TTK4150 Nonlinear Control Systems
Lecture 12

Feedback Linearization

NTNU
Innovation and Creativity

Previous lectures on nonlinear control design

Previous lectures on nonlinear control design:

- Lyapunov based control design
- Cascaded control: Lemma 4.7 allows for modular design
  (And background material, Sontag and Loria)
- Passivity-based control design

Outline I

Outline II

- Summary: Input-output linearization for stabilization
- Summary: Input-output linearization for tracking control

- Application example
- Advantages/shortcomings

Today's goals

After today you should...

- Know the concepts of relative degree, normal form, external dynamics, internal dynamics and zero dynamics.
- Be able to design a stabilizing control law using the input-output linearization method, including
  1) Finding the relative degree
  2) Writing the system in normal form
  3) Creating a linear input-output relation by feedback control
  4) Analyzing the zero dynamics
  5) Choosing the transformed input variable v to stabilize the origin of the system, locally or globally
- Be able to design a control law that solves the local tracking control problem, using the input-output linearization method
- Be able to discuss the advantages and the disadvantages of the input-output linearization method

Literature

Today's lecture is based on

Khalil **Chapter 13**
Sections 13.1,13.2 and 13.4
Example 13.16 - page 538 is additional material

Motivation

A number of methods exist for control design for linear systems.

It would therefore be nice if we could obtain a linear system instead of the nonlinear system we are dealing with.

Jacobian linearization: An approximation

Question

Is it possible to algebraically transform a nonlinear system dynamics into a (fully or partly) linear one?

Input-state linearization (Full-state linearization)

Given a nonlinear system

$$\boxed{\dot{x} = f(x) + G(x)u}$$

where $f(0) = 0$, and $f : \mathbb{D} \to \mathbb{R}^n$ and $G : \mathbb{D} \to \mathbb{R}^{n \times p}$ are sufficiently smooth on a domain $\mathbb{D} \subset \mathbb{R}^n$.

Find a state transformation

$$z = T(x)$$

and an input transformation (a feedback control)

$$u = \alpha(x) + \beta(x)v$$

such that the corresponding closed-loop system

$$\dot{x} = f(x) + G(x)\alpha(x) + G(x)\beta(x)v$$

written in the coordinates $z = T(x)$ is linear and controllable

Input-state linearization (cont.)

i.e.

$$\boxed{\dot{z} = \frac{d}{dt}T(x) = \frac{\partial T}{\partial x}\dot{x} = Az + Bv}$$

where

$$\left[\frac{\partial T}{\partial x}(f(x) + G(x)\alpha(x))\right]_{x=T^{-1}(z)} = Az$$

$$\left[\frac{\partial T}{\partial x}G(x)\beta(x)\right]_{x=T^{-1}(z)} = B$$

and

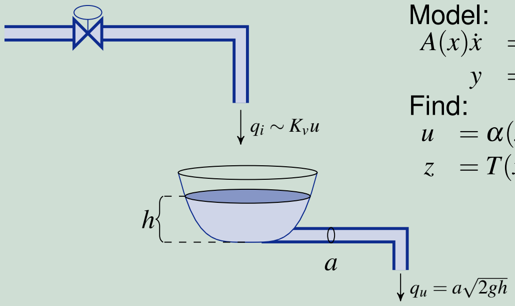
$$\text{rank}[B \quad AB \quad \cdots \quad A^{n-1}B] = n$$

Example: Pendulum equation

See pages 505 - 507

Example
Fluid level control

Fluid level control



Model:

$$\begin{aligned} A(x)\dot{x} &= -a\sqrt{2gx} + K_v u \\ y &= x \end{aligned}$$

Find:

$$\begin{aligned} u &= \alpha(x) + \beta(x)v \\ z &= T(x) \end{aligned}$$

Find a state transformation $z = T(x)$, an input transformation $u = \alpha(x) + \beta(x)v$, and a control law $v(z)$ which

a) asymptotically stabilizes $z_d = h_d = \text{const.}$
b) asymptotically tracks $z_d(t) = h_d(t)$

Input-output linearization
The system

Given a nonlinear system

$$\boxed{\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}}$$

where $f : \mathbb{D} \to \mathbb{R}^n$, $g : \mathbb{D} \to \mathbb{R}^n$ and $h : \mathbb{D} \to \mathbb{R}$ are sufficiently smooth on a domain $\mathbb{D} \subset \mathbb{R}^n$.

