TTK4150 Nonlinear Control Systems
Lecture 6

Stability analysis of nonautonomous system

NTNU
Innovation and Creativity

Previous lectures

Previous lectures:

Lyapunov's direct method for autonomous systems:

- Lyapunov's theorems for
  - stability
  - local and global asymptotic stability
  - local and global exponential stability
- La Salle's theorem
  - $\dot{V} \leq 0$ asymptotic stability of equilibrium points
  - Regions of attraction - find an estimate
  - Convergence to other invariant sets than equilibrium points
- Some methods for finding Lyapunov function candidates (LFCs)

Outline I

Outline II

- Stability theorems
- Estimate of Region of attraction
- Stability theorem: Exponential stability

6 Invariance-like results
- Barbalat's lemma

7 Next lecture

Today's goals

After today you should...

Know **Lyapunov's direct method for nonautonomous systems**. In particular,

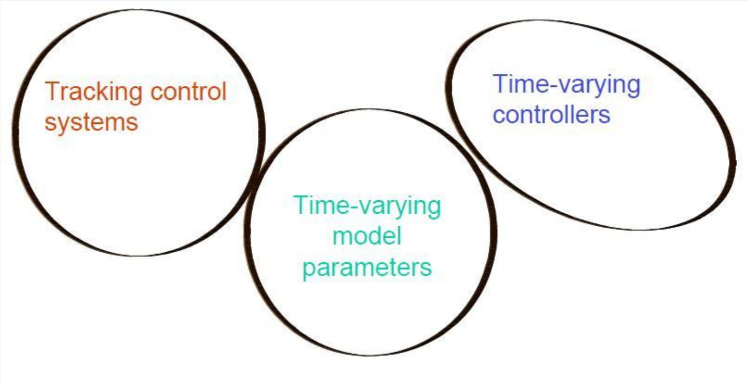
- Know comparison functions of class $\mathscr{K}$, $\mathscr{K}_\infty$ and $\mathscr{K}\mathscr{L}$
- Know the stability definitions of nonautonomous systems (and how they deviate from the stability definitions of autonomous systems)
- Be able to use Lyapunov's direct method to analyze the stability properties of an equilibrium point of a nonautonomous system.
- Be able to use Barbalat's lemma to analyze the convergence properties when $\dot{V}(t,x) \leq 0$

Literature

Today's lecture is based on

Khalil Sections 4.4-4.5
Section 8.3

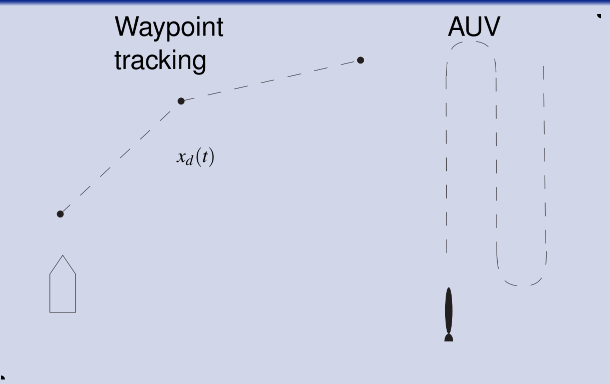
Nonautonomous systems



Need to analyze the stability properties of an equilibrium point of a nonautonomous system $\dot{x} = f(t,x)$

Tracking control systems

Tracking control systems



Time-varying model parameters

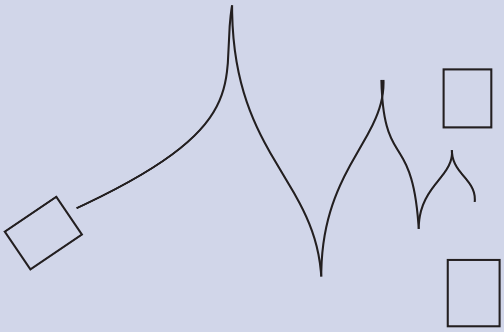
Time-varying model parameters

- Example: Spacecraft
  - At constant thrust, the mass $m(t)$ is decreasing at a constant rate

Time-varying controllers

Time-varying controllers

- Some systems cannot be stabilized by $u(x)$, but need $u(t,x)$
- Example: Point stabilization, i.e. parking, of car



