Cascaded nonlinear time-varying systems: analysis and design

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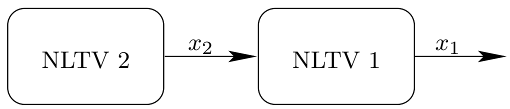
Contents

Abstract

These notes gather the material presented at a minicourse of 6 hrs at the conference mentioned above. The material we present here is not original and has been published in different papers. The adequate references are provided in the Bibliography. The general topic of study is Lyapunov stability of nonlinear time-varying cascaded systems. Roughly speaking these are systems in "open loop" as illustrated in the figure below.



The document is organised in three main sections. In the first, we will introduce the reader to several definitions and state our motivations to study time-varying systems. We will also state the problems of design and analysis of cascaded control systems. The second part contains the main stability results. All the theorems and propositions in this section are on conditions to guarantee Lyapunov stability of cascades. No attention is paid to the control design problem. Finally, the third section contains some selected practical applications where control design aiming at obtaining a cascaded system in closed loop reveals to be better than classical Lyapunov-based designs such as Backstepping.

For the sake of clarity and due to space constraints the technical proofs of the main stability results are omitted here but the interested readers are invited to see the cited references.

Keywords: Time-varying systems, tracking control, time-varying stabilisation, Lyapunov.

Notations. The solution of a differential equation, $\dot{x} = f(t, x)$, where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$, with initial conditions $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ and $x_\circ = x(t_\circ)$, is denoted $x(\cdot\,; t_\circ, x_\circ)$ or simply, $x(\cdot)$. We say that the system $\dot{x} = f(t, x)$, is uniformly globally stable (UGS) if the trivial solution $x(\cdot\,; t_\circ, x_\circ) \equiv 0$ is UGS. Respectively for UGAS. These properties will be precisely defined later. $\|\cdot\|$ stands for the Euclidean norm of vectors and induced norm of matrices, and $\|\cdot\|_p$, where $p \in [1, \infty]$, denotes the $\mathcal{L}_p$ norm of time signals. In particular, for a measurable function $\phi : \mathbb{R}_{\geq t_\circ} \to \mathbb{R}^n$, by $\|\phi\|_p$ we mean $(\int_{t_\circ}^{\infty} \|\phi(t)\|^p \, dt)^{1/p}$ for $p \in [1, \infty)$ and $\|\phi\|_\infty$ denotes the quantity ess $\sup_{t \geq t_\circ} \|\phi(t)\|$. We denote by B_r the open ball $B_r := \{x \in \mathbb{R}^n : \|x\| < r\}$. $\dot{V}_{(\#)}(t, x)$ is the time derivative of the Lyapunov function $V(t, x)$ along the solutions of the differential equation (#). When clear from the context we use the compact notation $V(t, x(t)) = V(t)$. We also use $L_\psi V = \frac{\partial V}{\partial x} \cdot \psi$ for a vector field $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R}^q \to \mathbb{R}^n$.

1 Preliminaries on time-varying systems

The first subject of study in this report are sufficient (and for some cases, necessary) conditions to guarantee *uniform* global asymptotic stability (UGAS) (see Def. 5) of the origin, for nonlinear ordinary differential equations (ODE)

$$\dot{x} = f(t, x) \qquad x(t_\circ) =: x_\circ \ . \tag{1}$$

Most of the literature for nonlinear systems in the last decades has been devoted to time-*invariant* systems nonetheless, the importance of nonautonomous systems cannot be over-estimated, these arise for instance as closed-loop systems in nonlinear trajectory tracking control problems; that is, where the goal is to design a control input $u(t, x)$ for the system

$$\dot{x} = f(x, u) \qquad x(t_\circ) =: x_\circ \tag{2a}$$
$$y = h(x) \tag{2b}$$

such that the output y follows asymptotically, a desired time-varying reference $y_d(t)$. For a "feasible" trajectory $y_d(t) = h(x_d(t))$, some "desired" state trajectory $x_d(t)$, satisfying $\dot{x}_d = f(x_d, u)$, the system (2) in closed loop with the control input $u = u(x, x_d(t), y_d(t))$, may be written as

$$\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}) \qquad \tilde{x}(t_\circ) = \tilde{x}_\circ \tag{3a}$$
$$\tilde{y} = \tilde{h}(t, \tilde{x}) \,, \tag{3b}$$

where $\tilde{x} = x - x_d$, and similarly for all the other variables. The tracking control problem so stated, applies to many physical systems, e.g. mechanical and electromechanical, for which there is a large body of literature (see [37] and references therein).

Another typical situation where closed loop systems of the form (3) arise, is in regulation problems (that is, when the desired set-point y_d is *constant*) in which the open-loop plant is not stabilisable by continuous time-invariant feedbacks $u = u(x)$. This is the case of driftless (e.g. nonholonomic) systems, $\dot{x} = g(x)u$. See e.g. [3, 20].

A classical approach to analyse the *stability* of the *nonautonomous* system (1) is to search for a so-called Lyapunov function with certain properties (see e.g. [61, 18]). Consequently, for the tracking control design problem above, one searches for a so-called Control Lyapunov Function (CLF) for the system (2) so that the control law u is derived from the CLF (see e.g. [26, 53]). In general, finding an adequate LF or CLF is very hard and one has to focus on systems with specific structural properties.

The second subject of study in this report, is a specific structure of systems, which are wide enough to cover a large number of applications, while simple enough to allow criteria for stability which are easier to verify than finding an LF. These are *cascaded* systems. We distinguish between two problems: stability *analysis* and control *design*. For the design problem, we do not offer a general methodology as in [26, 53] however, we show through different applications, that simple (mathematically speaking) controllers can be obtained by aiming at giving the closed loop system a cascaded structure.

1.1 Stability definitions

There are various types of asymptotic stability that can be pursued for time-varying nonlinear systems. As we shall see in this section, from a *robustness* viewpoint, the most useful are *uniform* (global) asymptotic stability and *uniform* (local) exponential stability (ULES). In this section we state the precise definitions we will use throughout this report.

To start with, we recall some basic concepts (see e.g. [18, 61, 9]).

Definition 1 A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to *class* $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. $\square$

Definition 2 A continuous function $\beta : [0, a) \times [0, a) \to [0, \infty)$ is said to belong to *class* $\mathcal{KL}$ if, for each fixed s $\beta(\cdot, s)$ is of class $\mathcal{K}$ and, for each fixed r, $\beta(r, \cdot)$ is strictly decreasing and $\beta(r, s) \to 0$ as $s \to \infty$. $\square$

For the system (1), we define the following.

Definition 3 (Uniform boundedness) We say that the solutions of (1) are (resp. globally) uniformly bounded if there exist a class $\mathcal{K}$ (resp. $\mathcal{K}_\infty$) function α and a number $c > 0$ such that

$$\|x(t, t_\circ, x_\circ)\| \le \alpha(\|x_\circ\|) + c \qquad \forall\, t \ge t_\circ \,. \tag{4}$$

$\square$

Definition 4 (Uniform stability) The origin of the system (1) is said to be uniformly stable (US) if there exist a constant $r > 0$ and $\gamma \in \mathcal{K}_\infty$ such that, for each $(t_\circ, x_\circ) \in \mathbb{R}_{\ge 0} \times B_r$

$$\|x(t, t_\circ, x_\circ)\| \le \gamma(\|x_\circ\|) \qquad \forall\, t \ge t_\circ \,. \tag{5}$$

If the bound (5) holds for all $(t_\circ, x_\circ) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^n$, then the origin is uniformly globally stable (UGS). $\square$

Remark 1

- Notice that the formulation above of uniform stability is equivalent to the classical one, i.e., the system (1) is uniformly stable in the sense defined above if and only if for each ϵ there exists $\delta(\epsilon)$ such that $\|x_\circ\| \le \delta$ implies that $\|x(t)\| \le \epsilon$ for all $t \ge t_\circ$. This is evident if we take $\delta(s) = \gamma^{-1}(s)$.

- It is clear from the above that uniform boundedness is a necessary condition for uniform stability, that is, (5) implies (4).

- Another common characterization of UGS and which we use in some proofs is the following (see e.g. [26]): "the system is UGS if it is US and uniformly globally bounded (UGB)". Indeed, observe that US implies that there exists $\gamma \in \mathcal{K}$ such that (5) holds then, using (4) it is easy to construct $\bar{\gamma} \in \mathcal{K}_\infty$ such that $\bar{\gamma}(s) \geq \alpha(s) + c$ for all $s \geq b > 0$ and $\bar{\gamma}(s) \leq \alpha(s)$ for all $s \leq b$ and hence (5) holds. □

Definition 5 (Uniform asymptotic stability) The origin of the system (1) is said to be uniformly asymptotically stable (UAS) if it is uniformly stable and uniformly attractive, i.e., for each pair of strictly positive real numbers (r, σ) there exists $T > 0$, such that

$$\|x_\circ\| \leq r \implies \|x(t, t_\circ, x_\circ)\| \leq \sigma \qquad \forall\, t \geq t_\circ + T\ , \tag{6}$$

if moreover the system is UGS then the origin is uniformly globally asymptotically stable (UGAS). □

Remark 2 (class-$\mathcal{KL}$ characterization of UGAS:) It is known (see, e.g., [9, Section 35] and [23, Proposition 2.5]) that the two properties characterizing uniform global asymptotic stability hold if and only if there exists a function $\beta \in \mathcal{KL}$ such that all solutions satisfy

$$\|x(t, t_\circ, x_\circ)\| \leq \beta(\|x_\circ\|, t - t_\circ) \qquad \forall\, t \geq t_\circ\, ;\ (t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n\, . \tag{7}$$

The local counterpart is that the system (1) is UAS if there exist a constant $r > 0$ and $\beta \in \mathcal{KL}$ such that for all $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times B_r$. □

Definition 6 (Exponential convergence) The system (1) is said to be exponentially convergent trajectory by trajectory, if there exists $r > 0$ such that *for each* pair of initial conditions $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times B_r$, there exist γ_1 and $\gamma_2 > 0$ such that the solution $x(t, t_\circ, x_\circ)$ of (1), satisfies

$$\|x(t, t_\circ, x_\circ)\| \leq \gamma_1 \|x_\circ\| e^{-\gamma_2 (t - t_\circ)}\ . \tag{8}$$

The system is said to be globally exponentially convergent, if the constants γ_i exist for each pair of initial conditions $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$. □

Definition 7 (Uniform exponential stability) The origin of the system (1) is said to be uniformly (locally) exponentially stable (ULES) if there exist constants γ_1, γ_2 and $r > 0$ such that for all $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times B_r$

$$\|x(t, t_\circ, x_\circ)\| \leq \gamma_1 \|x_\circ\| e^{-\gamma_2 (t - t_\circ)} \qquad \forall t \geq t_\circ. \tag{9}$$

If for each $r > 0$ there exist $\gamma_1,\ \gamma_2$ such that condition (9) holds for all $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times B_r$, then the system is said to be uniformly semiglobally exponentially stable[1].

[1] see, e.g. [5].

Finally, the system (1) is uniformly globally exponentially stable (UGES) if there exist $\gamma_1,\ \gamma_2 > 0$ such that (9) holds for all $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$. $\square$

We will also make use of the following

Definition 8 (UES in any ball) [51] We call the system (1) *uniformly exponentially stable* (UES) *in any ball* if for each $r > 0$ there exist $\gamma_1(r)$ and $\gamma_2(r) > 0$ (9) holds if $\|x_\circ\| \leq r$. $\square$

Remark 3 Note that UGES implies UES in any ball which in turn implies implies "UGAS and ULES". $\square$

1.2 Why uniform *stability?*

One of the main interests of the *uniform* forms of asymptotic stability, is *robustness* with respect to bounded disturbances.

