NTNU – Trondheim
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TTT4120 Digital Signal Processing
Fall 2017

Lecture: Discrete-Time Signals in Time-Domain

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Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 1.1 Signals, systems and signal processing
  - 1.2 Classification of signals
  - 1.3 The concept of frequency in continuous-time and…
  - 1.4.1 Sampling of analog signals
  - 2.1 Discrete-time signals

*Level of detail is defined by lectures and problem sets

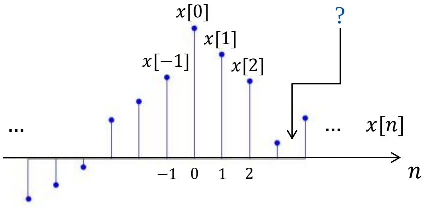
Contents and learning outcomes

- Discrete-time signals
- Power of digital signal processing (DSP)
- Properties, classification, and manipulations of sequences
- A few typical sequences
- Discrete-time sinusoids and sampling of continuous-time sinusoids

Discrete-time signals

- Continuous-time versus discrete-time signals?

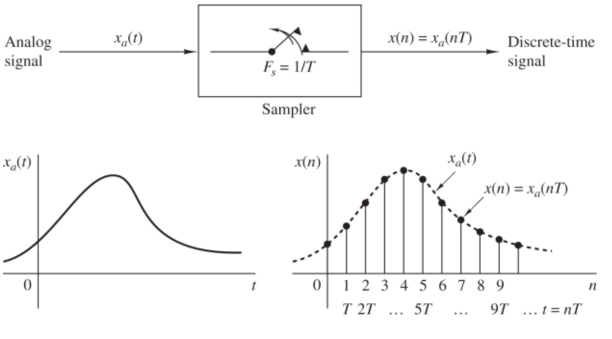
Discrete-time signals…



- A discrete-time signal $x[n]$ is represented by a sequence of numbers
- Sequence $x[n]$ can represent a discrete-time signal, where each number $x[n]$ corresponds to a signal amplitude at instant n

$$x[n] = \{\dots\ x[-2], x[-1], \underline{x[0]}, x[1], \dots\}$$

Discrete-time signals…

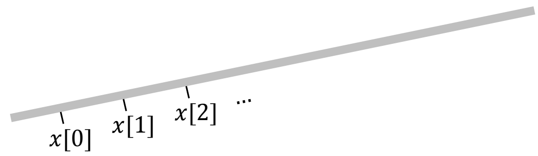


- Sometimes $x[n]$ is obtained from *sampling* an analog signal

$$x[n] \triangleq x_a(nT)$$

- Interval between samples $T = \frac{1}{F_s}$, where F_s is the sampling rate

Discrete-time signals…



- Note that interval T need not necessarily represent time
- For example, if $x_a(t)$ is the temperature along a metal rod, then if T is a length unit, $x[n] \triangleq x_a(nT)$ represents the temperature at uniformly placed sensors along this rod
- Different choices of T lead to different discrete-time sequences

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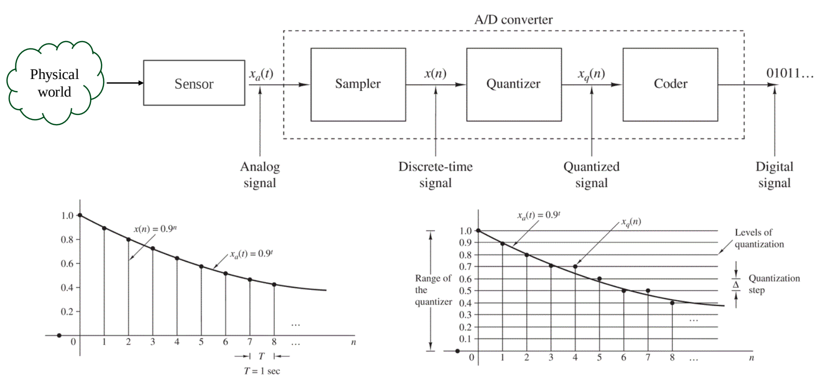
Characterization of signals



- Analog signal $x_a(nT)$: continuous in time and amplitude
- Sampled-data signal $x[n]$: discrete-time and continous-amplitude
- Digital signal $x_q[n]$: discrete in both time and amplitude

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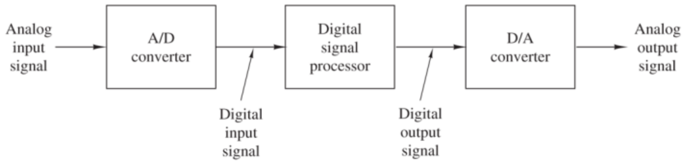
Power of digital signal processing



