned} \dot{V}(x) &= \psi(x_1)(-3x_1 + 2x_2) + x_2(-2\psi(x_1) - x_2 + \delta) = -3x_1 \psi(x_1) - x_2^2 + x_2 \delta \\ &\leq -3k_1 x_1^2 - x_2^2 + x_2 \delta \leq -x_2^2 + x_2 \delta = -y\rho(y) + \delta y \end{aligned}$$

where $\rho(y) = y$. Hence, it is output strictly passive.

(d) Using $V(x) = \int_0^{x_1} \psi(z)\, dz + \frac{1}{2} x_2^2$ we have that inequality (4.48) in Theorem 4.19 in Khalil is satisfied with $\alpha_1(\|x\|) = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} x_2^2$ and $\alpha_2(\|x\|) = \frac{1}{2} k_2 x_1^2 + \frac{1}{2} x_2^2$. Furthermore,

$$\begin{aligned} \dot{V}(x) &\leq -3k_1 x_1^2 - x_2^2 + x_2 u \\ &\leq -\min(3k_1, 1)\|x\|^2 + \|x\||\delta| \\ &= -\big(\min(3k_1, 1) - \theta\big)\|x\|^2 - \theta\|x\|^2 + \|x\||\delta| \qquad \text{for } 0 < \theta < \min(3k_1, 1) \\ &\leq -\big(\min(3k_1, 1) - \theta\big)\|x\|^2 \end{aligned}$$

when $-\theta\|x\|^2 + \|x\||\delta| \leq 0$, i.e. $\|x\| \geq \frac{|\delta|}{\theta}$. Then, by theorem 4.19 in Khalil, the system is ISS.

(e) The unforced system is given by

$$\begin{aligned}\dot{x}_1 &= -3x_1 + 2x_2 \\ \dot{x}_2 &= -2\psi(x_1) - x_2 \\ y &= x_2\end{aligned}$$

Consider

$$S = \{x \in \mathbb{R}^2 | y = 0\} = \{x \in \mathbb{R}^2 | x_2 = 0\}$$

then for every $(x_1, x_2) \in S$, i.e. $y(t) \equiv 0$, we have $x_2(t) \equiv 0 \implies \dot{x_2} \equiv 0 \implies -2\psi(x_1) - 0 \equiv 0 \implies \psi(x_1) \equiv 0 \implies x_1(t) \equiv 0, \dot{x}_1 \equiv 0$. Hence, no other solution can stay identically in S other than the zero solution. Thus the system is zero state observable.

6. The system is given by

$$h(s) = 1.5\frac{(1+2s)(1+s)}{(1+3s)(1+0.5s)}$$

which has poles with real parts less than zero.

(a) An upper bound on the magnitude of $h(j\omega)$ is found as

$$\begin{aligned}\left|h(j\omega)\right| &= \left|1.5\frac{(1+j2\omega)(1+j\omega)}{(1+j3\omega)(1+j0.5\omega)}\right| \\ &= 1.5\left|\frac{1+j2\omega}{1+j3\omega}\right|\left|\frac{1+j\omega}{1+j0.5\omega}\right| \\ &= 3\left|\frac{1+j2\omega}{1+j3\omega}\right|\left|\frac{1+j\omega}{2+j\omega}\right|\end{aligned}$$

The absolute value of $h(j\omega)$ is given as

$$\left|h(j\omega)\right| = h(j\omega)\, h(-j\omega) \tag{14}$$

which results in

$$\left|\frac{1+2j\omega}{1+j3\omega}\right| = \frac{1+2^2\omega^2}{1+3^2\omega^2} \leq 1$$

and

$$\left|\frac{1+j\omega}{2+j\omega}\right| = \frac{1+\omega^2}{2^2+\omega^2} \leq 1$$

An upper bound on $\left|h(j\omega)\right|$ is therefore given by

$$\left|h(j\omega)\right| \leq 3 = \frac{K_p\beta}{\alpha} \tag{15}$$

(b) We have that

$$\begin{aligned} h(j\omega) &= 1.5\frac{(1+j2\omega)(1+j\omega)}{(1+j3\omega)(1+j0.5\omega)} \\ &= 1.5\frac{(1+j2\omega)(1+j\omega)(1-j3\omega)(1-j0.5\omega)}{(1+3^2\omega^2)(1+0.5^2\omega^2)} \end{aligned}$$

where the numerator is calculated as

$$\begin{aligned} (1+j2\omega)(1+j\omega)(1-j3\omega)(1-j0.5\omega) &= \left(1+j3\omega+j^2 2\omega^2\right)\left(1-j3.5\omega+j^2 1.5\omega^2\right) \\ &= 1-j0.5\omega-j^2 7\omega^2-j^3 2.5\omega^3+j^4 3\omega^4 \\ &= \left(1+7\omega^2+3\omega^4\right)+j\left(2.5\omega^3-0.5\omega\right) \end{aligned}$$