Indeed, if the time-varying system (1) with $f(t, \cdot)$ locally Lipschitz uniformly in t, is ULAS or ULES then the system is locally Input-to-State Stable (ISS); that is, for this system, there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and a number δ such that $\forall\, t \geq t_\circ \geq 0$ (see e.g. [18, Definition 5.2])

$$\max\{\|x_\circ\|, \|u\|_\infty\} \leq \delta \implies \|x(t, t_\circ, x_\circ, u)\| \leq \beta(\|x_\circ\|, t - t_\circ) + \gamma(\|u\|_\infty)\,. \tag{10}$$

This fact can be verified invoking [18, Lemma 5.4], and the converse Lyapunov theorems in [21, 23]. The importance of this implication is that, in particular, local ISS implies total or robust stability, which can be defined as follows.

Definition 9 (Total stability[2]) The origin of of $\dot{x} = f(t, x, 0)$, is said to be totally stable if, for the system $\dot{x} = f(t, x, u)$ small bounded inputs $u(t, x)$ and small initial conditions $x_\circ = x(t_\circ)$, yield small state trajectories for all $t \geq t_\circ$, i.e., if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\max\{\|x_\circ\|, \|u\|_\infty\} \leq \delta \qquad \implies \qquad \|x(t, t_\circ, x_\circ, u)\| \leq \varepsilon \quad \forall\, t \geq t_\circ \geq 0\,. \tag{11}$$

$\square$

In contrast to this, weaker forms of asymptotic stability for time-varying systems, *do not* imply total stability. More precisely:

Proposition 1 Consider the system (1) and assume that $f(t, \cdot)$ is locally Lipschitz uniformly in t, and the origin is UGS. The following conditions are *not* sufficient for total stability:

1. The origin is globally attractive,
2. The system is exponentially convergent trajectory by trajectory and, $f(t, x)$ is globally Lipschitz in x, uniformly in t.

□

Proof. We present an interesting example of an UGS nonlinear time-varying system which satisfies items 1 and 2 of the proposition above, yet, is not totally stable.

Example 1 [42] Consider the system (1) with

$$f(t, x) = \begin{cases} -a(t)\mathrm{sgn}(x) & \text{if} \quad |x| \geq a(t) \\ -x & \text{if} \quad |x| \leq a(t) \end{cases} \tag{12}$$

and $a(t) = \dfrac{1}{t+1}$. This system has the following properties:

1. The function $f(t, x)$ is globally Lipschitz in x, uniformly in t and the system is UGS with linear gain equal to one.
2. For each $r > 0$ and $t_\circ \geq 0$ there exist strictly positive constants κ and λ such that for all $t \geq t_\circ$ and all $|x(t_\circ)| \leq r$
$$|x(t)| \leq \kappa |x(t_\circ)| e^{-\lambda(t-t_\circ)} \tag{13}$$
3. The origin is *not* totally stable. Furthermore, there always exist a bounded (arbitrarily small) additive perturbation and $t_\circ \geq 0$ such that the trajectories of the system grow *unboundedly* as $t \to \infty$.

The proof of these properties is provided in [42]. See also [33] for examples of linear time-varying systems proving the claim in Proposition 1 ■

This lack of total stability for GAS (but not UGAS) and LES (but not ULES) nonautonomous systems, and the unquestionable interest of total stability in time-varying systems arising in practical applications, is our main motivation to search for sufficient conditions that guarantee UGAS and ULES for nonlinear nonautonomous systems.

As it has been mentioned before, the stability analysis problem and hence, control design for time-varying systems is in general very hard to solve. By restricting the class of NLTV systems to cascades, we can establish simple-to-verify conditions for UGAS and UGES. The importance of these results will be evident from their application to specific control design problems that we will address.

2 Cascaded systems

2.1 Introduction

To put the topic of cascades in perspective, let us consider the two linear time-invariant systems

$$\dot{x}_1 = A_1 x_1 \tag{14a}$$
$$\dot{x}_2 = A_2 x_2 \tag{14b}$$

where A_1 and A_2 are stable matrices of equal dimension. Reconsider now the system (14a) as a system with an input and let these systems be interconnected in *cascade*, that is, redefine (14a) to be

$$\dot{x}_1 = A_1 x_1 + B x_2 \,. \tag{15}$$

It follows immediately from the property of *superposition* which is inherent to linear systems, that if each of the systems in (14) is exponentially stable, then the *cascade* (15) - (14b) is also exponentially stable. (Note that no other assumptions are imposed, not even controllability of A, B).

This property comes even more "handy" if we are confronted to designing an observer-based controller for the system $\dot{x} = Ax + Bu$ where (A, B) is controllable. Assume we can measure the output $y = Cx$ where the pair (A, C) is observable. Then, as it is studied in any automatic control textbook, this problem is easily solved by means of the control law $u := -K\hat{x}$ where $\hat{x}$ is the estimate of x, K is such that $(A - BK)$ is Hurwitz, and the observer

$$\dot{\hat{x}} = A\hat{x} - LC\bar{x} \,, \quad \bar{x} := x - \hat{x} \,. \tag{16}$$

Indeed, global exponential stability of the overall closed loop system

$$\dot{x} = (A - BK)x + BK\bar{x} \tag{17a}$$
$$\dot{\bar{x}} = (A - LC)\bar{x} \tag{17b}$$

follows if also $(A - LC)$ is Hurwitz, from the so called *separation principle* which is a direct consequence of the property of superposition. An alternative reasoning to infer GES for (17) starts with observing that this system has a so called *cascaded* structure as it does (15) - (14b). Roughly speaking, GES is concluded since both subsystems (17a), (17b) *separately* are GES and the interconnection term along the trajectories of the system, $BK\bar{x}(t)$, is obviously exponentially decaying.

Holding this simple viewpoint, it is natural to wonder whether a similar reasoning would hold to infer stability and convergence properties of cascaded *nonlinear (time-varying) systems*:

$$\dot{x}_1 = f_1(t, x_1) + g(t, x)x_2 \tag{18a}$$
$$\dot{x}_2 = f_2(t, x_2) \tag{18b}$$

where for now, let us simply assume that the functions $f_1(\cdot,\cdot)$, $f_2(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are such that the solutions exist and are unique on bounded intervals. The answer to this question is far from obvious even for autonomous partially linear systems, and has been object of study during the last 15 years at least. In particular, obtaining sufficient conditions for a nonlinear separation principle is of special interest. While a short literature review is provided in Section 3.1, let us briefly develop on our motivations and goals in the study of cascaded systems.

We identify two problems:

(Design) For the cascaded nonlinear time-varying system:

$$\dot{x}_1 = f_1(t, x_1, x_2) \tag{19a}$$
$$\dot{x}_2 = f_2(t, x_2, u) \tag{19b}$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$ and $u \in \mathbb{R}^l$, the function $f_1(t, x_1, x_2)$ is continuously differentiable in (x_1, x_2) uniformly in t, and measurable in t, find a control input $u = u(t, x_1, x_2)$ or $u = u(t, x_2)$ such that the cascade interconnection be uniformly globally asymptotically stable (UGAS) or uniformly globally stable (UGS).

(Analysis) Consider the cascade (19b), (18b). Assume that the perturbing system (19b) is uniformly globally exponentially stable (UGES), or UGAS. Assume further that the zero-input dynamics of the perturbed system (19a),

$$\dot{x}_1 = f_1(t, x_1, 0)$$

is UGAS (respectively UGS).
Find sufficient conditions under which the cascade (19) is UGAS (respectively UGS).

We will study both problems nevertheless, we take a deeper insight into the second problem. Concerning control design, we will not establish a general methodology as done for instance in [36, 26, 53] however, we show through diverse applications in Section 4 when the structural properties of the system in question allow it, one can design relatively simple controllers and observers, by simply aiming at obtaining a closed loop system with a cascaded structure.

2.2 Peaking: a technical obstacle to analysis

Let us come back to the linear cascade (15) - (14b). As discussed above, this system is GES if both subsystems in (14) are GES. As it shall become clear later this follows essentially because, for each fixed x_2 and large values of x_1 the drift term $f(x_1, x_2) := A_1 x_1$ (assuming A_1 Hurwitz) dominates over the "perturbation" $g(x_1, x_2) := B x_2$. Obviously, in general we do not have such property for nonlinear systems and, as a matter of fact, the growth rate of the functions $f(t, \cdot)$ and $g(t, x)$ for each fixed x_2 and t plays a major role in the stability properties of the cascade. For illustration let us consider the following example which we borrow from the celebrated paper [58] which first analysed in detail the so-called *peaking phenomenon*:

Example 2 (Peaking)

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{2}\left(x_1^3 + x_1^3 x_{2_2}\right) \\ \dot{x}_{2_1} &= x_{2_2} \\ \dot{x}_{2_2} &= -2a x_{2_1} - a^2 x_{2_2}\,, \quad a > 0 \end{aligned}$$

Clearly, the *linear* x_2-subsystem is GES. Indeed, the explicit solution for x_{2_2} is $x_{2_2}(t) = -a^2 t e^{-at}$ and it is illustrated in Fig. 1 below. Moreover, the subsystem $\dot{x}_1 = -0.5 x_1^3$ is obviously globally asymptotically stable (GAS).

Intuitively, one would conjecture that the faster, the perturbing input $x_{2_2}(t)$ vanishes, the better however, as it is clear from the figure this is at expense of larger *peaks* during the transient and which cannot be overestimated. Indeed, the explicit solution $x_1(t; t_0, x_1(t_0))$ with initial conditions $(t_\circ, x_1(t_\circ), x_2(t_\circ)) = (0, x_{1_\circ}, 0)$ is given by

$$x_1(t) = \left(-t - (1 + at)e^{-at} + \frac{1}{x_{1_0}^2} + 1\right)^{-1/2}. \tag{20}$$

It is clear from the expression above that there exists $t_e < \infty$ such that the sum in the parenthesis becomes zero at $t = t_e$ for *any* values of a and x_{1_0}. An approximation of the escape time t_e may be obtained as follows: approximate $e^{-at} \sim (1 - at)$ and substitute in (20) then,

$$t_e \sim \frac{1}{x_{1_0}^2} + \frac{1}{a^2}$$

which becomes smaller as a and x_{1_0} become larger. □

While the example above makes clear that the analysis and design problems described above are far from trivial, in the next section we present an example which illustrates the advantages of a cascades point of view in control design.

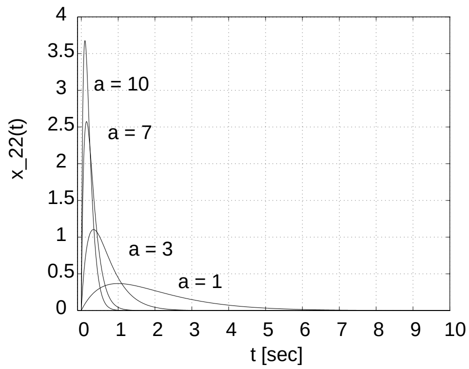


Figure 1: The peaking phenomenon.

