

TTT4120 Digital Signal Processing Fall 2017

Filter Structures

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Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
 - 9.1 Structures for the realization of discrete-time systems
 - 9.2 Structures for FIR Systems
 - 9.3 Structures for IIR Systems
- A compressed overview of topics treated in the lecture, see “[Filter implementation](#)” on Blackboard

*Level of detail is defined by lectures and problem sets

Contents and learning outcomes

- Filter structures
 - Direct-form
 - Cascade form
 - Parallel form
 - Transposed form

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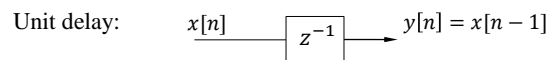
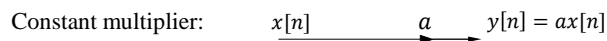
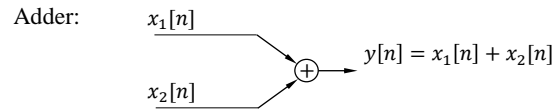
Background

- So far we have expressed a digital filter (system) using
 - System function
 - Frequency response
 - Impulse response
- How shall the filter be realized in practice?
 - Different filter structures dictate different design strategies
- Need to consider problems associated with quantization effects when finite-precision arithmetic is used in the implementation
 - Rounding errors in coefficients
 - Rounding errors in calculations

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Basic elements

- Three elements to describe digital filter structure (Lecture 2):



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Filter structures

- Consider rational system function

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}}, a_0 = 1$$

- Order of system is called N if $a_N \neq 0$
- Poles and zeros of $H(z)$ (depend on a_k, b_k) determine frequency response
- Difference equation associated with an IIR filter is

$$y[n] = \sum_{k=0}^M b_k x[n - k] - \sum_{k=1}^N a_k y[n - k]$$

- Arranging equations in different ways \Rightarrow different implementations
 - Computational complexity, memory requirements, finite-precision effects

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Filter structures...

- Computational complexity
 - Number of operations required to compute the output $y[n]$
- Memory requirements
 - Number of memory locations to store system parameters, inputs and outputs (past and present), intermediate values
- Finite-word-length effects (finite-precision)
 - Various structures are equivalent for infinite precision but behave differently with finite precision

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Filter structures...

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}},$$

$$y[n] = \sum_{k=0}^M b_k x[n-k] - \sum_{k=1}^N a_k y[n-k]$$

- Three different implementation structures
 - Direct form: implement difference equation (two versions)
 - Cascade form: factor $H(z)$ into products of 2nd-order sections
 - Parallel form: partial fraction of $H(z)$ into 2nd-order sections

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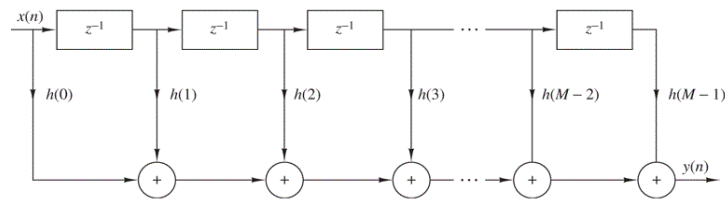
FIR filter

- Impulse response and system function

$$y[n] = \sum_{k=0}^{M-1} b_k x[n-k] = \sum_{k=0}^M h[k] x[n-k]$$

$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k}$$

- Direct-form structure follows immediately from the convolution sum



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Linear-phase FIR filter

- Linear-phase filter satisfies symmetry or asymmetry condition

$$h[n] = \pm h[M-1-n]$$

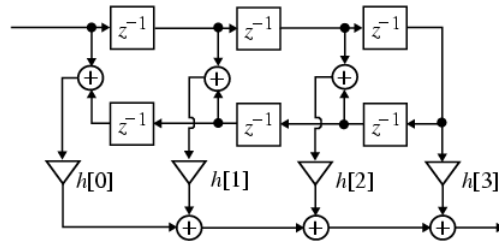
- Example: Consider a length-7 Type 1 FIR transfer function with a symmetric impulse response

$$\begin{aligned} H(z) &= h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[2]z^{-4} \\ &\quad + h[1]z^{-5} + h[0]z^{-6} \\ &= h[0](1 + z^{-6}) + h[1](z^{-1} + z^{-5}) \\ &\quad + h[2](z^{-2} + z^{-4}) + h[3]z^{-3} \end{aligned}$$

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Linear-phase FIR filter...

