NTNU – Trondheim
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TTT4120 Digital Signal Processing
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Lecture: Laplace - Review

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Contents and learning outcomes*

- Definition of Laplace transform and its existence
- Some properties of the Laplace transform
- System analysis and rational expressions

* For more details, read the document laplace_transform_final.pdf which is available on ItsLearning

Motivation

- Linear time-invariant system:

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) * x(t)$

$X(\Omega) \qquad\qquad\qquad Y(\Omega) = H(\Omega)X(\Omega)$

- What if $h(t) = e^{t}u(t)$?
  - System is unstable, $\int |h(t)| dt$ not finite
  - Fourier transform of $h(t)$ does not exists
- Can we analyze such systems using a transform method while retaining the good properties of the Fourier transform?

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Motivation…

- Consider the continuous-time Fourier transform pair

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\, dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{-j\Omega t}\, d\Omega$$

- What if $x(t) = e^{t}u(t)$?
  - Fourier transform of $x(t)$ does not exists
- Can we analyze such signals using a transform method while retaining the good properties of the Fourier transform?
- Very similar to our development of the z-transform!

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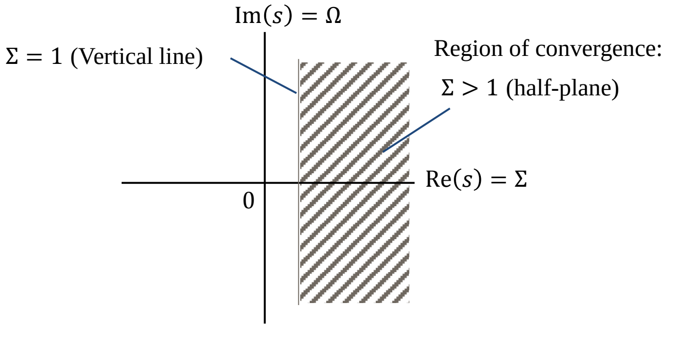
Basic idea

- Capture the source of instability or inapplicability of the FT
- Apply the FT to the modified (captured) signal
- Example: Suppose we have signal $x(t) = e^{t}u(t)$
  - Problem is due to the exponential growth
  - Capture the signal by multiplying it by a decaying exponential stronger than the growing one, i.e., $e^{-\Sigma t}x(t), \Sigma > 0$
  - What values of Σ allows for a FT of $e^{-\Sigma t}x(t)$?

$$\int_{-\infty}^{\infty} x(t)e^{-\Sigma t}e^{-j\Omega t}\,dt = \int_{0}^{\infty} e^{t}e^{-\overbrace{(\Sigma+j\Omega)}^{s}t}\,dt =$$

$$= \left.\frac{e^{(1-s)t}}{(1-s)}\right|_{t=0}^{\infty} = \frac{1}{s-1}, \text{ for } \Sigma > 1$$

Basic idea…



- Convergence has only to do with Σ and not Ω
- We have more general transform of the signal $x(t)$

Basic idea…

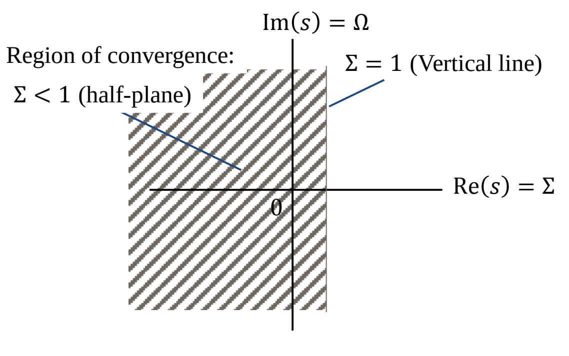
- Example: Suppose we have signal $x(t) = -e^{t}u(-t)$
  - What values of Σ allows for a FT for $e^{-\Sigma t}x(t)$?

$$\int_{-\infty}^{\infty} x(t)e^{-\Sigma t}e^{-j\Omega t}\,dt = \int_{-\infty}^{0} -e^{t}e^{-\overbrace{(\Sigma+j\Omega)}^{s}t}\,dt =$$

$$= \left.\frac{-e^{(1-s)t}}{(1-s)}\right|_{t=-\infty}^{0} = \frac{1}{s-1}, \text{ for } \Sigma < 1$$

- Same expression as before but different ROC

Basic idea…



Definition of Laplace transform

- The Laplace transform of a continuous-time signal $x(t)$ is

$$\boxed{X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt}$$

- Notation: $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s) \quad x(t) = \mathcal{L}^{-1}\{X(s)\}$
- Transforms $x(t)$ into its complex-plane representation $X(S)$
- Transform only exists whenever integral exists
- Region of convergence (ROC) of $X(s)$ is the set of all values of $s = \Sigma + j\Omega$ for which $X(s)$ attains a finite value

Most important properties

- Linearity:

$$x_3(t) = a_1x_1(t) + a_2x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_3(s) = a_1X_1(s) + a_2X_2(s)$$

ROC of $X_3(s)$ at least $\mathcal{R}_{X_1} \cap \mathcal{R}_{X_2}$ but can extend beyond intersection

- Differentiation:

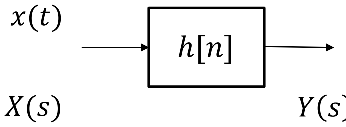
$$\frac{d^k x(t)}{dt^k} \overset{\mathcal{L}}{\leftrightarrow} s^k X(s) \text{ (for zero initial conditions)}$$

Comparison with z-transform

- ROC in z-transforms is a ring or a disc centered at zero
- ROC in Laplace transforms is a half plane or a strip parallel to the $j\Omega$-axis in the s-plane

System analysis

- Systems described by constant-coefficient differential equations



$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$Y(s) = H(s)X(s)$$

- Rational system function

$$H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = C\,\frac{\prod_{k=0}^{M}(s-z_k)}{\prod_{k=0}^{N}(s-p_k)}$$

System analysis

- Frequency response: $H(\Omega) = H(s)|_{s=j\Omega}$
- Stable system: imaginary axis part of region of convergence
- Causal system:
  - ROC: $\mathrm{Re}(s) > \max_k \mathrm{Re}(p_k)$
  - Stable if all poles in left half plane
- Anti-causal system (nonrealizable):
  - ROC: $\mathrm{Re}(s) < \min_k \mathrm{Re}(p_k)$
  - Stable if all poles in right half plane

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Summary

- Take-home messages
  - Concepts behind Laplace- and z-transforms similar
  - If you understand one, the other comes almost for free

- Reason for introducing Laplace in TTT4120 is its importance in IIR filter design

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