2.3 Control design from a cascades point of view

2.3.1 A synchronization example

Let us consider the problem of synchronizing the two pendula showed in Fig. 2 . That is, we consider the problem of making the "slave" pendulum oscillate at the frequency of the "master", while assuming that *no control action* is available *at the joints*. Instead, we desire to achieve our task by modifying on line the length of the slave pendulum thereby, its oscillating frequency:

$$\omega_1 = \sqrt{\frac{l_1}{9.81}} \,.$$

The dynamic equations are

$$\begin{aligned} \ddot{y} + 2\zeta_1\omega_1\dot{y} + \omega_1^2 y &= a_1 \cos\omega_1 t \\ \ddot{y}_d + 2\zeta_2\omega_2\dot{y}_d + \omega_2^2 y_d &= a_2 \cos\omega_2 t \end{aligned}$$

where the control input corresponds to the change in ω_1, i.e.,

$$\dot{\omega}_1 = u \,.$$

As it has been shown in [25] if $\omega_2 > 0$, $\zeta_1 = \zeta_2$, $a_1 = a_2$, using a cascades approach it is easy to prove that the linear control law $u = -k\tilde{\omega}$, with $k > 0$ makes that

$$\lim_{t\to\infty} \tilde{\omega}(t) = 0 \qquad \lim_{t\to\infty} \tilde{y}(t) = 0 \,.$$

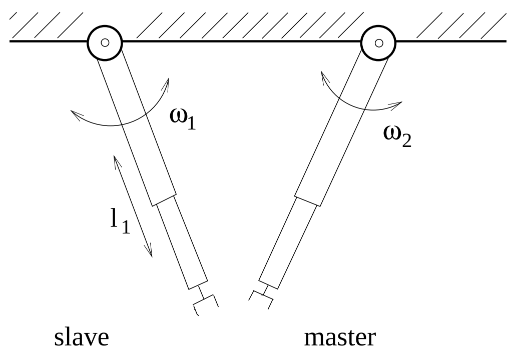


Figure 2: Synchronisation of two pendula.

where $\tilde{\omega}(t) := \omega_1(t) - \omega_2$ and $\tilde{y}(t) := y(t) - y_d(t)$. One only needs to observe that, defining

$$v := \ddot{y}_d + 2\zeta\omega_2\dot{y}_d + \omega_2^2 - a\cos\omega_2 t = 0$$

and $z :=$ col$[\tilde{y},\, \dot{\tilde{y}}]$, the two pendula dynamic equations

$$\ddot{y} + 2\zeta\omega_1\dot{y} + \omega_1^2 y = a\cos\omega_1 t + v$$
$$\ddot{y}_d + 2\zeta\omega_2\dot{y}_d + \omega_2^2 y_d = a\cos\omega_2 t$$

and the control law, are equivalent to

$$\left\{\begin{array}{rcl} \dot{z}_1 &=& z_2 \\ \dot{z}_2 &=& -2\zeta\omega_2 z_2 - \omega_2^2 z_1 + g_2(t, z, \tilde{\omega}) \end{array}\right.$$
$$\dot{\tilde{\omega}} = -k\tilde{\omega}$$

where $g_2(t, z, \tilde{\omega}) = 2\zeta\tilde{\omega}z_2 + 2\zeta\tilde{\omega}\dot{y}_d(t) - \tilde{\omega}^2(z_1 + y_d(t)) - 2\tilde{\omega}\omega_2 y_d + a(cos\omega_1 t - \cos\omega_2 t)$. Notice that this system is of the form

$$\dot{z} = f_1(z) + g(t,\, z,\, \tilde{\omega}) \tag{21a}$$
$$\dot{\tilde{\omega}} = -k\tilde{\omega} \tag{21b}$$

where $g(t, z, \tilde{\omega}) :=$ col$[0,\, g_2(t, z, \tilde{\omega})]$ and clearly, $\dot{z} = f_1(z)$ is exponentially stable. It occurs that, since the "growth" of the function $g(t, \cdot, \tilde{\omega})$ is linear for each fixed ω and uniformly in t that is, for each fixed $\tilde{\omega}$ there exists $c > 0$ such that $\|g(t, z, \tilde{\omega})\| \leq c\,\|z\|$ for all $t \geq 0$, the cascade (21) is globally asymptotically stable (for details, see [25]).

Roughly speaking, the difference between this system and the one in Example 2 is that the exponentially stable dynamics $\dot{z} = f_1(z)$ dominates over the interconnection term, no matter how large the input $\tilde{\omega}(t)$ gets. In Section 3 we will establish in a formal way sufficient conditions in terms of the growth rates.

2.3.2 From feedback interconnections to cascades

The objective of the application we discuss here is to illustrate that, under certain conditions, a *feedback interconnected* system can be viewed as a cascade (thereby, "neglecting the feedback interconnection!) if you twist your eyes. This property will be used in the design applications presented in Section 4. To introduce the reader to this technique we briefly discuss below the problem of tracking control of rigid-joint robot manipulators driven by AC motors. This problem was originally solved in [43].

To illustrate the main idea exploited in that reference, let

$$\dot{x}_1 = \phi_1(x_1) + \tau, \tag{22}$$

represent the rigid-joint robot dynamics and let τ be the (control) input torque. Assume that this torque is provided by an AC motor with dynamics

$$\dot{x}_2 = \phi_2(x_2, u), \tag{23}$$

and u is a control input to the AC motor. The control goal is to find u such that the robot generalized coordinates follow a specified time-varying reference $x_d(t)$. That is, we wish to design $u(t, x_1, x_2)$ such that the closed loop system be UGAS. The rational behind the approach undertaken in [43] can be summarized in two steps:

1. Design a control law $\tau_d(t, x_1)$ to render UGAS the robot closed loop dynamics,

$$\dot{x}_1 = \phi_1(x_1) + \tau_d(t, x_1) \tag{24}$$

   at the "equilibrium" $x_1 \equiv x_{1d}(t)$.

2. Design $u = u(t, x_1, x_2)$ so that the AC drive dynamics

$$\dot{x}_2 = \phi_2(x_2, u(t, x_1, x_2)) \tag{25}$$

   be uniformly globally exponentially stable[3] (UGES), uniformly in x_1 at the equilibrium $x_2 \equiv x_{2d}(t)$, so that the output error $\tau - \tau_d$ decay exponentially fast for any value of $x_1(t)$ and any $t \geq t_\circ \geq 0$.

[3]As we will show in this paper, in some cases, UGAS suffices.

It is important to remark that since the equilibrium $x_2 \equiv x_{2d}$ of (25) is GES, *uniformly* in x_1, the dynamics of this system can be considered along the trajectories and hence, we can write $f_2(t, x_2) := \phi_2(x_2, u(t, x_1(t), x_2))$ which is well defined for all $t \geq t_\circ \geq 0$ and all $x_2 \in \mathbb{R}^{n_2}$. The closed loop equation (24), represents the ideal case when the drives, provide the desired torque τ_d. Using this, we can write the real closed loop (22) as $\dot{x}_1 = f_1(t, x_1, x_2)$ with $f_1(t, x_1, x_2) := \phi_1(x_1) + \tau_d(t, x_1) + \tau(t, x_1) - \tau_d(t, x_1)$. Notice that by design, $x_2 \equiv x_{2d}(t) \Rightarrow \tau \equiv \tau_d$. This reveals the cascaded structure of the overall closed loop system[4].

This example suggests that the global stabilisation of nonlinear systems which allow a cascades decomposition, may be achieved by ensuring UGAS for both subsystems separately. The question remaining is to know whether the stability properties of both subsystems separately, remains valid under the cascaded interconnection (18b), (19a). The latter motivates us to study the *stability analysis problem* exposed above.

3 Stability of cascades

3.1 Literature review

The stability analysis problem for nonlinear *autonomous* systems

$$\Sigma_1' \quad : \quad \dot{x}_1 = f_1(x_1, x_2) \tag{26}$$

$$\Sigma_2' \quad : \quad \dot{x}_2 = f_2(x_2) \tag{27}$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$ and the functions $f_1(\cdot, \cdot)$, $f_2(\cdot)$ are sufficiently smooth in their arguments, was addressed for instance in [55] where the author used the "Converging Input - Bounded State" property:

CIBS: For each input $x_2(\cdot)$ on $[0, \infty)$ such that $\lim_{t\to\infty} x_2(t) = 0$, and for each initial state $x_{1_\circ}$, the solution of (26) with $x_1(0) = x_{1_\circ}$ exists for all $t \geq 0$ and it is bounded,

to prove that the cascaded system Σ_1', Σ_2' is GAS if the subsystems $\dot{x}_1 = f_1(x_1, 0)$ and (27), are GAS and CIBS holds. Also, based on Krasovskii-LaSalle's invariance principle, the authors of [52] showed that the composite system is GAS assuming that all solutions are bounded (in short, BS) and that both subsystems, (27) and $\dot{x}_1 = f_1(x_1, 0)$, are GAS.

Fact 1: $\text{GAS} + \text{GAS} + \text{BS} \Rightarrow \text{GAS}$.

[4] To explain the rationale of this cascades design approach, we have abbreviated $x_1(t) = x_1(t, t_0, x_0)$, however, due to the uniformity of the GES property, in x_1, it is valid to consider the closed loop as a cascade. See [43] for details.

For *autonomous* systems this fact is a fundamental result which has been used by many authors to prove GAS of the cascade (26), (27). The natural question which arises next, is "*how do we guarantee boundedness of the solutions?* ". One way is to use the now well known property of Input-to-State stability (ISS), introduced in [54]. For convenience we cite below the following characterization of ISS (see e.g. [57])

ISS: The system $\Sigma_1' : \dot{x}_1 = f_1(x_1, x_2)$ with input x_2, is Input-to-State Stable if and only if there exists a positive definite proper function $V(x_1)$, and two class $\mathcal{K}$ functions α_1 and α_2 such that, the implication

$$\{\|x_1\| \geq \alpha_1(\|x_2\|)\} \implies \left\{ \frac{\partial V}{\partial x_1} \cdot f_1(x_1, x_2) \leq -\alpha_2(\|x_1\|) \right\} \tag{28}$$

holds for each $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$.

For instance consider the system [56]

$$\dot{x}_1 = -x_1^3 + x_1^2 x_2 \tag{29}$$

with input $x_2 \in \mathbb{R}$ and the Lyapunov function candidate $V(x_1) = \frac{1}{2} x_1^2$. The time derivative of V is $\dot{V} = -x_1^4 + x_1^3 x_2$ clearly, if $\|x_1\| \geq 2\|x_2\|$ then $\dot{V} \leq -\frac{1}{2}\|x_1\|^4$.

Unfortunately, proving the ISS property as a condition to imply CIBS may appear in some cases very restrictive, for instance consider the one-dimensional system

$$\dot{x}_1 = -x_1 + x_1 x_2 \tag{30}$$

which is not ISS with respect to the input $x_2 \in \mathbb{R}$. While it may be already intuitive from the last two examples, we will see formally in this chapter that what makes the difference is that the terms in (30) have the *same* linear growth rate in the variable x_1, while in (29) the term x_1^3 dominates over $x_1^2 x_2$ for each fixed x_2 and "large" values of x_1.

Concerned by the control design problem, i.e., to stabilize the cascaded system Σ_1', Σ_2' by using feedback of the state x_2 only, the authors of [48] studied the case when Σ_2' is a linear controllable system. Assuming $f_1(x_1, x_2)$ in (26) to be continuously differentiable, rewrite (26) as

$$\dot{x}_1 = f_1^*(x_1) + g(x_1, x_2)x_2 \,. \tag{31}$$

In [48] the authors introduced the *linear* growth condition

$$\|g(x_1, x_2)x_2\| \leq \theta(\|x_2\|)\,\|x_1\| \tag{32}$$

where θ is $\mathcal{C}^1$, non-decreasing and $\theta(0) = 0$, together with the assumption that $x_2 = 0$ is GES, to prove boundedness of the solutions. Using such a condition one can deal with systems which are not ISS with respect to the input x_2.