- Digital signal
  - Discrete-time and discrete-valued sequence of numbers (last attribute less essential for DSP basics)
- Digital signal processing
  - Sequence is transformed to another sequence by means of arithmetic operations

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Power of digital signal processing…

- Analog signal processing:
  - Process a continuously varying quantity (analog signal)
  - Can be described by differential equations
- Digital signal processing:
  - Processes sequences of numbers (discrete-time signals) using some sort of digital hardware or software
  - Power of DSP is that once a sequence of numbers is available to an appropriate digital hardware we can carry out any form of numerical processing on it

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Power of digital signal processing…

- Example: Suppose we want to perform the following operation on a continuous-time signal $x(t)$:

$$y(t) = \frac{\cosh[\ln(|x(t)|)+x^3(t)+\cos^3(\sqrt{|x(t)|})]}{5x^5(t)+e^{x(t)}+\tan(x(t))}$$

- Difficult to implement using analog hardware!
- Alternatively, convert analog signal $x(t)$ into sequence $x[n]$, manipulate it on a digital computer, and generate sequence $y[n]$
- If the continuous-time signal $y(t)$ can be recovered from $y[n]$, then the desired processing has been successfully performed

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Power of digital signal processing…

- Previous example highlights two important points:
  1. How powerful digital signal processing is
  2. To process analog signals using DSP, we must have a way of converting a continuous-time signal into a discrete-time one, such that the continuous-time signal can be recovered from the discrete-time signal
- Many signals are originally discrete-time, and the results of their processing are only needed in digital form

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Basic properties of discrete-time signals

- A sequence $x[n]$ is causal if

$$x[n] = 0, n < 0$$

- A sequence $x[n]$ is periodic with period N if

$$x[n + N] = x[n], \forall n$$

where smallest N satisfying the above is the fundamental period

- A sequence that is not periodic is called non-periodic or aperiodic

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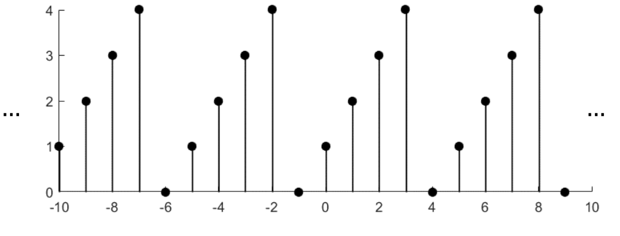
Basic properties of discrete-time signals…



- Is the above sequence periodic?
- If so, what is the fundamental period?

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Basic properties of discrete-time signals…

- A real-valued sequence $x_{\mathrm{e}}[n]$ is called even if

$$x_{\mathrm{e}}[n] = x_{\mathrm{e}}[-n], \forall n$$

- A real-valued sequence $x_{\mathrm{o}}[n]$ is called odd if

$$x[-n] = -x[n], \forall n$$

- Any real-valued sequence can be expressed as

$$x[n] = x_{\mathrm{e}}[n] + x_{\mathrm{o}}[n]$$

$$x_{\mathrm{e}}[n] = \frac{1}{2}(x[n] + x[-n]) \qquad x_{\mathrm{o}}[n] = \frac{1}{2}(x[n] - x[-n])$$

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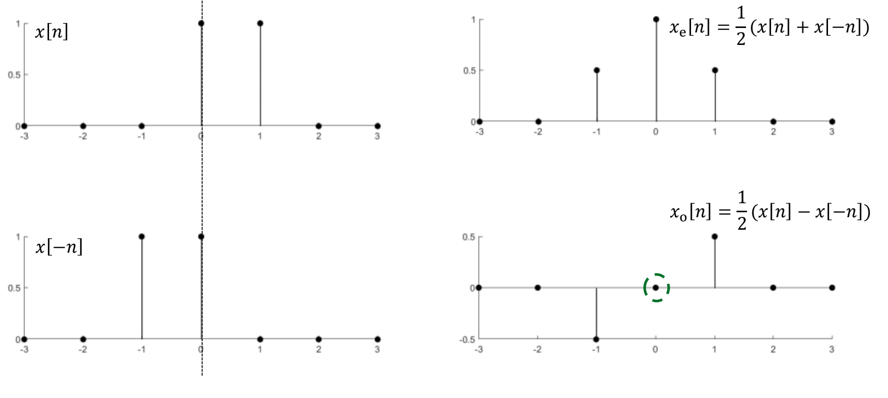
Basic properties of discrete-time signals…



