

TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2016 - Solution to Assignment 6

1. The system is given by

$$\begin{aligned} M\ddot{\delta} &= P - D\dot{\delta} - \eta_1 E_q \sin(\delta) \\ \tau \dot{E}_q &= -\eta_2 E_q + \eta_3 \cos(\delta) + E_{FD} \end{aligned}$$

which is rewritten in the form $\dot{x} = f(x) + g(x)u$ using

$$\begin{aligned} x_1 &= \delta \\ x_2 &= \dot{\delta} \\ x_3 &= E_q \\ u &= E_{FD} \end{aligned}$$

This results in the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{M} (P - Dx_2 - \eta_1 x_3 \sin(x_1)) \\ \dot{x}_3 &= \frac{1}{\tau} (-\eta_2 x_3 + \eta_3 \cos(x_1) + u) \end{aligned} \tag{1}$$

where it can be seen that

$$\begin{aligned} f(x) &= \begin{bmatrix} x_2 \\ \frac{1}{M} (P - Dx_2 - \eta_1 x_3 \sin(x_1)) \\ \frac{1}{\tau} (-\eta_2 x_3 + \eta_3 \cos(x_1)) \end{bmatrix} \\ g(x) &= \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\tau} \end{bmatrix} \end{aligned}$$

(a) The output is given by $y = \delta = x_1 = h(x)$. The relative degree is found as

$$\begin{aligned} y &= x_1 \\ \dot{y} &= \dot{x}_1 \\ &= x_2 \\ \ddot{y} &= \dot{x}_2 \\ &= \frac{1}{M} (P - Dx_2 - \eta_1 x_3 \sin(x_1)) \\ \ddot{y} &= -\frac{D}{M} \dot{x}_2 - \frac{\eta_1}{M} \dot{x}_3 \sin(x_1) - \frac{\eta_1}{M} x_3 \frac{\partial \sin(x_1)}{\partial x_1} \dot{x}_1 \\ &= -\frac{D}{M} \frac{1}{M} (P - Dx_2 - \eta_1 x_3 \sin(x_1)) \\ &\quad - \frac{\eta_1}{\tau M} \sin(x_1) (-\eta_2 x_3 + \eta_3 \cos(x_1) + u) \\ &\quad - \frac{\eta_1}{M} x_3 \cos(x_1) x_2 \end{aligned}$$

And the relative degree of the system is $\rho = 3$.

We have our system on the form

$$\dot{x} = f(x) + g(x)u \quad (2)$$

and would like to transform it to a system on the form

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A_c \xi + B_c \gamma(x) [u - \alpha(x)] \\ y &= C_c \xi \end{aligned}$$

where η is the internal dynamics and ξ the external dynamics. They are both given through the diffeomorphism

$$T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ \hline h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix} \triangleq \begin{bmatrix} \eta \\ \hline \xi \end{bmatrix} \quad (3)$$

where

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \text{ for } 1 \leq i \leq n - \rho, \forall x \in D_0$$

Since $\rho = n$, one only needs the external part of the system on normal form, ξ . External variables of the normal form is given by evaluating the Lie Derivative of h with respect to f

$$\begin{aligned} \xi_1 &= h(x) \\ &= x_1 \\ \xi_2 &= L_f h(x) \\ &= x_2 \\ \xi_3 &= L_f^2 h(x) \\ &= \frac{1}{M} (P - Dx_2 - \eta_1 x_3 \sin(x_1)) \end{aligned}$$

The system (1) can therefore be written on normal form as

$$\dot{\xi} = A_c \xi + B_c \gamma(x) [u - \alpha(x)] \quad (4)$$

$$y = C_c \xi \quad (5)$$

where

$$\gamma(x) = L_g L_f^{\rho-1} h(x) \quad \text{and} \quad \alpha(x) = - \frac{L_f^\rho h(x)}{L_g L_f^{\rho-1} h(x)} = - \frac{L_f^\rho h(x)}{\gamma(x)} \quad (6)$$

This transformation is therefore only valid when $\gamma(x) \neq 0$, which means that

$$\begin{aligned} L_g L_f^{\rho-1} h(x) &= L_g L_f^2 h(x) \\ &= -\frac{\eta_1}{\tau M} \sin(x_1) \\ &\neq 0 \quad \forall x \in D_0 \end{aligned}$$

where $D_0 = \{x \in R^3 \mid \sin(x_1) \neq 0\}$. Since the relative degree equals the dimension of the system, we have no internal dynamics and the system is minimum phase.

