TTK4150 Nonlinear Control Systems
Lecture 8

Stability of Perturbed Systems

NTNU
Innovation and Creativity

Previous lecture

Previous lecture:

- Introduced other stability concepts than Lyapunov stability.

In particular

- Motivation and definition of Input-to-State stability (ISS)
- ISS analysis using ISS-Lyapunov functions
- Relations between ISS and Lyapunov stability

- Definition of Input-Output Stability (IOS)
- How to analyze IOS using the definition
- Small-gain theorem

Outline

Today's goals

After today you should...

- Be able to analyze the stability properties of a system under the influence of disturbances
- Know the difference between
  - Vanishing perturbations
  - Nonvanishing perturbations
- Learn useful tools in order to study the stability of a stable system $\dot{x} = f(t,x)$ which is perturbed by another vanishing or nonvanishing vector field $g(t,x)$

Literature

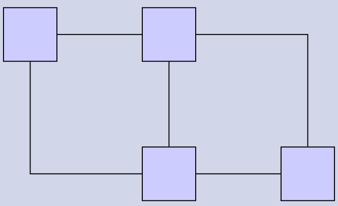
Today's lecture is based on

Khalil **Chapter 9**
Sections 9.1 and 9.2

Perturbed Systems

Interconnected Systems

- The complexity of the analysis grows rapidly as the order of the system increases
- Find a way to simplify the analysis of the system
- Model the system as an interconnection of lower order subsystems



Perturbed Systems

We want to analyse systems on the form

$$\dot{x} = f(t,x) + g(t,x) \quad (1)$$

- $D \subset \mathbb{R}^n$ is a domain that contains the origin $x^* = 0$
- f and $g : [0,\infty) \times D \to \mathbb{R}^n$, piecewise continuous in t and locally Lipschitz in x on $[0,\infty) \times D$
- Nominal system

$$\dot{x} = f(t,x). \quad (2)$$

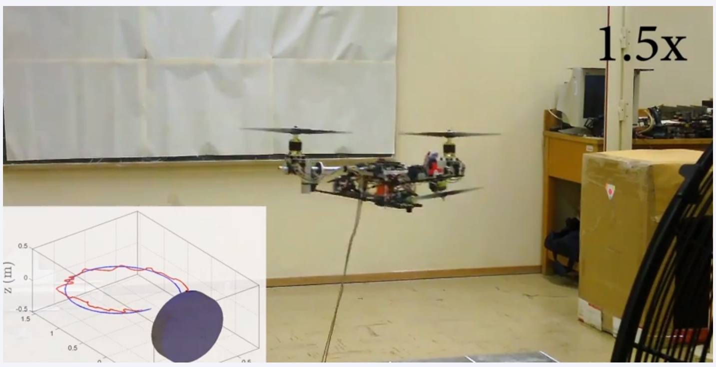
The Perturbation term $g(t,x)$

often unknown, but with a known upper bound on $||g(t,x)||$

- modeling errors, uncertainties, disturbances etc.

BigDog is the alpha male of the Boston Dynamics family of robots

Robust Predictive Flight Control



Worldwide first flight experiment with fully actuated robot arm mounted on an autonomous helicopter

Shark Attacks! - Not So Easy to Eat This Robot

Part I

Vanishing Perturbation

Equilibrium Point

Vanishing Additive Perturbations

- Suppose that $\dot{x} = f(t,x)$ has an exponentially stable equilibrium point at $x^* = 0$
- and suppose that $g(t,0) = 0$ for all t

if $x^* = 0$ is an equilibrium point for the nominal system

$$\dot{x} = f(t,x)$$

$\Rightarrow x^* = 0$ is an equilibrium point for the entire system

$$\dot{x} = f(t,x) + g(t,x)$$

Lyapunov Function

Suppose $x^* = 0$ is

- an exponentially stable equilibrium point of $\dot{x} = f(t,x)$,
- and let $V(t,x)$ be a Lyapunov function that satisfies

$$c_1 \|x\|^2 \le V(t,x) \le c_2 \|x\|^2 \tag{3}$$

$$\frac{\delta V}{\delta t} + \frac{\delta V}{\delta x} f(t,x) \le -c_3 \|x\|^2 \tag{4}$$

$$\left\| \frac{\delta V}{\delta x} \right\| \le c_4 \|x\| \tag{5}$$

for all $(t,x) \in [0,\infty) \times D$ for some positive constants c_1, c_2, c_3 and c_4.