- Example (cont.): $H(z) = h[0](1 + z^{-6}) + h[1](z^{-1} + z^{-5}) + h[2](z^{-2} + z^{-4}) + h[3]z^{-3}$

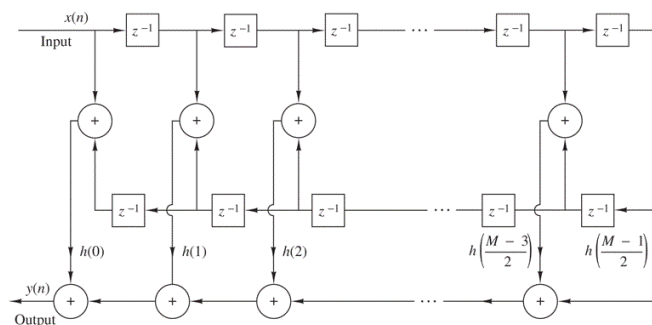


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Linear-phase FIR filter...

- Linear-phase filter satisfies symmetry or asymmetry condition

$$h[n] = \pm h[M - 1 - n]$$
- Reduce the number of multiplications by a factor of two (2)



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IIR filter

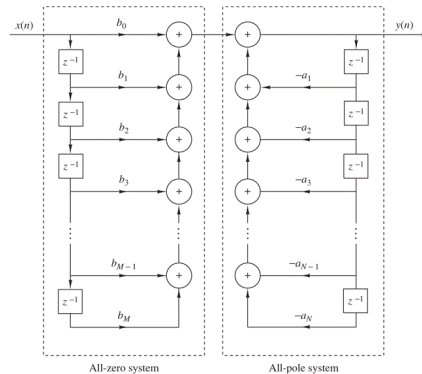
- Impulse response and system function obtained from

$$y[n] = \sum_{k=0}^M b_k x[n-k] - \sum_{k=1}^N a_k y[n-k]$$

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}} = \frac{B(z)}{A(z)}$$

$$= B(z) \frac{1}{A(z)}$$

- Direct-form structure I
 - $M + N + 1$ multiplications
 - $M + N + 1$ memory locations



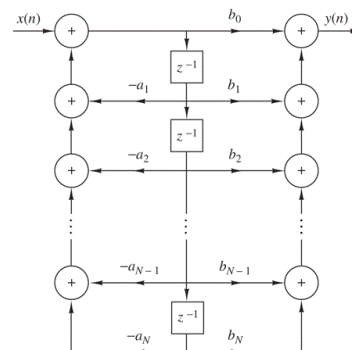
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IIR filter...

- Can reverse the order without changing the system response

$$B(z) \frac{1}{A(z)} = \frac{1}{A(z)} B(z)$$

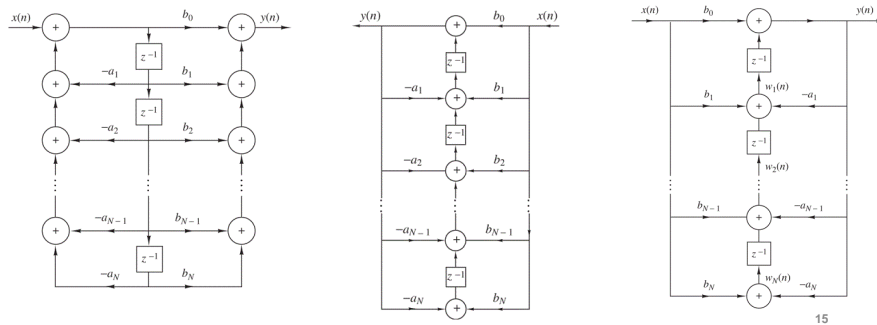
- Direct-form structure II
 - Only need to store past values of a single variable
 - Canonic $\Rightarrow \max(N, M)$ delays



