NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF ENGINEERING CYBERNETICS

Exam
TTK4150 Nonlinear Control Systems

Saturday December 10, 2016

SOLUTION

Problem 1 (17%) The system is given by

$$\begin{aligned}\dot{x}_1 &= -kx_1 \\ \dot{x}_2 &= -kx_2 - x_1^2 x_2\end{aligned}$$

a The equilibrium point is found by solving $f(x) = 0$. We have that

$$\begin{aligned}0 &= -kx_1 \\ &\Leftrightarrow x_1 = 0 \\ 0 &= -kx_2 \\ &\Leftrightarrow x_2 = 0\end{aligned}$$

from which we conclude that origin is the only equilibrium point of the system.

b To classify the qualitative behavior of the equilibrium point we calculate the Jacobian at the origin. The elements in the Jacobian is given by

$$\begin{aligned}\frac{\partial f_1}{\partial x_1} &= -k \\ \frac{\partial f_1}{\partial x_2} &= 0 \\ \frac{\partial f_2}{\partial x_1} &= -2x_1 x_2 \\ \frac{\partial f_2}{\partial x_2} &= -k - x_1^2\end{aligned}$$

Evaluating the Jacobian at the origin results in

$$\begin{aligned}A &= \left.\frac{\partial f}{\partial x}\right|_{x=0} \\ &= \begin{bmatrix} -k & 0 \\ 0 & -k \end{bmatrix}\end{aligned}$$

and the eigenvalues are calculated by solving

$$\begin{aligned}
\det\left(\lambda I - A\right) &= \det\left(\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}-k & 0\\ 0 & -k\end{bmatrix}\right)\\
&= \det\left(\begin{bmatrix}\lambda + k & 0\\ 0 & \lambda + k\end{bmatrix}\right)\\
&= \left(\lambda + k\right)^2\\
&= 0\\
&\Rightarrow \lambda + k = 0\\
&\Rightarrow \lambda = -k
\end{aligned}$$

From the eigenvalues we can conclude of the qualitative behavior of the system (p. 52 in Khalil)
Case 1: $k > 0$ results in a stable focus behavior near the origin
Case 2: $k = 0$ has no conclusion (a center for a linear system)
Case 3: $k < 0$ results in a unstable focus behavior near the origin

c From Corollary 4.3 we know that the origin is exponentially stable if and only if $A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$ is Hurwitz (Re $\left[eig\left(A\right)\right] < 0$). We conclude from the previously calculated eigenvalues that the system can only be exponential stable if $k > 0$.

d By considering the system in the form $\dot{x} = f\left(x\right)$ it can be recognized that $f\left(x\right)$ is continuous on $\mathbb{R}^2$. Furthermore, we have that

$$\frac{\partial f\left(x\right)}{\partial x} = \begin{bmatrix}-k & 0\\ -2x_1x_2 & -k - x_1^2\end{bmatrix}$$

is continuous on $\mathbb{R}^2$. Since the system is continuously differentiable on $\mathbb{R}^2$ it is locally Lipschitz on $\mathbb{R}^2$ (Lemma 3.2).

e First of all we require that $k\left(t\right)$ is such that the system is locally Lipschitz on $\mathbb{R}^2$. Let the Lyapunov function candidate be given by

$$\begin{aligned}
V &= \frac{1}{2}\left\|x\right\|_2^2\\
&= \frac{1}{2}\left(x_1^2 + x_2^2\right)
\end{aligned}$$

which is is a positive definite, continuously differentiable function. The time derivative of $V\left(x\right)$ along the solution of the system is found as

$$\begin{aligned}
\dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2\\
&= x_1\left(-k\left(t\right)x_1\right) + x_2\left(-k\left(t\right)x_2 - x_1^2x_2\right)\\
&= \left(x_1^2 + x_2^2\right)\left(-k\left(t\right)\right) - x_1^2x_2^2\\
&= -\left\|x\right\|_2^2 k\left(t\right) - x_1^2x_2^2\\
&= -k\left(t\right)\left\|x\right\|_2^2 - x_1^2x_2^2\\
&\leq -k\left(t\right)\left\|x\right\|_2^2
\end{aligned}$$

