NTNU – Trondheim
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TTT4120 Digital Signal Processing
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Lecture: Z-Transform – System Analysis

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Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 4.2.6 Relationship of the Fourier transform to the z-transform
  - 3.5.3 Causality and stability
  - 3.5.6 Stability of second-order systems
  - 5.2.2 Computation of the frequency response

*Level of detail is defined by lectures and problem sets

Contents and learning outcomes

- LTI systems: The system function, stability and causality
- Computation and sketching of frequency response function

Linear time-invariant systems

- Output of linear time-invariant system

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$$

$$X(z) \qquad\qquad\qquad Y(z) = H(z)X(z)$$

- By knowing $x[n]$ and observing $y[n]$, we can obtain

$$H(z) = \frac{Y(z)}{X(z)}$$

- Since $H(z) = \sum_n h[n]z^{-n}$, we obtain $h[n] = \mathcal{Z}^{-1}\{H(z)\}$
- Two equivalent descriptions of an LTI system

Linear time-invariant systems…

- Linear time-invariant systems described by constant-coefficient difference equations

$x[n]$ → $h[n]$ → $y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$

$X(z)$ $\quad$ $Y(z) = H(z)X(z)$

- Rational system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

- Special cases: $a_k = 0$ or $b_k = 0$ for $1 \leq k \leq N$

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Linear time-invariant systems…

- Example: $y[n] = \frac{1}{4} y[n-2] + x[n]$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{4}z^{-2}}$$

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Causality and stability

- Causal linear time-invariant system: $h[n] = 0$ for $n < 0$
- ROC of $H(z)$ must be the exterior of a circle
- Stability of LTI system in terms of system function

$$|H(z)| = \left|\sum_{n=-\infty}^{\infty} h[n]z^{-n}\right| \leq \sum_{n=-\infty}^{\infty} |h[n]||z^{-n}|$$

- If the system is BIBO stable, the unit circle, $z = e^{j\omega}$, is within ROC of $H(z)$. Converse is also true.
- ROC of $H(z)$ can provide information of whether a linear time-invariant system is causal and stable

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Causality and stability…

- In general, if system function is rational, and $N > M$

$$H(z) = b_0 \frac{\prod_{k=0}^{M}(1-z_k z^{-1})}{\prod_{k=0}^{N}(1-p_k z^{-1})} = \sum_{k=0}^{N} \frac{C_k}{1-p_k z^{-1}}$$

- Causal if ROC is the exterior of a circle, $|z| > \max|p_k|$

$$h[n] = \sum_{k=0}^{\infty} C_k p_k^n u[n]$$

- Stable if $\max|p_k| < 1$ (unit circle is included in ROC)

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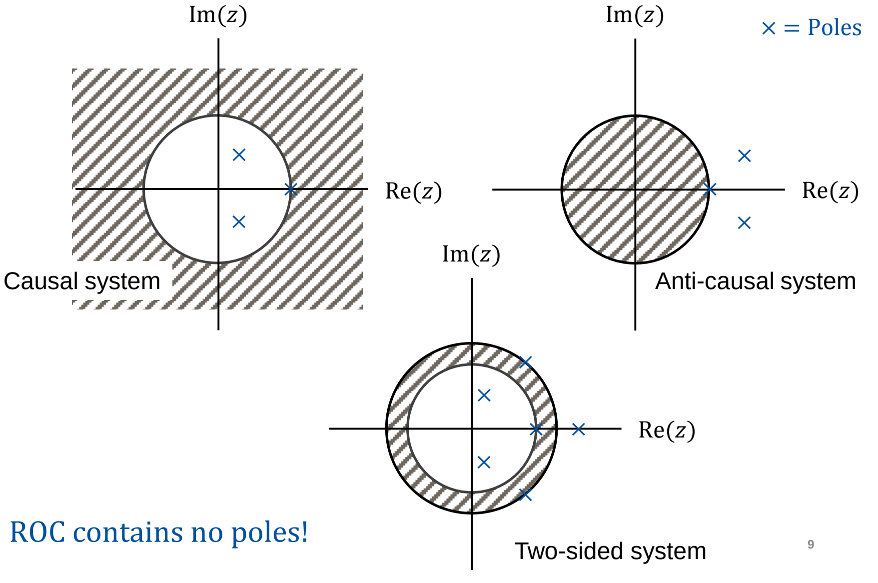
Causality and stability…



ROC contains no poles!