Note: $\dim u = \dim y = 1$

The method
Notation

Lie derivative of h

$$L_f h = \frac{\partial h}{\partial x} f$$

$$L_f^2 h = L_f(L_f h) = \frac{\partial L_f h}{\partial x} f$$

$$\vdots$$

$$L_f^0 h = h$$

$$L_f^i h = L_f(L_f^{i-1} h), \quad i = 1, 2, \ldots$$

The method
Step 1 - Find the relative degree

Given the system

$$\boxed{\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}}$$

where $f : \mathbb{D} \to \mathbb{R}^n$, $g : \mathbb{D} \to \mathbb{R}^n$ and $h : \mathbb{D} \to \mathbb{R}$ are sufficiently smooth on a domain $\mathbb{D} \subset \mathbb{R}^n$.

Example:

$$\begin{aligned}\dot{x}_1 &= -x_1 + 2u \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -x_3 + x_2 x_3 + u \\ y &= x_2\end{aligned}$$

1) Differentiate y until u appears

$$\begin{aligned}\dot{y} &= \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x} f + \frac{\partial h}{\partial x} g \cdot u \\ &= L_f h + L_g h \cdot u\end{aligned}$$

The method
Step 1 - Find the relative degree

Suppose $L_g h = 0 \quad \forall\, x \in \mathbb{D}_0 \subset \mathbb{D}$

$$\ddot{y} = \frac{\partial (L_f h)}{\partial x}\dot{x} = \underbrace{\frac{\partial (L_f h)}{\partial x} f}_{L_f^2 h} + \underbrace{\frac{\partial (L_f h)}{\partial x} g(x)u}_{L_g L_f h \cdot u}$$

$$\vdots$$

$$y^{(i)} = L_f^{(i)} h + L_g L_f^{(i-1)} h \cdot u$$

Relative degree

The system has relative degree ρ in a region $\mathbb{D}_0 \subset \mathbb{D} \subset \mathbb{R}^n$ if

$$\left.\begin{aligned} &L_g L_f^{(i-1)} h = 0, \quad 1 \le i \le \rho - 1 \\ &L_g L_f^{(\rho-1)} h \neq 0 \end{aligned}\right\} \forall\, x \in \mathbb{D}_0$$

The method
Step 1 - Find the relative degree

Terminology/Relation to linear systems

Let

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

with $\dim u = \dim y = 1$.
The system transfer function is

$$h(s) = C(sI - A)^{-1}B = K\frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

Then $\rho = n - m$

The method
Step 1 - Find the relative degree

When is input-output linearization possible?

Q When is it possible to perform an input-output linearization?

A If the relative degree is well defined in the region of interest $\mathbb{D}_0$ (Theorem 13.1), then the system can be input-output linearized.

The method
Step 1 - Find the relative degree

Example

Given
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + x_2 + u\end{aligned}$$
find the relative degree ρ at $x_0 = 0$ when

a) $y = x_1$

b) $y = x_2^2$

Example

Given
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= (1+x_1)u\\ \dot{x}_3 &= u\end{aligned}$$
find the relative degree ρ at $x_0 = 0$ when

a) $y = x_1$

The method
Step 2 - Write the system in normal form

Suppose ρ is well defined in $\mathbb{D}_0$

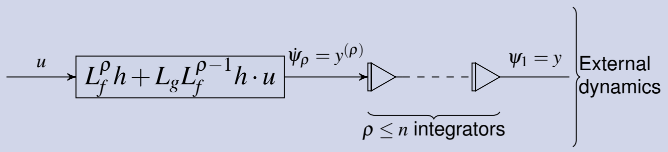
2a) Derive the external dynamics

2a) Let
$$\begin{aligned}\psi_1 &= y\\ \psi_2 &= \dot{y}\\ &\vdots\\ \psi_\rho &= y^{(\rho-1)}\end{aligned}$$

Then
$$\begin{aligned}\dot{\psi}_1 &= \psi_2\\ \dot{\psi}_2 &= \psi_3\\ &\vdots\\ \dot{\psi}_\rho &= L_f^\rho h + L_g L_f^{\rho-1} h \cdot u\end{aligned}$$

The method
Step 2a - Derive the external dynamics

The external dynamics



If $\rho < n$

$\dot\varphi = ?$ Internal dynamics

The method
Step 2b - Derive the internal dynamics

2b) Derive the internal dynamics

2b) Choose $n-\rho$ coordinates $\varphi_1, \ldots, \varphi_{n-\rho}$

$\rightarrow$ Coordinate transformation

$$z = T(x) = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_{n-\rho} \\ \psi_1 \\ \vdots \\ \psi_\rho \end{bmatrix}$$ normal coordinates/states

The method
Step 2b - Derive the internal dynamics

Choose $\varphi_1, \ldots, \varphi_{n-\rho}$ such that

T is a diffeomorphism and $L_g\varphi_1 = 0$, $\vdots$, $L_g\varphi_{n-\rho} = 0$ and $\varphi_i(0) = 0$

When are these conditions possible to fulfill?

