



NTNU – Trondheim
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TTT4120 Digital Signal Processing Fall 2016

Design of Digital Filters: IIR

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Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
 - 10.3.3 IIR filter design by the bilinear transformation
 - 10.3.4 Characteristics of commonly used analog filters
- A compressed overview of topics treated in the lecture, see “Design av digitale filtre” on Blackboard

*Level of detail is defined by lectures and problem sets

Contents and learning outcomes

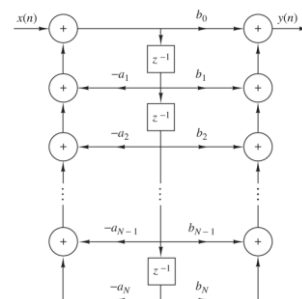
- IIR filter
- Bilinear transformation
- Examples

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IIR filters

- Moving and recursive averages
- Filter has both poles and zeros

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}}$$



- IIR filters designed, $\{a_k\}$ and $\{b_k\}$, by specifying poles and zeros in the z-plane
- In general IIR filters can, for a given filter order, satisfy a tighter specification than FIR filters (lower computational complexity)

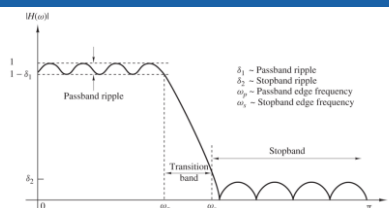
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IIR filters...

- In contrast to FIR filter design, IIR filters are typically designed by utilizing known analog filter design
 - Take an analog design and transform it to the digital domain
 - Nice thing: closed-form solutions exist
 - How to transform the analog solutions to discrete-time?
- Three ways of describing an analog filter
 - System function: $H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^M \beta_k s^k}{\sum_{n=0}^N \alpha_n s^n}$
 - Impulse response: $H_a(s) = \int_{-\infty}^{\infty} h_a(t) e^{-st} dt$
 - Differential equations: $\sum_{k=0}^N \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M \beta_k \frac{d^k x(t)}{dt^k}$
- We will be using the system function

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IIR filters...



- IIR filter design quite different from the FIR design
 - **Step 1:** State filter specs of digital filter, $\{\omega_p, \omega_s, \delta_1, \delta_2\}$
 - **Step 2:** Map the specs to analog domain, $\omega_p \rightarrow \Omega_p, \omega_s \rightarrow \Omega_s$
 - **Step 3:** Design an analog filter (for resistors, capacitors, and inductors) using the Laplace transform $H(s)$
 - **Step 4:** The design in digital domain is obtained using mapping $s = f(z)$, or $H(z) = H(s)|_{s=f(z)}$

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Transformation between s- and z-planes

- We need to transform an analog design into a digital design
 - How to go from the s-plane to the z-plane?

$$H(z) = H_a(s)|_{s=f(z)}$$

- Demands on the mapping?
 - Stable analog filters need to be mapped to stable digital filters

$$\operatorname{Re}\{s\} < 0 \Rightarrow |z| < 1$$

- Imaginary axis in s-plane mapped to unit circle in z-plane

$$\operatorname{Re}\{s\} = 0 \Rightarrow |z| = 1 \Leftrightarrow j\Omega \rightarrow e^{j\omega}$$

- The bilinear transformation satisfies these conditions
- Alternative method is to sample analog impulse response

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Impulse invariance method

- Sample analog impulse response

$$h[n] = h_a(t)|_{t=nT} \xleftrightarrow{\mathcal{F}} H(\Omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(\Omega - \frac{2\pi k}{T}\right)$$

- Frequency mapping: $\omega = \Omega T$
 - Simple and linear
 - Suffers from potential aliasing \Rightarrow not useful for highpass
- Transfer function:

$$H_a(s) = \sum_{k=1}^N \frac{c_k}{s-p_k} \rightarrow H(z) = \sum_{k=1}^N \frac{c_k}{1-e^{p_k T} z^{-1}}$$

- No explicit mapping $s = f(z)$, but mapping of poles

$$s_k = p_k \rightarrow z_k = e^{p_k T}$$

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Impulse invariance method...