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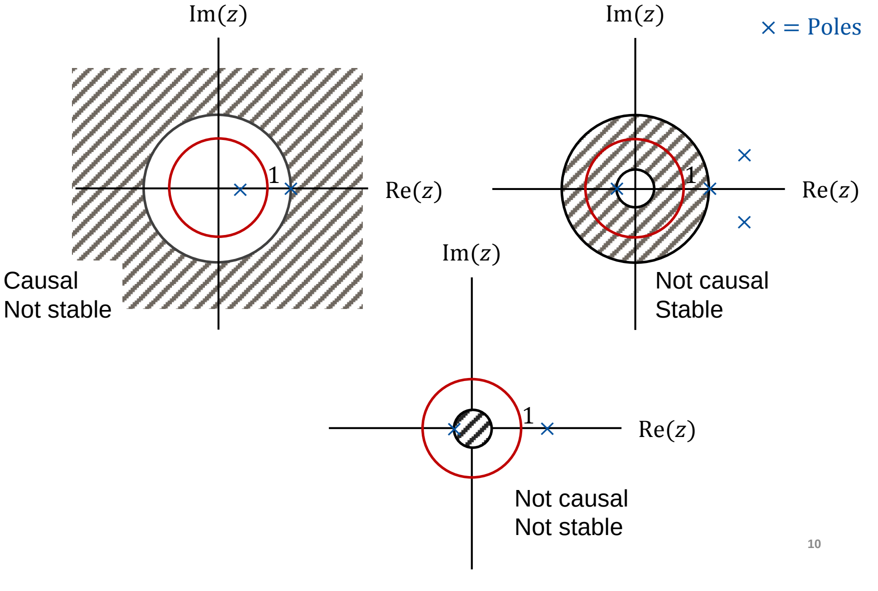
Causality and stability…



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Causality and stability…

- Example: $y[n] = \frac{1}{4}y[n-2] + x[n]$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{4}z^{-2}} = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}$$
$$= \frac{1/2}{1-\frac{1}{2}z^{-1}} + \frac{1/2}{1+\frac{1}{2}z^{-1}}$$

- Causal if $|z| > \frac{1}{2}$ and stable since ROC contains unit circle
- Not causal if $|z| < \frac{1}{2}$ and unstable since unit circle not in ROC

Causality and stability…

- Example: $H(z) = \frac{1}{1-0.5z^{-1}} + \frac{2}{1-3z^{-1}}$

Specify ROC and determine $h[n]$ when
1) system is stable
2) system is causal
3) system is anti-causal

Computation of the frequency response

- The z-transform expressed in polar form

$$X(z)|_{z=re^{j\omega}} = \sum_{n=-\infty}^{\infty}(x[n]r^{-n})e^{-j\omega n}, r_2 < r < r_1$$

- If unit circle, $z = e^{j\omega}$, is within ROC of $X(z)$ we have

$$X(\omega) = X(z)|_{z=re^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- If $X(z)$ does not converge for $|z| = 1$, Fourier transform does not exist, e.g., $r_2 > 1$

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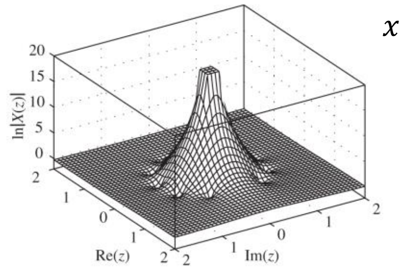
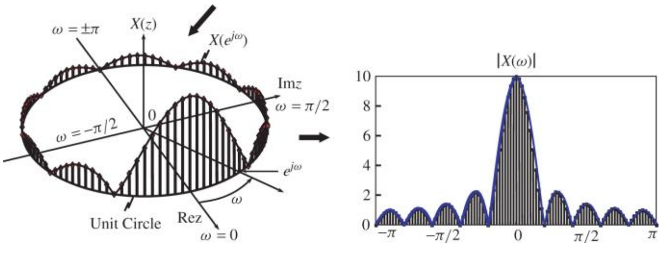
Computation of the frequency response…

$$x[n] = u[n] - u[n-10]$$





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Computation of the frequency response…