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Transposed structure

- Equivalent structure to the direct form obtained using **transposition**
 - All signal path arrows are reversed
 - All branch nodes are replaced by adder nodes, all adder nodes replaced by branch nodes
 - Input and output interchanged



Cascade structure

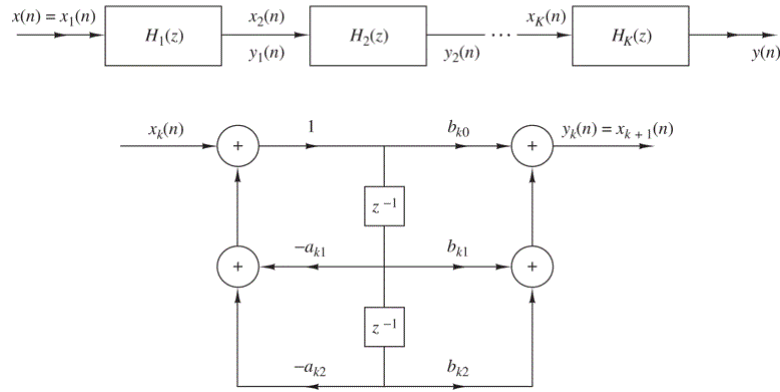
- Consider a high-order IIR system with system function ($N \geq M$)

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}} = \frac{b_0 \prod_{k=1}^M (1 - z_k z^{-1})}{\prod_{k=1}^N (1 - p_k z^{-1})}$$

- Filter coefficients are real-valued \Rightarrow poles and zeros are real-valued or comes in complex conjugated pairs
- Complex-conjugated poles/zeros together in one section, real-valued poles/zeros paired arbitrarily
- Write the system function as a cascade of second-order systems

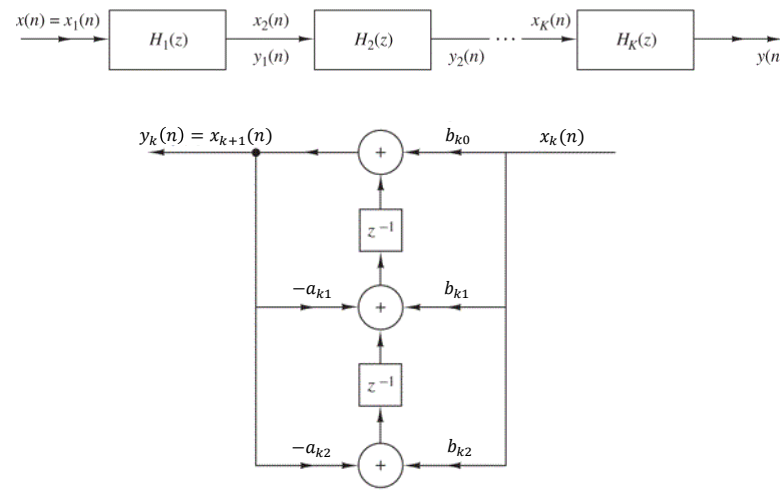
$$H(z) = \prod_{k=1}^K H_k(z) = \prod_{k=1}^K \frac{b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}}{a_{k0} + a_{k1} z^{-1} + a_{k2} z^{-2}}$$

Cascade structure...



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Cascade structure...



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Cascade structure...

