



NTNU – Trondheim
Norwegian University of
Science and Technology

TTT4120 Digital Signal Processing Fall 2017

Lecture: Filter Properties and Inverse Systems

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Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
 - 5.4.2 Lowpass, highpass, and bandpass filters
 - 5.4.3 Digital resonators
 - 5.4.4 Notch filters
 - 5.4.5 Comb filters
 - 5.4.6 All-pass filters
 - 5.4.1 Ideal filter characteristics
 - 10.2.1 Symmetric and antisymmetric FIR filters
 - 5.5 Inverse systems and deconvolution

*Level of detail is defined by lectures and problem sets

Contents and learning outcomes

- Some simple filter properties
- Why linear phase?
- Minimum-phase and inverse systems

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Ideal filter characteristics

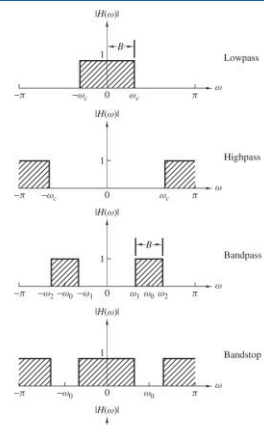
- Systems may be designed to amplify or attenuate parts of the input signal (e.g., remove noise, suppress interference)

$$\begin{array}{ccccc}
 x[n] & \longrightarrow & \boxed{h[n]} & \longrightarrow & y[n] = h[n] * x[n] \\
 X(\omega) & & & & Y(\omega) = H(\omega)X(\omega) \\
 X(z) & & & & Y(z) = H(z)X(z)
 \end{array}$$

- Frequency response $H(\omega)$ shapes the spectrum of the input signal to have a desired form

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Linear time-invariant systems...

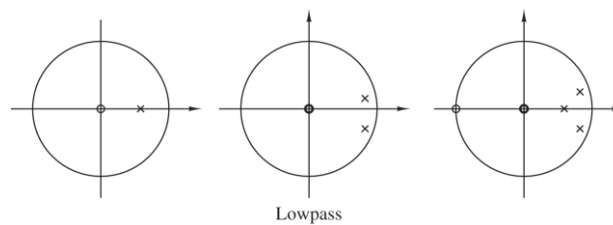


- Passband, stopband, cutoff frequencies
- Cannot get this kind of shapes using a causal impulse response with a finite number of coefficients (later)

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Lowpass

- Poles close(r) to $z = 1$ and zeros close(r) to $z = -1$. Why?

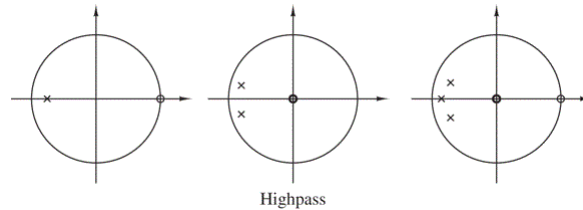


- Example: $H(z) = \frac{1-a}{2} \cdot \frac{1+z^{-1}}{1-az^{-1}}$

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Highpass

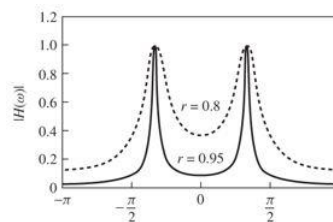
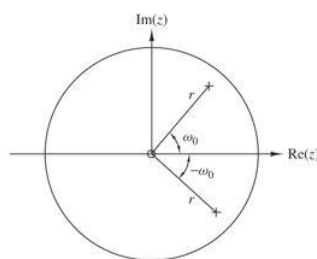
- Poles close(r) to $z = -1$ and zeros close(r) to $z = 0$



- Reflect poles-zeros of lowpass around imaginary axis
- Frequency translation: $H_{hp}(\omega) = H_{lp}(\omega - \pi)$
- Example: $H_{lp}(z) = \frac{1-a}{2} \cdot \frac{1+z^{-1}}{1-az^{-1}} \rightarrow H_{hp}(z) = \frac{1-a}{2} \cdot \frac{1-z^{-1}}{1+az^{-1}}$