NB

The existence of such a Lyapunov function is guaranteed by Th. 4.14.

Linear growth bound on the perturbation term

Assume that $g(t,x)$ satisfies the linear growth bound

$$\|g(t,x)\| \le \gamma \|x\| \tag{6}$$

It can be shown that

$$\begin{aligned} \dot{V}(t,x) &= \frac{\delta V}{\delta t} + \frac{\delta V}{\delta x} f(t,x) + \frac{\delta V}{\delta x} g(t,x) \\ &\le -c_3 \|x\|^2 + \left\| \frac{\delta V}{\delta x} \right\| \|g(t,x)\| \\ &\le -c_3 \|x\|^2 + c_4 \|x\| \, \gamma \|x\| \end{aligned} \tag{7}$$

$\dot{V}(t,x) < 0$ if

$$\gamma < \frac{c_3}{c_4} \tag{8}$$

Exponentially Stable Equilibrium Point

Lemma 9.1

- Let $x^* = 0$ be an exponentially stable equilibrium point of the nominal system $\dot{x} = f(t,x)$
- Let $V(t,x)$ be a Lyapunov function of the nominal system that satisfies

$$c_1 \|x\|^2 \le V(t,x) \le c_2 \|x\|^2 \quad \text{and} \quad \left\| \frac{\delta V}{\delta x} \right\| \le c_4 \|x\|$$

  in $[0,\infty) \times D$
- Suppose the perturbation term $g(t,x)$ satisfies

$$\|g(t,x)\| \le \gamma \|x\| \quad \text{and} \quad \gamma < \frac{c_3}{c_4}$$

Then, the origin is an exponentially stable equilibrium point of the perturbed system $\dot{x} = f(t,x) + g(t,x)$.

Global Exponentially Stable Equilibrium Point

Global Exponential Stability

If all assumptions hold globally $\Rightarrow$ the origin is globally exponentially stable.

Example: Exp. stable linear nominal system

Example

Consider the system

$$\dot{x} = Ax + g(t,x)$$

where A is Hurwitz and $\|g(t,x)\|_2 \le \gamma \|x\|_2$ for all $t \ge 0$ and all $x \in \mathbb{R}^n$.
Choose $Q = Q^T > 0$ and solve the Lyapunov equation $PA + A^T P = -Q$ for P and use $V(t,x) = x^T P x$.

Example: Exp. stable linear nominal system

Example

- The Lyapunov function satisfies

$$\lambda_{\min}(P)\|x\|_2^2 \le V(t,x) \le \lambda_{\max}(P)\|x\|_2^2 \tag{9}$$

$$\frac{\delta V}{\delta t} + \frac{\delta V}{\delta x} f(t,x) \le -\lambda_{\min}(Q)\|x\|_2^2 \tag{10}$$

$$\left\|\frac{\delta V}{\delta x}\right\| \le 2\lambda_{\max}(P)\|x\|_2 \tag{11}$$

- By Lemma 9.1, $x=0$ is a globally exponentially stable equilibrium point of $\dot{x} = Ax + g(t,x)$ if $\gamma < \frac{\lambda_{\min}(Q)}{2\lambda_{max}(P)}$. By choosing $Q = I$, this ratio is maximized.

Example: Exp. stable equilibrium point

Example

Consider the second-order system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -4x_1 - 2x_2 + \beta x_2^3$$

where the constant $\beta \ge 0$ is unknown. Show that the origin $x^* = 0$ is exponentially stable.

Uniformly Asymptotically Stable $x^* = 0$

Uniformly Asymptotic Stability

- Suppose $x^* = 0$ is a uniformly asymptotically stable equilibrium point of the nominal system $\dot{x} = f(t,x)$, and let $V(t,x)$ be a positive definite, decrescent Lyapunov function that satisfies

$$\frac{\delta V}{\delta t} + \frac{\delta V}{\delta x} f(t,x) \le -W_3(x)$$

for all $(t,x) \in [0,\infty) \times D$ where $W_3(x)$ is positive definite and continuous.

- The derivative of V is given by

$$\dot{V}(t,x) = \frac{\delta V}{\delta t} + \frac{\delta V}{\delta x} f(t,x) + \frac{\delta V}{\delta x} g(t,x) \le -W_3(x) + \left\|\frac{\delta V}{\delta x} g(t,x)\right\|$$

Uniformly Asymptotically Stable $x^* = 0$

Uniformly Asymptotic Stability cont.