- (b) The output is given by $y = \delta + \zeta \dot{\delta} = x_1 + \zeta x_2 = h(x)$ where $\zeta \neq 0$. The relative degree is obtained from

$$\begin{aligned} y &= x_1 + \zeta x_2 \\ \dot{y} &= \dot{x}_1 + \zeta \dot{x}_2 \\ &= x_2 + \zeta \frac{1}{M} (P - Dx_2 - \eta_1 x_3 \sin(x_1)) \\ &= \left(1 - \frac{\zeta D}{M}\right) x_2 - \frac{\zeta \eta_1}{M} x_3 \sin(x_1) + \zeta P \frac{1}{M} \\ \ddot{y} &= \frac{\partial \dot{y}}{\partial x} \dot{x} \\ &= \left[-\frac{\zeta \eta_1}{M} x_3 \cos(x_1) \quad \left(1 - \frac{\zeta D}{M}\right) \quad -\frac{\zeta \eta_1}{M} \sin(x_1) \right] \dot{x} \\ &= \frac{\zeta \eta_1}{M} x_3 \cos(x_1) x_2 \\ &\quad + \left(1 - \frac{\zeta D}{M}\right) \frac{1}{M} (P - Dx_2 - \eta_1 x_3 \sin(x_1)) \\ &\quad - \frac{\zeta \eta_1}{\tau M} \sin(x_1) (-\eta_2 x_3 + \eta_3 \cos(x_1) + u) \end{aligned}$$

And the system thus has relative degree $\rho = 2$.

The region D_0 where the transformation is valid is where $L_g L_f^{\rho-1} h(x) \neq 0$

$$\begin{aligned} L_g L_f^{\rho-1} h(x) &= L_g L_f^1 h(x) \\ &= -\frac{\gamma \eta_1}{\tau M} \sin(x_1) \\ &\neq 0 \quad \forall x \in D_0 \end{aligned}$$

where $D_0 = \{x \in R^3 \mid \sin(x_1) \neq 0\}$.

Since $\rho < n$, both internal and external dynamics are needed. The external variables of the normal form is found by evaluating the Lie Derivative of h with

respect to f

$$\begin{aligned}\xi_1 &= h(x) = x_1 + \zeta x_2 \\ \xi_2 &= L_f h(x) = \frac{\partial h(x)}{\partial x} f(x) = \begin{bmatrix} 1 & \zeta & 0 \end{bmatrix} f(x) \\ &= x_2 + \frac{\zeta}{M} (P - Dx_2 - \eta_1 x_3 \sin(x_1))\end{aligned}$$

The internal dynamics $\eta = \phi(x)$ is chosen to satisfy $\frac{\partial \phi(x)}{\partial x} g(x) = 0$ and the existence of $T^{-1}(x)$ in D_0 . It can be verified that $\phi(x) = x_1$ meets these conditions. With $\phi(x) = x_1$ we have that

$$\dot{\eta} = \dot{\phi}(x) = \dot{x}_1 = x_2 = \frac{1}{\zeta} (\xi_1 - \eta) = f_0(\eta, \xi)$$

The system on normal form is thus

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A_c \xi + B_c \gamma(x) [u - \alpha(x)] \\ y &= C_c \xi\end{aligned}$$

The system is said to be minimum phase if the zero dynamics, $\dot{\eta} = f_0(\eta, 0)$, has an asymptotically stable equilibrium point in the domain of interest. From $\dot{\eta} = f_0(\eta, 0) = -\frac{1}{\gamma}\eta$ it can be recognized that the origin of η is asymptotically stable.

