NTNU

Norwegian University of
Science and Technology

Department of Engineering Cybernetics

Examination paper for TTK4150 Nonlinear Control Systems

Academic contact during examination: Mikkel Sørensen

Phone: +45 60 95 23 59

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Informasjon om trykking av eksamensoppgave

Originalen er:

1-sidig □ **2-sidig** □

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Problem 1 (17%)

Consider the system

$$\begin{aligned}\dot{x}_1 &= -kx_1 \\ \dot{x}_2 &= -kx_2 - x_1^2 x_2\end{aligned}$$

In all the questions except e) k is assumed to be a constant parameter.

- **a [3%]** Verify that the origin is the only equilibrium point of the system.
- **b [4%]** Classify the equilibrium point as a function of k using Lyapunov indirect method and explain intuitively the qualitative behavior of trajectories near the equilibrium point.
- **c [3%]** Based on the eigenvalues of the system, find conditions on k under which we can conclude exponential stability of the origin.
- **d [3%]** Is the system locally Lipschitz on $\mathbb{R}^2$?
- **e [4%]** Suppose that $k(t)$ is a time-varying signal. Find conditions on $k(t)$ under which we can conclude global exponential stability using Lyapunov direct method.

Problem 2 (16%)

Consider the system

$$\begin{aligned}\dot{x}_1 &= -k_1 x_1 + x_2 \\ \dot{x}_2 &= -\phi(x_1) - (1 + \cos^2(t))x_2 + u \\ y &= x_2\end{aligned}$$

with $u \in \mathbb{R}$ and $y \in \mathbb{R}$ as the input and output, respectively, $x = [x_1, x_2]^T \in \mathbb{R}^2$ and $k_1 > 0$. The function ϕ satisfies the sector condition where $z\phi(z) \geq k_2 z^2$ for all $z \in \mathbb{R}$ and for some $k_2 > 0$, and $\phi(0) = 0$. *Hint:* use $V(x) = \int_0^{x_1} \phi(z)dz + \frac{1}{2}x_2^2$ and the fact that $\frac{d}{dv}\int_0^v \phi(z)dz = \phi(v)$.

- **a [3%]** For zero input (i.e. $u(t) = 0$ for all t) show that the origin is globally asymptotically stable.
- **b [3%]** Show that the system from u to y is strictly passive.
- **c [3%]** Show that the system from u to y is also output strictly passive.
- **d [3%]** Show that the system is zero state observable.
- **e [4%]** Using $V(x)$ as the Lyapunov function candidate, show that the system is input to state stable.

Problem 3 (12%)

Consider the system

$$\begin{aligned}\dot{x}_1 &= \varepsilon((-1 + 1.5\sin(t)\cos(t))x_1 + x_2)\\ \dot{x}_2 &= \varepsilon(-x_1 - (1 + 2\sin^2(t))x_2)\end{aligned}$$

which also can be written as

$$\dot{x} = \varepsilon \begin{bmatrix} (-1 + 1.5\sin(t)\cos(t))x_1 + x_2 \\ -x_1 - (1 + 2\sin^2(t))x_2 \end{bmatrix} = \varepsilon f(t, x)$$

Use the averaging method to show that for a sufficient small $\varepsilon > 0$ the origin is the exponential stable. *Hint:* Check the appendix for integration by substitution and note that the system is π-periodic in t.

Problem 4 (13%)

Consider dynamical system Σ_1 given by

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2\\ \dot{x}_2 &= -x_1 - x_2 - x_2^3 + u_1\\ y_1 &= x_2\end{aligned}$$

and the system Σ_2 given by

$$\begin{aligned}\dot{\omega} &= u_2\\ y_2 &= \omega\end{aligned}$$

a **[3%]** Show that the system Σ_1 from u_1 to y_1 is output strictly passive using $V_1 = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$

b **[1%]** Is the the system Σ_1 finite gain $\mathcal{L}_2$ stable? Justify your answer.

c **[3%]** Is the system Σ_1 input strictly passive using $V_1 = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$? Justify your answer.

d **[3%]** Now, suppose that we feedback connect Σ_1 and Σ_2 with

$$\begin{aligned}u_1 &= -y_2\\ u_2 &= y_1.\end{aligned}$$

Is the overall system finite gain $\mathcal{L}_2$ stable? Justify your answer. *Hint:* use $V_2 = \frac{1}{2}\omega^2$.

e **[3%]** What can be concluded about the stability of the feedback connected system Σ_1 and Σ_2. Justify your answer.

Problem 5 (32%)

Consider the following system

$$
\begin{aligned}
\dot{x}_1 &= x_3 - x_2^6 + u \\
\dot{x}_2 &= x_1 + x_3 \\
\dot{x}_3 &= x_3 + x_1 - |x_3|x_3 \\
y &= x_2
\end{aligned}
$$

a [3%] Find the relative degree r of this system. Is the system input-output linearizable?

b [10%] Transform the system into the normal form

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi} &= A_c\xi + B_c\gamma(x)\left[u - \alpha(x)\right]
\end{aligned}
$$

Specify the diffeomorphism $z = T(x) = \begin{bmatrix}\eta & \xi\end{bmatrix}^T$, the functions $\gamma(x)$, $\alpha(x)$ and $f_0(\eta, \xi)$ and the matrices A_c and B_c. In which domain is the transformation valid?

c [4%] Find an input-output linearizing controller on the form $u = \alpha(x) + \beta(x)v$.

d [4%] Find a controller $v = \delta(\xi)$ such that the external dynamics ξ is globally asymptotically stable at the origin.

e [4%] Is the system minimum phase?
Hint: If you were not able to solve **b** you may use the following equations for the internal dynamics: (It is not the correct internal dynamics equation, but it has the same property with respect to minimum phase)

$$
\dot{\eta} = -\eta + \xi_1^2 + \xi_2^2
$$

f [2%] Is the closed-loop system $[\eta, \xi]^\top$ asymptotically stable at the origin? Justify your answer.

g [2%] Is the closed-loop system $[\eta, \xi]^\top$ global asymptotically stable at the origin? Justify your answer.

h [3%] Write the overall controller (both the input-output linearizing controller $u = \alpha(x) + \beta(x)v$ and the stabilizing controller $v = \delta(\xi)$ in the form $u = \mathcal{U}(x)$. Does the controller $u = \mathcal{U}(x)$ make the system globally asymptotically stable at the origin? Justify your answer.

Problem 6 (10%)

Consider the system,

$$\begin{aligned}\dot{x}_1 &= x_1^5 + x_2 \\ \dot{x}_2 &= x_2^3 + x_1^2 + u\end{aligned}$$

Use backstepping method to design a feedback controller for the system such that x = 0 is asymptotically stable. *Hint:* you may start the first step by using $V_1(x_1) = \frac{1}{2}x_1^2$ and the virtual control $x_2 = \phi(x_1) = -2x_1^5 - x_1$ for $\dot{x}_1 = x_1^5 + x_2$.

Appendix: Formulae

$$\cos^2(t) = \frac{1}{2}(1 + \cos(2t))$$
$$\sin^2(t) = \frac{1}{2}(1 - \cos(2t))$$

Integral by substitution

$$\int \sin(\tau)\cos(\tau)d\tau = \int u du$$

where

$$u = \sin(\tau)$$
$$du = \cos(\tau)d\tau$$

Then

$$\int \sin(\tau)\cos(\tau)d\tau = \int u du = \left[\frac{1}{2}u^2\right] = \frac{1}{2}\sin^2(t)$$