TTK4150 Nonlinear Control Systems
Lecture 9

Perturbation Theory and Averaging

NTNU
Innovation and Creativity

Previous lecture

Previous lecture:

Passivity

- How to analyze the passivity properties of a system by using the definitions of passivity for
  - Memoryless functions
  - Dynamical systems
- Understand the relations between passivity and
  - Lyapunov stability
  - $\mathscr{L}_2$ stability (IOS)
- The passivity theorems
  (for feedback connections)

Outline

1. Introduction
2. Introduction
   - Previous lecture
   - Today's goals
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3. Periodic Perturbation of Autonomous Systems
   - Introduction
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4. Averaging Theory
   - Introduction to Averaging Theory
   - Averaging Method
   - Averaged System
   - Theorem 10.4
5. Examples
   - Linear System
   - The Suspended Pendulum
   - Snake Robot
6. Next lecture

Today's goals

After today you should...

- Analyze an autonomous system under the influence of a weak periodic perturbation
- Use Averaging method-Periodic averaging

Literature

Today's lecture is based on

Khalil **Chapter 10**
Sections 10.3 and 10.4

Motivation

Asymptotic Methods

- Exact closed-form analytical solutions of nonlinear differential equations are possible only for special cases
- An alternative
  - Numerical Methods
  - Asymptotic Methods
    - **Perturbation Method**
    - **Averaging Theory**
    - Singular Perturbation
- The goal of an asymptotic method is to obtain $\tilde{x}(t,\varepsilon)$ such that $x(t,\varepsilon) - \tilde{x}(t,\varepsilon)$ is small, for small $|\varepsilon|$ and the $\tilde{x}(t,\varepsilon)$ is expressed in terms of equations simpler than the original equation.

Part I

Periodic Perturbation of Autonomous Systems

Introduction

Consider the system

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon) \tag{1}$$

where

- f, g, and their first partial derivatives with respect to x are continuous and bounded for all $(t, x, \varepsilon) \in [0, \infty) \times D_0 \times [-\varepsilon_0, \varepsilon_0]$, for every compact set $D_0 \subset D$, where $D \subset R^n$ is a domain that contains the origin.
- Suppose the origin is an exponentially stable equilibrium point of the autonomous system

$$\dot{x} = f(x) \tag{2}$$

Introduction

We should show that

- there exist $r > 0$ and $\varepsilon_1 > 0$ such that for all $\|x(0)\| \leq r$ and $|\varepsilon| \leq \varepsilon_1$, the solution of

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon)$$

  is uniformly ultimately bounded with ultimate bound proportional to $|\varepsilon|$.
- all solutions approach an $O(\varepsilon)$ neighborhood of the origin as $t \to \infty$ and this is true for any bounded g.

Question

What happens inside that $O(\varepsilon)$ neighborhood when g is T-periodic in t?

Introduction

Possibility that a T- periodic solution might exist within an $O(\varepsilon)$ neighborhood of the origin.

Define $P_\varepsilon(x)$

Let $\phi(t;t_0,x_0,\varepsilon)$ be the solution of (1) that starts at (t_0,x_0); that is, $x_0 = \phi(t_0;t_0,x_0,\varepsilon)$. For all $\|x\| < r$, define a map $P_\varepsilon(x)$ by

$$P_\varepsilon(x) = \phi(T;0,x,\varepsilon)$$

$P_\varepsilon(x)$ is the state of the system at time T when the initial state at time zero is x.

T-periodic solution

Lemma 10.1

The system

$$\dot{x} = f(x) + \varepsilon g(t,x,\varepsilon)$$

has a T-periodic solution if and only if the equation

$$x = P_\varepsilon(x) \quad (3)$$

has a solution.

Lemma 10.2

There exist positive constants k and ε_2 such that

$$x = P_\varepsilon(x)$$

has a unique solution in $\|x\| = k|\varepsilon|$, for $|\varepsilon| < \varepsilon_2$.

Lemma 10.1 and Lemma 10.2

NB

- For sufficient small ε, the perturbed system

$$\dot{x} = f(x) + \varepsilon g(t,x,\varepsilon)$$

  has a T- periodic solution in an $O(\varepsilon)$ neighborhood of the origin
- The periodic solution has to be unique due to the uniqueness of the solution of equation

$$x = P_\varepsilon(x)$$

Exponentially stable $\bar{x}(t,\varepsilon)$

Lemma 10.3

If $\bar{x}(t,\varepsilon)$ is a T-periodic solution of

$$\dot{x} = f(x) + \varepsilon g(t,x,\varepsilon)$$

such that

$$\|\bar{x}(t,\varepsilon)\| \le k|\varepsilon|$$

then $\bar{x}(t,\varepsilon)$ is exponentially stable.