Autonomous Car Learns To Powerslide Into Parking Spot

Tightest Parallel Parking Record

Obstacle-aided locomotion

AIKO

Nonautonomous systems

Nonautonomous systems

$$\dot{x} = f(t,x) \quad f : [0,\infty) \times \mathbb{D} \to \mathbb{R}^n$$

- $f(t,x)$ Piecewise continuous in t
  locally Lipschitz in x on $[0,\infty) \times \mathbb{D}$
- $x = 0 \in \mathbb{D}$

Definition: Equilibrium point

x^* is an equilibrium point for $\dot{x} = f(t,x)$ at $t = 0$ iff

$$f(t,x^*) = 0 \quad \forall t \geq 0$$

Examples

Example

Find the equilibrium points x^* of the following systems

a) $\dot{x} = -\dfrac{a(t)x}{1+x^2} \qquad a(t) > 0$

b) $\dot{x} = -\dfrac{a(t)x}{1+x^2} + b(t) \qquad \begin{array}{l} a(t) > 0 \\ b(t) \neq 0 \quad \forall t > 0, \quad b(0) = 0 \end{array}$

Translate a nonzero equilibrium point to the origin

We will analyse the stability properties of $x^* = 0$

Translate the equilibrium point of interest to the origin

We can always translate a nonzero equilibrium point to the origin.

$$\dot{x} = f(t,x) \quad f(t,x^*) = 0 \quad \forall\, t \geq 0$$

Define the error variable

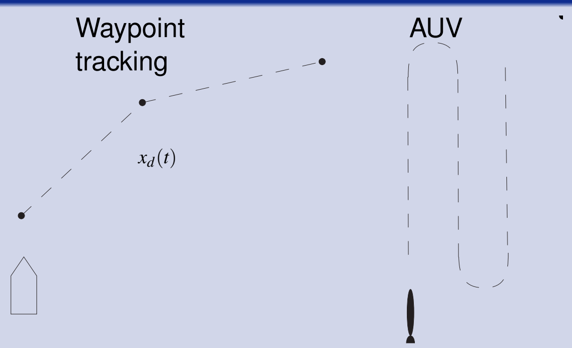
$$e = x - x^*$$
$$\dot{e} = \dot{x} - \dot{x}^* = f(t, e + x^*) = \bar{f}(t,e)$$

We can now analyse

$$\dot{e} = \bar{f}(t,e) \quad e^* = 0 \quad \text{is an equilibrium point}$$

Translate a nonzero solution of interest to the origin

Tracking control systems



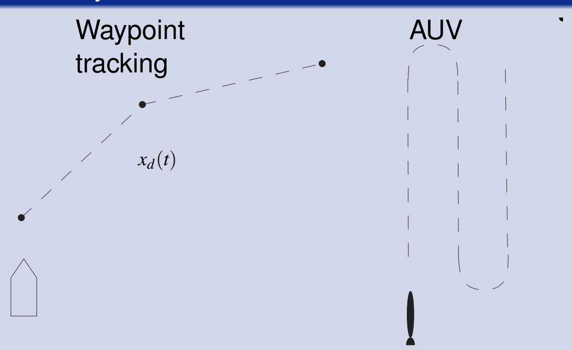
$$e = x - x_d(t) \quad \dot{x} = f(x)$$

$$\dot{e} = \dot{x} - \dot{x}_d(t) = f(e + x_d(t)) - \dot{x}_d(t) = \bar{f}(t, e)$$

Nonautonomous systems | Nonautonomous systems and equilibrium points

Translate a nonzero solution of interest to the origin

Tracking control systems



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Comparison functions

Class $\mathcal{K}$ function

A continuous function $\alpha : [0,a) \to [0,\infty)$

- is a class $\mathcal{K}$ function
- belongs to class $\mathcal{K}$

iff $\begin{cases} \alpha(0) = 0 \\ \alpha(r) \text{ it is strictly increasing, i.e. } \frac{\partial \alpha}{\partial r} > 0,\ r > 0 \end{cases}$