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Digital resonator



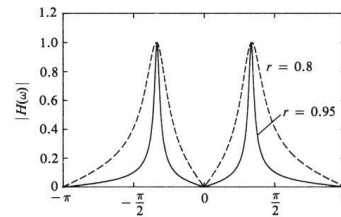
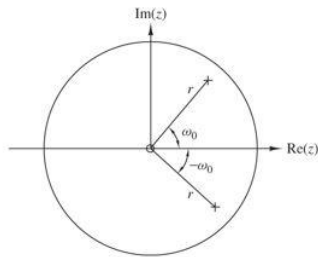
- Complex-conjugate poles $p_{1,2} = re^{\pm j\omega_0}$ close to $|z| = 1$

$$H(z) = \frac{b_0}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})}$$

- Resonant peak can be computed: $\omega_r = \cos^{-1}\left(\frac{1+r^2}{2r}\cos\omega_0\right)$
- For $r \approx 1$, $\omega_r \approx \omega_0$

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Digital resonator...



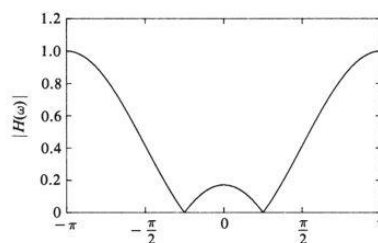
- Complex-conjugate poles $p_{1,2} = re^{\pm j\omega_0}$ and zeros $z_{1,2} = \pm 1$

$$H(z) = \frac{(1+z^{-1})(1-z^{-1})}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})}$$

- Exact location of resonant peak harder to find analytically

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Notch filter



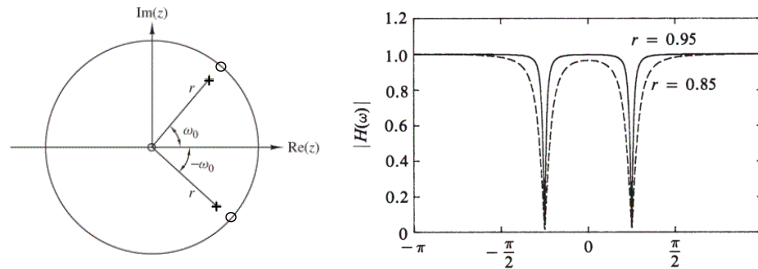
- A filter that contains deep notches in its frequency response
- Removing powerline frequency disturbance
- Create nulls by complex-conjugate zeros on the unit circle

$$H(z) = b_0(1 - e^{j\omega_0}z^{-1})(1 - e^{-j\omega_0}z^{-1})$$

- Large bandwidth is a problem with FIR notch filters

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Notch filter...

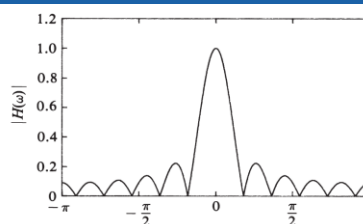


- Introduce poles close to unit circle reduces notch bandwidth

$$H(z) = \frac{b_0(1 - e^{j\omega_0}z^{-1})(1 - e^{-j\omega_0}z^{-1})}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})}$$

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Comb filter



- Notch filter with nulls periodically spaced across frequency
- Simple moving average (FIR) filter

$$H(z) = \sum_{k=0}^M z^{-k} = \frac{1 - z^{-(M+1)}}{1 - z^{-1}}$$

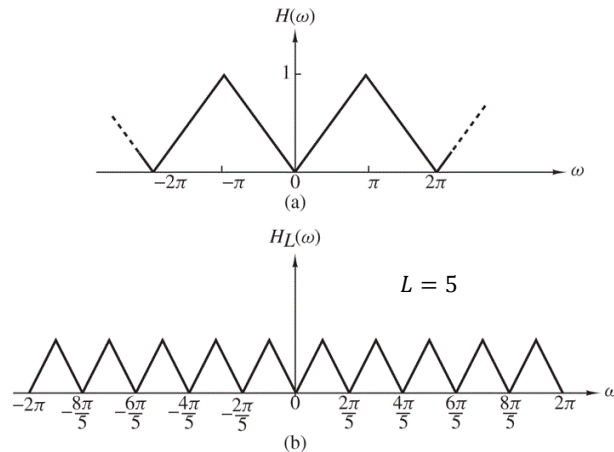
- We may also construct a comb filter by replacing z with z^L

$$H_L(z) = \sum_{k=0}^M h[k]z^{-Lk} \Leftrightarrow H_L(\omega) = H(L\omega)$$

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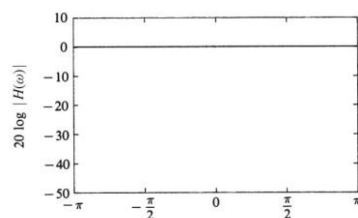
Comb filter...

