TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2016 - Assignment 3
Due date: Tuesday 11 October at 16.00.

1. Consider again the Duckmaze system from Assignment 1 and 2.

   (a) Use the transformed system from Assignment 2 (Exercise 1b) and the Lyapunov function candidate
   $$V = \frac{1}{2}\left(\tilde{x}_1^2 + m\tilde{x}_2^2\right)$$
   to derive a controller (find $\tilde{u}$) such that
   $$\dot{V} = -(d + k_2)\tilde{x}_2^2$$
   where k_2 is the controller gain.
   (Hint: The resulting closed-loop system should be linear)

   (b) Is the closed-loop system locally/globally asymptotically/exponentially stable at the origin? Investigate all four possibilities and motivate your answers.

   (c) What happens to the system dynamics as k_2 increases? Explain this physically.

   (d) By using the controller in part (a), is it possible to place the poles of the system arbitrarily?

2. For a real matrix Λ we denote $\Lambda \geq 0$ when we mean that the matrix Λ is positive semidefinite and $\Lambda \leq 0$ when it is negative semidefinite. For a real symmetric positive definite matrix P we denote $\lambda_{\min}$ and $\lambda_{\max}$ as its smallest and largest eigenvalue, respectively. Show that the following inequalities
   $$\lambda_{\min} I \leq P \leq \lambda_{\max} I$$
   hold for
   $$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$
   Furthermore show that
   $$\lambda_{\min} \|x\|_2^2 \leq x^T P x \leq \lambda_{\max} \|x\|_2^2$$
   for all x.

3. In checking radial unboundedness of a positive definite function $V(x)$, it may appear that it is sufficient to examine $V(x)$ as $||x|| \to \infty$ along the principal axes. This is not true, as shown in by the function
   $$V(x) = \frac{(x_1 + x_2)^2}{1 + (x_1 + x_2)^2} + (x_1 - x_2)^2$$

   (a) Show that $V(x) \to \infty$ as $||x|| \to \infty$ along the lines $x_1 = 0$ or $x_2 = 0$.

   (b) Show that $V(x)$ is not radially unbounded.

4. Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h_1(x_1) - x_2 - h_2(x_3), \qquad \dot{x}_3 = x_2 - x_3$$

where h_1 and h_2 are locally Lipschitz functions that satisfy $h_i(0) = 0$ and $yh_i(y) > 0$ for all $y \neq 0$. (Hint: $\frac{d}{dz}\int_0^z \psi(u)\,du = \psi(z)$).

   (a) Show that the system has a unique equilibrium point at the origin
   (b) Show that $V(x) = \int_0^{x_1} h_1(y)dy + x_2^2/2 + \int_0^{x_3} h_2(y)dy$ is positive definite for all $x \in R^3$
   (c) Show that the origin is asymptotically stable.
   (d) Under what conditions on h_1 and h_2, can you show that the origin is globally asymptotically stable?

5. Consider

$$\begin{aligned}\dot{x}_1 &= -(x_1 + 2x_2)(x_1 + 2)\\ \dot{x}_2 &= -8x_2(2 + 2x_1 + x_2)\end{aligned}$$

   (a) Using the indirect method (Theorem 4.7 of Khalil), show that the origin is asymptotically stable.
   (b) Using the direct method (Theorem 4.1 of Khalil), show that the origin is asymptotically stable.
   (Hint: use $\mathcal{D} = \{x \in R^2 | x_1 + 2x_2 + 1 \geq 0 \text{ and } 2x_1 + x_2 + 1 \geq 0\}$ and the Lyapunov function candidate $V = x_1^2 + x_2^2$)
   (c) Let $\Omega_c \triangleq \{x \in R^2 | V(x) \leq c\}$. Draw $\mathcal{D}$, $\Omega_{\frac{1}{9}}$ and $\Omega_{6.25}$ together on the plane (you may use `pplane` and select 'Plot level curves' from the 'Solutions' menu). Explain why the trajectory converges to the origin when $x(0) = \left(0, \frac{1}{3}\right)$? Explain also why the trajectory does not converge to the origin when $x(0) = \left(-\frac{4}{3}, 2\right)$ eventhough $x(0)$ belongs to D.

6. Let α be a class $\mathcal{K}$ function on $[0, a)$. Show that

$$\alpha(r_1 + r_2) \leq \alpha(2r_1) + \alpha(2r_2), \quad \forall\, r_1, r_2 \in [0, a/2)$$

7. Suppose that for each initial condition $x(0)$ the solution of $\dot{x} = f(x)$ satisfies

$$\|x(t)\| \leq \beta(\|x(0)\|, t)$$

for $t \geq 0$ where β is of class $\mathcal{KL}$.
Show that the origin of the system is globally asymptotically stable, i.e.

   (a) Show stability for $x = 0$ using the definition of stability and the definition of class-$\mathcal{KL}$ functions.
   (b) Show that every trajectory of the system converges to the origin.

