NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF ENGINEERING CYBERNETICS

Exam
TTK4150 Nonlinear Control Systems
Thursday December 5, 2013

SOLUTION

Problem 1 (10%)

a Setting $\dot{x} = 0$, the equilibrium points of the system are found by solving the following equations

$$0 = x_2^\star \tag{1}$$
$$0 = -\sin(x_1^\star) - \left(5 + x_2^{\star 2} + x_2^{\star 4}\right) x_2^\star \tag{2}$$

From (1) we have that $x_2^\star = 0$, which gives $\sin(x_1^\star) = 0$ in (2), and $x_1^\star = \pm k\pi$ where k is an integer. Hence the system has infinitely many equilibrium points $(\pm k\pi, 0)$, $k = \{0, 1, 2, 3, \cdots\}$.

To classify the qualitative behavior of the system near the equilibrium points, we first linearize the system about the equilibrium points

$$A = \frac{\partial f(x)}{\partial x}\bigg|_{x=x^\star} = \begin{bmatrix} 0 & 1 \\ -\cos(x_1^\star) & -5 - 3x_2^{\star 2} - 5x_2^{\star 4} \end{bmatrix} \tag{3}$$

For odd k we have

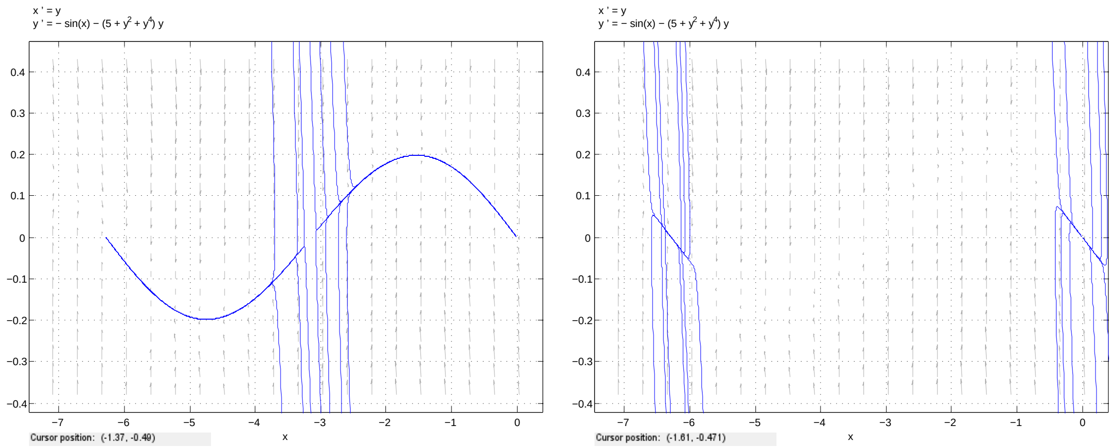
$$A = \begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix} \tag{4}$$

which has the eigenvalues $\lambda_1 = \frac{-5-\sqrt{29}}{2} \approx -5.19$, $\lambda_2 = \frac{-5+\sqrt{29}}{2} \approx 0.19$. Hence, the qualitative behavior of these equilibrium points corresponds to saddle points. See Fig. 1(a), where all marked solutions move away from $(\pi, 0)$ to either $(2\pi, 0)$ or $(0, 0)$.

For even k we have

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -5 \end{bmatrix} \tag{5}$$

which has the eigenvalues $\lambda_1 = \frac{-5-\sqrt{21}}{2} \approx -4.79$, $\lambda_2 = \frac{-5+\sqrt{21}}{2} \approx -0.21$. Hence, the qualitative behavior of these equilibrium points corresponds to a stable nodes. See Fig. 1(b), where all marked solutions are drawn to either $(2\pi, 0)$ or $(\pi, 0)$



(a) Odd k.

(b) Even k

Figure 1: Phase portrait of equilibrium points.

b Here we apply Lemma 2.2. in Khalil (Bendixon criterion). We have that

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -5 - 3x_2^2 - 5x_2^4 \tag{6}$$

Which is negative (and therefore does not change sign) for all $x \in \mathbb{R}^2$. Hence there are no periodic orbits in $\mathbb{R}^2$.