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Classifications of discrete-time signals

- A sequence is bounded if $|x[n]| \leq B_x < \infty$ for all n
- A sequence is absolutely summable if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- A sequence is square-summable if its energy

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

is bounded. Such signal is called an energy signal

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Classifications of discrete-time signals…

- Not all sequences are energy signals (e.g., periodic signals)
- Average power of sequence $x[n]$ is defined as

$$P_x = \lim_{n \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

- If P_x is finite, the signal is called a power signal

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Operations on discrete-time signals

- Scaling, addition, and multiplication of sequences

$$y[n] = ax[n]$$

$$y[n] = x_1[n] + x_2[n]$$

$$y[n] = x_1[n]x_2[n]$$

- Time shifts and folding

$$y[n] = x[n-k]$$

$$y[n] = x[-n]$$

- Time shifts plus folding

$$y[n] = x[-n+k]$$

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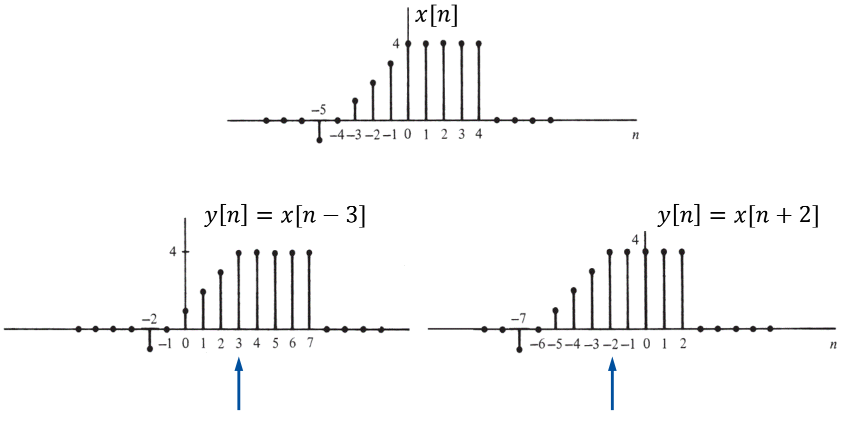
Operations on discrete-time signals…

- Example (time-shift): Given $x[n]$ below, plot $x[n-3]$ and $x[n+2]$



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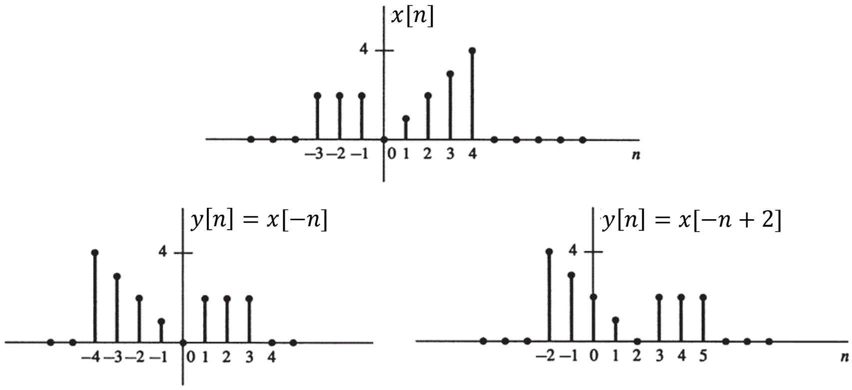
Operations on discrete-time signals…

- Example (folding): Given $x[n]$ below, plot $x[-n]$ and $x[-n+2]$



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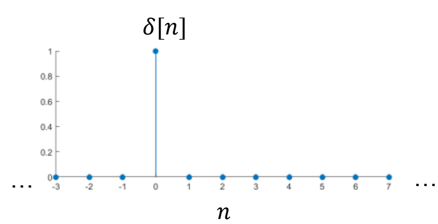
Basic types of sequences…

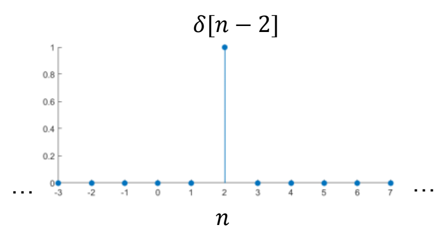
- Unit impulse:

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



- Delayed unit impulse:

$$\delta[n-k] = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$$



Matlab
```
k  = 2;
n = (-3:7)
delta = [(n-k)==0];
stem(n,delta)
```

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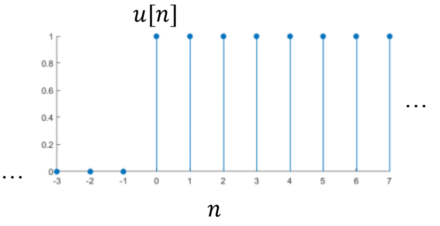
Basic types of sequences…

- Unit step:

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Matlab
```
n = (-3:7)
u = [n>=0];
stem(n,u)
```


Basic types of sequences…

- Relationship between $u[n]$ and $\delta[n]$:
  - Unit impulse is the first-order difference of the unit step

$$\delta[n] = u[n] - u[n-1]$$

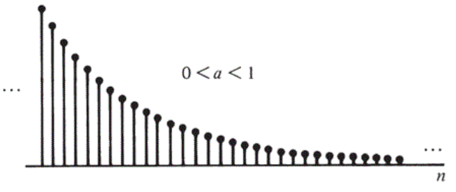
  - Unit step is the running sum of the unit impulse

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Basic types of sequences…

- Real-valued exponential function

$$x[n] = a^n, \forall n \text{ and } a \in \mathbb{R}$$



- What if a is complex-valued, i.e., $a \in \mathbb{C}$?

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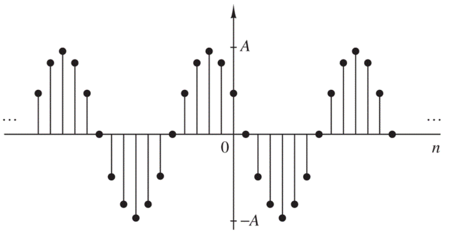
Discrete-time sinusoid



$$x[n] = A\cos[\omega n + \theta] = A\cos[2\pi f n + \theta]$$
$$= \frac{A}{2}\left(e^{j[\omega n+\theta]} + e^{-j[\omega n+\theta]}\right)$$

- What about the notion of frequency in discrete time?
- What about the notion of periodicity for discrete-time sinusoids?

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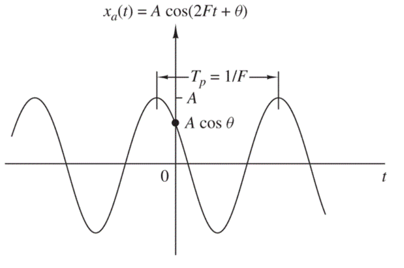
Discrete-time sinusoid…

- Continuous-time sinusoid:



- Consider two signals

$$x_1(t) = A\cos(\Omega_1 t) = A\cos(2\pi F_1 t)$$
$$x_2(t) = A\cos(\Omega_2 t) = A\cos(2\pi F_2 t)$$

where $F_2 > F_1$

- Signal $x_2(t)$ will oscillate faster than $x_1(t)$
- In general $x_2(t) \neq x_1(t)$, except at some possible points

Discrete-time sinusoid…

- Digital frequency: $x[n] = \cos[\omega n] = \cos[2\pi f n]$

$\omega = 0$, $f = 0$

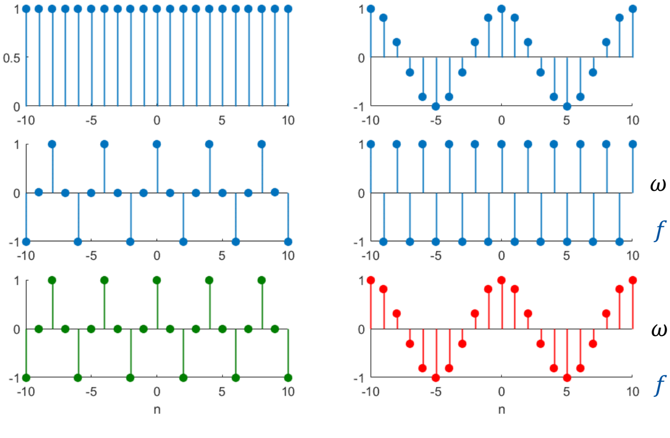
$\omega = \frac{\pi}{5}$, $f = \frac{1}{10}$

$\omega = \frac{\pi}{2}$, $f = \frac{1}{4}$

$\omega = \pi$, $f = \frac{1}{2}$

$\omega = \frac{3\pi}{2}$, $f = \frac{3}{4}$

$\omega = \frac{9\pi}{5}$, $f = \frac{9}{10}$



Discrete-time sinusoid…

- Discrete-time sinusoid is 2π-periodic in frequency

$$\cos[(\omega + 2k\pi)n] = \cos[\omega n + 2kn\pi] = \cos[\omega n]$$

$\Longrightarrow$ Any sinusoidal sequence with $|\omega| > \pi$ is identical to a sinusoidal sequence with $|\omega| \leq \pi$!