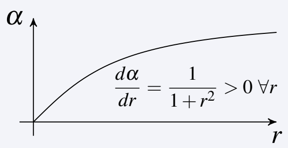

Figure: Example: $\alpha(r) = \arctan(r)$

Comparison functions

Class $\mathcal{K}_\infty$ function

If in addition

- $a \to \infty$
- $\alpha(r) \to \infty$ as $r \to \infty$

then

- α is a class $\mathcal{K}_\infty$ function / α belongs to class $\mathcal{K}_\infty$

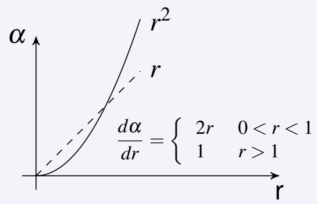

Figure: Example: $\alpha(r) = \min(r, r^2)$

Comparison functions | class $\mathcal{K}_\infty$ function

Comparison functions

Class $\mathcal{K}_\infty$ function

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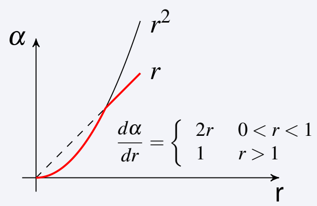

Figure: Example: $\alpha(r) = \min(r, r^2)$

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Comparison functions

Class $\mathcal{KL}$ function

A continuous function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$

- is a class $\mathcal{KL}$ function
- belongs to class $\mathcal{KL}$

if, for each fixed s

$$\beta(r,s) \text{ is a class } \mathcal{K} \text{ function with respect to } r$$

and, for each fixed r

- $\beta(r,s)$ is decreasing with respect to s
- $\beta(r,s) \to 0$ as $s \to \infty$

Properties

Read Lemma 4.2

System behavior depends on t_0

Initial value problem (IVP)

$$\begin{array}{lll} \dot{x} = f(x) \quad x(t_0) = x_0 & \} \text{ Aut. IVP} & x(t) = \varphi(t-t_0, x_0) \\ \dot{x} = f(t,x) \quad x(t_0) = x_0 & \} \text{ Nonaut. IVP} & x(t) = \varphi(t-t_0, x_0, t_0) \end{array}$$

NB

The solutions of nonautonomous systems in general depend on t_0

NB

The stability properties of nonautonomous systems in general depend on t_0

Stability, uniform stability and instability

Stability definitions

The equilibrium point $x^* = 0$ is

- Stable, iff

  $\forall\ \varepsilon > 0, \quad \exists \delta(\varepsilon, t_0) > 0$ such that

  $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall\ t \geq t_0 \geq 0$

- Uniformly stable, iff

  $\forall\ \varepsilon > 0, \quad \exists \delta(\varepsilon) > 0$ such that

  $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall\ t \geq t_0 \geq 0$

- Unstable, iff it is not stable

Asymptotic stability

Stability definitions cont.

The equilibrium point $x^* = 0$ is

- Asymptotically stable, iff
  - it is stable
  - $\exists\ c(t_0) > 0$ such that $\|x(t_0)\| < c \Rightarrow x(t) \xrightarrow{t\to\infty} 0$
- Uniformly asymptotically stable, iff
  - it is uniformly stable
  - $\exists\ c > 0$ such that $\|x(t_0)\| < c \Rightarrow x(t) \xrightarrow{t\to\infty} 0$ uniformly in t_0

Convergence vs Uniform convergence

Convergence

$$x(t) \xrightarrow{t\to\infty} 0$$

$$\Updownarrow$$

$$\forall\ \varepsilon > 0\ \exists\ T(\varepsilon, t_0) > 0 \ \text{ such that } \ \|x(t)\| < \varepsilon \quad \forall t \geq t_0 + T$$

Uniform convergence (in t_0)

$$x(t) \xrightarrow{t\to\infty} 0 \text{ uniformly in } t_0$$

$$\Updownarrow$$

$$\forall\ \varepsilon > 0\ \exists\ T(\varepsilon) > 0 \ \text{ such that } \ \|x(t)\| < \varepsilon \quad \forall t \geq t_0 + T$$

Convergence vs Uniform convergence cont.