$$H_L(z) = \sum_{k=0}^M h[k]z^{-Lk} \Leftrightarrow H_L(\omega) = H(L\omega)$$



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All-pass filters



- All-pass filter has constant magnitude response
- Can be used to compensate poor phase characteristics

$$H(z) = \frac{a_N + a_{N-1}z^{-1} + \dots + a_1z^{-N+1} + z^{-N}}{1 + a_1z^{-1} + \dots + a_{N-1}z^{-N+1} + a_Nz^{-N}} = \frac{z^{-N}A(z^{-1})}{A(z)}$$

- Assuming real coefficients

$$H(z) = \frac{z^{-N}A(z^{-1})}{A(z)} \Leftrightarrow |H(\omega)|^2 = H(z)H(z^{-1})|_{z=e^{j\omega}} = 1$$

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Linear phase filters...

- Why linear phase filters, i.e., $\angle H(\omega) = a + b\omega$?

$$H(\omega) = |H(\omega)|e^{j\angle H(\omega)}$$

- Compare the two ideal lowpass specifications

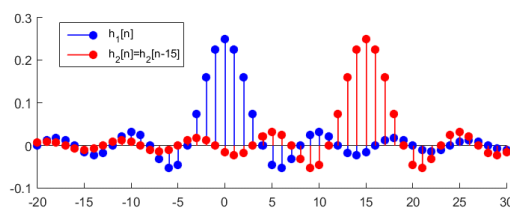
$$H_1(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \leq \pi \end{cases}$$

$$H_2(\omega) = \begin{cases} e^{-jn_d\omega}, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \leq \pi \end{cases}$$

- How about the time-domain pulses?

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Linear phase filters...



Matlab

```
n = -20:30;
wc = 2*pi*1/8;
x = wc/pi;
nd = 15;
stem(n, x*sinc(x*n), 'r')
hold on
stem(n, x*sinc(x*(n-nd)))
```

- How about the time-domain pulses?

$$h_1[n] = \frac{w_c}{\pi} \frac{\sin[w_c n]}{w_c n}$$

$$h_2[n] = \frac{w_c}{\pi} \frac{\sin[w_c(n-n_d)]}{[w_c(n-n_d)]} \Rightarrow h_2[n] = h_1[n - n_d]$$

- Delays the output signal with n_d samples, no signal distortion!

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Linear phase filters...

- Filter design in general (later in the course):

$$\min_{a,b} ||E(z)|| = \min_{a,b} \left\| H_{\text{des}}(z) - \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{\sum_{k=0}^{N-1} a_k z^{-k}} \right\|$$

- Consider FIR filters having a frequency response of the form

$$H(\omega) = H_r(\omega) e^{-j(\omega d + c)}, H_r(\omega) \text{ real-valued}$$

- We want a pure signal delay in passband
- Obtained by choosing $h[k]$ real and $h[k] = \pm h[M-1-k]$
 - Symmetric or antisymmetric

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Linear phase filters...

- Example: FIR with $M = 5 \Rightarrow N = 2$

$$\begin{aligned} H(z) &= h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} \\ &= z^{-2} \{ h[0]z^2 + h[1]z + h[2] \pm h[1]z^{-1} \pm h[0]z^{-2} \} \\ &= z^{-2} \{ h[2] + h[0][z^2 \pm z^{-2}] + h[1][z^1 \pm z^{-1}] \} \end{aligned}$$

- Frequency response symmetric filter (take the '+' signs):

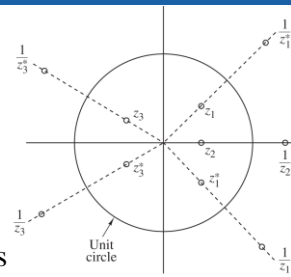
$$H(z) \Big|_{e^{j\omega}} = e^{-j2\omega} \{ h[2] + 2h[0] \cos 2\omega + 2h[1] \cos \omega \}$$

- Frequency response antisymmetric filter (take the '-' signs):

$$H(z) \Big|_{e^{j\omega}} = je^{-j2\omega} \{ 2h[0] \sin 2\omega + 2h[1] \sin \omega \}$$

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Linear phase filters...



