TTK4150 Nonlinear Control Systems
Lecture 10

Passivity

NTNU
Innovation and Creativity

Previous lecture

Previous lecture:

- Stability of Perturbed Systems

In particular, we learned to

- Analyze the stability properties of a system under the influence of disturbances
- Know the difference between
  - Vanishing perturbations
  - Nonvanishing perturbations
- Learn useful tools in order to study the stability of a stable system $\dot{x} = f(t,x)$ which is perturbed by another vanishing or nonvanishing vector field $g(t,x)$

Today's goals

After today you should...

- Be able to analyze the passivity properties of a system by using the definition of passivity for
  - Memoryless functions
  - Dynamical systems
- Understand the relations between passivity and
  - Lyapunov stability
  - $\mathscr{L}_2$ stability (IOS)
- Know the passivity theorems
  (for feedback connections)

Literature

Today's lecture is based on

Khalil **Chapter 6**
Sections 6.1 and 6.2
(Section 6.3 is additional material)
Sections 6.4 - 6.5, page 254
(Pages 254-259, incl. Ex. 6.12, is additional material)

Passivity

What is passivity?

- A tool (not a stability concept) for design and analysis of control systems
- Based on an Input-Output description of systems
- Has an interesting energy interpretation
  (allows the control engineer to relate a set of efficient mathematical tools to well known physical phenomena)

Main use:

- Relates nicely to
  - Lyapunov stability
  - $\mathscr{L}_2$ stability
- Can provide a somewhat systematic way to build Lyapunov functions
- Can give conclusions about properties of feedback connections (based on the properties of each subsystem)

Passivity for memoryless functions

Memoryless functions

$y = h(t,u) \qquad h : [0,\infty) \times \mathbb{R}^p \to \mathbb{R}^p$

Example: An electric circuit

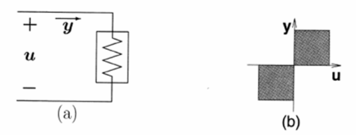


Figure 6.1: (a) A passive resistor; (b) u–y characteristic lies in the first–third quadrant.

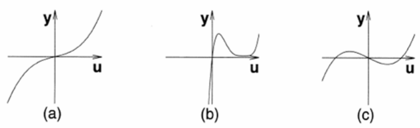


Figure 6.2: (a) and (b) are examples of nonlinear passive resistor characteristics; (c) is an example of a nonpassive resistor.

Passive elements

Passive elements in electric circuits

Passive element: The element cannot generate energy

$$P = u \cdot i = u \cdot y$$

$P > 0$ The element absorbs energy

$P < 0$ The element generates energy

$$\text{Passive} \quad \Leftrightarrow \quad P \geq 0 \quad \Updownarrow \quad u \cdot i \geq 0$$

Generalized definition

The system is passive if

$$u^T y \geq 0$$

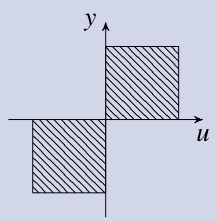
Passivity definition

Definition: Passive/Lossless

The memoryless system $y = h(t, u)$ is

- Passive if $u^T y \geq 0$

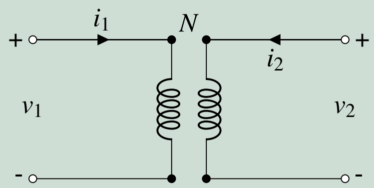
  i.e. $h \in$ sector $[0, \infty]$



- Lossless if $u^T y = 0$

Example

Ideal transformer



$$y = \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} \quad u = \begin{bmatrix} v_1 \\ i_2 \end{bmatrix} \quad y = Su \quad S = \begin{bmatrix} 0 & -N \\ N & 0 \end{bmatrix}$$

Analyse the passivity properties of the ideal transformer

Input strictly passive

Definition: Input strictly passive

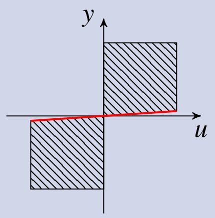
The memoryless system $y = h(t,u)$ is <u>input strictly passive</u> iff

$$u^T y \geq u^T \varphi(u)$$

and

$$u^T \varphi(u) > 0 \quad \forall\, u \neq 0$$

i.e. $h \in$ sector $(0, \infty]$



Output strictly passive

Output strictly passive

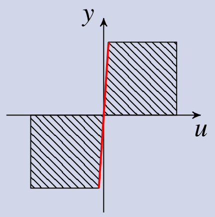
The memoryless system $y = h(t,u)$ is <u>output strictly passive</u> iff

$$u^T y \geq y^T \rho(y)$$

and

$$y^T \rho(y) > 0 \quad \forall\, y \neq 0$$

i.e. $h \in$ sector $[0, \infty)$



Passivity for dynamical systems

Dynamical systems

We consider dynamical systems

$$\Sigma \qquad \begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

$f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ locally Lipschitz
$h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ continuous