where we require $k(t) \geq c \; \forall t \geq 0$ for some positive constant c in order to conclude that $\dot{V}(x)$ is negative definite.In addition, since $V(x)$ is radially unbounded, the origin is globally asymptotically stable. Finally, by using Theorem 4.10 it can be concluded that the origin is globally exponentially stable with $a = 2$, $k_1 = k_2 = \frac{1}{2}$ and $k_3 = c$.

Problem 2 (16%)

a When $u = 0$ we have

$$\begin{aligned}\dot{V}(x) &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = \phi(x_1)(-k_1x_1 + x_2) + x_2(-\phi(x_1) - (1+\cos^2(t))x_2) \\ &= -k_1x_1\phi(x_1) - (1+\cos^2(t))x_2^2 \leq -k_1x_1\phi(x_1) - x_2^2 \\ &\leq -k_1k_2x_1^2 - x_2^2\end{aligned}$$

for all x. And since $V(x)$ is radially unbounded if follows that the origin is GAS.

b Using $V(x) = \int_0^{x_1}\phi(z)dz + \frac{1}{2}x_2^2$ we have

$$\begin{aligned}\dot{V}(x) &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = \phi(x_1)(-k_1x_1 + x_2) + x_2(-\phi(x_1) - (1+\cos^2(t))x_2 + u) \\ &= -k_1x_1\phi(x_1) - (1+\cos^2(t))x_2^2 + x_2u \leq -k_1x_1\phi(x_1) - x_2^2 + x_2u \\ &\leq -k_1k_2x_1^2 - x_2^2 + x_2u = -\psi(x) + yu\end{aligned}$$

where $\psi(x) = k_1k_2x_1^2 + x_2^2$ is positive definite. Hence it is strictly passive.

c Using $V(x) = \int_0^{x_1}\phi(z)dz + \frac{1}{2}x_2^2$ we have

$$\begin{aligned}\dot{V}(x) &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = \phi(x_1)(-k_1x_1 + x_2) + x_2(-\phi(x_1) - (1+\cos^2(t))x_2 + u) \\ &= -k_1x_1\phi(x_1) - (1+\cos^2(t))x_2^2 + x_2u \leq -k_1x_1\phi(x_1) - x_2^2 + x_2u \\ &\leq -k_1k_2x_1^2 - x_2^2 + x_2u \leq -x_2^2 + x_2u = -y\rho(y) + yu\end{aligned}$$

where $\rho(y) = y$. Hence it is output strictly passive.

d The unforced system is given by

$$\begin{aligned}\dot{x}_1 &= -k_1x_1 + x_2 \\ \dot{x}_2 &= -\phi(x_1) - (1+\cos^2(t))x_2 \\ y &= x_2\end{aligned}$$

Consider

$$S = \left\{x \in \mathbb{R}^2 | y = 0\right\} = \left\{x \in \mathbb{R}^2 | x_2 = 0\right\}$$

the for every $(x_1, x_2) \in S$ we have $0 = \dot{x}_2 = -\phi(x_1) - 0 \Rightarrow \phi(x_1) = 0 \Rightarrow x_1 = 0$, $\dot{x}_1 = -0 + 0 = 0$. Hence no other solution can stay identically in S other than the zero solution. Thus the system is zero-state observable.