From these examples one may conjecture that, in order to prove CIBS for the system (31) with decaying input $x_2(\cdot)$, some growth restrictions should be imposed on the functions $f_1^*(\cdot)$ and $g(\cdot,\cdot)$. For instance, for the NL system (31) one may impose a linear growth condition such as (32) or the ISS property with respect to the input x_2. As we will show later, for the latter it is "needed" that the function $f_1^*(x_1)$ grows faster than $g(x_1, x_2)$ as $\|x_1\| \to \infty$.

In the papers [28] and [11], the authors addressed the problem of global stabilisability of feedforward systems, by a systematic recursive design procedure, which leads to the construction of a Lyapunov function for the complete system. While the design procedures differ in both references, a common point is the stability analysis of cascaded systems. In order to prove that all solutions remain bounded under the cascaded interconnection, the authors of [11] used the linear growth restriction

$$\|g(x_1, x_2)x_2\| \leq \theta_1(\|x_2\|)\,\|x_1\| + \theta_2(\|x_2\|) \tag{33}$$

where $\theta_1(\cdot)$, $\theta_2(\cdot)$ are $\mathcal{C}^1$ and $\theta_i(0) = 0$, together with the growth rate condition on the Lyapunov function $V(x_1)$ for the zero-dynamics $\dot{x}_1 = f_1(x_1, 0)$: $\left\|\frac{\partial V}{\partial x_1}\right\| \|x_1\| \leq cV$ for $\|x_1\| \geq c_2$ (which holds e.g. for all polynomials $V(x_1)$) and a condition of exponential stability for Σ_2. In [28] the authors used the assumption on the existence of continuous nonnegative functions ρ, $\kappa : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, such that $\left|\frac{\partial V}{\partial x_1} g(x)x_2\right| \leq \kappa(x_2)[1 + \rho(V)]$ and $\frac{1}{1+\rho(V)} \notin \mathcal{L}_1$ and $\kappa(x_2) \in \mathcal{L}_1$. The choice of κ is restricted depending on the type of stability of Σ_2'. In other words, there is a tradeoff between the decay rate of $x_2(t)$ and the growth of $g(x)$.

Nonetheless, all the results mentioned above apply only to *autonomous* nonlinear systems whereas in these notes we are interested in trajectory tracking control problems and time-varying stabilisation therefore, non-autonomous systems deserve particular attention. Some of the first efforts made to extend the ideas exposed above for *time-varying* nonlinear cascaded systems are contained in [14, 40, 39, 29].

In [14] the stabilisation problem of a robust (vis-a-vis dynamic uncertainties) controller was considered, while in [40, 39] we established sufficient conditions for UGAS of cascaded nonlinear non autonomous systems based on a similar *linear* growth condition as in (33), and an *integrability* assumption on the input $x_2(\cdot)$ thereby, relaxing the exponential-decay condition used in other references. In [29] the results of [28] are extended to the nonautonomous case.

3.2 Nonautonomous cascades: problem statement

We will study cascaded NLTV systems

$$\begin{aligned} \Sigma_1 : \ \dot{x}_1 &= f_1(t,x_1) + g(t,x)x_2 && \text{(34a)} \\ \Sigma_2 : \ \dot{x}_2 &= f_2(t,x_2) && \text{(34b)} \end{aligned}$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$, $x := \operatorname{col}[x_1,\ x_2]$. The functions $f_1(t,x_1)$, $f_2(t,x_2)$ and $g(t,x)$ are continuous in their arguments, locally Lipschitz in x, uniformly in t, and $f_1(t,x_2)$ is continuously differentiable in both arguments. We also assume that there exists a nondecreasing function $G(\cdot)$ such that,

$$\|g(t,x)\| \leq G(\|x\|)\,. \tag{35}$$

Probably the main observation in these notes is that Fact 1 above holds for nonlinear time-varying systems and as a matter of fact, the implication holds in both senses. That is, uniform global boundedness (UGB) is a *necessary* condition for UGAS of cascades. See Lemma 1.

Then, we will present several statements of sufficient conditions for UGB. These statements are organised in the following two sections. In Section 3.4 we present a theorem and some lemmas which rely on a *linear* growth condition (in x_1) of the interconnection term $g(t,x)$ and the fundamental assumption that the perturbing input $x_2(\cdot)$ is *integrable.* In Section 3.5 we will enunciate sufficient conditions to establish UGAS for three classes of cascades: roughly speaking, we consider systems such that, for each fixed x_2, the following hold uniformly in t:

(i) the function $f_1(t,x_1)$ grows faster than $g(t,x)$ as $\|x_1\| \to \infty$,

(ii) both functions $f_1(t,x_1)$ and $g(t,x)$ grow at similar rate as $\|x_1\| \to \infty$,

(iii) the function $g(t,x)$ grows faster than $f_1(t,x_1)$ as functions of x_1.

In each case, we give sufficient conditions to guarantee that a UGAS nonlinear time-varying system

$$\dot{x}_1 = f_1(t,x_1) \tag{36}$$

remains UGAS when it is perturbed by the output of another UGAS system of the form Σ_2, that is, we establish sufficient conditions to ensure UGAS for the system (34).

From converse Lyapunov theorems (see e.g. [21, 18, 23]), since we consider here cascades for which (36) is UGAS, there exists a Lyapunov function $V(t, x_1)$. Thus consider the assumption below which we divide in two parts for ease of reference.

Assumption 1

a) The system (36) is UGAS.

b) There exists a known $\mathcal{C}^1$ Lyapunov function $V(t, x_1)$, $\alpha_1,\ \alpha_2 \in \mathcal{K}_\infty$, a positive *semidefinite* function $W(x_1)$, and a continuous non-decreasing function $\alpha_4(\cdot)$ such that

$$\alpha_1(\|x_1\|) \leq V(t, x_1) \leq \alpha_2(\|x_1\|) \tag{37}$$

$$\dot{V}_{(36)}(t, x_1) \leq -W(x_1) \tag{38}$$

$$\left\| \frac{\partial V}{\partial x_1} \right\| \leq \alpha_4(\|x_1\|). \tag{39}$$

Remark 4 We point out that, to verify Assumption 1a it is enough to have a Lyapunov function with only semi-negative time derivative. Yet, we have the following.

Proposition 2 Assumption 1a implies the existence of a Lyapunov function $\mathcal{V}(t, x_1)$, functions $\bar{\alpha}_1,\ \bar{\alpha}_2 \in \mathcal{K}_\infty$ and $\bar{\alpha}_4 \in \mathcal{K}$ such that,

$$\bar{\alpha}_1(\|x\|) \leq \mathcal{V}(t, x_1) \leq \bar{\alpha}_2(\|x\|) \tag{40}$$

$$\dot{\mathcal{V}}_{(36)}(t, x_1) \leq -\mathcal{V}(t, x_1) \tag{41}$$

$$\left\| \frac{\partial \mathcal{V}}{\partial x_1} \right\| \leq \bar{\alpha}_4(\|x\|)\,. \tag{42}$$

□

Sketch of proof. The inequalities in (40), as well as the existence of $\bar{\alpha}_3 \in \mathcal{K}$ such that,

$$\dot{\mathcal{V}}_{(36)}(t, x_1) \leq -\bar{\alpha}_3(\|x\|)\,, \tag{43}$$

follow from [23, Theorem 2.9]. The property (42) follows along the lines of proofs of [18, Theorems 3.12, 3.14] and [21], using the assumption that $f_1(t, x_1)$ is continuously differentiable and locally Lipschitz. Finally, (41) follows using (43) and [45, Proposition 13]. See also [59].

We stress the importance of formulating Assumption 1**b** with the less restrictive conditions (37), (38) since, for some applications, UGAS for (36) may be established with a Lyapunov function $V(t, x_1)$ with a negative semidefinite derivative. For autonomous systems, e.g., using invariance principles (such as Krasovskii-LaSalle's) or, for non-autonomous systems, via Matrosov's theorem [46]. See Section 4 and [22] for some examples.

Finally, we remark that the same arguments apply to [40, Theorem 2] where we overlooked this important issue, imposing the unnecessarily restrictive assumption of negative definiteness on $\dot{V}_{(36)}(t, x_1)$. This is implicitly assumed in the proof of that Theorem. In Section 3.4 we present a theorem which includes the same result. $\square$

Further, we assume that

(Assumption 2) the subsystem Σ_2 is UGAS.

Let us stress some direct consequences of Assumption 2 in order to introduce some notation. Firstly, it means that there exists $\beta \in \mathcal{KL}$ such that,

$$\|x_2(t; t_\circ, x_{2_\circ})\| \leq \beta(\|x_{2_\circ}\|, t - t_\circ), \quad \forall\, t \geq t_\circ, \tag{44}$$

and hence, for each $r > 0$

$$\|x_2(t; t_\circ, x_{2_\circ})\| \leq c := \beta(r, 0), \qquad \forall\, \|x_{2_\circ}\| < r\,. \tag{45}$$

Secondly, note that due to [28, Lemma B.1], (35) implies that there exist continuous functions $\theta_1 : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ and $\alpha_5 : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ such that $\|g(t, x)\| \leq \theta_1(\|x_2\|)\alpha_5(\|x\|)$ hence under Assumption 2, we have for each $r > 0$, and for all $t_\circ \geq 0$, that

$$\|g(t, x(t, t_\circ, x_\circ))\| \leq c_g(r)\alpha_5(\|x_1(t, t_\circ, x_\circ)\|), \qquad \forall\ \|x_{2_\circ}\| < r\,,\ \forall t \geq t_\circ \tag{46}$$

where $c_g(\cdot)$ is the class $\mathcal{K}$ function defined by $c_g(\cdot) := \theta_1(\beta(\cdot, 0))$.

We are now ready to present an auxiliary but fundamental result for our main theorems. The following lemma extends the fact that GAS + GAS + BS $\Rightarrow$ GAS, to the nonautonomous case. This is probably the most fundamental result of these notes and therefore we provide the proof of it.

Lemma 1 (UGAS + UGAS + UGB $\Leftrightarrow$ UGAS) The cascade (34) is UGAS *if and only if* the systems (34b) and (36) are UGAS and the solutions of (34) are uniformly globally bounded (UGB). $\square$

Proof. (*Sufficency*). By assumption (from UGB), for each $r > 0$ there exists $\bar{c}(r) > 0$ such that, if $\|x_\circ\| < r$ then $\|x(t, t_\circ, x_\circ)\| \leq \bar{c}(r)$. Consider the function $\mathcal{V}(t, x_1)$ as defined in Proposition 2. It's time derivative along the trajectories of (34a) yields, using (42), (41) and (46), and defining $v(t) := \mathcal{V}(t, x_1(t))$,

$$\dot{v}_{(34a)}(t) \leq -v(t) + c(r)\, \|x_2(t)\|\ , \tag{47}$$

where $c(r) := c_g(r)\bar{\alpha}_4(\bar{c}(r))\alpha_5(\bar{c}(r))$. Therefore, using (44) and defining $v_\circ := v(t_\circ)$, we obtain that for all $t_\circ \geq 0$, $\|x_\circ\| < r$ and $v_\circ < \bar{\alpha}_2(r)$,

$$\dot{v}_{(34a)}(t, t_\circ, v_\circ) \leq -v(t, t_\circ, v_\circ) + \tilde{\beta}(r, t - t_\circ) \tag{48}$$

where $\tilde{\beta}(r, t - t_\circ) := c(r)\beta(r, t - t_\circ)$.