 – Verify this for the green and red sinusoids in previous slide

- Lowest frequency at $\omega_k = 0 + 2\pi k$
- Highest frequency at $\omega_k = \pi + 2\pi k$

$\Longrightarrow$ Range of frequencies is finite

$$-\pi \leq \omega \leq \pi, \text{ or } -\frac{1}{2} \leq f \leq \frac{1}{2}$$

$$(0 \leq \omega \leq 2\pi, \text{ or } 0 \leq f \leq 1)$$

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Discrete-time sinusoid…

- Is a discrete-time sinusoid a periodic sequence?

$$x[n] = x[n+N]?$$

$$\cos[2\pi f\, n] = \cos[2\pi f(n+N)]?$$

- Answer: (Yes/No/Sometimes) [Tick your option]

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Discrete-time sinusoid…

- Answer: Sometimes
- A discrete-time sinusoid is periodic only if its frequency is a rational number

$$\cos[2\pi n] = \cos[2\pi f(n+N)]$$

$$\Rightarrow 2\pi f N = 2\pi k$$

$$\boxed{\Rightarrow f = \frac{k}{N}}$$

Discrete-time sinusoid…

- Example: Determine if the discrete-time signals below are periodic; if they are, determine their periods

1. $x[n] = \cos\left[\frac{12\pi}{5}n\right]$
2. $x[n] = \sin^2\left[\frac{7\pi}{12}n + \sqrt{2}\right]$
3. $x[n] = \cos[0.02n + 3]$

Complex exponential

- Complex exponential: $x[n] = Ae^{j[2\pi f n+\theta]}$
- Same properties as discrete-time sinusoids
  - 2π-periodic in (angular) frequency
  - Periodic sequence if frequency f is rational
- Used as building block for discrete-time Fourier representation

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Sampling a sinusoidal signal

- Consider sampling sinusoidal signal at intervals $nT = n/F_s$

$$x_a(t) = A\cos(\Omega t) = A\cos(2\pi F t)$$

- Discretized signal

$$x[n] = x_a(nT) = A\cos\left[2\pi \frac{F}{F_s} n\right] = A\cos[2\pi f n]$$

$$\Rightarrow f = \frac{F}{F_s} \text{ or } \omega = \Omega T \text{ (relative/normalized frequency)}$$

- For accurate representation we know from before

$$-\frac{1}{2} \leq f \leq \frac{1}{2} \quad \Leftrightarrow \quad -\frac{F_s}{2} \leq F \leq \frac{F_s}{2}$$

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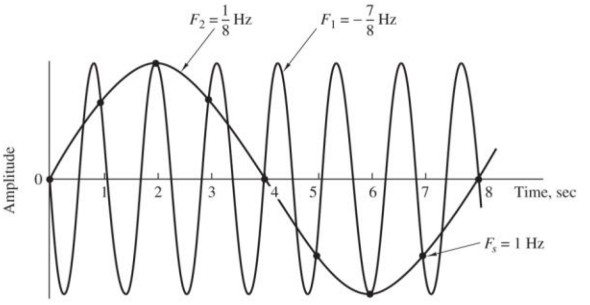
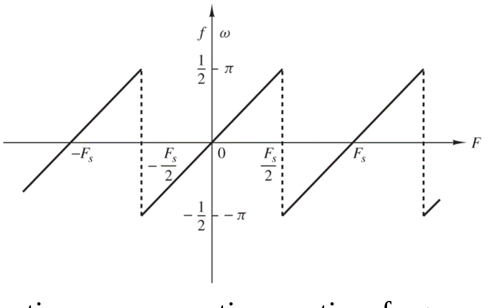
Aliasing



$$A\cos\left[2\pi\frac{F_1}{F_s}n\right] = A\cos\left[2\pi\frac{1}{8}n\right]$$

$$A\cos\left[2\pi\frac{F_2}{F_s}n\right] = A\cos\left[2\pi\frac{(-7)}{8}n\right] = A\cos\left[2\pi\underbrace{\left(\frac{(-7)}{8}+1\right)}_{=\frac{1}{8}}n\right]$$

Aliasing…



- Discrete-time versus continuous-time frequency variables in periodic sampling

Summary

- Today we discussed:
  – Discrete-time signals in time-domain
- Next:
  – Discrete-time systems in time-domain