Exponentially stable solution

Theorem 10.3

Suppose

- f, g and their first partial derivatives with respect to x are continuous and bounded for all $(t,x,\varepsilon) \in [0,\infty) \times D_0 \times [-\varepsilon_0, \varepsilon_0]$, for every compact set $D_0 \subset D$, where $D \subset R^n$ is a domain that contains the origin
- The origin is an exponentially stable equilibrium point of the autonomous system

$$\dot{x} = f(x)$$

- $g(t,x,\varepsilon)$ is T- periodic in t.

Exponentially stable solution

Theorem 10.3 cont.

Then, there exist positive constants ε^* and k such that for all $|\varepsilon| < \varepsilon^*$, the perturbed system

$$\dot{x} = f(x) + \varepsilon g(t,x,\varepsilon)$$

has a unique T-periodic solution $\bar{x}(t,\varepsilon)$ with the property that

$$\|\bar{x}(t,\varepsilon)\| \le k|\varepsilon|$$

Moreover, this solution is exponentially stable.

Using the Theorem 10.3

Perturbed system

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon)$$

- If $g(t, 0, \varepsilon) = 0$, the origin will be an equilibrium point of the perturbed system.
- By uniqueness of the periodic solution $\bar{x}(t, \varepsilon)$, it follows that $\bar{x}(t, \varepsilon)$ is the trivial solution $x = 0$.

NB

The Theorem 10.3 ensures that the origin is an exponentially stable equilibrium point of the perturbed system.

Part II

Averaging Theory

Introduction to Averaging Theory

The basic idea of averaging theory–deterministic or stochastic

is to approximate the **original system**

- time-varying and periodic
- almost periodic, or randomly perturbed

by **a simpler (average) system**

- time-invariant, deterministic

or some approximating **diffusion system**

- a stochastic system simpler than the original one

The averaging method has been developed as:

- **a practical tool** in mechanics/dynamics
- **a theoretical tool** in mathematics both for deterministic dynamics and for stochastic dynamics.

Averaging Method–Historical Information

Averaging method

- Averaging method is a useful computational technique
- **Lagrange** formulated the gravitational three-body problem as a perturbation of the two-body problem (1788)
- **Fatou** gave the first proof of the asymptotic validity of the method in 1928
- After the systematic researches done by Krylov, Bogoliubov, Mitropolsky etc, in 1930s, the averaging method gradually became **one of the classical methods in analyzing nonlinear oscillations**

Averaging method

Averaging method

The averaging method applies to a system of the form

$$\dot{x} = \varepsilon f(t,x,\varepsilon)$$

where ε is a small positive parameter and $f(t,x,\varepsilon)$ is T- periodic in t:

$$f(t+T,x,\varepsilon) = f(t,x,\varepsilon), \forall (t,x,\varepsilon) \in [0,\infty) \times D \times [0,\varepsilon_0]$$

for some domain $D \subset R^n$.

The averaging methond approximates

the solution of the system by the solution of **an "averaged system,"** obtained by averaging $f(t,x,\varepsilon)$ at $\varepsilon = 0$.

Averaged system

Consider the system

$$\dot{x} = \varepsilon f(t,x,\varepsilon) \tag{4}$$

- where f and its partial derivatives with respect to (x,ε) up to the second order are continuous and bounded for $(t,x,\varepsilon) \in [0,\infty) \times D \times [0,\varepsilon_0]$, for every compact set $D_0 \subset D$, where $D \subset R^n$ is a domain
- Moreover, $f(t,x,\varepsilon)$ is T- periodic in t for some $T > 0$ and ε is positive.

We associate with (4) an autonomous averaged system

$$\dot{x} = \varepsilon f_{av}(x) \tag{5}$$

where

$$f_{av}(x) = \frac{1}{T}\int_0^T f(\tau,x,0)d\tau \tag{6}$$

The basic problem in averaging

Nonautonomous System

$$\dot{x} = \varepsilon f(t, x, \varepsilon)$$

Autonomous System

$$\dot{x} = \varepsilon f_{av}(x)$$

NB

Determine in what sense the behavior of the

- **autonomous system**

approximates the behavior of the

- **nonautonomous system**.