Let $\tau_\circ \geq t_\circ$, multiplying by $e^{(t-\tau_\circ)}$ on both sides of (48) and rearranging the terms, we obtain

$$\frac{d}{dt}\left[v(t)e^{(t-\tau_\circ)}\right] \leq \tilde{\beta}(r, t - t_\circ)e^{(t-\tau_\circ)}\,, \quad \forall t \geq \tau_\circ \tag{49}$$

then, integrating on both sides and multiplying by $e^{-(t-\tau_\circ)}$ we obtain

$$v(t) \leq v(\tau_\circ)e^{-(t-\tau_\circ)} + \int_{\tau_\circ}^{t} \tilde{\beta}(r, s - t_\circ)e^{-(t-s)}ds\,, \quad \forall t \geq \tau_\circ \tag{50}$$

which in turn implies that

$$v(t) \leq v(t_\circ) + \tilde{\beta}(r, 0)\left(1 - e^{-(t-t_\circ)}\right) \leq \bar{\alpha}_2(r) + \tilde{\beta}(r, 0)\,, \quad \forall t \geq t_\circ \tag{51}$$

hence $\|x_1(t)\| \leq \bar{\alpha}_1^{-1}(\bar{\alpha}_2(r) + \tilde{\beta}(r, 0)) =: \gamma(r)$. Uniform global stability follows observing that $\gamma \in \mathcal{K}_\infty$ and that the subsystem Σ_2 is UGS by assumption. On the other hand, for each $0 < \varepsilon_1 < r$, let $T_1(\varepsilon_1, r) \geq 0$ be such that $\tilde{\beta}(r, T_1) = \varepsilon_1/2$ ($T_1 = 0$ if $\tilde{\beta}(r, 0) \leq \varepsilon_1/2$), then (50) also implies that

$$v(t) \leq v(t_\circ + T_1)e^{-(t-t_\circ-T_1)} + \int_{t_\circ+T_1}^{t} \tilde{\beta}(r, T_1)e^{-(t-s)}ds\,, \quad \forall t \geq t_\circ + T_1$$

which, in vue of (51), implies

$$v(t) \leq \left[\bar{\alpha}_2(r) + \tilde{\beta}(r, 0)\right]e^{-(t-t_\circ-T_1)} + \frac{\varepsilon_1}{2}\,, \quad \forall t \geq t_\circ + T_1\,.$$

It follows that $v(t) \leq \varepsilon_1$ for all $t \geq t_\circ + T$ with $T := T_1 + \ \ln\left[\frac{2[\bar{\alpha}_2(r)+\tilde{\beta}(r,0)]}{\varepsilon_1}\right]$. Finally, defining $\varepsilon := \bar{\alpha}_2(\varepsilon_1)$ we conclude that $\|x_1(t)\| \leq \varepsilon$ for all $t \geq t_\circ + T$. The result follows observing that ε_1 is arbitrary, $\bar{\alpha}_2 \in \mathcal{K}_\infty$, and that Σ_2 is UGAS by assumption.

(*Necessity*). By assumption there exists $\beta \in \mathcal{KL}$ such that $\|x(t)\| \leq \beta(\|x_\circ\|, t - t_\circ)$. UGB follows observing that $\|x(t)\| \leq \beta(\|x_\circ\|, 0)$. Also, notice that the solutions $x(t)$ restricted to $x_2(t) \equiv 0$ satisfy $\|x_1(t)\| \leq \beta(\|x_{1\circ}\|, t - t_\circ)$ which implies UGAS of (36). It is clear that (36) is UGAS only if (34b) is UGAS. ■

As discussed in the previous section, the next question is how to guarantee the global uniform boundedness. This can be established by imposing extra *growth rate* assumptions. In particular, in Section 3.5, under Assumptions 1 and 2, we shall consider the three previously mentioned cases according to the growth rates of $f_1(t, x_1)$ and $g(t, x)$. For this, we will make use of the following concepts.

Definition 10 (small order) Let $\varrho(x)$, $\varphi(t,x)$ be continuous functions of their arguments. We denote $\varphi(t,\cdot) = o(\varrho(cdot))$ (and say that "phi is of small order of rho") if there exists a continuous function $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\|\varphi(t,x)\| \leq \lambda(\|x\|)\,\|\varrho(x)\|$ for all $(t,\ x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ and $\lim\limits_{\|x\|\to\infty} \lambda(\|x\|) = 0$. □

A direct consequence of the definition above is that the following holds uniformly in t:

$$\lim_{\|x\|\to\infty} \frac{\|\varphi(t,x)\|}{\|\varrho(x)\|} = 0 \ .$$

Definition 11 Let $\varrho(x)$ and $\varphi(t,x)$ be continuous in their arguments. We say that the function $\varrho(x)$ majorises the function $\varphi(t,x)$ if

$$\overline{\lim_{\|x\|\to\infty}} \frac{\|\varphi(t,x)\|}{\|\varrho(x)\|} < +\infty \qquad \forall\, t \geq 0 \ .$$

□

Notice that, as a consequence of the definition above, it holds true that there exist finite positive constants η and λ such that, the following holds uniformly in t:

$$\|x\| \geq \eta \ \Rightarrow \ \frac{\|\varphi(t,x)\|}{\|\varrho(x)\|} < \lambda \,. \tag{52}$$

We may also refer to this property as "large order" or "order" and write $\phi(t\cdot) = \mathcal{O}(\rho(\cdot))$.

3.4 An integrability criterion

Theorem 1 Let Assumption 1**a** hold and suppose that the trajectories of (34b) are uniformly globally bounded. If moreover, Assumptions 3 – 5 below are satisfied, then the solutions $x(t,t_\circ,x_\circ)$ of the system (34) are uniformly globally bounded. If furthermore, the system (34b) is UGAS, then so is the origin of the cascade (34).

Assumption 3 There exist constants $c_1,\ c_2,\ \eta > 0$ and a Lyapunov function $V(t,x_1)$ for (36) such that $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is positive definite and radially unbounded, which satisfies:

$$\left\|\frac{\partial V}{\partial x_1}\right\| \|x_1\| \ \leq \ c_1 V(t,x_1) \qquad \forall\ \|x_1\| \geq \eta \tag{53}$$

$$\left\|\frac{\partial V}{\partial x_1}\right\| \ \leq \ c_2 \qquad \forall\ \|x_1\| \leq \eta \tag{54}$$

Assumption 4 There exist two continuous functions $\theta_1, \theta_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that $g(t,x)$ satisfies

$$\|g(t,x)\| \leq \theta_1(\|x_2\|) + \theta_2(\|x_2\|)\|x_1\| \tag{55}$$

Assumption 5 There exists a class $\mathcal{K}$ function $\alpha(\cdot)$ such that, for all $t_\circ \geq 0$, the trajectories of the system (34b) satisfy

$$\int_{t_\circ}^{\infty} \|x_2(t, t_\circ, x_2(t_\circ))\| dt \leq \alpha(\|x_2(t_\circ)\|)\,. \tag{56}$$

□

The following proposition is a local counterpart of the theorem above and establishes exponential stability of the cascade in any ball (see Def. 8, i.e., for each r there exists $\gamma_1(r) > 0$ and $\gamma_2(r) > 0$ such that for all $\|x_\circ\| \leq r$ the system satisfies (9). Notice that this concept differs from ULES in that the numbers γ_1 and γ_2 depend on the size of initial conditions however, the convergence is uniform.

Even though this proposition may seem obvious to some readers, we write it separately from Theorem 1 for ease of reference in Section 4, where it will be an instrumental tool in tracking control design of nonholonomic carts.

Proposition 3 If in addition to the assumptions in Theorem 1 the systems (34b) and (36) are exponentially stable in any ball B_r, then the cascaded system (34) is exponentially stable in the same ball. In particular if the subsystems are UGES the cascade is UGES. □

3.5 Growth rate theorems

In Theorem 1 we have imposed a linear growth condition on the interconnection term and used an integrability assumption on the solutions of (34b) to establish UGAS of the cascade. In this section we allow for different growth rates of the interconnection term, relative to the growth of the drift term.

3.5.1 Case 1: "The function $f_1(t, x_1)$ grows faster than $g(t, x)$"

The following theorem allows to deal with systems which are ISS[5] but which do not necessarily satisfy a linear growth condition such as (33). Roughly speaking, the stability induced by the drift $f_1(t, x_1)$ dominates over the "perturbations" induced by the trajectories $x_2(t)$ through the interconnection term $g(t, x)$.

Theorem 2 If Assumptions 1 and 2 hold, and

(Assumption 6) for each fixed x_2 and t the function $g(t, x)$ satisfies

$$\|[L_g V](t, x)\| = o(W(x_1)), \text{ as } \|x_1\| \to \infty \tag{57}$$

where $W(x_1)$ is defined in Assumption 1;

then, the cascade (34) is UGAS. □

Proposition 4 If in addition to the assumptions of Theorem 2 there exists $\alpha_3 \in \mathcal{K}$ such that $W(x_1) \geq \alpha_3(\|x_1\|)$ then the system (34a) is ISS with respect to the input $x_2 \in \mathbb{R}^m$. □

Remark 5 If for a particular (autonomous) system we have $W(x_1) = \left\|\frac{\partial V}{\partial x_1}\right\| \|f_1(x_1)\|$ then condition (57) reads simply $g(t, x) = o(f_1(x_1))$ however, it must be understood that in general, such relation of order between functions $f_1(t, x_1)$ and $g(t, x)$ is not implied by condition (57). This motivates the use of "quotes" in the phrase "Function $f_1(t, x_1)$ grows faster than $g(t, x)$". □

Remark 6 The function $W(x_1)$ depends on the choice of the Lyapunov function $V(t, x_1)$ for system (36). However, note that for any $\rho \in \mathcal{K}_\infty$, the relation $L_g V = o(W)$ is equivalent to $L_g \rho(V) = o(\frac{\partial \rho}{\partial V} W)$. This proves that as far as V is concerned, we have an assumption on the shape of the level set and not on its value. □

Example 3 Consider the ISS system $\dot{x}_1 = -x_1^3 + x_1^2 x_2$ with input $x_2 \in \mathbb{R}$ with an ISS-Lyapunov function $V(x_1) = \frac{1}{2}x_1^2$ which satisfies Assumption 6 with $\alpha_4(s) = s$ and $\alpha_3(s) = s^4$. □

[5] It is worth mentioning that the concept of ISS systems was originally proposed and is more often used in the context of autonomous systems, for definitions and properties of time-varying ISS systems see e.g. [26].

3.5.2 Case 2: "The function $f_1(t, x_1)$ majorizes $g(t, x)$"

We consider now systems like (34a), which are not necessarily ISS with respect to the input $x_2 \in \mathbb{R}^m$ but for which the following assumption on the growth rates of $V(t, x_1)$ and $g(t, x)$ holds.

Assumption 7 There exist a continuous non-decreasing function $\alpha_6 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and a constant $a \geq 0$, such that $\alpha_6(a) > 0$ and

$$\alpha_6(s) \geq \alpha_4(\alpha_1^{-1}(s))\alpha_5(\alpha_1^{-1}(s)) \tag{58}$$

where α_5 was defined in (46), and

$$\int_a^\infty \frac{ds}{\alpha_6(s)} = \infty \,. \tag{59}$$

Assumption 7 imposes a restriction on the growth rate in x_1, of the function $g(t, x)$. Indeed, notice that for (59) to hold it is seemingly needed that $\alpha_6(s) = \mathcal{O}(s)$ for large s thereby imposing a restriction on $g(t, x)$ for each t and x_2. The condition in (59) guarantees (considering that the "inputs" $x_2(t)$ are absolutely continuous on $[0, \infty)$) that the solutions of the overall cascaded system $x(t\,; t_\circ, x_\circ)$ do not escape in finite time. The formal statement which support our arguments, can be found for instance in [50].