Problem 2 (15%)

a Setting $\dot{x} = 0$, the equilibrium point(s) of the system are found by solving

$$0 = -x_1^\star x_2^{\star 2} - x_2^{\star 3} = -x_2^{\star 2}\left(x_1^\star + x_2^\star\right) \tag{7}$$

$$0 = -4x_2^\star - x_1^{\star 2} x_2^\star + x_1^{\star 3} \tag{8}$$

From (7) we have that $x_2^\star = 0$ or $x_2^\star = -x_1^\star$. With $x_2^\star = 0$, (8) gives $x_1^{\star 3} = 0$ and $(0,0)$ is one equilibrium point. With $x_2^\star = -x_1^\star$, we get from (8) that $x_1^\star\left(2x_1^{\star 2} + 4\right) = 0$, and hence $(0,0)$ is the only equilibrium point.

b First we find the Jacobian of the system:

$$J = \frac{\partial f(x)}{\partial x} = \begin{bmatrix} -x_2^2 & -2x_1x_2 - 3x_2^2 \\ -2x_1x_2 + 3x_1^2 & -4 - x_1^2 \end{bmatrix} \tag{9}$$

Inserting for $(x_1, x_2) = (0,0)$ gives

$$J\Big|_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix} \tag{10}$$

which gives $\lambda_1 = 0$ and $\lambda_2 = -4$. Since one of the eigenvalues is zero, nothing can be concluded about stability of $(0,0)$ using the indirect Lyapunov method.

c $V(x)$ is a positive definite, continuously differentiable function. We now find that

$$\dot{V}(x) = x_1^3\dot{x}_1 + x_2^3\dot{x}_2 = x_1^3\left(-x_1x_2^2 - x_2^3\right) + x_2^3\left(-4x_2 - x_1^2x_2 + x_1^3\right) \tag{11}$$
$$= -4x_2^4 - x_1^2x_2^2\left(x_1^2 + x_2^2\right) \le 0 \;\forall\; x \in \mathbb{R}^2 \tag{12}$$

$\dot{V}(x)$ is negative semidefinite, and LaSalle's theorem needs to be applied to prove asymptotic stability. Let $S = \left\{x \in \mathbb{R}^2 \middle| x_2 = 0\right\}$. Let $x(t)$ be a solution that belongs identically to S: $x_2 = 0 \Rightarrow \dot{x}_2 = 0 \Rightarrow x_1^3 = 0 \Rightarrow x_1 = 0$, which means that no solution can stay identically in S other than $(0,0)$. And by Corollary 4.2, $(0,0)$ is asymptotically stable. In addition, since $V(x)$ is radially unbounded, the origin is globally asymptotically stable.

Problem 3 (17%)

a The 1-norm of an m-dimentional vector a is

$$\|a\|_1 = \sum_{i=1}^{m} |a_i| \tag{13}$$

and L can be found in the following way

$$\left\|\frac{\partial f(t,z)}{\partial z}\right\|_1 = |-\left(1+g(t)\right)| + |1| \le |-1||1+g(t)| + 1 \le (1+g_0) + 1 = 2 + g_0 = L \tag{14}$$

which is valid on $[0,\infty) \times \mathbb{R}^2$. From Lemma 3.1 in Khalil we have that (14) gives

$$\left\|f(t,x) - f(t,y)\right\|_1 \le L\|x-y\|_1 \tag{15}$$

Since this inequality is valid on $\mathbb{R}^2$ with the same constant L, we have that $f(t,z)$ is globally Lipshitz in z, uniformly in t.

