TTK4150 Nonlinear Control Systems

Lecture 4

Stability analysis of equilibrium points:

Lyapunov's direct method

NTNU
Innovation and Creativity

Previous lecture

Previous lecture:

- The control problem for
  - Regulation
  - Tracking

  leads to the Asymptotic Stabilization Problem
- Definitions of stability for autonomous systems
  - Stability
  - Asymptotic stability
  - Exponential stability
  - Global vs. local
- **Lyapunov stability analysis**
  - Lyapunov's indirect method

Today:

- **Lyapunov stability analysis** cont.
  - Lyapunov's direct method

Outline I

Outline II

- Theorem: Exponential stability (local and global)
- Convergence rate

6. Summarizing the method

7. Methods for choosing Lyapunov function candidates
   - LFCs with quadratic terms $\frac{1}{2}x^T P x$

8. Next lecture

Today's goals

After this lecture you should...

- Be able to use Lyapunov's direct method to analyze the stability properties of an equilibrium point.
- Know Lyapunov's theorems for
  - stability
  - local and global asymptotic stability
  - local and global exponential stability

Literature

Today's lecture is based on

Khalil	Section 4.1
	Theorem 4.10, Section 4.5

Motivation

Energy function $V(x)$

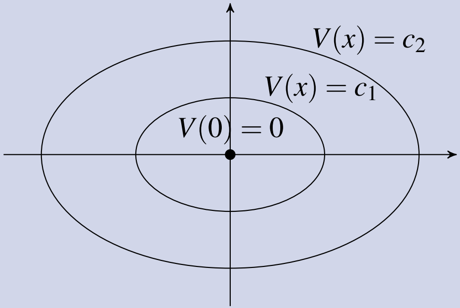
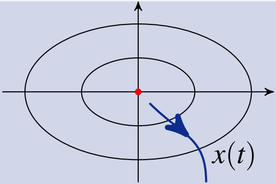


Figure: Level surfaces $V(x) = c_i \quad (0 < c_1 < c_2 < c_3 \cdots)$. Surfaces of constant energy.

Energy functions

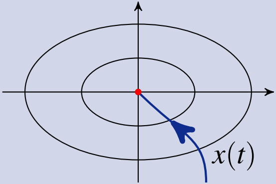
We consider the energy evolution of the system

$\dot{x} = f(x), \quad x = 0$ is an equilibrium point



Energy increases along $x(t)$

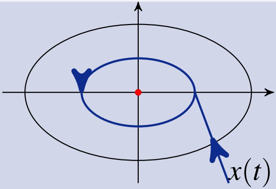
$$\frac{dV(x)}{dt} = \frac{dV}{dx} f(x) > 0$$



Energy decreases along $x(t)$

$$\frac{dV(x)}{dt} < 0$$

Energy functions cont.



Energy decreases or is constant

$$\frac{dV(x)}{dt} \leq 0$$

Aleksandr Lyapunov (1857-1918)

- Russian mathematician, mechanician and physicist
- Known for his
  - development of the stability theory of dynamical systems
  - many contributions to mathematical physics and probability theory

The general problem of the stability of motion (1892)

Lyapunov functions

The system

Consider the autonomous system

$$\dot{x} = f(x)$$

where $f : \mathbb{D} \to \mathbb{R}^n$ is locally Lipschitz.
$x = 0 \in \mathbb{D}$ is an equilibrium point of the system.

Lyapunov function candidate

Let $V : \mathbb{D} \to \mathbb{R}$ be a continuously differentiable (C^1) function

The derivative of V along the system trajectories is:

$$\dot{V} = \frac{dV(x)}{dt} = \frac{dV}{dx} f(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

Lyapunov functions

Definition (Lyapunov function)

V is a Lyapunov function for $x = 0$ iff

i) V is C^1

ii) $V(0) = 0$
$V(x) > 0$ in $\mathbb{D} \backslash \{0\}$

iii) $\dot{V}(0) = 0$
$\dot{V}(x) \leq 0$ in $\mathbb{D} \backslash \{0\}$

If, moreover,

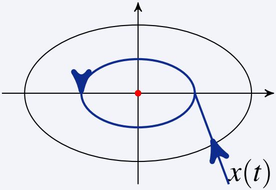
$\dot{V}(x) < 0$ in $\mathbb{D} \backslash \{0\}$

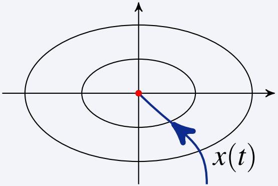
then V is a strict Lyapunov function.