- Procedure for transforming $H_a(s)$ to $H(z)$:

1. Find poles of $H_a(s)$, p_k
2. Express $H_a(s) = \sum_{k=1}^N \frac{c_k}{s-p_k}$
3. Finally, $H(z) = \sum_{k=1}^N \frac{c_k}{1-e^{p_k T} z^{-1}}$

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Bilinear transformation...

- The bilinear transform, a conformal mapping, provides an explicit mapping between s-plane and z-plane

$$s = \frac{2}{T} \frac{z-1}{z+1}, \text{ or } z = \frac{\frac{2}{T} + s}{\frac{2}{T} - s}$$

- Setting $s = \sigma + j\Omega$ and $z = e^{j\omega}$, we get the frequency mapping

$$\omega = 2 \arctan \frac{\Omega T}{2} \text{ or } \Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

- The discrete-time filter's system function is given by

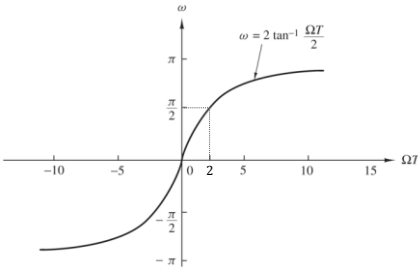
$$H(z) = H_a(s) \Big|_{s=\frac{2z-1}{Tz+1}}$$

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Bilinear transformation...

- Example: Fill in the table using $z = \frac{\frac{2}{T}+s}{\frac{2}{T}-s}$ and $\omega = 2 \arctan \frac{\Omega T}{2}$

s	z	ω
0		
∞		
$\frac{2j}{T}$		
$-\frac{2}{T}$		

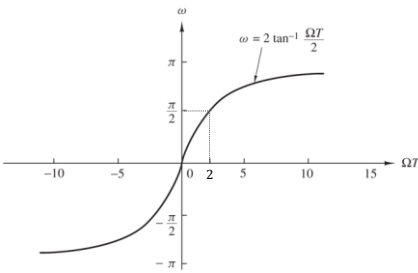


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Bilinear transformation...

- Example: Fill in the table using $z = \frac{\frac{2}{T}+s}{\frac{2}{T}-s}$ and $\omega = 2 \arctan \frac{\Omega T}{2}$

s	z	ω
0	1	0
∞	-1	π
$\frac{2j}{T}$	j	$\pi/2$
$-\frac{2}{T}$	0	N/A



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Bilinear transformation...

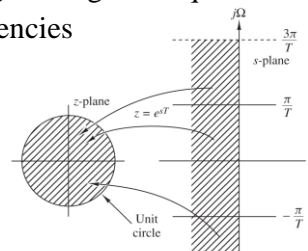
- Substitute $s = \sigma + j\Omega$ in $|z| = \left| \frac{\frac{2}{T} + s}{\frac{2}{T} - s} \right|$, look at $\sigma < 0, \sigma = 0, \sigma > 0$
 - For $\sigma < 0 \Rightarrow |z| = \left| \frac{\frac{2}{T} + \sigma + j\Omega}{\frac{2}{T} - \sigma - j\Omega} \right| < 1$
 - For $\sigma = 0 \Rightarrow |z| = \left| \frac{\frac{2}{T} + j\Omega}{\frac{2}{T} - j\Omega} \right| = 1$
 - For $\sigma > 0 \Rightarrow |z| = \left| \frac{\frac{2}{T} + \sigma + j\Omega}{\frac{2}{T} - \sigma - j\Omega} \right| > 1$
- Entire left half-plane maps into the inside of unit circle
- Imaginary axis maps onto the unit circle

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Bilinear transformation...

- The mapping satisfies the conditions for stability and mapping of the $j\Omega$ -axis to the unit circle
- Reversible mapping of frequency axis, i.e.,

$$\Omega \in (-\infty, \infty) \leftrightarrow \omega \in (-\pi, \pi]$$
- Nonlinear relation between analog and digital frequencies
 - Need to **pre-warp** digital frequencies
- Magnitude levels **unaffected**
- No aliasing
 - Can design all filter types

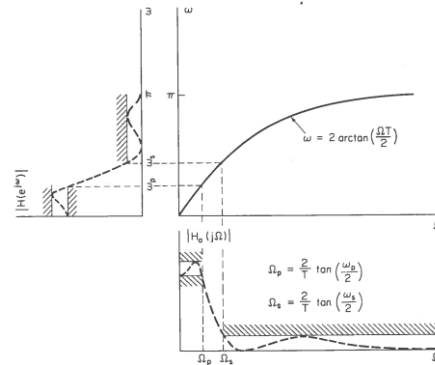


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Bilinear transformation...