b Looking at the system $f(t,x_1,0)$, the quadratic Lyapunov function, $V_1(x) = \frac{1}{2}x_1^2$, is used to prove exponential stability through Theorem 4.10 in Khalil. First we have that

$$\frac{1}{2}x_1^2 = k_1\|x_1\|^2 \le V_1(x) \le \frac{1}{2}x_1^2 = k_2\|x_1\|^2 \tag{16}$$

then

$$\dot{V}_1 = x_1\dot{x}_1 = x_1\left(-\left(1+g(t)\right)x_1\right) = -x_1^2\left(1+g(t)\right) \tag{17}$$
$$\le -x_1^2 = -k_3\|x_1\|^2 \;\;\forall\; t \ge 0, x \in \mathbb{R}^2 \tag{18}$$

Since k_1, k_2 and k_3 all are positive constants, and all the assumptions hold globally, $x_1 = 0$ is globally exponentially stable.

c Since $f_1(t,x_1,u)$ is continuously differentiable and globally Lipschitz in $[x_1,u]^T$, uniformly in t, and since $f_1(t,x_1,0)$ has a globally exponentially stable equilibrium point in $x_1 = 0$, then by Lemma 4.6 in Khalil the system $\dot{x}_1 = f_1(t,x_1,u)$ is input-to-state stable.

d First it is shown that $x_2 = 0$ is a globally uniformly asymptotically stable equilibrium point for the system $\dot{x}_2 = f_2(t, x_2)$, by using $V_2 = \frac{1}{2}x_2^2$ and Theorem 4.9 in Khalil

$$W_1(x) = \frac{1}{2}x_2^2 \leq V_2(x) \leq \frac{1}{2}x_2^2 = W_2(x) \tag{19}$$

$$\dot{V}_2 = x_2\left(-2x_2 - x_2\sin(t)\right) = -x_2^2\left(2 + \sin(t)\right) \leq -x_2^2 \;\forall\; t \geq 0, x_2 \in \mathbb{R} \tag{20}$$

Since $\dot{V}_2$ is negative definite and $W_1(x)$ is radially unbounded, $x_2 = 0$ is globally uniformly asymptotically stable.

We can now apply Lemma 4.7 in Khalil. Since $\dot{x}_1 = f_1(t, x_1, x_2)$ is input-to-state stable and the origin of $\dot{x}_2 = f_2(t, x_2)$ is globally uniformly asymptotically stable, the cascade system is globally uniformly asymptotically stable.

Problem 4 (20%)

a We will apply the storage function $V_1(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ and Definition 6.3 in Khalil to show that the system is output strictly passive with input u and output y. We find that

$$\begin{aligned}
\dot{V}_1 &= x_1^3\dot{x}_1 + x_2\dot{x}_2 = x_1^3\left(-x_1^2x_2^2 + x_2\right) + x_2\left(-x_1^3 + x_1^5x_2 - x_2 + u\right) && (21)\\
&= -x_1^5x_2^2 + x_1^3x_2 - x_1^3x_2 + x_1^5x_2^2 - x_2^2 + x_2u && (22)\\
&= -y^2 + yu && (23)
\end{aligned}$$

Which gives

$$uy = \dot{V} + y^2 \quad \text{where} \quad y^2 > 0 \;\forall\; y \neq 0$$

By Definiton 6.3 in Khalil, the system is output strictly passive with $\rho(y) = y$.

b Writing our system as

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

the system is said to be zero-state observable if no solution of $\dot{x} = f(x, 0)$ can stay identically in $S = \left\{x \in \mathbb{R}^2 \mid h(x, 0) = 0\right\}$, other than the trivial solution $x(t) \equiv 0$. With $u = 0$, we have that

$$h(x(t), 0) \equiv 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow x_1^3(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

Hence the only solution that can stay identically in S is $x(t) \equiv 0$, and by Definition 6.5 in Khalil the system is zero-state observable.

c Since the system is output strictly passive with a radially unbounded storage function, and also zero-state observable, Theorem 14.4 in Khalil states that it can be globally stabilized by $u = -\phi(y)$, as long as $\phi(y)$ is a locally Lipschitz function such that $\phi(0) = 0$ and $y^T\phi(y) > 0$ for all $y \neq 0$. Any function that is locally Lipschitz and has a graph which lies in the first and third quadrant will fulfill the requirements of ϕ. When the input is bounded, we can choose $\phi(y) = k \cdot \text{sat}(y)$ or $\phi(y) = \frac{2k}{\pi}\arctan(y)$, where k is the upper bound on u.