$f(0,0) = 0$ and $h(0,0) = 0$

Motivating example

Motivating example: Electric circuit

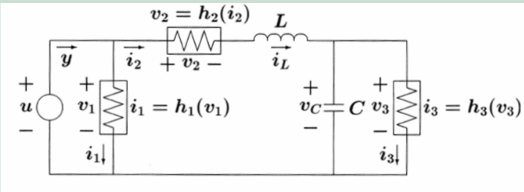


Figure 6.8: RLC circuit of Example 6.1.

$$x = \begin{bmatrix} i_L \\ v_c \end{bmatrix}$$

Kirchoff's laws give

$$\begin{aligned} L\dot{x}_1 &= u - h_2(x_1) - x_2 \\ C\dot{x}_2 &= x_1 - h_3(x_2) \\ y &= x_1 + h_1(u) \end{aligned}$$

Passive circuit

Passive electric circuits

Passive circuits cannot generate electric energy i.e.

change of stored energy $\leq$ energy supplied

$$V(x(t)) - V(x(0)) \leq \int_0^t u(s)y(s)ds$$

Generalized definition

The system is passive iff

$$u(t)y(t) \geq \dot{V}(x(t), u(t)) \quad \forall\, t \geq 0$$

Passivity definitions for dynamical systems

Definition

The dynamical system is

- passive if
  $\exists C^1$ positive semidefinite function $V(x) : \mathbb{R}^n \to \mathbb{R}$
  (Storage function)
  such that

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x,u) \qquad \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^p$$

Moreover, it is

- lossless if

$$u^T y = \dot{V}$$

Example

Example: Electric circuit

Given the electric circuit

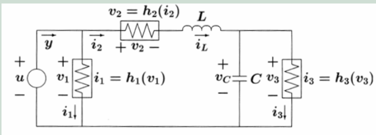


Figure 6.8: RLC circuit of Example 6.1.

The energy stored in the RLC network is

$$V(x) = \frac{1}{2} L x_1^2 + \frac{1}{2} C x_2^2$$

Choose input = input voltage u
output = current $y = x_1 + h_1(u)$

Analyse the passivity properties of the RLC network

Input strictly passive

Definition continued

- Input strictly passive if

$$u^T y \geq \dot{V} + u^T \varphi(u), \quad u^T \varphi(u) > 0 \quad \forall u \neq 0$$

- Output strictly passive if

$$u^T y \geq \dot{V} + y^T \rho(y), \quad y^T \rho(y) > 0 \quad \forall y \neq 0$$

- (State) Strictly passive if

$$u^T y \geq \dot{V} + \psi(x), \quad \psi(x) \quad \text{positive definite function}$$

Relations between Passivity properties and (Lyapunov) stability

Dynamical systems

$$\Sigma \qquad \begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

$f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ locally Lipschitz
$h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ continuous

$f(0,0) = 0$ and $h(0,0) = 0$

Lemma 6.6 (Lyapunov stable (0-stable))

If Σ is passive with a *positive definite* storage function $V(x)$, then

the origin of $\dot{x} = f(x,0)$ is stable

Relations between Passivity properties and $\mathscr{L}_2$ stability

Lemma 6.5 (Finite-gain $\mathscr{L}_2$ stable)

If Σ is output strictly passive with $\rho(y) = \delta y$, $\delta > 0$ then

Σ is finite-gain $\mathscr{L}_2$ stable
with $\mathscr{L}_2$-gain $\gamma \leq \frac{1}{\delta}$

Asymptotic stability of passive systems

Definition: Zero-state observability

Σ is zero-state observable iff
no solution of $\dot{x} = f(x,0)$ can stay identically in
$S = \{x \in \mathbb{R}^n | h(x,0) = 0\}$ other than the trivial solution $x(t) = 0$.

Lemma 6.7 (Asymptotically stable (0-AS))

The origin of $\dot{x} = f(x,0)$ is asymptotically stable if Σ is either

- state strictly passive

or

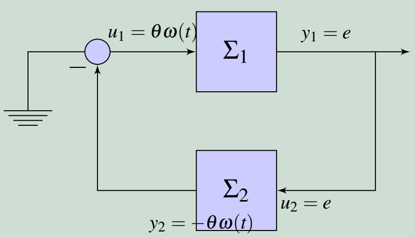
- output strictly passive
  zero-state observable

If furthermore $V(x)$ is radially unbounded, then the origin is globally asymptotically stable

Example: Adaptive control system

Adaptive control system

$$\dot{e} = -e + \theta\omega(t)$$
$$\dot{\theta} = -e\omega(t)$$



Subsystem Σ_1

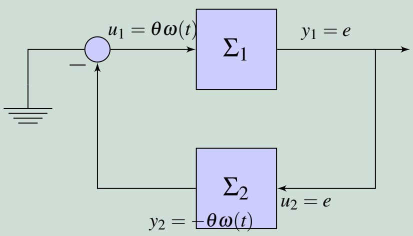
$$\dot{x}_1 = -x_1 + u_1$$
$$y_1 = ?$$

- Investigate the passivity properties of subsystem Σ_1
- What can thus be concluded about the stability properties of subsystem Σ_1

Example: Adaptive control system cont.