The real value of $h(j\omega)$ is

$$\begin{aligned} \text{Re}\left[h(j\omega)\right] &= 1.5\frac{1+7\omega^2+3\omega^4}{(1+3^2\omega^2)(1+0.5^2\omega^2)} \\ &= \frac{1.5+10.5\omega^2+4.5\omega^4}{1+9.25\omega^2+1.5\omega^4} \\ &= \frac{1+9.25\omega^2+1.5\omega^4+1.25\omega^2+3\omega^4}{1+9.25\omega^2+1.5\omega^4} \end{aligned}$$

and it can be recognized that to prove

$$\text{Re}\left[h(j\omega)\right] \geq 1 = K_p \tag{16}$$

is the same as proving

$$\left(1.25\omega^2+3\omega^4\right) \geq 0$$

which is true.

(c) To prove that h is passive is the same as proving Re $\left[h(j\omega)\right] \geq 0\ \forall\omega$ (see Appendix A in Assignment 5). Since Re $\left[h(j\omega)\right] \geq K_p > 0\ \forall\omega$ we conclude that the control law is passive.

(d) To prove that h is input strictly passive is the same as proving that Re $\left[h(j\omega)\right] \geq \delta \geq 0\ \forall\omega$ for some positive δ (see Appendix A in Assignment 5). Since Re $\left[h(j\omega)\right] \geq K_p > 0\ \forall\omega$ we conclude that the control law is input strictly passive.

(e) To prove that the system is output strictly passive is the same as proving that Re $\left[h(j\omega)\right] \geq \varepsilon\left|h(j\omega)\right|^2\ \forall\omega$ for some positive ε (see Appendix A in Assignment 5). From the assumptions in the exercise we know that

$$\begin{aligned} \left|h(j\omega)\right| &\leq \frac{K_p\beta}{\alpha} \\ \text{Re}\left[h(j\omega)\right] &\geq K_p \end{aligned}$$

Using these inequalities, an upper bound on $\left|h\left(j\omega\right)\right|^2$ is found as

$$\begin{aligned}\left|h\left(j\omega\right)\right|^2 &\leq \left(\frac{K_p\beta}{\alpha}\right)^2 \\ &= \frac{K_p\beta^2}{\alpha^2}K_p \\ &\leq \frac{K_p\beta^2}{\alpha^2}\text{Re}\left[h\left(j\omega\right)\right]\end{aligned}$$

which is rewritten as

$$\begin{aligned}\text{Re}\left[h\left(j\omega\right)\right] &\geq \frac{\alpha^2}{K_p\beta^2}\left|h\left(j\omega\right)\right|^2 \\ &= \varepsilon\left|h\left(j\omega\right)\right|^2\end{aligned}$$

and output strictly passivity of the control law is concluded.

(f) The system is given by

$$h\left(s\right) = \frac{u\left(s\right)}{e\left(s\right)} = K_p\beta\frac{\left(1+T_i s\right)\left(1+T_d s\right)}{\left(1+\beta T_i s\right)\left(1+\alpha T_d s\right)}$$

where e is the input and u is the output. When investigating if a system is zero-state observable, the system is analyzed with inputs set to zero, $e = 0$. This leads to the equation

$$\begin{aligned}\frac{u\left(s\right)}{e\left(s\right)} &= K_p\beta\frac{\left(1+T_i s\right)\left(1+T_d s\right)}{\left(1+\beta T_i s\right)\left(1+\alpha T_d s\right)} \\ &\Leftrightarrow u\left(s\right)\left(1+\beta T_i s\right)\left(1+\alpha T_d s\right) = K_p\beta\left(1+T_i s\right)\left(1+T_d s\right)e\left(s\right) \\ &\Rightarrow u\left(s\right)\left(1+\beta T_i s\right)\left(1+\alpha T_d s\right) = 0 \text{ when } e\left(s\right) = 0 \\ &\Leftrightarrow u\left(s\right)\left(1+\beta T_i s+\alpha T_d s+\beta T_i\alpha T_d s^2\right) = 0 \\ &\Leftrightarrow u+\beta T_i\dot{u}+\alpha T_d\dot{u}+\beta T_i\alpha T_d\ddot{u} = 0\end{aligned}$$