- Example (from Mitra):

$$H(z) = \frac{0.44z^{-1} + 0.362z^{-2} + 0.02z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$

$$= \left(\frac{0.44 + 0.362z^{-1} + 0.02z^{-2}}{1 + 0.8z^{-1} + 0.5z^{-2}} \right) \left(\frac{z^{-1}}{1 - 0.4z^{-1}} \right)$$

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Parallel-form structure

- Consider a high-order IIR system with system function

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1z^{-1} + \dots + b_Mz^{-M}}{1 + a_1z^{-1} + \dots + a_Nz^{-N}}$$

$$= \underbrace{\frac{\tilde{b}_0 + \tilde{b}_1z^{-1} + \dots + \tilde{b}_{N-1}z^{-(N-1)}}{1 + a_1z^{-1} + \dots + a_Nz^{-N}}}_{\text{Proper rational part}} + \underbrace{\sum_{k=0}^{M-N} C_k z^{-k}}_{\text{Only if } M \geq N}$$

- Assume distinct poles and make a partial fraction expansion

$$H(z) = \sum_{k=1}^N \frac{R_k}{(1 - p_k z^{-1})} + \underbrace{\sum_{k=0}^{M-N} C_k z^{-k}}_{\text{Only if } M \geq N}$$

- Rewrite first sum using 2nd-order sections
 - complex-conjugated poles, real-valued poles paired arbitrarily

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Parallel-form structure...

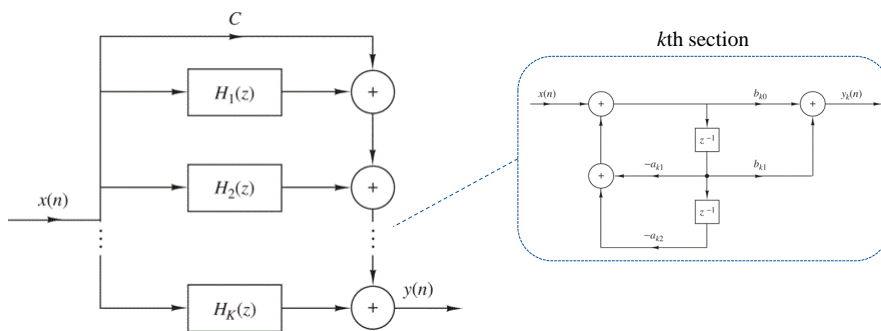
- Final form:

$$H(z) = \sum_{k=1}^K \frac{b_{k0} + b_{k1}z^{-1}}{a_{k0} + a_{k1}z^{-1} + a_{k2}z^{-2}} + \underbrace{\sum_{k=0}^{M-N} C_k z^{-k}}_{\text{Only if } M \geq N}$$

- Filter input available to all biquad sections and polynomial section
- Output from all sections summed to form filter output
 - A **parallel structure** can be built to realize $H(z)$
 - Biquad section can be implemented using, e.g., direct form II

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Parallel-form structure...



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Parallel structure...

- Example (from Mitra):

$$H(z) = \frac{0.44z^{-1} + 0.362z^{-2} + 0.02z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$

$$= -0.1 + \frac{0.6}{1 - 0.4z^{-1}} + \frac{-0.5 - 0.2z^{-1}}{1 + 0.8z^{-1} + 0.5z^{-2}}$$

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Example

- Determine the cascade and parallel realizations for system

$$H(z) = \frac{10\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{2}{3}z^{-1}\right)(1 + 2z^{-1})}{\left(1 - \frac{3}{4}z^{-1}\right)\left(1 - \frac{1}{8}z^{-1}\right)\left(1 - \left[\frac{1}{2} - j\frac{1}{2}\right]z^{-1}\right)\left(1 - \left[\frac{1}{2} + j\frac{1}{2}\right]z^{-1}\right)}$$

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Example...