2. The system is given by

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 - x_3 \\ \dot{x}_2 &= -x_1 x_3 - x_2 + u \\ \dot{x}_3 &= -x_1 + u \\ y &= x_3\end{aligned}$$

Rewriting this model on the form $\dot{x} = f(x) + g(x)u$ results in

$$\begin{aligned}f(x) &= \begin{bmatrix} -x_1 + x_2 - x_3 \\ -x_1 x_3 - x_2 \\ -x_1 \end{bmatrix} \\ g(x) &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\end{aligned}$$

(a) The relative degree is obtained from

$$\begin{aligned} y &= x_3 \\ \dot{y} &= \dot{x}_3 \\ &= -x_1 + u \end{aligned}$$

which shows that the system has relative degree 1 in R^3 . Hence, the system is input-output linearizable.

(b) The external part of the normal form is given by

$$\xi_1 = h(x) = x_3$$

To find the internal dynamics we start by setting up the requirements on $\frac{\partial \phi_i}{\partial x}$

$$\begin{aligned} \frac{\partial \phi_1}{\partial x} g(x) &= \left[\frac{\partial \phi_1}{\partial x_1} \quad \frac{\partial \phi_1}{\partial x_2} \quad \frac{\partial \phi_1}{\partial x_3} \right] g(x) \\ &= \frac{\partial \phi_1}{\partial x_2} + \frac{\partial \phi_1}{\partial x_3} \\ &= 0 \\ \frac{\partial \phi_2}{\partial x} g(x) &= \left[\frac{\partial \phi_2}{\partial x_1} \quad \frac{\partial \phi_2}{\partial x_2} \quad \frac{\partial \phi_2}{\partial x_3} \right] g(x) \\ &= \frac{\partial \phi_2}{\partial x_2} + \frac{\partial \phi_2}{\partial x_3} \\ &= 0 \end{aligned}$$

By choosing

$$\begin{aligned} \phi_1(x) &= x_1 \\ \phi_2(x) &= x_2 - x_3 \end{aligned}$$

we obtain a global diffeomorphism

$$\begin{aligned} T(x) &= \begin{bmatrix} x_1 \\ x_2 - x_3 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} x \end{aligned}$$

which is invertible. The system on normal form is

$$\begin{aligned} \dot{\eta}_1 &= \dot{x}_1 \\ &= -\eta_1 + \eta_2 \\ \dot{\eta}_2 &= \dot{x}_2 - \dot{x}_3 \\ &= -x_1 x_3 - x_2 + u + x_1 - u \\ &= -\eta_1 \xi_1 - (\eta_2 + x_3) + \eta_1 \\ &= \eta_1 - \eta_2 - \xi_1 - \eta_1 \xi_1 \\ \dot{\xi}_1 &= -\eta_1 + u \end{aligned}$$

Since $L_g L_f^0 h(x) = L_g h(x) = 1$, the transformation is valid in \mathcal{R}^3 .

(c) To investigate if the system is minimum phase, we analyze the zero dynamics

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi)|_{\xi=0} \\ &= \left[\begin{array}{cc} -\eta_1 + \eta_2 & \\ \eta_1 - \eta_2 - \xi_1 - \eta_1 \xi_1 & \end{array} \right] \Big|_{\xi=0} \\ &= \left[\begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array} \right] \eta = A\eta\end{aligned}$$

where it can be seen that $\text{eig}(A) = [-2 \ 0]^T$. Hence, the origin is not asymptotically stable, and the system is therefore not minimum phase.