Let $z_1 = u$, $z_2 = \dot{u}$ and $y = z_1$, then the control law with zero input can be expressed as

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \frac{1}{\beta T_i\alpha T_d}\left(-z_1-\left(\beta T_i+\alpha T_d\right)z_2\right) \\ y &= z_1\end{aligned}$$

To show that the system is zero-state observable we require that no solution can stay identical in $y = 0$ other than the trivial solution $z \equiv 0$ (see Definition 6.5 on p. 243 in Khalil). This is done as

$$\begin{aligned}y\left(t\right) &\equiv 0 \Leftrightarrow z_1\left(t\right) \equiv 0 \\ \dot{z}_1\left(t\right) &= 0 \Rightarrow z_2\left(t\right) \equiv 0 \\ \dot{z}_2 &= 0 \Rightarrow z_2 = \frac{1}{\left(\beta T_i+\alpha T_d\right)}z_1 = 0\end{aligned}$$

by which we conclude that the PID control law is zero-state observable.

7. The system is given by

$$
\begin{aligned}
J_1\dot{\omega}_1 &= (J_2 - J_3)\,\omega_2\omega_3 + u_1 \\
J_2\dot{\omega}_2 &= (J_3 - J_1)\,\omega_3\omega_1 + u_2 \\
J_3\dot{\omega}_3 &= (J_1 - J_2)\,\omega_1\omega_2 + u_3
\end{aligned}
$$

where $u = [u_1\ u_2\ u_3]^T$ and $\omega = [\omega_1\ \omega_2\ \omega_3]$.

(a) Let $V(\omega) = \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2$ be a candidate for a storage function. The time derivative along the trajectories of the system is found as

$$
\begin{aligned}
\dot{V}(\omega) &= J_1\dot{\omega}_1\omega_1 + J_2\dot{\omega}_2\omega_2 + J_3\dot{\omega}_3\omega_3 \\
&= \big((J_2 - J_3)\,\omega_2\omega_3 + u_1\big)\,\omega_1 \\
&\quad + \big((J_3 - J_1)\,\omega_3\omega_1 + u_2\big)\,\omega_2 \\
&\quad + \big((J_1 - J_2)\,\omega_1\omega_2 + u_3\big)\,\omega_3 \\
&= (J_2 - J_3)\,\omega_1\omega_2\omega_3 + (J_3 - J_1)\,\omega_1\omega_2\omega_3 + (J_1 - J_2)\,\omega_1\omega_2\omega_3 \\
&\quad + u_1\omega_1 + u_2\omega_2 + u_3\omega_3 \\
&= (J_2 - J_3 + J_3 - J_1 + J_1 - J_2)\,\omega_1\omega_2\omega_3 + u_1\omega_1 + u_2\omega_2 + u_3\omega_3 \\
&= u^T\omega
\end{aligned}
$$

which shows that the map from u to ω is lossless with the storage function $V(\omega)$.

(b) With $u = -K\omega + v$ where $K = K^T$where we have that

$$
\begin{aligned}
\dot{V}(\omega) &= u^T\omega \\
&= (-K\omega + v)^T\,\omega \\
&= -\omega^T K^T\omega + v^T\omega \\
&= v^T\omega - \omega^T K\omega \\
&\leq v^T\omega - \lambda_{\min}(K)\,\omega^T\omega \\
&\Rightarrow v^T\omega \geq \dot{V}(\omega) + \lambda_{\min}(K)\,\omega^T\omega
\end{aligned}
$$

From the last equation it can be seen that the system is output strictly passive from v to ω with $v^T\omega \geq \dot{V}(\omega) + \lambda_{\min}(K)\,\omega^T\omega$. Hence, the map from v to ω is finite gain L_2 stable with L_2 gain less than or equal to $\frac{1}{\lambda_{\min}(K)}$ (Lemma 6.5).

(c) With $u = -K\omega$, we have that

$$
\dot{V}(\omega) \leq -\lambda_{\min}(K)\,\omega^T\omega
$$

for the system $\dot{\omega} = f(\omega, -K\omega) = f'(\omega)$. Since $V(\omega)$ is positive definite and radially unbounded and $\dot{V}(\omega)$ is negative definite, we conclude that the system is globally asymptotically stable.