Adaptive control system, cont.

$$\dot{e} = -e + \theta\omega(t)$$
$$\dot{\theta} = -e\omega(t)$$



Subsystem Σ_2

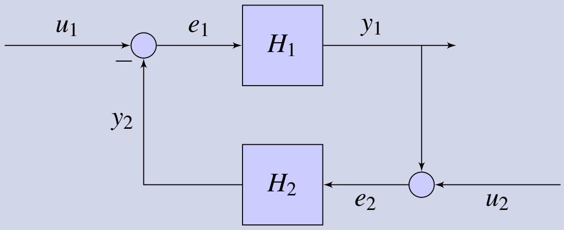
$$\dot{x}_2 = -u_2\omega(t)$$
$$y_2 = ?$$

- Investigate the passivity properties of subsystem Σ_2
- What can thus be concluded about the stability properties of subsystem Σ_2

Passivity theorems
Feedback systems

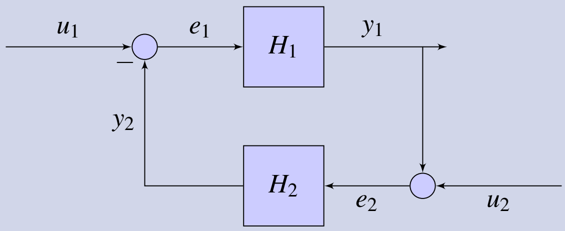
Feedback systems

Σ



$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Passivity theorems
Lyapunov stability of feedback connection

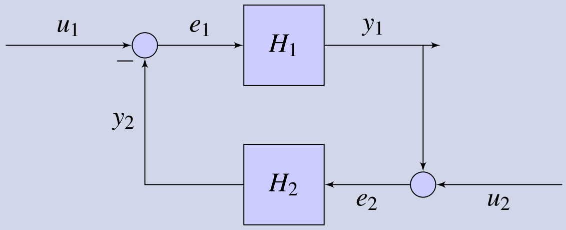


Theorem 6.1: Lyapunov stability of feedback connection

H_1 passive and H_2 passive $\Rightarrow \Sigma$ passive (with $V = V_1 + V_2$)

Passivity theorems

$\mathscr{L}_2$-stability of feedback connection



Theorem 6.2: $\mathscr{L}_2$-stability of feedback connection

If H_1 and H_2 satisfy

$$e_i^T y_i \geq \dot{V}_i + \varepsilon_i e_i^T e_i + \delta_i y_i^T y_i$$

and

$$\varepsilon_1 + \delta_2 > 0 \quad \text{and} \quad \varepsilon_2 + \delta_1 > 0$$

then Σ is finite-gain $\mathscr{L}_2$-stable.

Passivity theorems

Asymptotic stability of feedback connection

Theorem 6.3: Asymptotic stability of feedback connection

If

- H_1 and H_2 state strictly passive

or

- H_1 and H_2 output strictly passive and zero-state observable

or

- H_1 state strictly passive
  H_2 output strictly passive and zero-state observable
  or opposite

then Σ is 0-AS

If furthermore V_1 and V_2 are radially unbounded then Σ is 0-GAS.

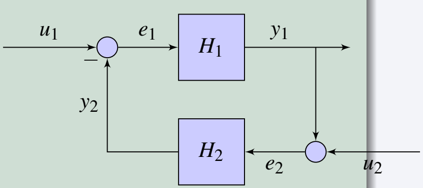
Example: DC Motor control system

Example: DC Motor control system (Boyd, 1997)

A DC motor is characterized by

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -\omega + u$$

where θ is the shaft angle and u is the input voltage.



The dynamic controller

$$\dot{z} = 2(\theta - z) - \mathrm{sat}(\theta - z)$$
$$u = z - 2\theta$$

is used to control the shaft position. Use passivity analysis to prove that $\theta(t)$ and $\omega(t)$ converge to zero as $t \to \infty$

Hint

Using the state transformation $x = z - \theta$ the dynamic controller can be rewritten as

$$\dot{x} = -2x + \mathrm{sat}(x) - \omega$$
$$u = x - \theta$$

Next lecture

Next lecture: Perturbation Theory and Averaging

Khalil **Chapter 10**
Sections 10.3 and 10.4