Theorem 10.4

- Let $f(t,x,\varepsilon)$ and its partial derivatives with respect to (x,ε) up to the second order be continious and bounded for $(t,x,\varepsilon) \in [0,\infty) \times D \times [0,\varepsilon_0]$, for every compact set $D_0 \subset D$, where $D \subset R^n$ is a domain.
- Suppose f is T- periodic in t for some $T > 0$ and ε is a positive parameter.
- Let $x(t,\varepsilon)$ and $x_{av}(\varepsilon t)$ denote the solutions of (4) and (5) respectively.

if $x_{av}(\varepsilon t) \in D \quad \forall t \in [0, b/\varepsilon]$ and $x(0,\varepsilon) - x_{av}(0) = O(\varepsilon)$, then there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, $x(t,\varepsilon)$ is defined and

$$x(t,\varepsilon) - x_{av}(\varepsilon t) = O(\varepsilon) \qquad \text{on} \qquad [0, b/\varepsilon]$$

continue ...

Theorem 10.4

If the origin $x = 0 \in D$ is an exponentially stable equilibrium point of the average system (5), $\Omega \in D$ is a compact subset of its region of attraction, $x_{av}(0) \in \Omega$, and $x(0,\varepsilon) - x_{av}(0) = O(\varepsilon)$, then there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, $x(t,\varepsilon)$ is defined and

$$x(t,\varepsilon) - x_{av}(\varepsilon t) = O(\varepsilon) \qquad \text{for all} \qquad t \in [0,\infty)$$

If the origin $x = 0 \in D$ is an exponentially stable equilibrium point of the average system (5), then there exist positive constant ε^* and k such that, for all $0 < \varepsilon < \varepsilon^*$, (4) has a unique, exponentially stable, T- periodic solution $\bar{x}(t,\varepsilon)$ with the property

$$\|\bar{x}(t,\varepsilon)\| \le k\varepsilon$$

NB

- If $f(t,0,\varepsilon)=0$ for all $(t,\varepsilon)\in[0,\infty)\times[0,\varepsilon_0]$, the origin will be an equilibrium point of
$$\dot{x}=\varepsilon f(t,x,\varepsilon)$$
- By uniqueness of the T-periodic solution $\bar{x}(t,\varepsilon)$, it follows that $\bar{x}(t,\varepsilon)$ is the trivial solution $x=0$.
- In this case, the theorem ensures that the origin is an exponentially stable equilibrium point of
$$\dot{x}=\varepsilon f(t,x,\varepsilon)$$

Part III

Examples

Example: Linear System

Consider the linear system

$$\dot{x}=\varepsilon A(t)x$$

where $A(t+T)=A(t)$ and $\varepsilon>0$. Let

$$\bar{A}=\frac{1}{T}\int_0^T A(\tau)d\tau$$

The average system is given by

$$\dot{x}=\varepsilon\bar{A}x$$

It has an equilibrium point at $x=0$.

Example: Linear System

Suppose that the matrix $\bar{A}$ is Hurwitz

Then, it follows from the Theorem 10.4 that:

- For sufficient small ε, $\dot{x} = \varepsilon A(t)x$ has unique T- periodic solution in an $O(\varepsilon)$ neighborhood of the origin $x = 0$.
- $x = 0$ is an equilibrium point for the system. Hence, the periodic solution is the trivial solution $x(t) = 0$.
- For sufficient small, ε, $x = 0$ is an exponentially stable equilibrium point for the nonautonomous system

$$\dot{x} = \varepsilon A(t)x$$

Example: The suspended pendulum

- Suspension point: vertical vibrations with $a \sin \omega t$, where a is the amplitude and ω is the frequency.
- $a/l \ll 1$ and $\omega_0/\omega \ll 1$, where $\omega_0 = \sqrt{g/l}$

The equation of the system is given by

$$m(l\ddot{\theta} - a\omega^2 \sin \omega t \sin \theta) = -mg \sin \theta - k(l\dot{\theta} + a\omega \cos \omega t \sin \theta)$$

- Let $\varepsilon = a/l$ and $\omega_0/\omega = \alpha\varepsilon$, where $\alpha = \omega_0 l/\omega a$.
- Let $\beta = k/m\omega_0$ and changing the time scale from t to $\tau = \omega t$

The equation of motion can be written as

$$\frac{d^2\theta}{d\tau^2} + \alpha\beta\varepsilon\frac{d\theta}{d\tau} + (\alpha^2\varepsilon^2 - \varepsilon \sin \tau) \sin \theta + \alpha\beta\varepsilon^2 \cos \tau \sin \theta = 0$$