e Using $V(x) = \int_0^{x_1} \phi(z)dz + \frac{1}{2}x_2^2$ we have

$$\begin{aligned}
\dot{V}(x) &\leq -k_1k_2x_1^2 - x_2^2 + x_2u = -(k_1k_2 - \theta)x_1^2 - (1-\theta)x_2^2 - \theta||x||_2^2 + x_2u \\
&\leq -(k_1k_2 - \theta)x_1^2 - (1-\theta)x_2^2 - \theta||x||_2^2 + |x_2||u| \\
&\leq -(k_1k_2 - \theta)x_1^2 - (1-\theta)x_2^2 - \theta||x||_2^2 + ||x_2||_2|u| \\
&\leq -(k_1k_2 - \theta)x_1^2 - (1-\theta)x_2^2
\end{aligned}$$

for $-\theta||x||_2^2 + ||x_2||_2|u| \leq 0$, or equivalently for $|u| \leq \theta||x||_2$. The positive constant θ is chose such that $k_1k_2 - \theta > 0$ and $1 - \theta > 0$.

Problem 3 (12%)

The given system can be rewritten as

$$\begin{aligned}
\dot{x}_1 &= \varepsilon((-1 + 1.5\sin(t)\cos(t))x_1 + x_2) \\
\dot{x}_2 &= \varepsilon(-x_1 - (1 + 2\sin^2(t))x_2)
\end{aligned}$$

$$\dot{x} = \varepsilon \begin{bmatrix} (-1 + 1.5\sin(t)\cos(t))x_1 + x_2 \\ -x_1 - (1 + 2\sin^2(t))x_2 \end{bmatrix} = \varepsilon f(t,x) \tag{1}$$

This can be associated with an autonomous average system

$$\dot{x} = \varepsilon f_{av}(x)$$

where

$$f_{av}(x) = \frac{1}{T}\int_0^T f(\tau, x)d\tau$$

Taking the integral of the system (1) becomes

$$\begin{aligned}
f_{av}(x) &= \frac{1}{T}\int_0^T \begin{bmatrix} (-1 + 1.5\sin(\tau)\cos(\tau))x_1 + x_2 \\ -x_1 - (1 + 2\sin^2(\tau))x_2 \end{bmatrix} d\tau \\
&= \frac{1}{T}\int_0^T \begin{bmatrix} (-1 + 1.5\sin(\tau)\cos(\tau))x_1 + x_2 \\ -x_1 - (1 + 2(\frac{1}{2}(1 - \cos(2\tau))))x_2 \end{bmatrix} d\tau \\
&= \frac{1}{T}\int_0^T \begin{bmatrix} (-1 + 1.5\sin(\tau)\cos(\tau))x_1 + x_2 \\ -x_1 - (2 - \cos(2\tau))x_2 \end{bmatrix} d\tau
\end{aligned}$$

To take the integral of

$$\int \sin(\tau)\cos(\tau)d\tau$$

we do integral by substitution, which means that

$$\int \sin(\tau)\cos(\tau)d\tau = \int u du$$

where

$$u = \sin(\tau)$$
$$du = \cos(\tau)d\tau$$

Then

$$\int \sin(\tau)\cos(\tau)d\tau = \int u du = \left[\frac{1}{2}u^2\right] = \frac{1}{2}\sin^2(t)$$

It now possible to take the integral of the system (1) which becomes

$$f_{av}(x) = \frac{1}{T}\begin{bmatrix} (-T + 0.75\sin^2(T))x_1 + Tx_2 \\ -Tx_1 - (2T - \frac{1}{2}\sin(2T))x_2 \end{bmatrix}$$

Since the function (1) is π-periodic in t, the average function is given by

$$\begin{aligned} f_{av}(x) &= \frac{1}{\pi}\begin{bmatrix} (-\pi + 0.75\sin^2(\pi))x_1 + \pi x_2 \\ -\pi x_1 - (2\pi - \frac{1}{2}\sin(2\pi))x_2 \end{bmatrix} \\ &= \frac{1}{\pi}\begin{bmatrix} -\pi x_1 + \pi x_2 \\ -\pi x_1 - 2\pi x_2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

The average system has an equilibrium point at the origin and matrix $\begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix}$ is Hurwitz. Thus, the origin of the average system is exponentially stable. It follows that, for sufficiently small ε, the original system has a unique exponentially stable periodic solution in the neighbourhood of the origin. But the origin is an equilibrium point of the original system. Hence, the periodic solution is the trivial solution, which shows that the origin is exponentially stable.