8. Consider the system

$$\begin{aligned}\dot{x}_1 &= -\phi(t)\,x_1 + a\phi(t)\,x_2\\ \dot{x}_2 &= b\phi(t)\,x_1 - ab\phi(t)\,x_2 - c\psi(t)\,x_2^3\end{aligned}$$

where a, b and c are positive constants and $\phi(t)$ and $\psi(t)$ are nonnegative, continuous, bounded functions that satisfy

$$\phi(t) \geq \phi_0 > 0, \quad \psi(t) \geq \psi_0 > 0, \quad \forall t \geq 0$$

Show that the origin is globally uniformly asymptotically stable.
(Hint: $V = 0.5\,(bx_1^2 + ax_2^2)$)

9. An RCL circuit with time-varying elements is represented by

$$\dot{x}_1 = \frac{1}{L(t)} x_2, \qquad \dot{x}_2 = -\frac{1}{C(t)} x_1 - \frac{R(t)}{L(t)} x_2$$

Suppose that $L(t)$, $C(t)$, and $R(t)$ are continuously differentiable and satisfy the inequalities $k_1 \leq L(t) \leq k_2$, $k_3 \leq C(t) \leq k_4$, and $k_5 \leq R(t) \leq k_6$ for all $t \geq 0$, where k_1, k_3, and k_5 are positive. Consider a Lyapunov function candidate

$$V(t,x) = \left[R(t) + \frac{2L(t)}{R(t)C(t)}\right] x_1^2 + 2x_1x_2 + \frac{2}{R(t)} x_2^2$$

(Hint: use the completion of squares)

(a) Show that $V(t,x)$ is positive definite and decrescent

(b) Find conditions on $\dot{L}(t)$, $\dot{C}(t)$, and $\dot{R}(t)$ that will ensure exponential stability of the origin.

10. Consider the system

$$\dot{x}_1 = h(t)x_2 - g(t)x_1^3, \qquad \dot{x}_2 = -h(t)x_1 - g(t)x_2^3$$

where $h(t)$ and $g(t)$ are bounded, continuously differentiable functions and $g(t) \geq k > 0$, for all $t \geq 0$. (Hint: use $V = 0.5\,(x_1^2 + x_2^2)$)

(a) Is the equilibrium point $x = 0$ uniformly asymptotically stable?

(b) Is it exponentially stable?

(c) Is it globally uniformly asymptotically stable?

(d) Is it globally exponentially stable?

11. Consider the system $\dot{x} = f(x)$ with $f(0) = 0$, where it is assumed that $f(x)$ is continuously differentiable and its Jacobian $A(x) \triangleq [\partial f/\partial x]$. The generalized Krasovskii's theorem then states that a sufficient condition for the origin to be asymptotically stable is that the matrix $F(x) = A^\top P + PA$ is negative semi definite in some neighbourhood D of the origin and $P = P^\top > 0$. In addition, if $D \in R^n$ and $V(x) \triangleq f^\top(x)Pf(x)$ is radially unbounded, then the system is globally asymptotically stable.

Apply Krasovskii's theorem to analyze the stability behaviour of the following system

$$\begin{aligned}\dot{x}_1 &= -6x_1 \\ \dot{x}_2 &= 2x_1 - 6x_2 - 2x_2^3.\end{aligned}$$

12. Let

$$
\begin{aligned}
V_1(x_1,x_2,t) &= x_1^2 + \left(1+e^t\right)x_2^2 \\
V_2(x_1,x_2,t) &= \frac{x_1^2+x_2^2}{1+t} \\
V_3(x_1,x_2,t) &= \left(1+\cos^4 t\right)\left(x_1^2+x_2^2\right)
\end{aligned}
$$

For each of the functions $V_i(x_1,x_2,t)\,,\ i \in \{1,2,3\}$ investigate the properties of positive definite and decrescent.

13. Consider the system

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - c(t)\,x_2
\end{aligned}
$$

where the function $c(t)$ is continuous differentiable and satisfies

$$k_1 \leq c(t) \leq k_2 \text{ and } |\dot{c}(t)| \leq k_3\ \forall t \geq 0$$

and k_i are constants and $k_1 > 0$. Use the Lyapunov function candidate

$$V(x) = \frac{1}{2}\left(x_1^2+x_2^2\right)$$

to show that the origin is uniformly stable and that $x_2 \to 0$ as $t \to \infty$.

14. **Optional exercise:** Consider the system $\dot{x} = f(x)$ with $f(0)=0$. Assume that $f(x)$ is continuously differentiable and its Jacobian $[\partial f/\partial x]$ satisfies

$$P\left[\frac{\partial f}{\partial x}(x)\right] + \left[\frac{\partial f}{\partial x}(x)\right]^T P \leq -I, \quad \forall\, x \in R^n, \quad \text{where } P = P^T > 0$$

(a) Using the representation $f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)x\ d\sigma$, show that

$$x^T P f(x) + f^T(x) P x \leq -x^T x, \quad \forall\, x \in R^n$$

(b) Show that $V(x) = f^T(x)Pf(x)$ is positive definete for all $x \in R^n$ and radially unbounded.

(c) Show that the origin is globally asymptotically stable.