Example: The suspended pendulum

Choosing

$$x_1 = \theta, \qquad x_2 = \frac{1}{\varepsilon}\frac{d\theta}{d\tau} + \cos \tau \sin \theta$$

as state variables, the state equation is given by

$$\frac{dx}{d\tau} = \varepsilon f(\tau, x) \tag{7}$$

where

$$f_1(\tau, x) = x_2 - \sin x_1 \cos \tau$$
$$f_2(\tau, x) = -\alpha\beta x_2 - \alpha^2 \sin x_1 + x_2 \cos x_1 \cos \tau - \sin x_1 \cos x_1 \cos^2 \tau$$

The function $f(\tau, x)$ is 2π- periodic in τ.

Example: The suspended pendulum

The average system is given by

$$\frac{dx}{d\tau} = \varepsilon f_{av}(x) \qquad (8)$$

where

$$f_{av1}(x) = \frac{1}{2\pi}\int_0^{2\pi} f_1(\tau,x)d\tau = x_2$$
$$f_{av2}(x) = \frac{1}{2\pi}\int_0^{2\pi} f_2(\tau,x)d\tau = -\alpha\beta x_2 - \alpha^2\sin x_1 - \frac{1}{4}\sin 2x_1$$

Example: The suspended pendulum

Equilibrium point of the systems

- Both the original system (7) and the average system (8) have equilibrium points at
  - $(x_1 = 0, x_2 = 0)$
  - $(x_1 = \pi, x_2 = 0)$

  which correspond to the equilibrium positions
  - $\theta = 0$
  - $\theta = \pi$

With a fixed suspension point,

- the equilibrium $\theta = 0$ is **exponentially stable**
- while the equilibrium position $\theta = \pi$ is **unstable**.

Question

What a vibrating suspension point will do to the system?

Example: The suspended pendulum

Applying the Theorem 10.4, analyze

the stability properties of the equilibrium points of the average system (8) via linearization.

- The Jacobian of $f_{av}(x)$ is given by

$$\frac{\partial f_{av}}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\alpha^2\cos x_1 - 0.5\cos 2x_1 & -\alpha\beta \end{bmatrix}$$

- At the equilibrium point $(x_1 = 0, x_2 = 0)$, the Jacobian

$$\begin{bmatrix} 0 & 1 \\ -\alpha^2 - 0.5 & -\alpha\beta \end{bmatrix}$$

  is Hurwitz for all positive values of α and β.

Example: The suspended pendulum

By Theorem 10.4:

- For sufficiently small ε, the original system (7) has a unique exponential stable 2π- periodic solution in an $O(\varepsilon)$ neighborhood of the origin.
- The periodic solution is the trivial solution $x = 0$ because the origin is an equilibrium point for the original system.
- For sufficiently small ε, the origin is an exponentially stable equilibrium point for the original system (7).

Which means that:

Exponential stability of the $\theta = 0$ is preserved under

- small-amplitude
- high-frequency

vibration of the suspension point.

Example: The suspended pendulum

At the equilibrium point $(x_1 = \pi, x_2 = 0)$

The Jacobian

$$\begin{bmatrix} 0 & 1 \\ \alpha^2 - 0.5 & -\alpha\beta \end{bmatrix}$$

is Hurwitz for $0 < \alpha < 1/\sqrt{2}$ and $\beta > 0$.

NB

- $(x_1 = \pi, x_2 = 0)$ is an equilibrium point for the original system
- and applying Theorem 10.4

we are led to the conclusion that if $\alpha < 1/\sqrt{2}$, then $\theta = \pi$ is an **exponentially stable** equilibrium point for the original system (7) for sufficiently small ε.

Example: The suspended pendulum

NB

The unstable upper equilibrium position of the pendulum can be stabilized by vibrating the suspended point vertically with **small amplitude** and **high frequency**.

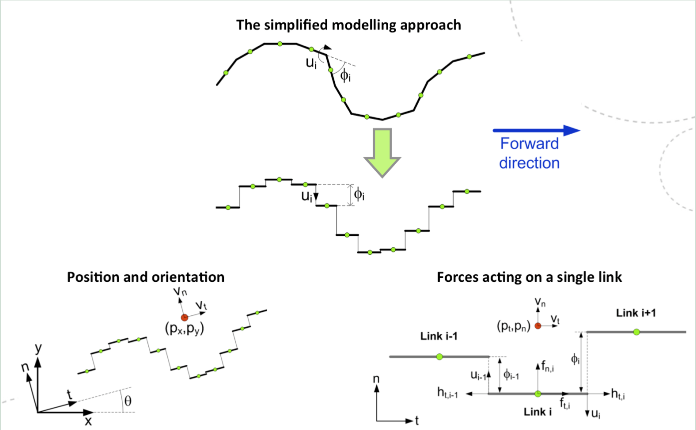
The idea of introducing

- high-frequency
- zero-mean vibration

in the parameters of a dynamic system in order to modify the properties of the system in a desired manner has been generalized into a **principle of vibrational control**.