- Transformation of $H_a(s)$ to $H(z)$:

$$H(z) = H_a(s) \Big|_{s = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}}$$



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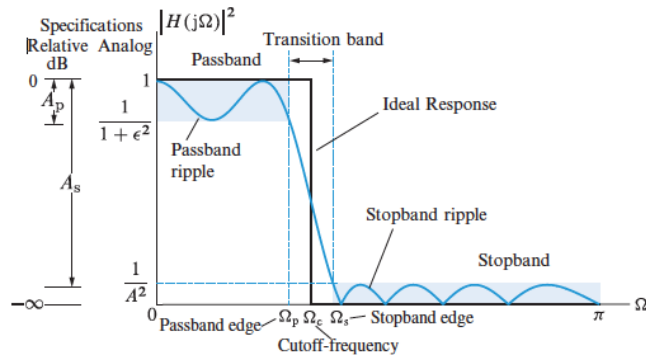
Bilinear transformation...

- Example: Transform $H_a(s) = \frac{s+1}{s^2+5s+6}$ into a digital filter using the bilinear transformation. You may choose $T = 1$

$$\begin{aligned} H(z) &= H_a(s) \Big|_{s = \frac{2}{1+z^{-1}}} = \frac{2 \frac{1-z^{-1}}{1+z^{-1}} + 1}{\left(2 \frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 5 \left(2 \frac{1-z^{-1}}{1+z^{-1}}\right) + 6} \\ &= \frac{2 \frac{1-z^{-1}}{1+z^{-1}} + 1}{\left(2 \frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 5 \left(2 \frac{1-z^{-1}}{1+z^{-1}}\right) + 6} = \frac{3+2z^{-1}-z^{-2}}{20+4z^{-1}} \end{aligned}$$

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Analog filter specifications



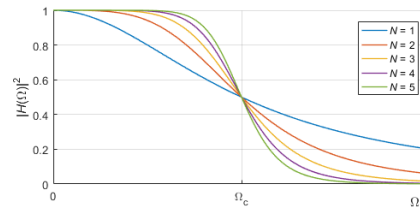
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Three classes of IIR filters

- Butterworth filters
 - In Matlab: `butter`
 - No ripples (oscillations) in $|H(\omega)|$, **maximally flat**
 - Smoothest transition from passband to stopband
- Chebyshev filters (two types)
 - `cheby1` and `cheby2` commands in Matlab
 - Ripples in either passband or stopband
- Elliptic filters
 - `ellip` in Matlab
 - Ripples in both passband and stopband
 - Sharpest transition from passband to stopband for a given order

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Butterworth filter



- Frequency response:

$$|H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} = \frac{1}{1+\epsilon^2(\Omega/\Omega_p)^{2N}}$$

- N poles lying at a circle with radius Ω_c in the s -plane
- Notice: $|H(0)|^2 = 1$, $|H(\Omega_c)|^2 = 0.5$ for all N
 $|H(\Omega)|^2$ monotonically decreasing
- Choose filter order depending on flatness of passband and how rapid decay in stopband

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Butterworth filter...

- How to find $H(s)$:

$$|H(\Omega)|^2 = H(\Omega)H^*(\Omega) = H(s)H(-s)|_{s=j\Omega} = \frac{1}{1+(-s^2/\Omega_c^2)^N} \Big|_{s=j\Omega}$$