3. The system is rewritten as

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}$$

where

$$\begin{aligned}f(x) &= \begin{bmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{bmatrix} \\ g(x) &= \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix} \\ h(x) &= x_3\end{aligned}$$

(a) The relative degree is found by derivative y with respect to time

$$\begin{aligned}y &= x_3 \\ \dot{y} &= \dot{x}_3 = x_2 \\ \ddot{y} &= \dot{x}_2 = x_1 x_2 + u\end{aligned}$$

where it can be seen that the system has a relative degree $\rho = 2$ in $x \in R^2$. The relative degree holds as long as $L_g L_f^{\rho-1} h(x) \neq 0$.

$$L_g L_f^{\rho-1} h(x) = L_g L_f h(x) = 1 \neq 0 \ \forall x \in \mathcal{R}^2$$

The relative degree thus holds over the entire \mathcal{R}^3 space.

(b) The system has a well defined relative degree ρ in the entire \mathcal{R}^3 , and is therefore input-output linearizable in \mathcal{R}^3 .

(c) The variables for the external dynamics are found according to

$$\begin{aligned}\xi_1 &= h(x) = x_3 \\ \xi_2 &= L_f h(x) = \frac{\partial h(x)}{\partial x} f = x_2\end{aligned}$$

The coordinates for the internal dynamics is chosen such that $T(x)$ is diffeomorphism on \mathcal{R}^3 and $\frac{\partial \phi(x)}{\partial x} g(x) = 0$ on \mathcal{R}^3 , where $[\eta, \xi^T]^T = [\phi(x), \psi(x)] = T(x)$. In addition to this we require $\phi(0) = 0$ in order to have the origin as equilibrium. We start by calculating

$$\begin{aligned}\frac{\partial \phi(x)}{\partial x} g(x) &= \begin{bmatrix} \frac{\partial \phi(x)}{\partial x_1} & \frac{\partial \phi(x)}{\partial x_2} & \frac{\partial \phi(x)}{\partial x_3} \end{bmatrix} \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix} \\ &= \frac{\partial \phi(x)}{\partial x_1} e^{x_2} + \frac{\partial \phi(x)}{\partial x_2} \\ &= 0\end{aligned}$$

and based on these calculations we try

$$\begin{aligned}\frac{\partial \phi(x)}{\partial x_1} &= 1 \\ \frac{\partial \phi(x)}{\partial x_2} &= -e^{x_2}\end{aligned}$$

which implies that

$$\phi(x) = x_1 - e^{x_2} + c$$

where c is some constant. This constant is chosen to satisfy our requirement $\phi(0) = 0$

$$\begin{aligned}\phi(0) &= -e^0 + c \\ &= -1 + c \\ &\Rightarrow c = 1\end{aligned}$$

Our resulting coordinate transformation is now given by

$$\begin{bmatrix} \eta \\ \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} x_1 - e^{x_2} + 1 \\ x_3 \\ x_2 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \eta + e^{\xi_2} - 1 \\ \xi_2 \\ \xi_1 \end{bmatrix}$$

Consequently the inverse transformation exists. It follows that $T(x)$ and $T^{-1}(x)$ are continuously differentiable. Hence, $T(x)$ is a diffeomorphism on \mathcal{R}^3 and $T(0) = T^{-1}(0) = 0$.

(d) The system may be rewritten as

$$\begin{aligned}
\dot{\eta} &= \dot{x}_1 - \frac{\partial e^{x_2}}{\partial x_2} \dot{x}_2 \\
&= -x_1 + e^{x_2} u - e^{x_2} (x_1 x_2 + u) \\
&= -x_1 - e^{x_2} x_1 x_2 \\
&= -(\eta + e^{x_2} - 1) - e^{x_2} (\eta + e^{x_2} - 1) x_2 \\
&= (1 - \eta - e^{\xi_2}) + (1 - \eta - e^{\xi_2}) e^{\xi_2} \xi_2 \\
&= (1 - \eta - e^{\xi_2}) (1 + e^{\xi_2} \xi_2)
\end{aligned}$$

and

$$\begin{aligned}
\dot{\xi} &= A_c \xi + B_c \gamma(x) (u - \alpha(x)) \\
y &= C_c \xi
\end{aligned}$$

where

$$\begin{aligned}
A_c &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
B_c &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
C_c &= \begin{bmatrix} 1 & 0 \end{bmatrix} \\
\gamma(x) &= L_g L_f h(x) \\
&= 1 \\
\alpha(x) &= -\frac{L_f^2 h(x)}{L_g L_f h(x)} \\
&= -\frac{x_1 x_2}{1} \\
&= -x_1 x_2
\end{aligned}$$