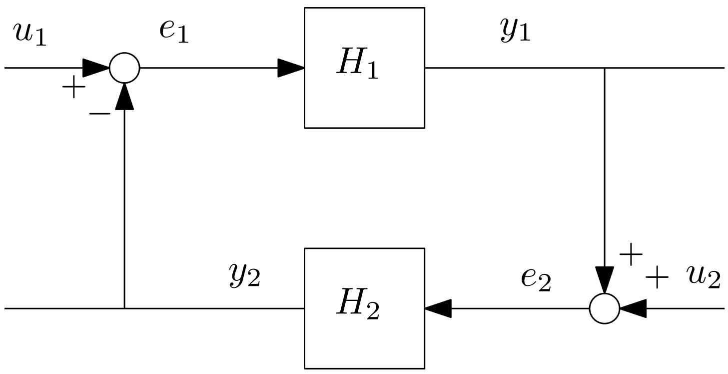


Figure 2: Feedback connection.

d Here we apply Theorem 6.3 in Khalil. First we rewrite our system to coincide with the terms used in Figure 2.

With our system as H_1, the dynamic controller as H_2 and

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u \\ y \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ -u \end{bmatrix}$$

we know from **a** and **b** that H_1 is output strictly passive and zero-state observable with input e_1 and output y_1. We now need to show that H_2 is either

- strictly passive
- or output strictly passive and zero-state observable

with input $e_2 = y$ and output $y_2 = -u$.

We rewrite our dynamic controller as

$$\begin{aligned} \dot{x}_3 &= -4x_3 + x_3 e^{-|x_3|} + e_2 \\ y_2 &= x_3 \end{aligned}$$

and apply the storage function $V_2 = \frac{1}{2}x_3^2$.

$$\begin{aligned} \dot{V}_2 = x_3\dot{x}_3 &= x_3\left(-4x_3 + x_3 e^{-|x_3|} + e_2\right) \\ &= -x_3^2\left(4 - e^{-|x_3|}\right) + x_3 e_2 \end{aligned}$$

Since $3 \leq 4 - e^{-|x_3|} \leq 4$, we have that

$$\dot{V}_2 \leq -3x_3^2 + y_2 e_2$$

By Definition 6.3 in Khalil, the system is strictly passive with the positive definite function $\psi(x_3) = x_3^2$.

This means that we have a feedback connection of one output strictly passive and zero-state observable system and one strictly passive system, and by theorem 6.3 in Khalil the origin of the unforced feedback connection (that is $u_1 = u_2 = 0$) is asymptotically stable. Moreover, since both V_1 and V_2 are radially unbounded, the origin is globally asymptotically stable.

Problem 5 (28%)

a Rewrite the system to

$$\dddot{x} + 4\ddot{x} + 4\dot{x} = -12x|x| = 12u \tag{24}$$

with $y = x$. Assuming that the initial state is zero, we take the Laplace transformation

$$s^3x + s^2 4x + s4x = 12u \tag{25}$$

$$x\left(s^3 + 4s^2 + 4s\right) = 12u \tag{26}$$

$$\downarrow$$

$$G(s) = \frac{y}{u} = \frac{x}{u} = \frac{12}{s^3 + 4s^2 + 4s} \tag{27}$$

b The transfer function must have a low pass character (strictly proper). The need for a low pass character is due to the simplification of evaluating only the first harmonic when deriving the describing function.