Problem 4 (13%)

a Using

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

we have

$$\begin{aligned}
\dot{V}_1(x) =&(-x_1 + x_2)x_1 + \left(x_1 - x_2 - x_2^3 + u\right)x_2 \\
=& -x_1^2 + x_1x_2 - x_2x_1 - x_2^2 - x_2^4 + u_1x_2 \\
=& -x_1^2 - x_2^2 - x_2^4 + u_1x_2 \\
\leq& -x_1^2 - x_2^2 + u_1x_2 \\
\leq& -x_2^2 + u_1x_2 \\
\leq& -y_1^2 + u_1y_1
\end{aligned}$$

Hence the system Σ_1 from u_1 to y_1 is output strictly passive.

b It follows, by applying Lemma 6.5, that it is finite gain $\mathcal{L}_2$ stable with gain at most one, since the system Σ_1 from u_1 to y_1 is output strictly passive.

c Considering the result of problem 4a and the definition of input strictly passive (Definition 6.3). It can be conclude that the system Σ_1 is not input strictly passive with the storage function V_1

d Now suppose that we feedback connect Σ_1 and Σ_2 with

$$\begin{aligned}
u_1 &= -y_2 \\
u_2 &= y_1.
\end{aligned}$$

Using $V_2 = \frac{1}{2}\omega^2$ we have

$$\begin{aligned}
\dot{V}_2(x) =&\omega(\dot{\omega}) \\
=&\omega u_2 = yu
\end{aligned}$$

which means that Σ_2 from u_2 to y_2 is now lossless and from Theorem 6.2 we can conclude that the feedback connected system is not finite gain $\mathcal{L}_2$ stable since Σ_2 is only lossless and Σ_1 is not both input strictly passive and output strictly passive.

e By applying Theorem 6.1 it can be concluded that the feedback connected system is passive. Additionally by applying Lemma 6.6 it can be concluded that the feedback connected system is 0-stable.

Problem 5 (32%)

a We differentiate the output y until the input appears:

$$\begin{aligned}
\dot{y} &= \dot{x}_2 = x_1 + x_3 \\
\ddot{y} &= \dot{x}_1 + \dot{x}_3 = x_3 - x_2^6 + u + x_3 + x_1 - |x_3|x_3 = \gamma(x)\big(u - \alpha(x)\big) \\
&\text{where} \\
\gamma(x) &= 1 \\
\alpha(x) &= -\big(x_1 - x_2^6 + 2x_3 - |x_3|x_3\big)
\end{aligned}$$

Hence, the relative degree is $r = 2$.
The function $\gamma(x) \neq 0 \; \forall \; x$, which means that the relative degree is defined for all $x \in \mathbb{R}^3$.

b Since $r = 2$ on $\mathcal{D}$, choose $\xi_1 = y = x_2$; $\xi_2 = \dot{y} = \dot{x}_2 = x_1 + x_3$. With this choice of ξ_1 and ξ_2 we obtain (see problem 5a):

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \gamma(x)(u - \alpha(x)) \\
&\text{where} \\
\gamma(x) &= 1 \\
\alpha(x) &= -\big(x_1 - x_2^6 + 2x_3 - |x_3|x_3\big)
\end{aligned}$$

Then the internal dynamics η are found. Since ξ is of dimension 2, η is of dimension $3 - 2 = 1$. With $\eta = \phi(x)$ we have that

$$\frac{\partial \phi(x)}{\partial x} g(x) = 0$$

Inserting for $g(x)$ we have that

$$\frac{\partial \phi(x)}{\partial x_2} = 0$$

A $\phi(x)$ which satisfies this is

$$\phi(x) = x_3$$

With this choice we obtain

$$\dot{\eta} = \dot{x}_3 = \underbrace{-|x_3|x_3}_{-|\eta|\eta} + \underbrace{x_1 + x_3}_{\xi_2} = -|\eta|\eta + \xi_2 =: f_0(\xi, \eta)$$