(e) The zero dynamics is given by

$$\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi)|_{\xi=0} \\
&= (1 - \eta - e^{\xi_2}) (1 + e^{\xi_2} \xi_2)|_{\xi=0} \\
&= (1 - \eta - 1) (1 + 0) \\
&= -\eta
\end{aligned}$$

which has a globally asymptotically stable equilibrium at the origin.

(f) The external dynamics are given by

$$\dot{\xi} = A_c \xi + B_c \gamma(x) (u - \alpha(x))$$

By choosing

$$u = \gamma^{-1}(x) v + \alpha(x)$$

the external dynamics are given by

$$\dot{\xi} = A_c \xi + B_c v$$

Since the system is controllable, $\text{rank}([B, AB]) = 2$, it can be stabilized (asymptotically stable) by a control input $v = -K\xi$ where K is chosen such that $(A_c - B_c K)$ is Hurwitz. u is now given by

$$u = -\gamma^{-1}(x) K\xi + \alpha(x)$$

Since $\dot{\eta} = f_0(\eta, \xi)|_{\xi=0}$ is asymptotically stable, the origin of the entire system is asymptotically stable.

(g) Let

$$\begin{aligned} \mathcal{R} &= \begin{bmatrix} r \\ \dot{r} \end{bmatrix} \\ e &= \xi - \mathcal{R} \end{aligned}$$

Then we can calculate

$$\begin{aligned} \dot{e} &= \dot{\xi} - \dot{\mathcal{R}} \\ &= (A_c \xi + B_c v) - (A_c \mathcal{R} + B_c \ddot{r}) \\ &= A_c (\xi - \mathcal{R}) + B_c (v - \ddot{r}) \\ &= A_c e + B_c (v - \ddot{r}) \\ &= A_c e + B_c (\gamma(x) [u - \alpha(x)] - \ddot{r}) \end{aligned}$$

where the simplified structure $\dot{\mathcal{R}} = A_c \mathcal{R} + B_c \ddot{r}$ is found using the known values of A_c and B_c (see part d).

We can choose the state feedback control

$$u = \gamma^{-1}(x) (v + \ddot{r}) + \alpha(x)$$

The resulting system is

$$\begin{aligned} \dot{\eta} &= f_0(\eta, e + \mathcal{R}) \\ \dot{e} &= A_c e + B_c v \end{aligned}$$

and since (A_c, B_c) is controllable, the loop is closed with $v = -Ke$ where K is chosen such that $(A_c - B_c K)$ is Hurwitz. This makes the external dynamics for e exponentially stable.

Since $\dot{\eta} = f_0(\eta, \xi)|_{\xi=0}$ is asymptotically stable, the origin of the overall closed-loop system is such that for sufficiently small initial conditions $e(0), \eta(0)$ and for $\mathcal{R}(t)$ with sufficiently small $\sup_{t \geq 0} \|\mathcal{R}(t)\|$, all solutions $(\eta(t), e(t))$ of the closed-loop system are bounded and $e(t) \rightarrow 0$ as $t \rightarrow \infty$.