c Since $\psi(y)$ is odd, time-invariant and memoryless, we can use

$$\Psi(a) = \frac{2}{\pi a}\int_0^\pi \psi(a\sin\theta)\sin\theta \ d\theta \tag{28}$$

$$= \frac{2}{\pi a}\int_0^\pi a\sin\theta|a\sin\theta|\sin\theta \ d\theta \tag{29}$$

$$= \frac{2a^2}{\pi a}\int_0^\pi \sin^2\theta|\sin\theta| \ d\theta \tag{30}$$

$$= \frac{2a}{\pi}\cdot 2\int_0^{\pi/2}\sin^3\theta \ d\theta = \frac{4a}{\pi}\cdot\frac{2}{3} = \frac{8a}{3\pi} \tag{31}$$

d If the harmonic balance equation $G(j\omega)\Psi(a) + 1 = 0$ has a solution $(\omega^\star, a^\star)$, then there exists a periodic solution with frequency $\omega^\star$ and amplitude $a^\star$.

$$G(j\omega) = \frac{12}{-j\omega^3 - 4\omega^2 + 4j\omega} \tag{32}$$

$$= \frac{12}{-4\omega^2 + j\omega\left(4 - \omega^2\right)} \tag{33}$$

$$= \frac{-48\omega^2 - 12j\omega\left(4 - \omega^2\right)}{16\omega^4 + \omega^2\left(4 - \omega^2\right)^2} \tag{34}$$

where

$$\text{Re}\left(G(j\omega)\right) = \frac{-48\omega^2}{16\omega^4 + \omega^2\left(4 - \omega^2\right)^2} \tag{35}$$

$$\text{Im}\left(G(j\omega)\right) = \frac{-12\omega\left(4 - \omega^2\right)}{16\omega^4 + \omega^2\left(4 - \omega^2\right)^2} \tag{36}$$

Since $\Psi(a)$ is strictly real, Im $\left(G(j\omega^\star)\right) = 0$ and Re $\left(G(j\omega^\star)\right)\Psi(a^\star) + 1 = 0$.

$$\text{Im}\left(G(j\omega^\star)\right) = \frac{-12\omega^\star\left(4-\omega^{\star 2}\right)}{16\omega^{\star 4} + \omega^{\star 2}\left(4-\omega^{\star 2}\right)^2} = 0 \rightarrow \omega^{\star 2} - 4 = 0 \rightarrow \omega^\star = \pm 2 \tag{37}$$

Since the frequency must be positive, we have $\omega^\star = 2$. This results in

$$\text{Re}\left(G(j\omega^\star)\right)\Psi(a^\star) + 1 = 0 \tag{38}$$

$$\frac{-48\cdot 2^2}{16\cdot 2^4}\cdot\frac{8a^\star}{3\pi} + 1 = 0 \tag{39}$$

$$\frac{-3}{4}\cdot\frac{8a^\star}{3\pi} + 1 = 0 \tag{40}$$

$$\downarrow$$

$$a^\star = \frac{\pi}{2} \tag{41}$$

Hence we have a periodic solution with frequency $\omega^\star = 2$ and amplitude $a^\star = \frac{\pi}{2}$.

e First we see that $-\frac{1}{\Psi(a)}$ and $G(j\omega)$ intersect at $a^\star \approx 1.58$, which is the same as we concluded using the analytic approach. We call this intersection point A. From the Nichols plot we see that when a is higher than $a^\star$, $G(j\omega)$ is lying to the left of A, the system is unstable and a will keep increasing. When a is lower than $a^\star$, $G(j\omega)$ is lying to the right of A, the system is stable, and a will keep decreasing. Hence the periodic solution is unstable.

Problem 6 (10%)

a Backstepping cannot be applied directly since the input affects both states and this system is hence not on strict feedback form. The states need to be selected such that the input affects only one state directly, and then is mediated through a cascade of other states.

b By change of variables, our system now looks like

$$\dot{\bar{x}}_1 = \bar{x}_1^2 + (\bar{x}_1 + 1)\bar{x}_2 \tag{42}$$

$$\dot{\bar{x}}_2 = -\bar{x}_2 + u \tag{43}$$

where $\bar{x} = [\bar{x}_1 \ \ \bar{x}_2]^T$. This system is now on strict feedback form, and backstepping may be applied.