Example: Snake Robot

Simplified Modeling Approach



Example: Snake Robot

The equation of motion is given by

Joint velocities → $\dot{\boldsymbol{\phi}} = \boldsymbol{v}_\phi$

Rotational velocity → $\dot{\theta} = v_\theta$

Translational velocity → $\dot{p}_x = v_t \cos\theta - v_n \sin\theta$, $\dot{p}_y = v_t \sin\theta + v_n \cos\theta$

Joint accelerations → $\dot{\boldsymbol{v}}_\phi = -\frac{c_1}{m}\boldsymbol{v}_\phi + \frac{c_2}{m} v_t \boldsymbol{A}\boldsymbol{D}^T\boldsymbol{\phi} + \frac{1}{m}\boldsymbol{D}\boldsymbol{D}^T\boldsymbol{u}$

Rotational acceleration → $\dot{v}_\theta = -c_3 v_\theta + \frac{c_4}{N-1} v_t \bar{\boldsymbol{e}}^T\boldsymbol{\phi}$

Translational acceleration →

$$\dot{v}_t = -\frac{c_1}{m} v_t + \frac{2c_2}{Nm} v_n \bar{\boldsymbol{e}}^T\boldsymbol{\phi} - \frac{c_2}{Nm}\boldsymbol{\phi}^T \boldsymbol{A}\bar{\boldsymbol{D}}\boldsymbol{v}_\phi$$

$$\dot{v}_n = -\frac{c_1}{m} v_n + \frac{2c_2}{Nm} v_t \bar{\boldsymbol{e}}^T\boldsymbol{\phi}$$

Example: Snake Robot

Study snake robot locomotion based on averaging theory

Lateral undulation: $\phi_{i,\text{ref}} = \alpha \sin(\omega t + (i-1)\delta) + \phi_o$

Simplified model

Averaging $\dot{\boldsymbol{v}} = \frac{1}{T}\int_0^T \cdots$

Forward velocity during lateral undulation: $\bar{v}_t^* = \alpha^2 \omega k_\delta \frac{N c_1 c_2}{2\left(N^2 c_1^2 - (4N^2 - 8N + 4) c_2^2 \phi_o^2\right)}$

$$v_t^* \propto \alpha^2$$
$$v_t^* \propto \omega$$
$$v_t^* \propto k_\delta = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} a_{ij}\sin((j-i)\delta)$$

Example: Snake Robot

The velocity dynamics in standard form of averaging is given by

$$\frac{dv}{d\tau} = \varepsilon f(\tau, v) \quad (9)$$

where

$$f(\tau,v) = \begin{bmatrix} -\frac{c_1}{m}v_t + \frac{2c_2}{Nm}v_n f_1(\tau) - \frac{c_2}{Nm}f_2(\tau) \\ -\frac{c_1}{m}v_n + \frac{2c_2}{Nm}v_t f_1(\tau) \\ -c_3 v_\theta + \frac{c_4}{N-1}v_t f_1(\tau) \end{bmatrix}$$

$$f_1(\tau) = (N-1)\,\phi_o + \sum_{i=1}^{N-1} \alpha \sin(\tau + (i-1)\,\delta)$$

$$f_2(\tau) = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\left[\frac{k_{\alpha\omega}}{\alpha}\phi_o a_{ij}\cos(\tau+(j-1)\,\delta)\right.$$
$$\left. + k_{\alpha\omega}a_{ij}\sin(\tau+(i-1)\,\delta)\cos(\tau+(j-1)\,\delta)\right]$$

Study the stability of the velocity dynamics.