Always possible when the relative degree is well-defined $\rho \le n$ (Theorem 13.1)

Useful fact

The Jacobian matrix

$$\left.\frac{\partial T}{\partial x}\right|_{x_0}$$

is nonsingular $\Rightarrow$ T is a diffeomorphism in a neighborhood of x_0

The method
Step 2b - Derive the internal dynamics

$$\begin{aligned}\dot{\varphi}_j &= \frac{\partial \varphi_j}{\partial x}\dot{x} = L_f\varphi_j + \underbrace{L_g\varphi_j}_{=0}\cdot u\\ &= f_{0_j}(\varphi_i,\psi_i)\end{aligned}$$

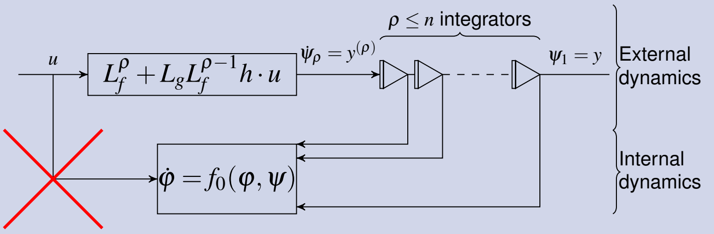
Write the system in normal form:

$$\left.\begin{aligned}\dot{\varphi}_1 &= f_{0_1}(\varphi_i,\psi_i)\\ &\vdots\\ \dot{\varphi}_{n-\rho} &= f_{0_{n-\rho}}(\varphi_i,\psi_i)\end{aligned}\right\}\ \text{Internal dynamics}\ \dot{\varphi} = f_0(\varphi,\psi)$$

$$\left.\begin{aligned}\dot{\psi}_1 &= \psi_2\\ \dot{\psi}_2 &= \psi_3\\ &\vdots\\ \dot{\psi}_\rho &= L_f^\rho h + L_gL_f^{\rho-1}h\cdot u\end{aligned}\right\}\ \text{External dynamics}$$

The method
Step 2 - Write the system in normal form

The system in normal form



The method
Step 3 - Choose u to cancel the nonlinearities

3) Choose u to cancel the nonlinearities

Create a linear input-output relation by feedback control:

Choose u to cancel the nonlinearities
(an input transformation/ a linearizing inner feedback control loop)

$$u = \frac{1}{L_gL_f^{\rho-1}h}(-L_f^\rho h + v)$$

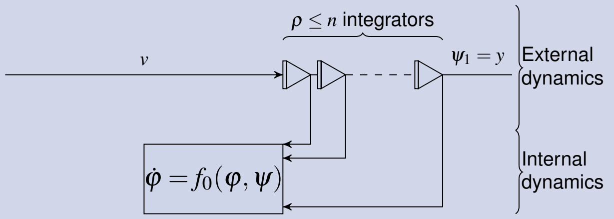
$$\Downarrow$$

$$\dot{\varphi} = f_0(\varphi,\psi)$$

$$\left.\begin{aligned}\dot{\psi}_1 &= \psi_2\\ &\vdots\\ \dot{\psi}_\rho &= v\end{aligned}\right\}\ y^{(\rho)} = v$$

The method
Step 3 - Choose u to cancel the nonlinearities

A linear input-output relationship



The method
Step 4 - Analyze the zero-dynamics

Definition: Zero-dynamics

The zero-dynamics is the internal dynamics when the system output is kept at zero by the input

i.e.

$$y \equiv 0$$
$$\vdots$$
$$y^{(\rho)} \equiv 0 \qquad \text{(i.e.} \quad u_0(x) = -\frac{L_f^{\rho} h(x)}{L_g L_f^{\rho-1} h(x)})$$
$$\Updownarrow$$
$$\psi_i = 0 \quad i = 1, \dots, \rho$$

The zero-dynamics:

$$\boxed{\dot{\varphi} = f_0(\varphi, 0)}$$

The method
Step 4 - Analyze the zero-dynamics

Definition: Minimum phase systems

The system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

is minimum phase if (the origin of) the zero-dynamics is asymptotically stable.