- If $V(t,x)$ is positive definite, decrescent, and satisfies

$$\frac{\delta V}{\delta t} + \frac{\delta V}{\delta x} f(t,x) \le -c_3 \phi^2(x) \quad (12)$$

$$\left\| \frac{\delta V}{\delta x} \right\| \le c_4 \phi(x) \quad (13)$$

for all $(t,x) \in [0,\infty) \times D$ for some positive constants c_3 and c_4 and $\phi : \mathbb{R}^n \to \mathbb{R}$ that is positive definite and continuous.

- If the perturbation term satisfies

$$\|g(t,x)\| < \gamma \phi(x) \quad (14)$$

$$\gamma < \frac{c_3}{c_4} \quad (15)$$

Uniformly Asymptotically Stable $x^* = 0$

Uniformly Asymptotic Stability cont.

Then,

$$\dot{V}(t,x) \le -(c_3 - c_4\gamma)\phi^2(x) \quad (16)$$

is negative definite and the perturbed system

$$\dot{x} = f(t,x) + g(t,x)$$

is asymptotically stable.

Example: GAS nominal system

Example

$$\dot{x} = -x^3 + g(t,x).$$

Show that $x^* = 0$ is a GUAS equilibrium point of the perturbed system. Consider $V(t,x) = x^4$ as a Lyapunov function for the nominal system.

Example: Unstable Origin

NB

A nominal system with UAS origin is not robust to smooth perturbations with arbitrarily small linear growth bounds

$$\|g(t,x)\| \leq \gamma \|x\|$$

Example

$$\dot{x} = -x^3 + \gamma x.$$

Show that $x^* = 0$ is unstable.

Part I

Nonvanishing Perturbation

Nonvanishing Additive Perturbations

Nominal system $\dot{x} = f(t,x)$
Perturbed system $\dot{x} = f(t,x) + g(t,x)$, $g(t,0) \neq 0$

- In this case, $x^* = 0$ may not be an equilibrium point of the perturbed system
- It can no longer be study the stability of the origin or expect that the solution of the perturbed system approaches the origin as $t \to \infty$.

The best we can do is find a bound on the size of $g(t,x)$ that ensures $x(t)$ remains close to the origin.

Uniform Ultimate Boundedness (UUB)

Nonvanishing Perturbations $\Leftrightarrow$ Uniform Ultimate Boundedness

Definition

Solutions of the nominal system $\dot{x} = f(t,x)$ are uniformly ultimately bounded (UUB) if there exists positive constants b and c and for all $\alpha \in (0,c)$ there is positive constant $T = T(\alpha)$ such that

$$\|x(t_0)\| < \alpha \Longrightarrow \|x(t)\| \le b \qquad \text{for all} \qquad t \ge t_0 + T$$

NB

The constant b is called the **ultimate bound**.

Exponential stable origin of $\dot{x} = f(t,x)$

Lemma 9.2

- Let $x^* = 0$ be an exponentially stable equilibrium point of the nominal system $\dot{x} = f(t,x)$
- Let $V(t,x)$ be a Lyapunov function of the nominal system that satisfies

$$c_1 \|x\|^2 \le V(t,x) \le c_2 \|x\|^2$$
$$\frac{\delta V}{\delta t} + \frac{\delta V}{\delta x} f(t,x) \le -c_3 \|x\|^2$$
$$\left\| \frac{\delta V}{\delta x} \right\| \le c_4 \|x\|$$

in $[0,\infty) \times D$, where $D = \{x \in \mathbb{R}^n | \, \|x\| < r\}$

Exponential stable origin of $\dot{x} = f(t,x)$

Lemma 9.2 cont.

- Suppose the perturbation term $g(t,x)$ satisfies

$$\|g(t,x)\| \le \delta < \frac{c_3}{c_4} \sqrt{\frac{c_1}{c_2}} \theta r$$

for all $t \ge 0$, all $x \in D$ and some positive constant $\theta < 1$.