Example

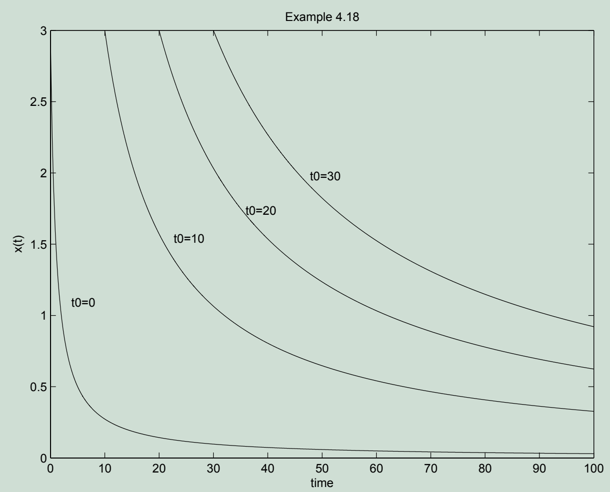
Given

$$\dot{x} = -\frac{x}{1+t} \qquad x(t_0) = x_0$$

Equilibrium point $x^* = 0$
Stability properties? Convergence properties?

Example: Non-uniform convergence

Note: The convergence rate depends on t_0. $(x_0 = 3)$



Global uniform asymptotic stability

Stability definitions cont.

The equilibrium point $x^* = 0$ is

- Globally uniformly asymptotically stable, iff
  - it is uniformly stable, with $\delta(\varepsilon) \xrightarrow{\varepsilon\to\infty} \infty$
  - $\forall\, c > 0 \quad \|x(t_0)\| < c \Rightarrow x(t) \xrightarrow{t\to\infty} 0$ uniformly in t_0

    i.e.
    $\forall\, c > 0, \varepsilon > 0 \quad \exists\, T(\varepsilon, c) > 0$ such that
    $\|x(t)\| < \varepsilon \quad \forall\, t \geq t_0 + T \quad \forall\ \|x(t_0)\| < c$

Global uniform asymptotic stability

Stability definitions cont.

The equilibrium point $x^* = 0$ is

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Equivalent stability definitions

Equivalent stability definitions using class $\mathscr{K}$ and $\mathscr{KL}$ functions

The equilibrium point $x^* = 0$ is

- uniformly stable, iff

$\exists$ class $\mathscr{K}$ function α such that $\exists c > 0$

$$\|x(t)\| \leq \alpha(\|x(t_0)\|)$$

$$\forall t \geq t_0 \geq 0, \quad \forall \ \|x(t_0)\| < c$$

- uniformly asymptotically stable, iff

$\exists$ class $\mathscr{KL}$ function β such that $\exists c > 0$

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$$

$$\forall t \geq t_0 \geq 0, \quad \forall \ \|x(t_0)\| < c$$

- globally uniformly asymptotically stable, iff

$\exists$ class $\mathscr{KL}$ function β such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$$

$$\forall t \geq t_0 \geq 0, \quad \forall \ \|x(t_0)\|$$

Exponential stability

Definition (Exponential stability)

The equilibrium point $x^* = 0$ is exponentially stable, iff

$$\exists c, k, \lambda > 0 \quad \text{s.t.} \quad \|x(t)\| \leq k \|x(t_0)\| e^{-\lambda(t-t_0)} \quad \begin{array}{l} t \geq t_0 \geq 0 \\ \|x(t_0)\| \leq c \end{array}$$

Exponential stability $\Rightarrow$ Uniform asymptotic stability

Special case of uniform asymptotic stability when

$$\beta(r, s) = kre^{-\lambda s}$$

Global exponential stability

If satisfied $\forall c$, then globally exponentially stable

GES $\Rightarrow$ GUAS

Time-varying Lyapunov function candidates

Time-varying generalized energy function $V(t,x)$

Definition: Positive definite

- $V(t,x)$ is positive definite iff

$$\left.\begin{array}{l} V(t,0) = 0 \\ V(t,x) \geq W_1(x) \end{array}\right\} \forall t \geq 0, \text{ for some positive definite } W_1(x)$$

- $V(t,x)$ is positive semidefinite if $W_1(x)$ positive semidefinite
- $V(t,x)$ is radially unbounded if $W_1(x)$ is radially unbounded