Hence, the coordinate transformation

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \eta \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 + x_3 \\ x_3 \end{bmatrix} =: T(x)$$

transforms the system into the normal form

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \gamma(x)(u - \alpha(x)) \\
\dot{\eta} &= -|\eta|\eta + \xi_2
\end{aligned}$$

The coordinate transformation $T(x) = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ is a diffeomorphism, since it is continuously differentiable and $T^{-1}\begin{pmatrix} \xi \\ \eta \end{pmatrix}$, which equals to

$$\begin{array}{rcl} x_1 & = & \xi_2 - \eta \\ x_2 & = & \xi_1 \\ x_3 & = & \eta \end{array} \quad = T^{-1}\begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

is also continuously differentiable. Hence $T(x)$ is a diffeomorphism.

c We choose $u = \alpha(x) + \frac{1}{\gamma(x)}v$, where $\alpha(x) = -\big(x_1 - x_2^6 + 2x_3 - |x_3|x_3\big)$ and $\gamma(x) = 1$ (see problems 5a–5b).
The controller is defined in the domain $\mathcal{D} := \{x : x_1, x_2, x_3 \in \mathbb{R}\}$.

d After applying the input-output linearizing controller, the ξ-dynamics has the form

$$\begin{array}{rcl} \dot{\xi}_1 & = & \xi_2 \\ \dot{\xi}_2 & = & v \end{array}$$

By choosing $v = -k_1\xi_1 - k_2\xi_2$, $k_1 > 0$, $k_2 > 0$, the closed-loop dynamics become

$$\begin{array}{rcl} \dot{\xi}_1 & = & \xi_2 \\ \dot{\xi}_2 & = & -k_1\xi_1 - k_2\xi_2 \end{array}$$

which is a globally exponentially stable (GES) linear system (its system matrix is Hurwitz). Thus, the controller $v = -k_1\xi_1 - k_2\xi_2$, $k_1 > 0$, $k_2 > 0$ makes external ξ-dynamics GES (and, therefore, also GAS).

e The system is minimum phase if the zero-dynamics $\dot{\eta} = f_0(\eta, 0)$ has an asymptotically stable equilibrium point at $\eta = 0$ (see Khalil p. 517).
In our case, the zero dynamics have the form $\dot{\eta} = -|\eta|\eta$ which is globally asymptotically stable (GAS). This can be shown, for example, with the quadratic Lyapunov function $V = \frac{1}{2}\eta^2$.

$$\dot{V} = \eta\dot{\eta} = \eta f_0(\eta, 0) = -|\eta|\eta^2$$

Since V is continuously differentiable and positive definite, and $\dot{V}$ is negative definite, $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable at the origin. Clearly $T(0) = 0$, and the system as a whole is therefore minimum phase.

f The system is minimum phase. Hence, the origin of the closed-loop system

$$\begin{array}{rcl} \dot{\xi}_1 & = & \xi_2 \\ \dot{\xi}_2 & = & -k_1\xi_1 - k_2\xi_2 \\ \dot{\eta} & = & f_0(\eta, \xi) \end{array}$$

is asymptotically stable (see Lemma 13.1 in Khalil p. 531).

g The internal dynamics $\dot{\eta} = f_0(\eta, \xi) = -|\eta|\eta + \xi_2$ with ξ_2 as input are input-to-state stable (ISS) which can be shown by

$$
\begin{aligned}
V(\eta) &= \frac{1}{2}\eta^2 \\
\dot{V}(\eta) &= -|\eta|\eta^2 + \eta\xi_2 \\
&= -(1-\theta)|\eta|\eta^2 - \theta|\eta|\eta^2 + \eta\xi_2 \\
&\leq -(1-\theta)|\eta|\eta^2 - \theta|\eta||\eta|^2 + |\eta||\xi_2| \\
&\leq -(1-\theta)|\eta|\eta^2 - \theta|\eta||\eta|^2 + |\eta||\xi_2| \\
&\leq -(1-\theta)|\eta|\eta^2, \quad \forall\, |x| \geq \sqrt{\frac{|\xi_2|}{\theta}}
\end{aligned}
$$