Lyapunov's direct method

Theorem 4.1

- If $\exists$ Lyapunov function for $x = 0$, then $x = 0$ is stable
- If $\exists$ strict Lyapunov function for $x = 0$, then $x = 0$ is asymptotically stable



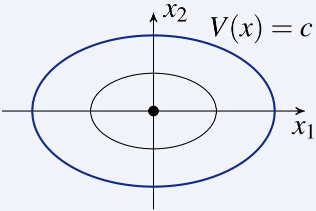


Notation

Level surfaces (curves)

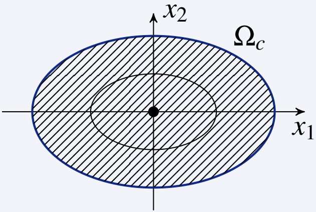
Lyapunov surfaces

$$V(x) = c$$



Level sets

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$$



When V is a Lyapunov function then Ω_c is a (positively) invariant set for the system $\dot{x} = f(x)$

How to apply Lyapunov's direct method

How to apply Lyapunov's direct method

1) Choose a Lyapunov function **candidate** $V(x)$
   - Electrical/mechanical systems
     - $V(x)$ = total energy
   - Others
     - $V(x) = \frac{1}{2}x^T Px$
     - $V(x) = \frac{1}{2}(x_1^2 + a_2 x_2^2 + \ldots + a_n x_n^2)$
     - *some* methods exist for choosing $V(x)$
2) Determine whether $V(x)$ is a Lyapunov function/a strict Lyapunov function for the equilibrium point.
3) If the answer is yes:

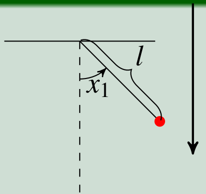
   The equilibrium point is Stable/Asymptotically stable

   If the answer is no:

Application of Lyapunov's direct method

Pendulum without friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1$$



$g = 9.81$

Pendulum with friction

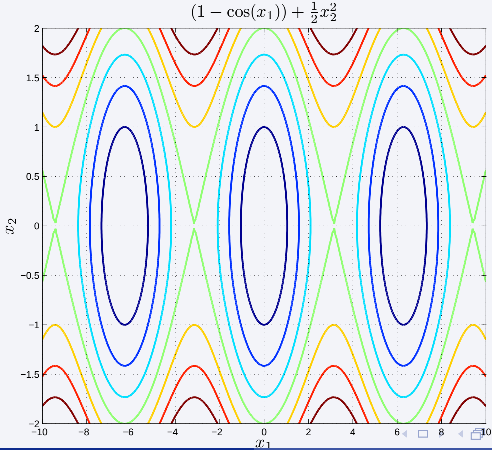
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$

Investigate the stability properties of $x = 0$ using Lyapunov's direct method

Pendulum without friction: Level curves (contour plot)

Matlab

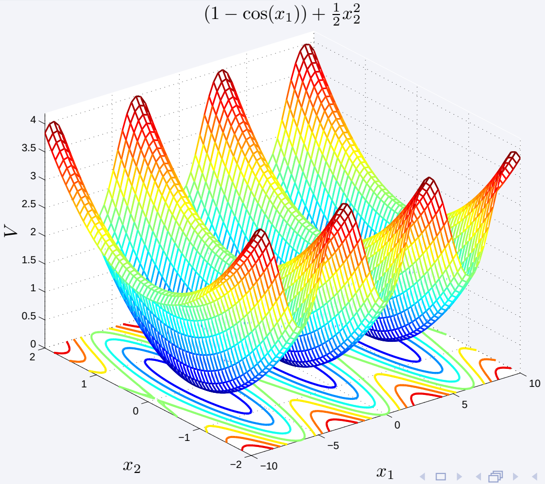
```
V=(1-cos(x1))+1/2*x2*x2
ezcontour(V,[-10,10,-2,2])
```