8. Two systems

$$H_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - h_1(x_2) + e_1 \\ y_1 = x_2 \end{cases}$$

and

$$H_2 : \begin{cases} \dot{x}_3 = -x_3 + e_2 \\ y_2 = h_2(x_3) \end{cases}$$

are connected as shown in Figure 6.11 in Khalil. The functions $h_i(\cdot)$ are locally Lipschitz and $h_i(\cdot) \in (0, \infty]$. Further, the function $h_2(z)$ satisfies $\left|h_2(z)\right| \geq \frac{|z|}{\left(1+z^2\right)}$.

(a) First the passivity properties of H_1 is investigated. Let $V_1(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$ be a candidate for a storage function. The time derivative along the trajectories of the system is found as

$$\begin{aligned} \dot{V}_1(x_1, x_2) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1x_2 + x_2\left(-x_1 - h_1(x_2) + e_1\right) \\ &= x_1x_2 - x_1x_2 - h_1(x_2)x_2 + e_1x_2 \\ &= -h_1(x_2)x_2 + e_1x_2 \\ &= -h_1(y_1)y_1 + e_1y_1 \\ &\Rightarrow e_1y_1 = \dot{V}_1(x_1, x_2) + y_1h_1(y_1) \end{aligned}$$

Since $h_1 \in (0, \infty]$, we know that $y_1h_1(y_1) > 0 \quad \forall y_1 \neq 0$ (See Definition 6.2 in Khalil on pp. 232–233). Thus, H_1 is output strictly passive.
The passivity properties of H_2 is investigated by using $V_2(x_3) = \int_0^{x_3} h_2(z)\,dz$ as a candidate for a storage function. The time derivative along the trajectories of the system is found as

$$\begin{aligned} \dot{V}_2(x_3) &= \frac{\partial}{\partial x_3}\left(\int_0^{x_3} h_2(z)\,dz\right)\dot{x}_3 \\ &= h_2(x_3)(-x_3 + e_2) \\ &= -x_3h_2(x_3) + h_2(x_3)e_2 \\ &= -x_3h_2(x_3) + y_2e_2 \\ &\Rightarrow y_2e_2 = \dot{V}_2(x_3) + x_3h_2(x_3) \end{aligned}$$

Since $h_2 \in (0, \infty]$, we know that $x_3h_2(x_3) > 0 \quad \forall x_3 \neq 0$ (See Definition 6.2 in Khalil on pp. 232–233). Thus, H_2 is strictly passive.
By Theorem 6.1 we conclude that the feedback connection is passive.

(b) Asymptotic stability of the unforced system is shown by using Theorem 6.3 from Khalil. Since we have one strictly passive system and one output strictly passive system, we need to show that the system which is output strictly passive also is zero-state observable. It can be recognized that no solution can stay identical in $S = \{x_2 = 0\}$ other than the trivial solution $(x_1, x_2) = (0, 0)$. That is

$$\begin{aligned} y_1 &\equiv 0 \Leftrightarrow x_2 \equiv 0 \\ \dot{x}_2 &= 0 \Rightarrow x_1 = -h_1(x_2) = 0 \end{aligned}$$

Hence, the unforced system is asymptotically stable. To prove global results, we need to show that the storage functions are radially unbounded. The first storage function is given by

$$\begin{aligned} V_1(x_1, x_2) &= \frac{1}{2}\left(x_1^2 + x_2^2\right) \\ &= \frac{1}{2}\left\|(x_1, x_2)\right\|_2 \end{aligned}$$

which clearly is radially unbounded. The second storage function is given by

$$\begin{aligned} V_2(x_3) &= \int_0^{x_3} h_2(z)\, dz \\ &\geq \int_0^{x_3} \frac{|z|}{(1+z^2)} dz \\ &= \int_0^{x_3} \frac{z}{(1+z^2)} dz \\ &= \frac{1}{2}\ln\left(1 + x_3^2\right) \end{aligned}$$

where it can be recognized that $V_2(x_3) \to \infty$ as $|x_3| \to \infty$. Hence, the unforced system is globally asymptotically stable.