Terminology - Relation to linear systems

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \rightarrow \quad h(s) = K\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

Zero-dynamics:

$$\dot{\varphi} = Q\varphi, \qquad \lambda_j(Q) = z_j, \quad j = 1, \dots, m = n - \rho$$

The method
Step 5 - Choose v to solve the control problem

Asymptotic stabilization

Control objective:

$$x = 0 \text{ asymptotically stable}$$
$$\Updownarrow$$
$$z = \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = 0 \text{ asymptotically stable}$$

We have

$$\begin{aligned} \dot\varphi &= f_0(\varphi, \psi) \\ \dot\psi_1 &= \psi_2 \\ &\vdots \\ \dot\psi_\rho &= v \end{aligned}$$

Special case of

$$\begin{aligned} \dot\varphi &= f_0(\varphi, \psi) \\ \dot\psi &= A\psi + Bv \end{aligned}$$

(A, B) controllable

The method
Step 5 - Choose v to solve the control problem

Recall: We have a special case of

$$\begin{aligned} \dot\varphi &= f_0(\varphi, \psi) \\ \dot\psi &= A\psi + Bv \end{aligned}$$

(A, B) controllable

$v = -K\psi$ $\quad$ K is chosen such that $(A - BK)$ is Hurwitz

$$\Downarrow$$

$$\dot\varphi = f_0(\varphi, \psi)$$

$$\dot\psi = (A - BK)\psi \ \}\text{Exponentially stable}$$

The method
Step 5 - Choose v to solve the control problem

Recall: $\dot\varphi = f_0(\varphi, \psi)$

$\dot\psi = (A - BK)\psi \ \}$ Exponentially stable

Lemma 13.1

If the origin $\varphi = 0$ of $\dot\varphi = f_0(\varphi, 0)$ is asymptotically stable, then the origin $(\varphi, \psi) = (0, 0)$ of

$$\begin{aligned} \dot\varphi &= f_0(\varphi, \psi) \\ \dot\psi &= (A - BK)\psi \end{aligned}$$

is asymptotically stable.

NB

Only *local* asymptotic stability can be concluded

The method
Step 5 - Choose v to solve the control problem

Summary Step 5 - Asymptotic stabilization

Choose

$$v = -K\psi$$

such that $(A - BK)$ is Hurwitz.

Choose for instance

$$\begin{aligned} v &= -k_0\psi_1 - k_1\psi_2 - \cdots - k_{\rho-1}\psi_\rho \\ &= -k_0 y - k_1\dot{y} - \cdots - k_{\rho-1}y^{(\rho-1)} \end{aligned}$$

such that

$$s^\rho + k_{\rho-1}s^{\rho-1} + \cdots + k_1 s + k_0$$

has all its roots strictly in the left-half plane.

The method
Summary: Input-output linearization for stabilization

If the system

$$\boxed{\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}}$$

where $f : \mathbb{D} \to \mathbb{R}^n$, $g : \mathbb{D} \to \mathbb{R}^n$ and $h : \mathbb{D} \to \mathbb{R}$ are sufficiently smooth on a domain $\mathbb{D} \subset \mathbb{R}^n$, has a well-defined relative degree $\rho \in \mathbb{D}_0 \subset \mathbb{D}$, $1 \leq \rho \leq n$ and is minimum phase then the control law

$$\boxed{u = \frac{1}{L_g L_f^{\rho-1} h}(-L_f^\rho h + v)}$$

makes $x = 0$ locally asymptotically stable.

The method
Input-output linearization for stabilization - global result

Global result (Lemma 13.2)

If $\varphi = f_0(\varphi, \psi)$ is ISS then the origin $(\varphi, \psi) = (0, 0)$ of

$$\begin{aligned} \dot{\varphi} &= f_0(\varphi, \psi) \\ \dot{\psi} &= (A - BK)\psi \end{aligned}$$

is globally asymptotically stable GAS.

Proof: Satisfies conditions of Lemma 4.7 (Cascaded control)

The method
Step 5 - Choose v to solve the control problem

Tracking control

Reference trajectory: $y_d(t), \dot{y}_d(t), \dots, y_d^{(\rho)}$ bounded
$y_d^{(\rho)}$ piecewise continuous

Define

$$e = \begin{bmatrix} \psi_1 - y_d \\ \vdots \\ \psi_\rho - y_d^{(\rho-1)} \end{bmatrix} = \begin{bmatrix} y - y_d \\ \dot{y} - \dot{y}_d \\ \vdots \\ y^{(\rho-1)} - y_d^{(\rho-1)} \end{bmatrix} = \psi - R, \quad R = \begin{bmatrix} y_d \\ \dot{y}_d \\ \vdots \\ y_d^{(\rho-1)} \end{bmatrix}$$

Choose

$$\boxed{v = -Ke + y_d^{(\rho)}}$$

where K is chosen such that $(A - BK)$ is Hurwitz.