Definition: Negative definite

- $V(t,x)$ is negative (semi-)definite iff $-V(t,x)$ is positive (semi-)definite

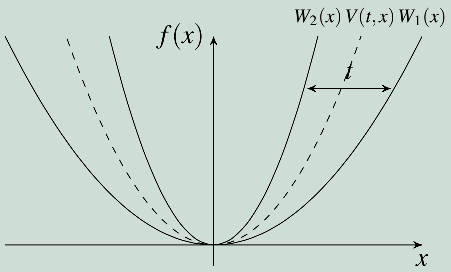
Time-varying Lyapunov function candidates

Definition: Decrescent

- $V(t,x)$ is decrescent iff

$$\left.\begin{array}{l} V(t,0)=0 \\ V(t,x)\le W_2(x) \end{array}\right\} \forall t\ge 0, \text{ for some positive definite } W_2(x)$$

Positive definite and decrescent $V(t,x)$



Examples

Example

a) $V_A(t,x)=(t+1)(x_1^2+x_2^2)$

b) $V_B(t,x)=e^{-t}(x_1^2+x_2^2)$

c) $V_C(t,x)=\frac{1}{1+\cos^2 t}(x_1^2+x_2^2)$

Q: Positive definite? Positive semidefinite? Radially unbounded? Decrescent?

Stability theorems

$\dot{x}=f(t,x)$ $\quad f:[0,\infty)\times\mathbb{D}\to\mathbb{R}^n$ piecewise continuous in t, locally Lipschitz

Stability theorem (Theorem 4.8 - 4.9)

Let $V:[0,\infty)\times\mathbb{D}\to\mathbb{R}$ $\quad C^1$

The equilibrium point $x^*=0$ is

	Stable	Uniformly stable	Uniformly as. st.	GUAS
V	Pos.def.	Pos.def. Decrescent	Pos.def. Decrescent	Pos.def. Decrescent Rad. unb.
$\dot{V}$	Neg.semidef	Neg.semidef.	Neg.def.	Neg.def.
	$\forall x\in\mathbb{D}$	$\forall x\in\mathbb{D}$	$\forall x\in\mathbb{D}$	$\forall x\in\mathbb{D}=\mathbb{R}^n$

Region of attraction

When $x^* = 0$ is Uniformly asymptotically stable

Estimate of Region of attraction

Choose r, c such that

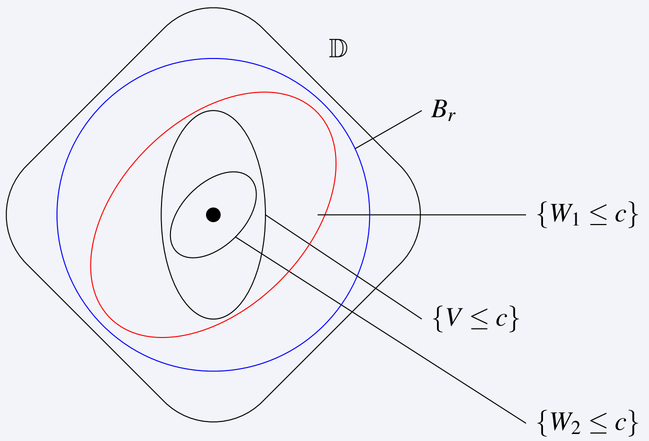
$$B_r = \{x \in \mathbb{R}^n : \|x\| < r\} \subset \mathbb{D}$$
$$c < \min_{\|x\|=r} W_1(x)$$

then

$$\{x \in B_r : W_2(x) \leq c\}$$

is a region of attraction for $x^* = 0$.

Region of attraction



Exponential stability

Exponential stability (Theorem 4.10)

Let $V : [0, \infty) \times \mathbb{D} \to \mathbb{R}$ $\quad C^1$

If there exists constants $a, k_1, k_2, k_3 > 0$ such that

- $k_1 \|x\|^a \leq V(t,x) \leq k_2 \|x\|^a, \quad \forall t \geq 0, \quad \forall\, x \in \mathbb{D}$
- $\dot{V}(t,x) \leq -k_3 \|x\|^a, \qquad \forall t \geq 0, \quad \forall\, x \in \mathbb{D}$

then $x^* = 0$ is exponentially stable.

Global exponential stability

If the conditions in the theorem are satisfied with

$$\mathbb{D} = \mathbb{R}^n$$

then $x^* = 0$ is globally exponentially stable.