- The frequency response

$$H(\omega) = H(z)|_{z=re^{j\omega}} = b_0 \frac{\prod_{k=0}^{M}(1-z_k e^{-j\omega})}{\prod_{k=0}^{N}(1-p_k e^{-j\omega})}$$

$$= b_0 e^{j(N-M)\omega} \frac{\prod_{k=0}^{M}(e^{j\omega}-z_k)}{\prod_{k=0}^{N}(e^{j\omega}-p_k)}$$

- Product of frequency-dependent distance-vectors in z-plane

$$e^{j\omega} - z_k = V_k e^{j\Theta_k(\omega)}$$

$$e^{j\omega} - p_k = U_k e^{j\Phi_k(\omega)}$$

- If we know z_k and p_k we can plot/sketch the frequency response and phase response

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Computation of the frequency response…

- The magnitude of frequency response

$$|H(\omega)| = |b_0| \frac{\prod_{k=0}^{M}|e^{j\omega}-z_k|}{\prod_{k=0}^{N}|e^{j\omega}-p_k|} = |b_0| \frac{\prod_{k=0}^{M} V_k}{\prod_{k=0}^{N} U_k}$$

- Phase response:

$$\angle H(\omega) = \angle b_0 e^{j(N-M)\omega} \frac{\prod_{k=0}^{M} V_k e^{j\Theta_k(\omega)}}{\prod_{k=0}^{N} U_k e^{j\Phi_k(\omega)}}$$

$$= \angle b_0 + (N-M)\omega + \sum_{k=0}^{M} \Theta_k(\omega) - \sum_{k=0}^{N} \Phi_k(\omega)$$

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Computation of the frequency response…

- Example:

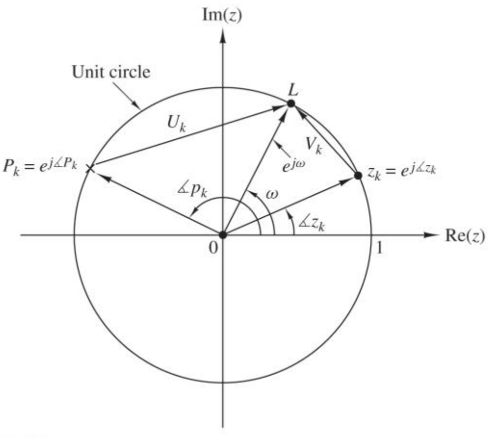

Figure 5.2.2 A zero on the unit circle causes $|H(\omega)| = 0$ and $\omega = \measuredangle z_k$. In contrast, a pole on the unit circle results in $|H(\omega)| = \infty$ at $\omega = \measuredangle p_k$.

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Computation of the frequency response…

- Example: Sketch the frequency response of systems from the pole-zero plot

$$H_1(z) = \frac{1}{1 - 0.5z^{-1}}$$

Matlab

```
B = 1;
A = [1 -0.5];
figure(1)
zplane(B,A)

figure(2)
[H,W]=freqz(B,A);
plot(W/pi,abs(H));
```

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Computation of the frequency response…

- Another Matlab example:
  Sketch the frequency response of system using `zplane(B,A)`

$$H(z) = \frac{B(z)}{A(z)}$$

with `B = fircls1(8,0.3,0.02,0.008);`
and `A = [1]`

- Verify using `[H,W]=freqz(B,A), plot(W/pi,abs(H))`

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Summary

Today:

- LTI systems: causality and stability
- System function
- Computation and sketch of frequency response from the system function

Next:

- Some simple filters and properties
- Why do we want linear phase filters?
- Minimum-phase and inverse systems

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