The method
Step 5 - Choose v

We obtain

$$\boxed{\begin{aligned} &\dot{\varphi} = f_0(\varphi, e + R) \\ &\dot{e} = (A - BK)e \quad \text{exponentially stable} \end{aligned}}$$

If the origin $\varphi = 0$ of $\dot{\varphi} = f_0(\varphi, 0)$ is asymptotically stable, then, for sufficiently small $e(0), \varphi(0)$ and $R(t)$

$$\boxed{\begin{aligned} &e(t) \xrightarrow{\exp} 0 \\ &\varphi(t) \text{ is bounded} \end{aligned}}$$

i.e. solves local tracking control problem.

The method
Step 5 - Choose v to solve the control problem

Summary Step 5 - Tracking control

Choose

$$v = -Ke + y_d^{(\rho)}$$

such that $(A - BK)$ is Hurwitz.

Choose for instance

$$\begin{aligned} v &= -k_0 e_1 - k_1 e_2 - \cdots - k_{\rho-1} e_\rho + y_d^{(\rho)} \\ &= -k_0(y - y_d) - k_1(\dot{y} - \dot{y}_d) - \cdots - k_{\rho-1}(y^{(\rho-1)} - y_d^{(\rho-1)}) + y_d^{(\rho)} \end{aligned}$$

such that

$$s^\rho + k_{\rho-1} s^{\rho-1} + \cdots + k_1 s + k_0$$

has all its roots strictly in the left-half plane.

The method

Summary: Input-output linearization for tracking control

If the system

$$\boxed{\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}}$$

where $f : \mathbb{D} \to \mathbb{R}^n$, $g : \mathbb{D} \to \mathbb{R}^n$ and $h : \mathbb{D} \to \mathbb{R}$ are sufficiently smooth on a domain $\mathbb{D} \subset \mathbb{R}^n$, has a well-defined relative degree $\rho \in \mathbb{D}_0 \subset \mathbb{D}$, $1 \le \rho \le n$ and is minimum phase then the control law

$$\boxed{u = \frac{1}{L_g L_f^{\rho-1} h}(-L_f^{\rho} h + v)}$$

ensures that if $e(0)$ and $\varphi(0)$ and $R(t)$ are sufficiently small, then

$$\boxed{\begin{aligned}& e(t) \xrightarrow{\exp} 0 \\ & \varphi(t) \text{ is bounded}\end{aligned}}$$

The method

Input-output linearization for tracking control - Global results

If $\rho = n$ then no internal dynamics

$\Downarrow$

System dynamics is then

$$\dot{e} = (A - BK)e \quad \text{GES}$$

If $\dot{\varphi} = f_0(\varphi, \psi)$ is ISS then

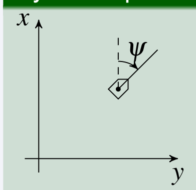
$$\boxed{u = \frac{1}{L_g L_f^{\rho-1} h}(-L_f^{\rho} h + v)}$$

gives

$$\boxed{\begin{aligned}& e(t) \xrightarrow{\exp} 0 \\ & \varphi(t) \text{ is bounded}\end{aligned} \qquad \forall\ e(0), \varphi(0), R(t)}$$

Example: Dynamic positioning system for ships

Dynamic positioning system for ships



$$\eta = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix}$$

System model:

$$\begin{aligned} M(\eta)\ddot{\eta} + C(\eta,\dot{\eta})\dot{\eta} + D(\eta)\dot{\eta} &= \tau \\ y &= \eta \end{aligned}$$

System properties:

$$\begin{aligned} M = M^T &> 0 \\ z^T D z &> 0 \quad z \neq 0 \\ z^T(\tfrac{1}{2}\dot{M} - C)z &= 0 \quad \forall\ z \in \mathbb{R}^3 \end{aligned}$$

Control problem: Design a control law $\tau = g(t, (\eta, \dot{\eta}))$ using input-output linearization, that makes the origin $(\eta, \dot{\eta}) = (0, 0)$ an asymptotically stable equilibrium point.

Advantages/shortcomings

Advantages/shortcomings

- Cancels all dynamics $L_f h$
  - ÷ Does not take advantage of stabilizing terms
  - ÷ Robustness to modelling errors is questionable
- Requires well-defined relative degree
- Requires minimum phase system
- \+ Exponential convergence
- \+ We can use linear control design methods
- \+ Easy tuning

Next lecture

Backstepping

Recommended reading:

Khalil **Chapter 14**
Sections 14.3 pages 589-598