4. (Khalil 13.25) The system is given by

$$\begin{aligned}\dot{x}_1 &= x_2 + 2x_1^2 \\ \dot{x}_2 &= x_3 + u \\ \dot{x}_3 &= x_1 - x_3 \\ y &= x_1\end{aligned}$$

Since

$$y = x_1 \Rightarrow \dot{y} = x_2 + 2x_1^2 \Rightarrow \ddot{y} = x_3 + u + 2x_1(x_2 + x_1^2)$$

Therefore, the system has a relative degree 2 in R^3 . Let us check the minimum-phase property.

$$y(t) \equiv 0 \Rightarrow \dot{x}_3 = -x_3.$$

Hence, the system is minimum.-phase. Let $e = y - r$.

$$\ddot{e} = \ddot{y} - \ddot{r} = x_3 + u + 2x_1(x_2 + x_1^2) - \ddot{r}$$

Choosing

$$\begin{aligned}u &= -x_3 - 2x_1(x_2 + x_1^2) + \ddot{r} - k_1 e - k_2 \dot{e} \\ &= -x_3 - 2x_1(x_2 + x_1^2) + \ddot{r} - k_1(y - r) - k_2(x_2 + 2x_1^2 - \dot{r}),\end{aligned}$$

where k_1 and k_2 are positive constants. the tracking error e satisfies the equation $\ddot{e} + k_2 \dot{e} + k_1 e = 0$, which shows that $e(t) \rightarrow 0$ as $t \rightarrow \infty$.

5. (Khalil 14.31) The system is given by

$$\begin{aligned}\dot{x}_1 &= x_2 + a + (x_1 - a^{1/3})^3 \\ \dot{x}_2 &= x_1 + u\end{aligned}$$

where the first system equation has the virtual input x_2 . Choose

$$x_2 = -a - x_1 - (x_1 - a^{1/3})^3$$

such that $\dot{x}_1 = -x_1$. Then the Lyapunov function candidate $V_1 = \frac{1}{2}x_1^2$ (which is positive definite and radially unbounded) will have $\dot{V}_1 = x_1\dot{x}_1 = -x_1^2$ which is negative definite.

Augment the virtual input with z , such that

$$x_2 = -a - x_1 - (x_1 - a^{1/3})^3 + z$$

then

$$z = x_1 + x_2 + a + (x_1 - a^{1/3})^3$$

and

$$\begin{aligned}
\dot{z} &= \dot{x}_1 + \dot{x}_2 + 3(x_1 - a^{1/3})^2 \dot{x}_1 \\
&= \left(x_2 + a + (x_1 - a^{1/3})^3\right) + (x_1 + u) + 3(x_1 - a^{1/3})^2 \left(x_2 + a + (x_1 - a^{1/3})^3\right) \\
&= x_1 + u + \left(1 + 3(x_1 - a^{1/3})^2\right) \left(x_2 + a + (x_1 - a^{1/3})^3\right)
\end{aligned}$$

Calculate

$$\dot{V}_1 = x_1 \dot{x}_1 = x_1(-x_1 + z) = -x_1^2 + x_1 z$$

We may choose a Lyapunov function candidate for the overall system as $V_c = V_1 + \frac{1}{2}z^2$, then

$$\begin{aligned}
\dot{V}_c &= \dot{V}_1 + z\dot{z} \\
&= -x_1^2 + x_1 z + z \left\{ x_1 + u + \left(1 + 3(x_1 - a^{1/3})^2\right) \left(x_2 + a + (x_1 - a^{1/3})^3\right) \right\} \\
&= -x_1^2 + z \underbrace{\left\{ 2x_1 + u + \left(1 + 3(x_1 - a^{1/3})^2\right) \left(x_2 + a + (x_1 - a^{1/3})^3\right) \right\}}_{\text{choose } -z} \\
&= -x_1^2 - z^2
\end{aligned}$$

where we have chosen

$$-z = 2x_1 + u + \left(1 + 3(x_1 - a^{1/3})^2\right) \left(x_2 + a + (x_1 - a^{1/3})^3\right)$$

i.e.

$$\begin{aligned}
u &= -z - 2x_1 - \left(1 + 3(x_1 - a^{1/3})^2\right) \left(x_2 + a + (x_1 - a^{1/3})^3\right) \\
&= -\overbrace{\left(x_1 + x_2 + a + (x_1 - a^{1/3})^3\right)}^z - 2x_1 - \left(1 + 3(x_1 - a^{1/3})^2\right) \left(x_2 + a + (x_1 - a^{1/3})^3\right) \\
&= -3x_1 - \left(2 + 3(x_1 - a^{1/3})^2\right) \left(x_2 + a + (x_1 - a^{1/3})^3\right)
\end{aligned}$$

such that \dot{V}_c is negative definite. We already know that V_c is positive definite and radially unbounded. Hence, the overall system is globally asymptotically stable (GAS).