- Poles can be found from

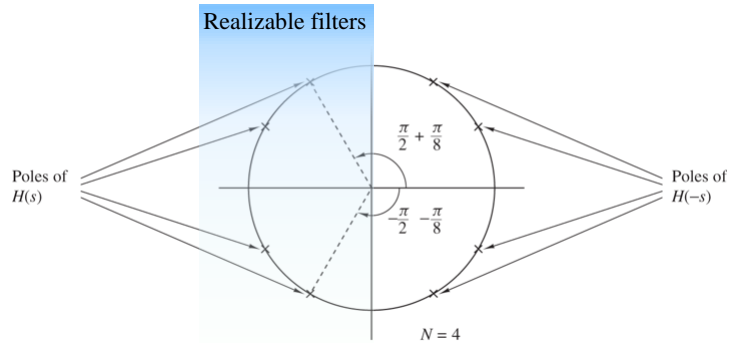
$$1 + (-p_k^2/\Omega_c^2)^N = 0 \Rightarrow p_k = \Omega_c e^{\frac{j\pi}{2}} e^{\frac{j(2k+1)\pi}{2N}}, k = 0, \dots, 2N-1$$

- Poles in $H(s)$: p_k in the left half-plane $k = 0, \dots, N-1$

$$H(s) = \frac{1}{(s-p_0)(s-p_1)\dots(s-p_{N-1})}$$

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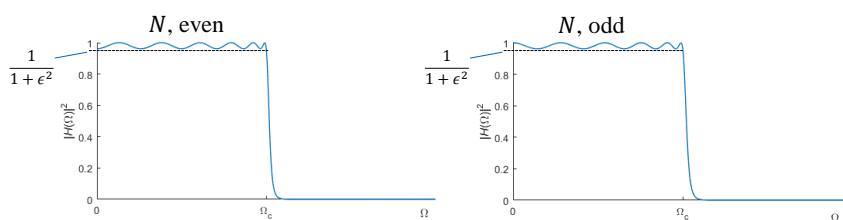
Butterworth filter...



$$p_k = \Omega_c e^{\frac{j\pi}{2}} e^{\frac{j(2k+1)\pi}{2N}}, k = 0, \dots, N-1$$

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Chebyshev I



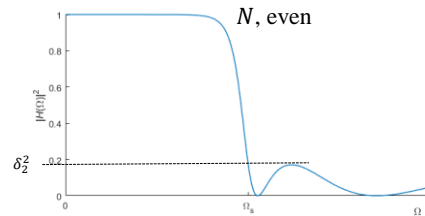
- Frequency response:

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_c)}, (T_N(x) \text{ } N\text{th-order Chebyshev pol.})$$

- Parameter ϵ decides ripple in passband
- Poles lying on an ellipse in the s -plane

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Chebyshev II



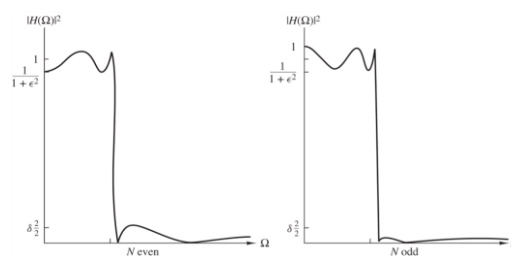
- Frequency response:

$$|H(\Omega)|^2 = \frac{1}{1 + \epsilon^2 [T_N^2(\Omega_s/\Omega_p) / T_N^2(\Omega_s/\Omega)]}$$

- Parameter ϵ decides ripple in **stopband**
- Poles on an ellipse in s -plane
- Zeros on the imaginary axis ($j\Omega$ -axis)

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Elliptic filter



- Frequency response:

$$|H(\Omega)|^2 = \frac{1}{1 + \epsilon^2 U_N^2(\Omega/\Omega_c)} \quad (U_N(x) \text{ Nth-order Jacobi elliptic function})$$

- Parameter ϵ decides ripple in **passband**
- Sharpest transition from passband to stopband among discussed filters
- Zeros on the imaginary axis ($j\Omega$ -axis)

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Example: Bandpass filter

- Design three digital IIR bandpass filters with resonance frequencies

$$\omega_{r_1} = \frac{\pi}{4}, \omega_{r_2} = \frac{\pi}{2}, \text{ and } \omega_{r_3} = \frac{3\pi}{4}$$

by converting the analog filter with system function

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+9}$$

using the the bilinear transformation

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Example: Bandpass filter...