- Cascade form: Group pairwise, e.g.,

$$H(z) = \frac{10 \left(1 - \frac{1}{2}z^{-1}\right) \left(1 - \frac{2}{3}z^{-1}\right) (1 + 2z^{-1})}{\left(1 - \frac{3}{4}z^{-1}\right) \left(1 - \frac{1}{8}z^{-1}\right) \left(1 - \left[\frac{1}{2} - j\frac{1}{2}\right]z^{-1}\right) \left(1 - \left[\frac{1}{2} + j\frac{1}{2}\right]z^{-1}\right)}$$

$$H_1(z) = \frac{1 - \frac{2}{3}z^{-1}}{\left(1 - \frac{3}{4}z^{-1}\right) \left(1 - \frac{1}{8}z^{-1}\right)} = \frac{1 - \frac{2}{3}z^{-1}}{1 - \frac{7}{8}z^{-1} + \frac{3}{32}z^{-2}}$$

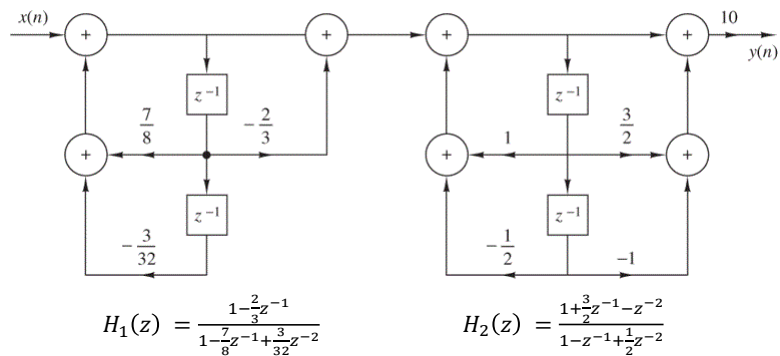
$$H_2(z) = \frac{\left(1 - \frac{1}{2}z^{-1}\right) (1 + 2z^{-1})}{\left(1 - \left[\frac{1}{2} - j\frac{1}{2}\right]z^{-1}\right) \left(1 - \left[\frac{1}{2} + j\frac{1}{2}\right]z^{-1}\right)} = \frac{1 + \frac{3}{2}z^{-1} - z^{-2}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

$$\Rightarrow H(z) = 10H_1(z)H_2(z)$$

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Example...

- Cascade form: $H(z) = 10H_1(z)H_2(z)$



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Example...

- Parallel form: Expand $H(z)$ in partial fraction expansion

$$H(z) = \frac{10\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)(1+2z^{-1})}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)\left(1-\left[\frac{1}{2}-j\frac{1}{2}\right]z^{-1}\right)\left(1-\left[\frac{1}{2}+j\frac{1}{2}\right]z^{-1}\right)}$$

$$= \frac{R_1}{1-\frac{3}{4}z^{-1}} + \frac{R_2}{1-\frac{1}{8}z^{-1}} + \frac{R_3}{1-\left[\frac{1}{2}-j\frac{1}{2}\right]z^{-1}} + \frac{R_3^*}{1-\left[\frac{1}{2}+j\frac{1}{2}\right]z^{-1}}$$

- Residue calculus

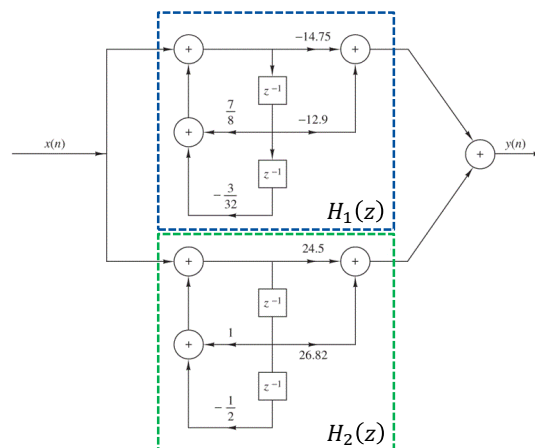
$$R_1 = 2.93, R_2 = -17.68, R_3 = 12.25 + j14.57$$

- Group terms into biquads

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Example...

- Parallel form: $H(z) = \frac{-14.75-12.90z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}} + \frac{24.50+26.82z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}}$



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Summary

- Today we discussed:
 - Filter implementations
- Next:
 - Finite-precision and roundoff effects