Example: Snake Robot

The averaged model is given by

$$\dot{v} = \mathscr{A}v + b \quad (10)$$

where

$$\mathscr{A} = \mathscr{A}(\phi_o) = \begin{bmatrix} -\frac{c_1}{m} & \frac{2(N-1)}{Nm}c_2\phi_o & 0 \\ \frac{2(N-1)}{Nm}c_2\phi_o & -\frac{c_1}{m} & 0 \\ c_4\phi_o & 0 & -c_3 \end{bmatrix}$$

$$b = b(\alpha,\omega,\delta) = \begin{bmatrix} \frac{c_2}{2Nm}k_{\alpha\omega}k_\delta \\ 0 \\ 0 \end{bmatrix}$$

Example: Snake Robot

Stability Analysis

By performing coordinate transformation $z = v + \mathscr{A}^{-1}b$ we have

$$\dot{z} = \dot{v} = \mathscr{A}\left(z - \mathscr{A}^{-1}b\right) + b = \mathscr{A}z$$

The eigenvalues of $\mathscr{A}$ are easily calculated as

$$\operatorname{eig}(\mathscr{A}) = \begin{bmatrix} -\frac{c_1}{m} - \frac{2(N-1)}{Nm}c_2\phi_o \\ -\frac{c_1}{m} + \frac{2(N-1)}{Nm}c_2\phi_o \\ -c_3 \end{bmatrix}$$

Stability condition

$$|\phi_o| < \frac{N}{2(N-1)}\frac{c_1}{c_2}$$

Example: Snake Robot

Using the Theorem 10.4

- There exist $k > 0$ and $\omega^* > 0$ such that for all $\omega > \omega^*$,

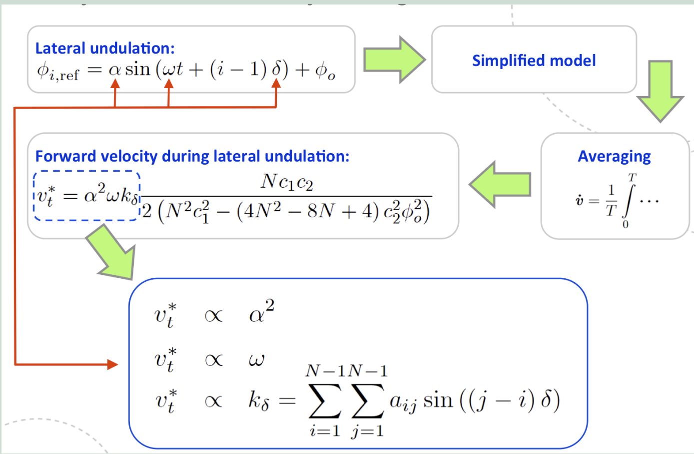
$$\|\boldsymbol{v}(t) - \boldsymbol{v}_{av}(t)\| \leq \frac{k}{\omega} \quad \text{for all } t \in [0, \infty)$$

  where $\boldsymbol{v}(t)$ denotes the exact velocity of the snake robot and $\boldsymbol{v}_{av}(t)$ denotes the average velocity.
- Furthermore, the average velocity $\boldsymbol{v}_{av}(t)$ of the snake robot will converge exponentially fast to the steady state velocity $\bar{\boldsymbol{v}}$ given by

$$\bar{\boldsymbol{v}} = -\mathscr{A}^{-1}\boldsymbol{b} = \begin{bmatrix} \bar{v}_t & \bar{v}_n & \bar{v}_\theta \end{bmatrix}^T$$

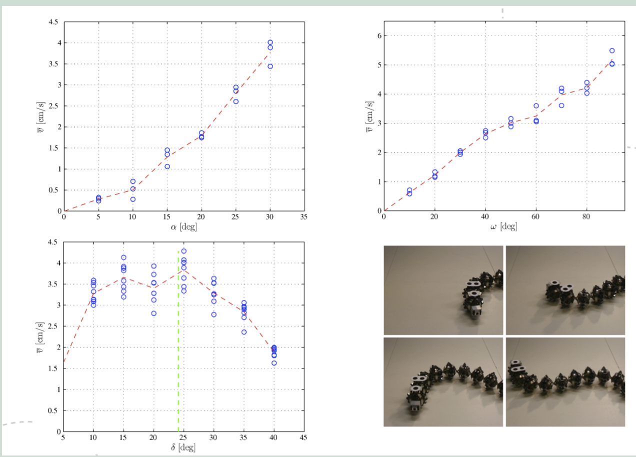
Example: Snake Robot

Properties derived based on averaging theory



Example: Snake Robot

Experimental Results



Next lecture

Next lecture: Passivity-based control

Khalil	**Chapter 6**
	Sections 6.4 and 6.5
	(Pages 254-259, including Ex. 6.12, is additional material)
	Chapter 14
	Section 14.4
Lozano et al.	**Dissipative Systems Analysis and Control**
	Section 2.3-2.4