Remark 7 Notice that the assumptions on the growth rates of $g(t, x)$ and $V(t, x_1)$ considered in [11] are a particular case of Assumption 7. Also, this assumption is equivalent to the hypothesis made in Assumption A3.1 of [28], on the existence of a nonnegative function ρ such that $\frac{1}{1+\rho} \notin \mathcal{L}_2$. □

Theorem 3 Let Assumptions 1, 2 and 7 hold, and suppose that

(Assumption 8) The function $g(t, x)$ is majorized by the function $f_1(t, x_1)$ in the following sense: for each $r > 0$ there exist $\lambda,\ \eta > 0$ such that, for all $t \geq 0$ and all $\|x_2\| < r$

$$\|[L_g V](t, x)\| \leq \lambda W(x_1) \quad \forall\ \|x_1\| \geq \eta \,. \tag{60}$$

where $W(x_1)$ is defined in Assumption 1.

Then, the cascade (34) is UGAS. □

Example 4 The system $\dot{x}_1 = -x_1 + x_1 x_2$ with input x_2, satisfies Assumptions 1 and 8 with a quadratic Lyapunov function $V(x_1) = \frac{1}{2}x_1^2$, $W(x_1) = x_1^2$ and $\alpha_4(s) = s$. Assumption 7 is also satisfied with $\alpha_5(s) = s$, and $\alpha_1 = \frac{1}{2}s^2$ hence from (58), with $\alpha_6(s) = 2s$. □

It is worth remarking that the practical problems of tracking control of robot manipulators with induction motors [43], and controlled synchronization of two pendula [25] mentioned in Section 2.3 fit into the class of systems considered in Theorem 3.

3.5.3 *Case 3: "The function $f_1(t, x_1)$ grows slower than $g(t, x)$"*

Theorem 4 Let Assumptions 1, 2 and 7 hold and suppose that

(Assumption 9) there exists $\alpha \in \mathcal{K}$ such that, the trajectory $x_2(t; t_\circ, x_2(t_\circ))$ of Σ_2 satisfies (56) for all $t_\circ \geq 0$.

Then, the cascade (34) is UGAS. $\square$

Example 5 Let us define the saturation function $\operatorname{sat} : \mathbb{R} \to \mathbb{R}$ as a $\mathcal{C}^2$ non-decreasing function that satisfies $\operatorname{sat}(0) = 0$, $\operatorname{sat}(\zeta)\zeta > 0$ for all $\zeta \neq 0$ and $|\operatorname{sat}(\zeta)| < 1$. For instance, we can take $\operatorname{sat}(\zeta) := \tanh(\omega\zeta)$, $\omega > 0$, or $\operatorname{sat}(\zeta) = \frac{\zeta}{1+\zeta^p}$ with p being an even integer. Consider

$$\dot{x}_1 = -\operatorname{sat}(x_1) + x_1 \ln(|x_1| + 1)x_2 \tag{61}$$

$$\dot{x}_2 = f_2(t, x_2) \tag{62}$$

where $x_1 \in \mathbb{R}$ and the system $\dot{x}_2 = f_2(t, x_2)$ is UGAS and satisfies Assumption 9. The zero input dynamics of (61), $\dot{x}_1 = -\operatorname{sat}(x_1)$, is UGAS with Lyapunov function $V(x_1) = \frac{1}{2}x_1^2$ hence, let $\alpha_1(s) = \frac{1}{2}s^2$ and $\alpha_4(s) = s$, while the function $\alpha_5(s) = s\ln(s+1)$. It is easy to verify that condition (59) holds with $\alpha_6(s) = [\ln(\sqrt{2s}+1) + 1](2s + \sqrt{2s})$. $\square$

The last example of this section illustrates the importance of the integrability condition and shows that, in general, it does *not* hold that GAS + GAS + Forward Completeness[6] $\Rightarrow$ GAS.

Example 6 Consider the autonomous system

$$\dot{x}_1 = -\operatorname{sat}(x_1) + x_1 x_2 \tag{63}$$

$$\dot{x}_2 = -x_2^3 \tag{64}$$

where the function $\operatorname{sat}(x_1)$ is defined as follows: $\operatorname{sat}(x_1) = \sin(x_1)$ if $|x_1| < \pi/2$, $\operatorname{sat}(x_1) = 1$ if $x_1 \geq \pi/2$, and $\operatorname{sat}(x_1) = -1$ if $x_1 \leq -\pi/2$. Notice that, even though Assumptions 1, 2 and 7 are satisfied and the system is forward complete, the trajectories may grow unbounded. The latter follows observing that the set $S := \{(x_1, x_2) : z \geq 0,\ x_1 \geq 2,\ 1/2 \geq x_2 \geq 0\}$ with $z = -\operatorname{sat}(x_1) + x_1 x_2 - 1$, is positively invariant. On the other hand, the solution of (64), $x_2(t) = \left(2t + \frac{1}{x_{2_\circ}^2}\right)^{-1/2}$, does not satisfy (56). $\square$

[6] That is, that the solutions $x(\cdot)$ be defined over the infinite interval.

3.5.4 Discussion

Each of the examples above, illustrates a class of systems that one can deal with using Theorems 2, 3 and 4. In this respect it shall be noticed that, even though the three theorems presented here, cover a large group of dynamical non-autonomous systems, our conditions are not necessary, hence, our main results can be improved in several directions.

Firstly, for clarity of exposition, we have assumed that the interconnection term in (34a) can be factorised as $g(t,x)x_2$; in some cases, this may be unnecessarily restrictive. With an abuse of notation, let us redefine $g(t,x)x_2 =: g(t,x)$, i.e., consider a cascaded system of the form

$$\dot{x}_1 = f_1(t,x_1) + g(t,x) \tag{65}$$
$$\dot{x}_2 = f_2(t,x_2) \tag{66}$$

where $g(t,x)$ satisfies

$$\|g(t,x)\| \leq \alpha_5'(\|x_1\|)\gamma'(\|x_2\|) + \alpha_5''(\|x_1\|)\gamma''(\|x_2\|) \tag{67}$$

where α_5', α_5'', γ' and γ'' are nondecreasing functions such that $\gamma''(s) \to 0$ as $s \to 0$ and $\alpha_5''(\|x_1\|) \leq c_1\alpha_5'(\|x_1\|)$ for all $\|x_1\| \geq c$.

Secondly, notice that Assumption 2 does not impose any condition on the convergence rate of the trajectory (input) $x_2(t,t_\circ,x_{2_\circ})$. In this respect, let $V_2(t,x_2)$ be a Lyapunov function for system (66), and consider the following Corollary which, under the assumptions of Theorems 3 and 4, establishes UGAS of the cascade.

Corollary 1 Consider the cascaded system (65), (66), (67), and suppose that Assumptions 1 —with a Lyapunov function $V_1(t,x)$ –, 2 and 7 hold with $\alpha_6(V_1) = \alpha_4(\alpha_1^{-1}(V_1))\alpha_5'(\alpha_1^{-1}(V_1))$. Assume further that $\alpha_5''(\|x_1\|)$ is majorised by the function $\frac{W(x_1)}{\alpha_4(\|x\|)}$ and the function $\gamma'(\|x_2\|)$ satisfies either of the following:

$$\gamma'(\|x_2\|) \leq \kappa(V_2(x_2))U(x_2) \tag{68}$$

where $\dot{V}_2(t,x_2) \leq -U(x_2)$ with $\kappa : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ continuous; or there exists $\phi \in \mathcal{K}$, s.t.

$$\int_{t_\circ}^{\infty} \gamma'(\|x_2(t)\|)dt \leq \phi(\|x_{2_\circ}\|)\,. \tag{69}$$

Under these conditions, the cascaded system (65), (66) is UGAS.

Further relevant remarks on the relation of the bound (68) with the order of zeroes of the *input* $x_2(t)$ in (65) and (66) were given in [28].

A last important observation concerns the restrictions on the growth rate of $g(t, x)$ with respect to x_1. Coming back to Example 5, we have seen that Theorem 4 and Corollary 1 apply to the cascaded system (65), (66) for which the interconnection term $g(t, x)$ grows slightly faster than linearly in x_1. Allowing for high order growth rates in x_1 is another interesting direction in which our contributions may be extended. This issue has already been studied for instance in [12] for *not*-ISS *autonomous* systems, where the authors established conditions (assuming that Σ_2' is a linear controllable system) under which global and semiglobal stabilisation of the cascade are *impossible*.

In this respect, it is also worth mentioning that in [30] the authors study a *feedback* interconnected *autonomous* system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g(x) & (70) \\ \dot{x}_2 &= f_2(x_1, x_2) & (71) \end{aligned}$$

under the assumption that

$$\|g(x)\| \leq \theta_1(\|x_2\|) \, \|x_1\|^k + \theta_2(\|x_2\|) \quad k \geq 1 \ . \tag{72}$$

Then, global asymptotic stability of (70), (71) can be proven by imposing the following condition on the derivative of the Lyapunov function $V_2(x_2)$, for (71):

$$\dot{V}_{2_{(71)}}(x_1, x_2) \leq -\gamma_1(x_2) - \gamma_2(x_2) \, \|x_1\|^{k-1}$$

with γ_1 and γ_2 positive definite functions.

4 Practical applications

Cascaded systems may appear in many control applications, most remarkably, in some cases a system can be decomposed in two subsystems for which control inputs can be designed with the aim that the closed loop have a cascaded structure. In this direction the results in [43] suggest that the global stabilisation of nonlinear systems which allow a cascades decomposition, may be achieved by ensuring UGAS for both subsystems separately. The question remaining is to know whether the stability properties of both subsystems separately, remains valid under the cascaded interconnection.

See also [22] and references therein, for an extensive study and a very complete work on cascaded-based control of applications including ships and nonholonomic systems. Indeed, in that reference some of the theorems presented here have been successfully used to design *simple* controllers. The term “simple” is used relative to the mathematical complexity of the expression of the control laws. Even though it is not possible to show it in general, there exists a good number of applications where controllers based on a cascaded approach are simpler than highly nonlinear Lyapunov-based control laws.

The mathematical simplicity is of obvious importance from a practical point of view since it is directly translated into "lighter" computational load in engineering implementations. See for instance [44] for an experimental comparative study of cascaded-based and backstepping-based controllers. See also [37].

In this section we present some practical applications of our theorems. These works were originally reported in [38, 41, 24]. It is not our intention to repeat these results here but to treat in more detail than our previous examples, two control *synthesis* applications. In contrast to [28, 53, 26] we do not give a design methodology, yet, we illustrate with these examples, that the control design with a closed loop cascaded structure in mind, may simplify considerably both, the control laws and the stability analysis.

4.1 Output feedback dynamic positioning of a ship

The problem we discuss here, was originally solved in [24] using the results previously proposed in [40].