Alternative solution:

The system is in the form of (14.53)-(14.54) in Khalil with

$$\begin{aligned}
f &= a + (x_1 - a^{1/3})^3 \\
g &= 1 \\
f_a &= x_1 \\
g_a &= 1
\end{aligned}$$

Take

$$\begin{aligned}\phi(x_1) &= -a - (x_1 - a^{1/3})^3 - x_1 \\ V &= \frac{1}{2}x_1^2\end{aligned}$$

and use (14.56) in Khalil.

6. Consider $\dot{x}_1 = x_1x_2 + x_1^2$ with x_2 as a virtual input. Choose a Lyapunov function candidate $V_1(x) = \frac{1}{2}x_1^2$ and calculate

$$\dot{V}_1 = x_1\dot{x}_1 = x_1(x_1x_2 + x_1^2)$$

We can enforce $\dot{V}_1 = -x_1^4$ which is negative definite, by choosing the input $x_2 = -x_1 - x_1^2$ (actually, any choice $x_2 = -x_1^{2k} - x_1^2$, $k = 1, 2, 3, \dots$ will be possible, to get a negative definite \dot{V}_1 , but for simplification we choose $k=1$).

Augment the input with z , such that we have $x_2 = -x_1 - x_1^2 + z$ (i.e. $z = x_2 + x_1 + x_1^2$), then

$$\dot{V}_1 = x_1(x_1x_2 + x_1^2) = x_1(x_1(-x_1 - x_1^2 + z) + x_1^2) = -x_1^4 + x_1^2z$$

Now choose a Lyapunov function candidate for the complete system $V_c = V_1 + \frac{1}{2}z^2$, which is positive definite and radially unbounded. Then

$$\begin{aligned}\dot{V}_c &= \dot{V}_1 + z\dot{z} \\ &= -x_1^4 + x_1z + z(u + (2x_1 + 1)(x_1x_2 + x_1^2)) \\ &= -x_1^4 + z \underbrace{(x_1 + u + (2x_1 + 1)(x_1x_2 + x_1^2))}_{\text{choose } -z}\end{aligned}$$

We can enforce $\dot{V}_c = -x_1^2 - z^2$ (then \dot{V}_c is negative definite), by choosing

$$\begin{aligned}-z &= x_1 + u + (2x_1 + 1)(x_1x_2 + x_1^2) \\ u &= -x_1 - (2x_1 + 1)(x_1x_2 + x_1^2) - z\end{aligned}$$

By inserting z , we get the expression for the stabilizing input

$$\begin{aligned}u &= -x_1 - (2x_1 + 1)(x_1x_2 + x_1^2) - x_2 + x_1 + x_1^2 \\ &= -(2x_1 + 1)(x_1x_2 + x_1^2) - x_2 + x_1^2\end{aligned}$$

Since $V_c(x_1, z)$ is continuously differentiable and positive definite, and $\dot{V}_c(x_1, z)$ is negative definite, u asymptotically stabilizes x_1 and z at the origin. Since $z = 0 \rightarrow x_2 = -x_1 - x_1^2$ and $x_1 = 0 \rightarrow -x_1 - x_1^2 = 0$, this means that also x_2 is asymptotically stabilized at the origin. In addition, since $V_c(x_1, z)$ is radially unbounded and there are no singularities in u , the equilibrium point $x = (0, 0)$ is globally asymptotically stable.