- Zeros of $H(z)$ occur in reciprocal pairs
- Example (cont.): Symmetric FIR with $M = 5$ ($N = 2$)

$$\begin{aligned}
 H(z) &= h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} \\
 &= z^{-2}\{h[0]z^2 + h[1]z + h[2] + h[1]z^{-1} + h[0]z^{-2}\} \\
 &= z^{-2}\{h[2] + h[0][z^2 + z^{-2}] + h[1][z + z^{-1}]\} \\
 &= z^{-4}H(z^{-1})
 \end{aligned}$$

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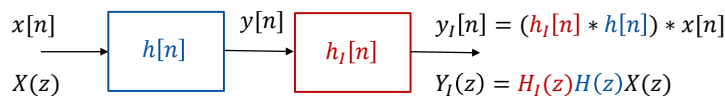
Inverse and minimum-phase systems

$$\begin{array}{ccc}
 x[n] & \longrightarrow & \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n] \\
 X(z) & & Y(z) = H(z)X(z)
 \end{array}$$

- What if we are given $y[n]$ and want to determine $x[n]$?
 - Information signal passing through communication channel

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Inverse and minimum-phase systems



- If system \mathcal{T} is invertible, $x[n]$ can be recovered from $y[n]$

$$x[n] = \mathcal{T}^{-1}\{y[n]\} = \mathcal{T}^{-1}\{\mathcal{T}[x[n]]\}$$

- Linear time-invariant systems

$$h[n] * h_I[n] = \delta[n] \stackrel{\mathcal{Z}}{\Leftrightarrow} H(z)H_I(z) = 1$$

- Solving for $h_I[n]$ usually simpler in z-domain, especially if $H(z)$ is rational, i.e., $H(z) = B(z)/A(z)$

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Inverse and minimum-phase systems...

- Example: Determine inverse system $h_I[n] = \delta[n] - \frac{1}{3}\delta[n-1]$
- Time-domain solution ($h_I[n]$ causal and stable)

$$h[n] * h_I[n] = \delta[n] \Leftrightarrow \sum_{k=0}^n h[k]h_I[n-k] = \delta[n]$$

- Solution via z-transform

$$H(z)H_I(z) = 1 \Leftrightarrow H_I(z) = 1/H(z)$$

Remember to specify ROC of $H_I(z)$ (two possibilities)!

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Inverse and minimum-phase systems...

- Solution via z-transform

$$H(z)H_I(z) = 1 \Leftrightarrow H_I(z) = 1/H(z)$$

Remember to specify ROC of $H_I(z)$ (two possibilities)!

$$H(z) = 1 - \frac{1}{3}z^{-1} \text{ ROC: } |z| \neq 0 \rightarrow H_I(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}$$

- Corresponds to either

$$h_I[n] = -\left(\frac{1}{3}\right)^n u[-n-1], \text{ ROC: } |z| < \frac{1}{3} \text{ (anti-causal unstable) or}$$

$$h_I[n] = \left(\frac{1}{3}\right)^n u[n], \text{ ROC: } |z| > \frac{1}{3} \text{ (causal stable)}$$

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Inverse and minimum-phase systems...

- In general we have that if $H(z)$ is stable and causal then poles $|p_k| < 1 \forall k$ and ROC: $|z| > \max_k |p_k|$

\Rightarrow There exists a stable and causal inverse $H_I(z) = 1/H(z)$ if zeros of $H(z)$ are within the unit circle, i.e., $|z_k| < 1 \forall k$

- Definition: A system is called **minimum-phase** if all zeros and poles are inside the unit circle
 \Rightarrow a stable pole-zero system that is minimum phase has a stable inverse that is also minimum phase

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Summary

Today:

- Some simple filter types and their properties
- Linear phase systems
- Inverse and minimum-phase systems

Next:

- Correlation and energy spectrum density

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