Then, the solution $x(t)$ of the perturbed system for all $\|x(t_0)\| < \sqrt{c_1/c_2}\, r$ satisfies

$$\|x(t)\| \le k \exp\left[-\gamma(t - t_0)\right] \|x(t_0)\|, \qquad \forall t_0 \le t < t_0 + T$$
$$\|x(t)\| \le b, \qquad \forall t \ge t_0 + T$$

for some finite T, where

$$k = \sqrt{\frac{c_2}{c_1}} \qquad \gamma = \frac{(1-\theta)c_3}{2c_2} \qquad b = \frac{c_4}{c_3} \sqrt{\frac{c_2}{c_1}} \frac{\delta}{\theta}$$

Example: Nonvanishing Perturbation

Example

Consider the second-order system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -4x_1 - 2x_2 + \beta x_2^3 + d(t)\end{aligned}$$

where $\beta \geq 0$ is unknown and $d(t)$ is a uniformly bounded disturbance that satisfies $|d(t)| \leq \delta$ for all $t \geq 0$.
Using the Lyapunov function $V(x) = x^T P x$ show that all solutions of the perturbed system are uniformly bounded.

NB

The results are similar for the case when the origin is Uniformly Asymptotically Stable equilibrium.

Uniformly asymptotically stable origin of $\dot{x} = f(t,x)$

Lemma 9.3

- Let $x^* = 0$ be a uniformly asymptotically stable equilibrium point of the nominal system $\dot{x} = f(t,x)$.
- Let $V(t,x)$ be a Lyapunov function of the nominal system that satisfies inequalities

$$\begin{aligned}\alpha_1(\|x\|) \leq V(t,x) &\leq \alpha_2(\|x\|)\\ \frac{\delta V}{\delta t} + \frac{\delta V}{\delta x} f(t,x) &\leq -\alpha_3(\|x\|)\\ \left\|\frac{\delta V}{\delta x}\right\| &\leq \alpha_4(\|x\|)\end{aligned}$$

in $[0,\infty) \times D$, where $D = \{x \in \mathbb{R}^n | \, \|x\| < r\}$ and $\alpha_i(\cdot)$ are class $\mathcal{K}$ functions.

Uniformly asymptotically stable origin of $\dot{x} = f(t,x)$

Lemma 9.3 cont.

- Suppose the perturbation term $g(t,x)$ satisfies

$$\|g(t,x)\| \leq \delta < \frac{\theta \alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{\alpha_4(r)}$$

for all $t \geq 0$, all $x \in D$ and some positive constant $\theta < 1$.

Then, for all $\|x(t_0)\| < \alpha_2^{-1}(\alpha_1(r))$, the solution $x(t)$ of the perturbed system satisfies

$$\begin{aligned}\|x(t)\| &\leq \beta(\|x(t_0)\|, t - t_0), \quad && \forall t_0 \leq t < t_0 + T\\ \|x(t)\| &\leq \rho(\delta), && \forall t \geq t_0 + T\end{aligned}$$

for some $\mathcal{KL}$ function β and some finite T, where ρ is a class $\mathcal{K}$ function of δ defined by $\rho(\delta) = \alpha_1^{-1}\left(\alpha_2\left(\alpha_3^{-1}\left(\frac{\delta\alpha_4(r)}{\theta}\right)\right)\right)$.

Uniformly asymptotically stable origin of $\dot{x} = f(t,x)$

NB

Lemma 9.3 is similar to Lemma 9.2 in the special case of exponential stability

In the case of exponential stability, δ is required to satisfy

$$\|g(t,x)\| \leq \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r$$

It can be seen that the right-hand side of the equation approaches ∞ as $r \to \infty$.

Therefore, if the assumptions hold globally, we can conclude that for all uniformly bounded disturbances, the solution of the perturbed system will be uniformly bounded.

Uniformly asymptotically stable origin of $\dot{x} = f(t,x)$

In the case of UAS, δ is required to satisfy

$$\|g(t,x)\| \leq \delta < \frac{\theta \alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{\alpha_4(r)}$$

We can not say anything about the right-hand side as $r \to \infty$.

Therefore, we can not conclude that uniformly bounded perturbations of a nominal system with a UAS equilibrium at the origin will have bounded solutions irrespective to the size of the perturbation.

Next lecture

Next lecture: Passivity

Khalil **Chapter 6**
Sections 6.1 and 6.2
(Section 6.3 is additional material)
Sections 6.4 - 6.5, page 254
(Pages 254-259, incl. Ex. 6.12, is additional material)