An interconnection of ISS for the internal dynamics and GES for external ξ-dynamics

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= -k_1\xi_1 - k_2\xi_2
\end{aligned}
$$

is globally asymptotically stable (GAS) at the origin (see, for example, Lemma 13.2 in Khalil p. 532).

h The overall controller (both the input-output linearizing controller $u = \alpha(x) + \beta(x)v$ and the stabilizing controller $v = \delta(\xi)$) can be written as

$$
\mathcal{U}(x) = \underbrace{-\left(x_1 - x_2^6 + 2x_3 - |x_3|x_3\right)}_{\alpha(x)} + \underbrace{1}_{\beta(x)} \left(-k_1 \underbrace{x_2}_{\xi_1} - k_2 \underbrace{(x_1 + x_3)}_{\xi_2} \right)
$$

This controller is defined globally since et only has a equilibrium point at $x = 0$. For this reason it can globally stabilize the original system.

Problem 6 (10%)

Defining $V_1(x_1) = \frac{1}{2}x_1^2$ we have

$$\dot{V}_1 = x_1\dot{x_1} = x_1^6 + x_1x_2$$

As the first step we use $x_2 = \phi(x_1) = -2x_1^5 - x_1$ where $\phi(0) = 0$. Then

$$\dot{V}_1 = x_1\dot{x}_1 = x_1^6 + x_1\phi(x_1) = x_1^6 + x_1(-2x_1^5 - x_1) = -x_1^6 - x_1^2 \leq -W(x_1)$$

where $W(x_1)$ is positive definite in x_1. For the second step we define $z = x_2 - \phi(x_1)$ then

$$\dot{V}_1 = x_1\dot{x_1} = x_1^6 + x_1(\phi(x_1) + z) = -x_1^6 - x_1^2 + = -x_1^6 - x_1^2 + x_1z \leq -W(x_1) + x_1z$$

and

$$
\begin{aligned}
\dot{z} &= x_2^3 + x_1^2 + u - \dot{\phi}(x_1) \\
\dot{z} &= x_2^3 + x_1^2 + u + (1 + 10x_1^4)\dot{x}_1 \\
\dot{z} &= x_2^3 + x_1^2 + u + (1 + 10x_1^4)(-x_1^5 - x_1 + z)
\end{aligned}
$$

Defining $V_c(x_1, z) = V_1(x_1) + \frac{1}{2}z^2$ then

$$\begin{aligned}
\dot{V}_c &= \dot{V}_1 + z\dot{z} \leq -W(x_1) + z(x_1 + \dot{z}) \\
\dot{V}_c &= \dot{V}_1 + z\dot{z} \leq -W(x_1) + z(x_1 + x_2^3 + x_1^2 + u + (1 + 10x_1^4)(x_1^5 + x_2))
\end{aligned}$$

It follows that $\dot{V}_c \leq -W(x_1) - kz^2$ if

$$x_1 + x_2^3 + x_1^2 + u + (1 + 10x_1^4)(x_1^5 + x_2) = -kz$$

for any positive constant k. Which means that

$$u = -(1 + 10x_1^4)(x_1^5 + x_2) - x_1 - kz - x_2^3 - x_1^2$$

Remark: The answer until this step is considered to be correct. For complete expression of the feedback in terms of state x we have

$$\begin{aligned}
u &= -(1 + 10x_1^4)(x_1^5 + x_2) - x_1 - k(x_2 - \phi(x_1)) - x_2^3 - x_1^2 \\
&= -(1 + 10x_1^4)(x_1^5 + x_2) - x_1 - k(x_2 - 2x_1^5 - x_1) - x_2^3 - x_1^2
\end{aligned}$$