- Poles of analog filter

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+9} = \frac{s+0.1}{(s-[-0.1+j3])(s-[-0.1-j3])} = \frac{s+0.1}{(s-p_r)(s-p_r^*)}$$

reveals analog resonance frequency $\Omega_r = 3 \text{ rad/s}$

- Use frequency relation between Ω and ω_{r_i} to obtain T_i

$$\Omega = \frac{2}{T_i} \tan \frac{\omega_i}{2} \Rightarrow T_i = \frac{2}{\Omega} \tan \frac{\omega_i}{2}$$

$$T_1 = \frac{2}{3} \tan \frac{\pi}{8}, T_2 = \frac{2}{3}, \text{ and } T_3 = \frac{2}{3} \tan \frac{\pi}{8}$$

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Example: Bandpass filter...

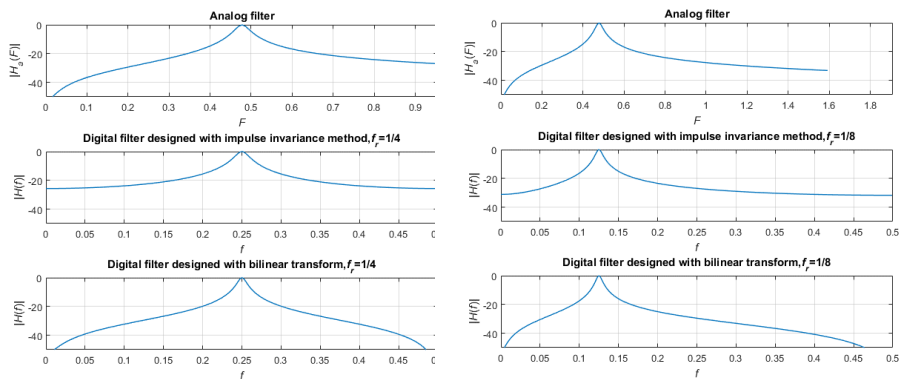
- For each T_i apply the bilinear transform:

$$H_i(z) = H_a(s) \Big|_{s=\frac{2}{T_i} \frac{1-z^{-1}}{1+z^{-1}}} = \frac{\left(\frac{2}{T_i} \frac{1-z^{-1}}{1+z^{-1}}\right) + 0.1}{\left(\frac{2}{T_i} \frac{1-z^{-1}}{1+z^{-1}} + 0.1\right)^2 + 9}$$

$$= \frac{(2T_i + 0.1T_i^2) + 0.2T_i^2 z^{-1} + (0.1T_i^2 - 2T_i)}{(4 + 0.4T_i + 9.01T_i^2) + (18.02T_i^2)z^{-1} + (4 - 0.4T_i + 9.01T_i^2)z^{-1}}$$

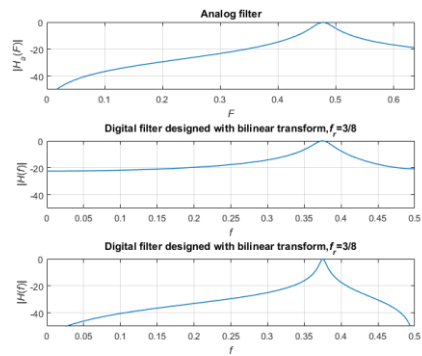
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Example: Bandpass filter...



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Example: Bandpass filter...



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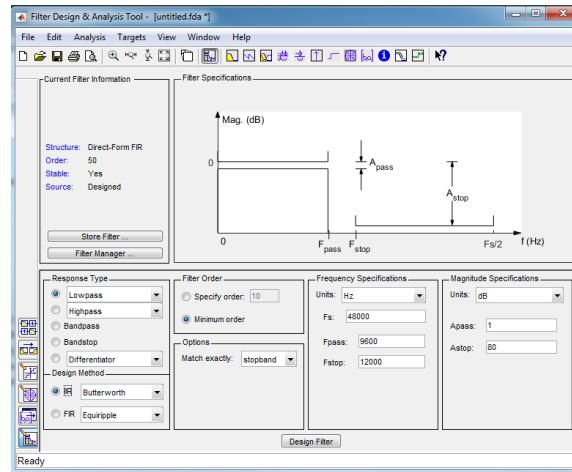
Summary

- Today we discussed:
 - IIR filter design
- Next:
 - Wiener filters

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Matlab: fdatool

- Type `fdatool` at Matlab command prompt:



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