9. Two systems

$$H_1 : \begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = -x_1^3 - x_2 + e_1 \\ y_1 = x_2 \end{cases}$$

and

$$H_2 : \begin{cases} \dot{x}_3 = -x_3 + e_2 \\ y_2 = x_3^3 \end{cases}$$

are connected as shown in Figure 6.11 in Khalil.

(a) First the passivity properties of H_1 is investigated. Let $V_1(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ be a candidate for a storage function. The time derivative along the trajectories of the system is found as

$$\begin{aligned}
\dot{V}_1(x_1, x_2) &= x_1^3\dot{x}_1 + x_2\dot{x}_2 \\
&= x_1^3(-x_1 + x_2) + x_2\left(-x_1^3 - x_2 + e_1\right) \\
&= -x_1^4 + x_1^3x_2 - x_1^3x_2 - x_2^2 + x_2e_1 \\
&= -x_1^4 - x_2^2 + e_1y_1 \\
&\Rightarrow \ e_1y_1 = \dot{V}_1(x_1, x_2) + x_1^4 + x_2^2
\end{aligned}$$

Hence, H_1 is strictly passive with storage function $V_1(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.
The passivity properties of H_2 is investigated by using $V_2(x_3) = \frac{1}{4}x_3^4$ as a candidate for a storage function. The time derivative along the trajectories of the system is found as

$$\begin{aligned}
\dot{V}_2(x_3) &= x_3^3\dot{x}_3 \\
&= x_3^3(-x_3 + e_2) \\
&= -x_3^4 + x_3^3e_2 \\
&= -x_3^4 + e_2y_2
\end{aligned}$$

Hence, H_2 is strictly passive with storage function $V_2(x_3) = \frac{1}{4}x_3^4$, and the feedback connection is passive.

(b) Since both systems are strictly passive with radially unbounded storage functions, it follows from Theorem 6.3 that the origin of the unforced system is asymptotically stable.

10. Consider the system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1^3 + v \\
y &= x_2
\end{aligned}$$

With $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$

$$\dot{V} = x_1^3x_2 - x_1^3x_2 + x_2v = yv$$

Hence the system is passive. With $v = 0$

$$y(t) \equiv 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

Thus the system is zero-state observable. Therefore, we can globally stabilize the system by $v = -\phi(y)$ for any $\phi(y)$ such that

$$\begin{gathered}
\phi(y) \ \text{ is locally Lipschitz} \\
\phi(0) = 0 \\
y\phi(y) > 0 \text{ for all } y \neq 0
\end{gathered}$$

Pick $\phi(y) = -\psi(-y)$, then

$$\begin{gathered}\phi(y) \text{ is locally Lipschitz} \\ \phi(0) = -\psi(-0) = 0 \\ y\phi(y) = -y\psi(-y) > 0 \text{ for all } -y \neq 0\end{gathered}$$

Thus $v = \psi(-y)$ is the stabilizing feedback which means $u = -y$.

11. Let the input to the system be denoted $\tilde{u}$ and the output be denoted $\tilde{y}$. From the block diagram we have the following relations

$$\begin{aligned}\tilde{y} &= h(t,u) - K_1 u \\ \tilde{u} + \tilde{y} &= K u\end{aligned}$$

From the sector condition we have that

$$\begin{gathered}\left(h(t,u) - K_1 u\right)^T \left(h(t,u) - K_2 u\right) \leq 0 \\ K = K_2 - K_1 = K^T > 0\end{gathered} \tag{17}$$

Evaluating the block diagram it can be seen that

$$h(t,u) - K_1 u = \tilde{y} \tag{18}$$

and that

$$\begin{aligned}h(t,u) - K_2 u &= h(t,u) - K_2 u - K_1 u + K_1 u \\ &= \tilde{y} - (K_2 - K_1) u \\ &= \tilde{y} - K u \\ &= \tilde{y} - \tilde{u} - \tilde{y} \\ &= -\tilde{u}\end{aligned} \tag{19}$$

Using (18), (19) and the sector condition (17) we have

$$\begin{aligned}\left(h(t,u) - K_1 u\right)^T \left(h(t,u) - K_2 u\right) &= \tilde{y}^T(-\tilde{u}) \\ &= -\tilde{u}^T \tilde{y} \\ &\leq 0 \\ &\Rightarrow \tilde{u}^T \tilde{y} \geq 0\end{aligned}$$

which implies that the system is passive from $\tilde{u}$ to $\tilde{y}$, which corresponds to being in sector $[0, \infty]$.