We consider the following model of a surface marine vessel as in [7]:

$$\begin{aligned} M\dot{\nu} + D\nu &= \tau + J^\top(y)b && (73)\\ \dot{\eta} &= J^\top(y)\nu && (74)\\ \dot{\zeta} &= \Omega\zeta && (75)\\ \dot{b} &= Tb && (76) \end{aligned}$$

where $M = M^\top > 0$ is the constant mass of the ship, $\eta \in \mathbb{R}^3$ is the position and orientation vector with respect to an Earth-fixed reference frame, similarly to the example above. The only available state is the noisy measurement $y = \eta + \zeta$ where ζ is a noise vector obeying the slowly convergent dynamics (75). It is supposed that the ship rotates only about the yaw axis (perpendicular to the water surface) hence the rotation matrix $J(y)$ is orthogonal, i.e. $J(y)^\top J(y) \equiv I$. The matrix $D \geq 0$ is a natural damping and the bias b represents the force of environmental perturbations, such as wind, waves, etc. The dynamics (76) is also assumed to be asymptotically stable, but slowly-converging.

The goal is to design an *output feedback* control law τ, to maintain the vessel stable at the origin, while filtering out noise and disturbances. The approach followed in [24] was to design a state observer based on the output measurement y and a state feedback controller of a classical PD-type as used in the robotics literature. As in the previous example, to avoid redundancy we give here only the main ideas where the cascades approach, via theorems like those presented here have been fundamental.

Let us firstly introduce the notation $x_1 = \ \mathrm{col}[\nu,\ \eta,\ \zeta,\ b]$ for the position error state, that is, the desired set-point (hence $\nu \equiv 0$) is the origin $\ \mathrm{col}[\eta,\ \zeta,\ b] = 0$, therefore the error and actual state are taken to be the same. With this in mind, notice that the system

(73)–(76) is linear, except for the Jacobian matrix $J(y)$, thus the dynamic equations can be written in the compact form

$$\dot{x}_1 = A(y)x_1. \tag{77}$$

One can also verify that the closed loop system (77) with the *state* feedback $\tau = -K(y)x_1$, or in expanded form,

$$\tau = -J^\top(y)b - K_d\nu - J^\top(y)K_p\eta \tag{78}$$

with K_p, $K_d > 0$, is GAS. This follows by writing the closed loop equations in the compact form $\dot{x}_1 = A_c(y)x_1$ with $A_c(y) := (A(y) - BK(y))$ and using the Lyapunov function candidate $V_1(x_1) = x_1^\top P_c x_1$ (with P_c constant and positive definite) whose time derivative is negative *semi*definite. not necessarily (41).

On the other hand, in [8], an observer of the form

$$\dot{\hat{x}}_1 = A(y)\hat{x}_1 - L(y - \hat{y}) + B\tau\,, \tag{79}$$

where L is a design parameters matrix of suitable dimensions and $\hat{(\cdot)}$ stands for the "estimate of" $(\cdot)$, was proposed. This observer in closed loop with (77) yields

$$\underbrace{(\dot{x}_1 - \dot{\hat{x}}_1)}_{\dot{x}_2} = \underbrace{(A(y) - LC)}_{\bar{A}_o(y)}\underbrace{(x_1 - \hat{x}_1)}_{x_2} \tag{80}$$

where $C := [0,\, I,\, I,\, 0]$. In [8], using the Kalman-Yacubovich-Popov lemma, it is shown that (80) can be made globally exponentially stable, uniformly in the trajectories $y(t)$, hence, (after proving completeness of the x_1-subsystem) the estimation error equation (80) can be rewritten as a linear time-varying system $\dot{x}_2 = A_o(t)x_2$ where $A_o(t) := \bar{A}_o(y(t))$.

Finally, since it is not desirable to implement the controller (78) using state measurements, we use

$$\tau = -J^\top(y)b - K_d\hat{\nu} - J^\top(y)K_p\hat{\eta}\,. \tag{81}$$

In summary we have that the overall controller-observer-boat closed loop system (77), (79) and (81) has the following desired cascaded structure:

$$\begin{aligned} \Sigma_1 \quad &: \quad \dot{x}_1 = A_c(x_1)x_1 + g(x_1)x_2 \\ \Sigma_2 \quad &: \quad \dot{x}_2 = A_o(t)x_2 \end{aligned}$$

where $g(x_1)x_2 = J^\top(y)\bar{b} - K_d\bar{\nu} - J^\top(y)K_p\bar{\eta}$ where $\bar{(\cdot)} := (\cdot) - \hat{(\cdot)}$, which is uniformly bounded in x_1 since $J(\cdot)$ is uniformly bounded and $y := h(x_1)$.

Therefore, GAS for the closed loop system can be concluded invoking any of the three theorems of Section 3.5, based on the stabilisation property of the *state*-feedback (78) and the filtering properties of the observer (79). An immediate interesting consequence is that both, the observer and the controller can be tuned *separately*.

For further details and experimental results, see [24].

We also invite the reader to consult [22] for other simple cascaded-based controllers for ships. Specifically, one must remark the mathematical simplicity of controllers obtained using the cascades approach in contrast to the complexity of some backstepping designs. This has been clearly put in perspective in [22, Appendix A] where the 2782 (!) terms of a backstepping controller are written explicitly.

4.2 Pressure stabilisation of a turbo-diesel engine

4.2.1 Model and problem formulation

To further illustrate the utility of our main results we consider next the set-point control problem for the simplified emission VGT-EGR diesel engine depicted in Figure 3.

The simplified model structure consists of two dynamic equations derived by differentiation of ideal gas flow ([32]); they describe the intake pressure dynamics p_1 and the exhaust pressure p_2 dynamics under the assumption of time-independent temperatures. The third equation describes the dynamics of the compressor power P_c:

$$\dot{p}_1 = k_1(w_c + w_{egr} - k_{1e}p_1) \tag{82}$$

$$\dot{p}_2 = k_2(k_{1e}p_1 + w_f - w_{egr} - w_{turb}) \tag{83}$$

$$\dot{P}_c = \frac{1}{\tau_c}(-P_c + K_t(1 - p_2^{-\mu})w_{turb}) \tag{84}$$

where the fuel flow rate w_f and $k_1,\ k_2,\ k_{1e},\ K_t,\ \tau_c$ are assumed to be positive constants. The control inputs are the back flow rate to the intake manifold w_{egr} and the flow rate through the turbine w_{turb}. The outputs to be controlled are the EGR flow rate w_{egr} and the compressor flow rate $w_c = K_c\frac{P_c}{p_1^{\mu}-1}$, where the constants $K_c > 0$ and $0 < \mu < 1$. The compressor flow rate, intake and exhaust pressures are supposed to be the measurable outputs of the system, i.e. $z = col(p_1, p_2, w_c)$.

From practical considerations it's reasonable to assume that $p_1, p_2 > 1$ [32]. As a matter of fact, the region of possible initial conditions $p_1(0) > 1,\ p_2(0) > 1,\ P_c(0) > 0$ is always known in practice, hence one can always choose the controller gains to ensure that $p_1, p_2 > 1$ for all $t \geq 0$ and thus to relax our practical assumption.

Under these conditions, the control objective is to assure asymptotic stabilisation of the desired setpoint $y_d = \operatorname{col}(w_{c,d},\, w_{egr,d})$ for the controlled output $y = \operatorname{col}(w_c, w_{egr})$.

4.2.2 Controller design

The approach undertaken here consists on performing a suitable change of coordinates and designing decoupling control laws in order to put the controlled system into a cascaded form. Then, instead of looking for a Lyapunov function for the overall system we investigate the stability properties of both subsystems separately and exploit the structure of the interconnection, we do this by verifying the conditions of Theorem 2 which allow us to claim global uniform asymptotic stability.

First it should be noted that, as shown in [32], fixing the outputs to their desired values $w_{c,d}$, $w_{egr,d}$ corresponds to the following equilibrium position of the diesel engine

$$p_{1*} = \frac{1}{k_{1e}}(w_{c,d} + w_{egr,d}) \tag{85}$$

$$p_{2*} = \left(1 - \frac{w_{c,d}}{K_c K_t (w_{c,d} + w_f)}(p_{1*}^{\mu} - 1)\right)^{-\frac{1}{\mu}} \tag{86}$$

$$P_c^* = \frac{w_{c,d}}{K_c}(p_{1*}^{\mu} - 1), \tag{87}$$

in other words, the stabilisation problem of the output y to y_d reduces to the problem of stabilising the equilibrium position p_{1*}, p_{2*}, P_{c*}.

Next let us introduce the following change of variables

$$\begin{aligned} \tilde{p}_1 &= p_1 - p_{1*} \\ \tilde{p}_2 &= p_2 - p_{2*} \\ \tilde{p}_c &= P_c - P_c^* \end{aligned} \tag{88}$$

$$\begin{aligned} w_{egr} &= w_{egr,d} + u_1 \\ w_{turb} &= w_{c,d} + w_f + u_2 \end{aligned} \tag{89}$$

which will appear more suitable for our analysis. In these new coordinates and using (85)–(87) the system (82) –(84) with the new measurable output $z' =$ col$[\tilde{p}_1 - p_{1*}, \tilde{p}_2 - p_{2*}, w_c - w_{c,d}]$ takes the form

$$\dot{\tilde{p}}_1 \;=\; k_1(z_3' - k_{1e}\tilde{p}_1 + u_1) \tag{90}$$

$$\dot{\tilde{p}}_2 \;=\; k_2(k_{1e}\tilde{p}_1 - u_1 - u_2) \tag{91}$$

$$\dot{\tilde{p}}_c \;=\; \frac{1}{\tau_c}\Big[-\tilde{p}_c + K_t\left(p_{2*}^{-\mu} - (\tilde{p}_2 + p_{2*})^{-\mu}\right)(w_{c,d} + w_f) + \tag{92}$$

$$K_t\left(1 - (\tilde{p}_2 + p_{2*})^{-\mu}\right)u_2\Big]. \tag{93}$$

In order to apply our cascaded approach, notice that system (90)–(91) has the required cascaded form with Σ_1 being equation (92) and Σ_2 the pressure subsystem represented by equations (90) and (91).

Let us first consider the subsystem Σ_2 and let the control input u be

$$u_1 = -z_3' - \gamma_1 \tilde{p}_1 - \gamma_2 \tilde{p}_2 \tag{94}$$
$$u_2 = z_3' + \gamma_1 \tilde{p}_1 + \gamma_2' \tilde{p}_2 \tag{95}$$

where $\gamma_1, \gamma_2, \gamma_2'$ are arbitrary constants with the property $\gamma_2 < \gamma_2'$. Using the Lyapunov function candidate $V(\tilde{p}_1, \tilde{p}_2) = \frac{1}{2}\tilde{p}_1^2 + \frac{c}{2}\tilde{p}_2^2$ one can easily show that the closed loop system Σ_2 with (94,95) is globally exponentially stable uniformly in $\tilde{p}_c$.

To this point it is important to remark that this closed loop system actually depends on the variable $\tilde{p}_c$ due to the presence of z_3' in the control inputs. However, the *uniform* character of the stability property established above implies that for any initial conditions, the signal $w_c(t)$ "inside" z_3' simply adds a time-varying character to the closed loop system Σ_2 with (94,95) and hence it becomes non-autonomous. This allows us to consider these feedback interconnected systems as a *cascade* of an autonomous and a non autonomous nonlinear system.