Examples

Example

Consider the system

$$\dot{x}_1 = -x_1 - e^{-2t}x_2$$
$$\dot{x}_2 = x_1 - x_2$$

Determine the stability properties of $x^* = 0$ using

$$V(t,x) = x_1^2 + (1+e^{-2t})x_2^2$$

Read: Ex 4.19 and Ex 4.20

Inavariance-like theorems (Sec. 8.3)

Autonomous systems

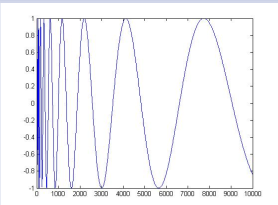
$\dot{V} \leq 0 \Rightarrow$ LaSalle $E = \{x \in \Omega_c : \dot{V}(x) = 0\}$
$x(t) \rightarrow$ largest invariant set in E.

Nonautonomous systems

$\dot{V} \leq 0 \Rightarrow$?

Note

- $\dot{f} \rightarrow 0 \not\Rightarrow f$ converges to a limit
  Ex: $f(t) = \sin(10\log t)$
- f converges to a limit $\not\Rightarrow \dot{f} \rightarrow 0$
  Ex: $f(t) = e^{-t}\sin(e^{2t})$



Inavariance-like theorems (Sec. 8.3)

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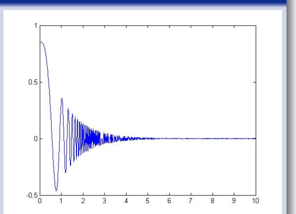

Barbalat's lemma

General result

If $f(t)$ is lower bounded and $\dot{f} \leq 0$

then

f converges to a limit.

Barbalat's lemma

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $[0, \infty)$

If $\lim_{t\to\infty} \int_0^t \varphi(\tau) d\tau$ exists and is finite, then

$$\varphi(t) \to 0 \text{ as } t \to \infty$$

Barbalat's lemma cont.

Barbalat's lemma 2

Let $\varphi = \dot{f}$. We can then rephrase the lemma:

Let $\dot{f} : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $[0, \infty)$
If $\lim_{t\to\infty} f(t)$ exists and is finite, then

$$\dot{f} \to 0 \text{ as } t \to \infty$$

Barbalat's lemma

Definition: Uniformly continuous

$\varphi : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $[0, \infty)$ iff

$$\forall\, \varepsilon\ \exists\ \delta(\varepsilon) \text{ s.t. } |t - t_1| < \delta \Rightarrow |\varphi(t) - \varphi(t_1)| < \varepsilon\ \forall\, t, t_1 \in [0, \infty)$$

Sufficient condition

$\frac{dg}{dt}$ is bounded (uniformly in t) $\Rightarrow$ g is uniformly continuous on $[0, \infty)$

Barbalat's lemma

With

$$f(t) = V(t, x(t)) \quad C^1$$
$$\varphi(t) = \dot{V}(t, x(t))$$

Barbalat's lemma gives

Barbalat's lemma 3

If

- V is lower bounded (e.g. $V \geq 0$)
- $\dot{V} \leq 0$
- $\ddot{V}$ is uniformly bounded

then

$$\dot{V} \to 0 \quad \text{as} \quad t \to \infty$$

Example

Adaptive control example

Consider the system

$$\dot{e} = -e + \theta\omega(t) \qquad e = y - y_d(t) \quad \text{tracking error}$$
$$\dot{\theta} = -e\omega(t) \qquad \theta = \text{parameter estimation error}$$
$$\omega(t) = \text{continuous, bounded function}$$

Analyse the stability properties of the system.

Theorem 8.4

Barbalat's lemma gives Theorem 8.4. Read on your own

Summary

Lyapunov's direct method for nonautonomous systems

- Time-varying Lyapunov functions candidates
- Lyapunov's theorems for
  - stability
  - uniform stability (US)
  - uniform asymptotic stability (UAS)
  - global uniform asymptotic stability (GUAS)
  - local and global exponential stability (GES $\Rightarrow$ GUAS)
- Barbalat's lemma

Next lecture

- Learn that there also exist other stability concepts than Lyapunov stability, and get a taste of these.
- Recommended reading
  Khalil Section 4.9
  Sections 5.1 and 5.4
  (5.2 - 5.3 and Ex. 5.14 are additional material)