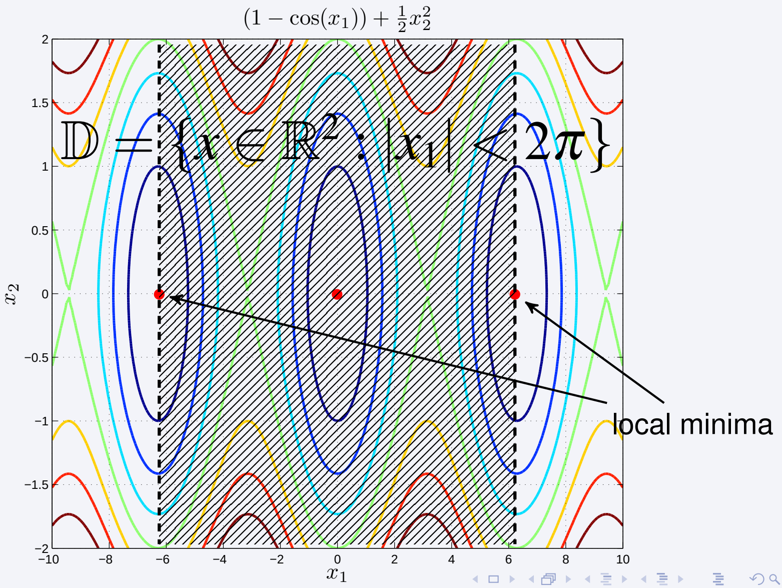
Pendulum without friction: Level curves (surface plot)

Matlab

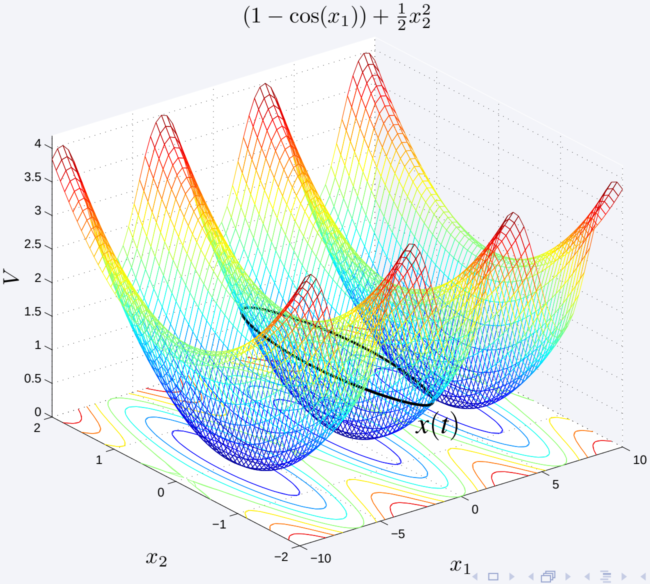
```
ezmeshc(V,[-10,10,-2,2])
```



Pendulum without friction: Domain of analysis



Pendulum without friction: System trajectory



Examples cont.

Example

Given

$$\dot{x} = -x^3$$

Analyze the stability properties of the equilibrium point $x = 0$ using Lyapunov's direct method.

Global asymptotic stability

Theorem 4.2: Global asymptotic stability

If $\exists$ strict Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ for $x = 0$

and

V is radially unbounded

then $x = 0$ is globally asymptotically stable.

Radial unboundedness

Definition

$V(x)$ is radially unbounded iff

$$\|x\| \to \infty \Rightarrow V(x) \to \infty$$

Necessary for global results

For C^1 functions V:

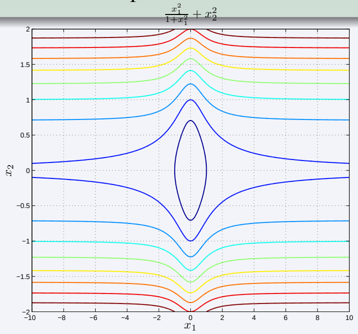
- Positive definite $\Rightarrow$ Level surfaces are closed for small values of c
- Radial unboundedness $\Rightarrow$ Level surfaces are closed $\forall c$

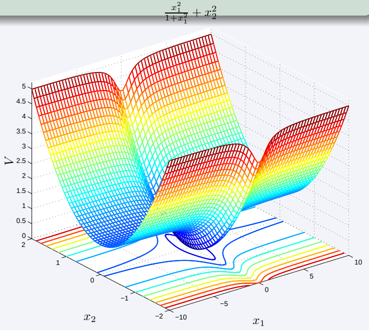
If the level surfaces are not closed, we may have that $\|x\| \to \infty$ even if $\dot{V} < 0$

