NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF ENGINEERING CYBERNETICS

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Exam
TTK4150 Nonlinear Control Systems

Friday December 18, 2015

Hours: 09.00 – 13.00

Aids:	D - No printed or written materials allowed. NTNU type approved calculator with an empty memory allowed.
Language:	English
No. of pages:	5
Grades available:	January 19, 2016

This exam counts for 100% of the final grade.

Problem 1 (13%)

Consider the system

$$\dot{x}_1 = -x_2^3$$
$$\dot{x}_2 = -x_1^2 x_2 + x_1^3 - x_2$$

where $x \in \mathbb{R}^2$

a **[3%]** Show that the origin is the only equilibrium point.

b **[4%]** Using Lyapunov's indirect method, what is the strongest conclusion you can make about the stability properties of the origin?

c **[6%]** Use the Lyapunov function

$$V(x) = \frac{1}{4}x_1^4 + \frac{1}{4}x_2^4$$

to show that the origin is an asymptotically stable equilibrium point. Is it globally asymptotically stable? Justify your answer.

Problem 2 (20%)

Consider the subsystems Σ_1 and Σ_2 of a cascaded system given by

$$\Sigma_1 \qquad \dot{x}_1 = -|x_1|x_1 \cos^2(t) - x_1^3 + x_2$$

$$\Sigma_2 \qquad \begin{aligned} \dot{x}_2 &= -(1 + \sin^2(t))x_2 + x_2^2 x_3 \\ \dot{x}_3 &= -x_2^3 - x_3^5. \end{aligned}$$

a **[4%]** Show that for $x_2 = 0$ the equilibrium point $x_1^* = 0$ is uniformly globally asymptotically stable (UGAS) for the subsystem Σ_1.

b **[5%]** Show that the subsystem Σ_2 is UGAS in the origin.

c **[6%]** Show that the subsystem Σ_1 is input to state stable (ISS) when x_2 is viewed as the input.

d **[5%]** Is the origin of the cascaded system UGAS? Justify your answer.

Problem 3 (12%)

Consider the perturbed system

$$\dot{x} = -\alpha(t)x + g(t, x)$$

where $\alpha(t) > 0 \quad \forall\, t \geq 0$.
Hint: For **a** and **b** it is assumed that $\alpha(t) = \alpha_1 > 0 \quad \forall\, t \geq 0$, where α_1 is constant.

a [2%] Use both Lyapunov's indirect method and Lyapunov's direct method to show that the origin of unperturbed system is exponential stable.

b [5%] Show that the origin of perturbed system is exponential stable where it is assumed that

$$\begin{aligned} ||g(t,x)|| &\leq \gamma ||x||, \quad \forall\, t \geq 0, \quad \forall\, x \in \mathbb{R} \\ \gamma &> 0 \end{aligned}$$

c [5%] Now, it is stated that

$$\begin{aligned} \alpha(t) &= \varepsilon(1 + \sin^2(t)) \\ g(t,x) &= \frac{\varepsilon}{2} x, \end{aligned}$$

and the scalar system can be rewritten as

$$\dot{x} = \varepsilon \left(-(1 + \sin^2(t))x + \frac{1}{2}x \right).$$

Use the averaging method to show that for a sufficient small $\varepsilon > 0$ that the origin the exponential stable. *Hint:* The system is π-periodic in t.

Problem 4 (15%)

Consider the system

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1 + x_2 \\ \dot{x}_2 &= -k_1 x_1 - k_2 x_3 - \frac{k_3}{\sqrt{k_4 + x_1^2}} x_2 \\ \dot{x}_3 &= k_5 x_2 - k_5 x_3 + \frac{k_5}{k_2} u \\ y &= x_3 \end{aligned}$$

where $k_1 > 0$, $k_2 > 0$, $k_3 > 0$, $k_4 > 0$, $k_5 > 0$ and $x \in \mathbb{R}^3$

a [4%] Show that the system from u to y is strictly passive (state strictly passive). *Hint:* $V(x) = \frac{k_1}{2} x_1^2 + \frac{1}{2} x_2^2 + \frac{k_2}{2k_5} x_3^2$

b [3%] Show that the system from u to y is also output strictly passive.

c [4%] Show that the system is zero state observable when u is viewed as the input.

d [4%] What can be concluded about the stability from passivity properties of the system.
(i) Is the origin AS? Can you comment on the global result?
(ii) Is the system finite-gain $\mathcal{L}_2$ stable?
Justify your answer.

Problem 5 (30%)

Consider the following system

$$\begin{aligned}
\dot{x}_1 &= x_1 + x_2 - u \\
\dot{x}_2 &= x_3 - x_1 \\
\dot{x}_3 &= -x_3 - 2x_1 + u \\
y &= x_2
\end{aligned}$$

a [3%] Find the relative degree of this system. Is the system input-output linearizable?

b [10%] Transform the system into the normal form

$$\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi} &= A_c \xi + B_c \gamma(x) \left[u - \alpha(x)\right]
\end{aligned}$$

Specify the diffeomorphism $z = T(x) = \begin{bmatrix} \eta & \xi \end{bmatrix}^T$, the functions $\gamma(x)$, $\alpha(x)$ and $f_0(\eta, \xi)$ and the matrices A_c and B_c. In which domain is the transformation valid?

c [4%] Find an input-output linearizing controller on the form $u = \alpha(x) + \beta(x) v$.

d [4%] Find a controller v such that the external dynamics ξ is asymptotically stable at the origin.

e [6%] Is the system minimum phase?
Hint: If you were not able to solve **b** you may use the following equations for the internal dynamics: (It is not the correct internal dynamics equation, but it has the same property with respect to minimum phase)

$$\dot{\eta} = -\eta + \xi_1^2 + \xi_2^2$$

f [3%] Is the closed-loop system $[\eta, \xi]^\top$ asymptotically stable at the origin?

Problem 6 (10%)

Consider the surge-motion model of a ship,

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
m\dot{x}_2 + d(x_2)x_2 &= u,
\end{aligned}$$

where for notational simplicity, the time t is omitted and

$$d(x_2) > 0.$$

u is the input. It is assume that the position $x_1(t)$ and the velocity $x_2(t)$ can be measured. Use the backstepping method to design a controller to globally stabilize the origin (0,0) of the system .

Appendix: Formulae

$$\cos^2(t) = \frac{1}{2}(1 + \cos(2t))$$
$$\sin^2(t) = \frac{1}{2}(1 - \cos(2t))$$