- First the stabilizing virtual input is found using the following continuously differentiable and positive definite Lyapunov function

$$V_1 = \frac{1}{2}\bar{x}_1^2 \tag{44}$$

$$\dot{V}_1 = \bar{x}_1^3 + \bar{x}_2\left(\bar{x}_1^2 + \bar{x}_1\right) \tag{45}$$

With $\bar{x}_2 = -\bar{x}_1$, $\bar{x}_1$ is asymptotically stable:

$$\dot{V}_1 = \bar{x}_1^3 + (-\bar{x}_1)\left(\bar{x}_1^2 + \bar{x}_1\right) = -\bar{x}_1^2 < 0 \ \forall \ \bar{x}_1 \neq 0 \tag{46}$$

- Then $\phi(\bar{x}_1)(\bar{x}_1+1) = -\bar{x}_1(\bar{x}_1+1)$ is added and subtracted from $\dot{\bar{x}}_1$

$$\dot{\bar{x}}_1 = \bar{x}_1^2 + (\bar{x}_1+1)\bar{x}_2 + \phi(\bar{x}_1)(\bar{x}_1+1) - \phi(\bar{x}_1)(\bar{x}_1+1) \tag{47}$$
$$= -\bar{x}_1 + (\bar{x}_1+1)\big(\bar{x}_2 - \phi(\bar{x}_1)\big) \tag{48}$$

  And the new variable z is defined as $z = \bar{x}_2 - \phi(\bar{x}_1)$.

- Next, the expression for $\dot{z}$ is found

$$\dot{z} = \dot{\bar{x}}_2 - \frac{\partial\phi(\bar{x})}{\partial t} \tag{49}$$
$$= -\bar{x}_2 + u + \dot{\bar{x}}_1 \tag{50}$$
$$= -\bar{x}_2 + u + \bar{x}_1^2 + \bar{x}_2(\bar{x}_1+1) \tag{51}$$
$$= u + \bar{x}_1^2 + \bar{x}_1\bar{x}_2 \tag{52}$$

- Finally, the overall stabilizing input u is found using the following continuously differentiable and positive definite Lyapunov function

$$V_2 = \frac{1}{2}\bar{x}_1^2 + \frac{1}{2}z^2 \tag{53}$$
$$\dot{V}_2 = -\bar{x}_1^2 + (\bar{x}_1+1)\bar{x}_1 z + z\big(u + \bar{x}_1^2 + \bar{x}_1\bar{x}_2\big) \tag{54}$$
$$= -\bar{x}_1^2 + z\big(u + \bar{x}_1(1 + 2\bar{x}_1 + \bar{x}_2)\big) \tag{55}$$

  u is chosen such that

$$u + \bar{x}_1(1 + 2\bar{x}_1 + \bar{x}_2) = -z \tag{56}$$

  which means that

$$u = -z - \bar{x}_1(1 + 2\bar{x}_1 + \bar{x}_2) \tag{57}$$

  and

$$\dot{V}_2 = -\bar{x}_1^2 - z^2 < 0 \ \forall \ (\bar{x}_1, z) \neq (0,0) \tag{58}$$

  Since $V_2(\bar{x}_1, z)$ is continuously differentiable and positive definite, and $\dot{V}_2(\bar{x}_1, z)$ is negative definite, u asymptotically stabilizes $\bar{x}_1$ and z at the origin. Since $z = 0 \rightarrow \bar{x}_2 = \phi(\bar{x}_1)$ and $\bar{x}_1 = 0 \rightarrow \phi(\bar{x}_1) = 0$, this also means that $\bar{x}_2$ is asymptotically stabilized at the origin. In addition, since $V_2(\bar{x}_1, z)$ is radially unbounded and there are no singularities in u, $(\bar{x}_1, \bar{x}_2) = (0,0)$ is globally asymptotically stable. Since $(\bar{x}_1, \bar{x}_2) = (0,0) \rightarrow (x_1, x_2) = (0,0)$, the controller also makes the origin of the original system globally asymptotically stable.