Necessary for global results

Example

$$V(x) = \frac{x_1^2}{(1+x_1^2)} + x_2^2$$

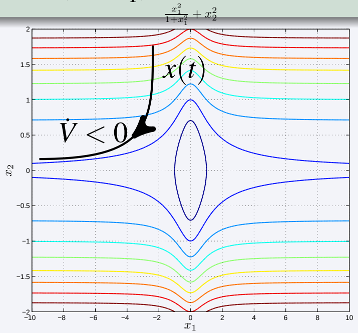


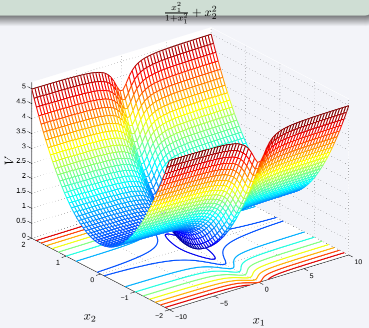


Necessary for global results

Example

$$V(x) = \frac{x_1^2}{(1+x_1^2)} + x_2^2$$





Exponential stability

Theorem 4.10: Exponential stability

If there exists constants $a, k_1, k_2, k_3 > 0$ such that

i) V is C^1

ii) $k_1 \|x\|^a \leq V(x) \leq k_2 \|x\|^a \quad \forall x \in \mathbb{D}$

iii) $\dot{V}(x) \leq -k_3 \|x\|^a \quad \forall x \in \mathbb{D}$

then $x = 0$ is exponentially stable.

Global exponential stability

If the conditions in the theorem are satisfied with

$$\mathbb{D} = \mathbb{R}^n$$

then $x = 0$ is globally exponentially stable.

Exponential stability cont.

Convergence rate

Using the comparison lemma we can show that

$$\|x(t)\| \leq \left(\frac{k_2}{k_1}\right)^{\frac{1}{a}} \|x(t_0)\| \, e^{-\frac{k_3}{k_2 a}(t-t_0)}$$

Example

Given

$$\dot{x} = -x - x^3$$

Analyze the stability properties of the equilibrium point $x = 0$ using Lyapunov's direct method.

How to apply Lyapunov's direct method

How to apply Lyapunov's direct method - revisited

1) Choose a Lyapunov function **candidate** $V(x)$
   - Electrical/mechanical systems
     - $V(x)$ = total energy
   - Others
     - $V(x) = \frac{1}{2}x^T Px$
     - $V(x) = \frac{1}{2}(x_1^2 + a_2 x_2^2 + \ldots + a_n x_n^2)$
     - *some* methods exist for choosing $V(x)$
2) Determine whether $V(x)$ satisfies the conditions of any of the Lyapunov theorems.
3) If the answer is yes:

   The equilibrium point is Stable/Asymptotically stable/Exponentially stable

   If the answer is no:

Methods for choosing Lyapunov function candidates

Methods for choosing LFCs

- Total energy
- LFCs with quadratic terms $\frac{1}{2}x^T Px$
  - $V(x) = \frac{1}{2}(x_1^2 + x_2^2 + \cdots + x_n^2)$
  - $V(x) = \frac{1}{2}(x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2)$
  - $V(x) = \frac{1}{2}x^T Px$
- $V(x) = \frac{1}{2}\ln(1 + x_1^2 + \cdots + x_n^2)$
- The variable gradient method
- LFCs for linear time-invariant systems
- Krasovskii's method (Assignment)
- $\vdots$

Examples: LFCs with quadratic terms $\frac{1}{2}x^T Px$

Example

Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 - x_2 \\ \dot{x}_2 &= x_1 - x_2 \end{aligned}$$

Analyze the stability properties of $x = 0$ using Lyapunov's direct method

Examples: LFCs with quadratic terms $\frac{1}{2}x^T Px$

Example

Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 - 3x_2 \\ \dot{x}_2 &= x_1 - x_2 \end{aligned}$$

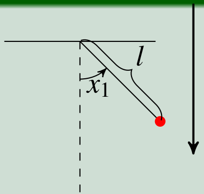
Analyze the stability properties of $x = 0$ using Lyapunov's direct method

Example: LFCs with quadratic terms $\frac{1}{2}x^T Px$

Pendulum with friction

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$



$g = 9.81$

Use the generalized energy function

$$V(x,p) = \frac{g}{l}(1-\cos x_1) + \frac{1}{2}x^T Px$$

as Lyapunov function candidate

Summary

Lyapunov's direct method

- Lyapunov functions - a generalization of energy functions
- Lyapunov's theorems for
  - stability
  - local and global asymptotic stability
  - local and global exponential stability
- How to apply Lyapunov's direct method

Advantages and disadvantages

+ General

÷ No general way to find $V(x)$

+ Can give global results

- Some methods for choosing Lyapunov Function candidates

Next lecture

- La Salle's theorem
  - $\dot{V} \leq 0$ asymptotic stability of equilibrium points
  - Convergence to other invariant sets than equilibrium points
  - Regions of attraction - find an estimate
- More methods for finding Lyapunov function candidates (LFCs)
- Recommended reading

Khalil	Section 4.1 p. 120-122
	Sections 4.2-4.3
	Section 8.2