Having established the stability property of system Σ_2 we proceed to investigate the properties of Σ_1 in closed loop with u_2. Substituting u_2 defined by (95) in (92) and after some lengthy but straightforward calculations involving the identity

$$1 - p_{2*}^{-\mu} = \frac{1}{K_c K_t} \frac{w_{c,d}}{w_{c,d} + w_f} (p_{1*}^{\mu} - 1)$$

we obtain

$$\begin{aligned}\dot{\tilde{p}}_c \;=\; & \underbrace{-\frac{1}{\tau_c}\frac{w_f}{w_{c,d}+w_f}\tilde{p}_c}_{f(x_1)} + \frac{1}{\tau_c}\left[\frac{w_{c,d}}{w_{c,d}+w_f}\frac{[\tilde{p}_c + P_{c*}][p_{1*}^{\mu} - (\tilde{p}_1 + p_{1*})^{\mu}]}{(\tilde{p}_1 + p_{1*})^{\mu} - 1} + \right.\\ & \left. k_t\left(1 - (\tilde{p}_2 + p_{2*})^{-\mu}\right)(\gamma_1 \tilde{p}_1 + \gamma_2' \tilde{p}_2) + K_t(w_{c,d} + w_f + z_3')\frac{(\tilde{p}_2 + p_{2*})^{\mu} - p_{2*}^{\mu}}{p_{2*}^{\mu}(\tilde{p}_2 + p_{2*})^{\mu}}\right]\end{aligned}$$

which in compact form can be written as

$$\dot{x}_1 = f(x_1) + g(x_1, x_2)$$

where we recall that $x_1 = \tilde{p}_c$, $x_2 = \text{col}(\tilde{p}_1, \tilde{p}_2)$. Notice that $g(x_1, x_2) \equiv 0$ if $x_2 = 0$ and $\dot{x}_1 = f(x_1)$ is GES with a quadratic Lyapunov function satisfying Assumption 3.

Since $(\tilde{p}_1 + p_{1*}) > 1$, $(\tilde{p}_2 + p_{2*}) > 1$ for all $t \geq 0$ and $0 < \mu < 1$ one can show that $g(x_1, x_2)$ is continuously differentiable and moreover notice that it grows linearly in x_1 (i.e., $\tilde{p}_c$) hence it satisfies the bound (55). Since $x_2 = 0$ is GES $x_2(t)$ satisfies (56) with $\alpha(s) := \alpha s$, $\alpha > 0$. Thus all the conditions of Theorem 1 and Proposition 3 are satisfied and therefore the overall system is UGES.

4.3 Nonholonomic systems

In recent years the control of nonholonomic dynamic systems has received considerable attention, in particular the stabilisation problem. One of the reasons for this is that no smooth stabilizing state-feedback control law exists for these systems, since Brockett's necessary condition for smooth asymptotic stabilisability is not met [3]. For an overview we refer to the survey paper [20] and references cited therein. In contrast to the stabilisation problem, the tracking control problem for nonholonomic control systems has received little attention. In [6, 17, 31, 34, 35] tracking control schemes have been proposed based on linearisation of the corresponding error model. In [1, 47] the feedback design issue was addressed via a dynamic feedback linearisation approach. All these papers solve the local tracking problem for some classes of nonholonomic systems. Some global tracking results that we are presented in [49, 16, 13].

Recently, the results in [16] have been extended to arbitrary chained form nonholonomic systems [15]. The proposed backstepping-based recursive design turned out to be useful for simplified dynamic models of such chained form systems, see [16, 15]. However, it is clear that the technique used in [16] (and [15]) does not exploit the physical structure behind the model, and then the controllers may become quite complicated and computationally demanding when computed in original coordinates.

The purpose of this section is to show that the nonlinear controllers proposed in [16] can be simplified into *linear* controller for both the kinematic model and an 'integrated' simplified dynamic model of the mobile robot. Our approach is based on cascaded systems. As a result, instead of exponential stability for small initial errors as in [16], the controllers proposed here yield exponential stability for initial errors in a ball of arbitrary radius.

For a more detailed study on tracking control of nonholonomic systems based on the stability theorems for cascades presented here, see [22]. Indeed, the material presented in this section was originally reported in [38] and later in [22].

4.3.1 Model and problem-formulation

A kinematic model of a wheeled mobile robot with two degrees of freedom is given by the following equations

$$\begin{aligned} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega \end{aligned} \tag{96}$$

where the forward velocity v and the angular velocity ω are considered as inputs, (x, y) is the center of the rear axis of the vehicle, and θ is the angle between heading direction and x-axis (see Fig. 4).

Consider the problem of tracking a reference robot as done by Kanayama et al [17]:

$$\begin{aligned}\dot{x}_r &= v_r \cos\theta_r\\ \dot{y}_r &= v_r \sin\theta_r\\ \dot{\theta}_r &= \omega_r\,.\end{aligned}$$

Following [17] we define the error coordinates (see Fig. 5)

$$\begin{bmatrix} x_e\\ y_e\\ \theta_e\end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} x_r - x\\ y_r - y\\ \theta_r-\theta\end{bmatrix}.$$

It is easy to verify that in these new coordinates the error dynamics become

$$\begin{aligned}\dot{x}_e &= \omega y_e - v + v_r\cos\theta_e\\ \dot{y}_e &= -\omega x_e + v_r\sin\theta_e\\ \dot{\theta}_e &= \omega_r - \omega\,.\end{aligned} \tag{97}$$

Our aim is to find appropriate velocity control laws v and ω of the form

$$\begin{aligned} v &= v(t, x_e, y_e, \theta_e)\\ \omega &= \omega(t, x_e, y_e, \theta_e)\end{aligned} \tag{98}$$

such that the closed-loop trajectories of (97,98) are exponentially stable in any ball.

4.3.2 Controller design

The approach used in [16] is based on the integrator backstepping idea [19, 4, 60, 26] which consists of searching a stabilizing function for a subsystem of (97), assuming the remaining variables to be controls. Then new variables are defined, describing the difference between this desired dynamics and the real dynamics. Subsequently a stabilizing controller for this 'new system' is looked for.

This approach has the advantage that it can lead to globally stabilizing controllers. A disadvantage, however, is that they may cancel or compensate for high order nonlinearities yielding unnecessarily complicated control laws. The main reason for this is that the stability of a 'new system' is studied using a Lyapunov function expressed in 'new coordinates'. A result of this is that the controller also is expressed in these 'new coordinates'. When written in the original coordinates usually complex expressions are obtained.

To arrive to simple controllers our approach is different. We find our inspiration in the potentiality of our theorems for cascades. The goal is to subdivide the tracking control problem into two simpler and 'independent' problems: for instance, positioning and orientation. More precisely, we search for a subsystem of the form $\dot{y} = f_2(t, y)$ that is asymptotically stable. In the remaining dynamics we then can replace the appearance of this y by 0, leading to the system $\dot{x} = f_1(t, x)$. If this system is asymptotically stable we might be able to conclude asymptotic stability of the overall system.

Consider the error dynamics (97):

$$\begin{aligned} \dot{x}_e &= \omega y_e - v + v_r \cos\theta_e && (99) \\ \dot{y}_e &= -\omega x_e + v_r \sin\theta_e && (100) \\ \dot{\theta}_e &= \omega_r - \omega && (101) \end{aligned}$$

Firstly, we can easily stabilize mobile car's orientation change rate, that is the linear equation (101), by using the linear controller

$$\omega = \omega_r + c_1\theta_e \tag{102}$$

which yields GES for θ_e, provided $c_1 > 0$.

If we now replace θ_e by 0 in (99,100) we obtain

$$\begin{aligned} \dot{x}_e &= \omega_r y_e - v + v_r \\ \dot{y}_e &= -\omega_r x_e \end{aligned} \tag{103}$$

where we used (102). Concerning the positioning of the cart, if we choose the linear controller

$$v = v_r + c_2 x_e \tag{104}$$

where $c_2 > 0$, we obtain for the closed-loop system (103,104):

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \end{bmatrix} = \begin{bmatrix} -c_2 & \omega_r(t) \\ -\omega_r(t) & 0 \end{bmatrix} \begin{bmatrix} x_e \\ y_e \end{bmatrix} \tag{105}$$

which, as it is well known in the literature of adaptive control (see e.g. [2, 10]) is asymptotically stable if $\omega_r(t)$ is persistently exciting (PE), i.e., there exist T, $\mu > 0$ such that $\omega_r(t)$ satisfies

$$\int_t^{t+T} \omega_r(\tau)^2 d\tau \geq \mu \quad \forall\, t \geq 0$$

and $c_2 > 0$. The following proposition makes this result rigorous.

Proposition 5 Consider the system (97) in closed-loop with the controller

$$\begin{aligned} v &= v_r + c_2 x_e \\ \omega &= \omega_r + c_1\theta_e \end{aligned} \tag{106}$$

where $c_1 > 0$, $c_2 > 0$. If $\omega_r(t)$, $\dot{\omega}_r(t)$, and $v_r(t)$ are bounded and ω_r is PE then, the closed-loop system (97,106) is exponentially stable in any ball. □

Proof. Observing that

$$\sin\theta_e = \theta_e \int_0^1 \cos(s\theta_e)ds \qquad \text{and} \qquad 1-\cos\theta_e = \theta_e \int_0^1 \sin(s\theta_e)ds$$

we can write the closed-loop system (97,106) as

$$\begin{aligned} \begin{bmatrix} \dot{x}_e \\ \dot{y}_e \end{bmatrix} &= \begin{bmatrix} -c_2 & \omega_r(t) \\ -\omega_r(t) & 0 \end{bmatrix} \begin{bmatrix} x_e \\ y_e \end{bmatrix} + \begin{bmatrix} v_r \int_0^1 \sin(s\theta_e)ds + c_1 y_e \\ v_r \int_0^1 \cos(s\theta_e)ds - c_1 x_e \end{bmatrix} \theta_e \\ \dot{\theta}_e &= -c_1\theta_e \end{aligned} \tag{107}$$

which is of the form (34) with $x_1 := (x_e, y_e)^\top$, $x_2 := \theta_e$, $f_2(t, x_2) = -c_1\theta_e$,

$$f_1(t, x_1) = \begin{bmatrix} -c_2 & \omega_r(t) \\ -\omega_r(t) & 0 \end{bmatrix} \begin{bmatrix} x_e \\ y_e \end{bmatrix}, \; g(t, x) = \begin{bmatrix} v_r \int_0^1 \sin(s\theta_e)ds + c_1 y_e \\ v_r \int_0^1 \cos(s\theta_e)ds - c_1 x_e \end{bmatrix}$$

To be able to apply Theorem 1 we need to verify the following three conditions

- Assumption on Σ_1: Due to the assumptions on $\omega_r(t)$ we have that $\dot{x} = f_1(t, x_1)$ is GES and therefore UGAS. From converse Lyapunov theory (see e.g. [18] or the proof of Proposition 3 in [38]) the existence of a suitable $V(t, x_1)$ is guaranteed.
- Assumption on the interconnection term: Since $|v_r(t)| \le v_r^{max}$ for all $t \ge 0$ we have:
$$\|g(t, x)\| \le v_r^{max}\sqrt{2} + c_1\|x_1\|.$$
- Assumption on Σ_2: Follows from GES of (101) in closed loop with (102).

Therefore, we can conclude UGAS from Theorem 1. Since both Σ_1 and Σ_2 are GES, Proposition 3 gives the desired result. ■

Remark 8 It is interesting to notice the link between the tracking condition that the reference trajectory should not converge to a point and the well known persistence-of-excitation condition in adaptive control theory. More precisely, we could think of (105) as a controlled system with state x_e, parameter estimation error y_e and regressor, the *reference* trajectory ω_r. □

Remark 9 It is important to remark that the cascaded decomposition used above is not unique. One may find other ways to subdivide the original system, for which different control laws may be found. Notice however that the structure we have used has an interesting physical interpretation: roughly speaking we have proved that the positioning and the orientation of the cart can be controlled *independently* of each other. □

4.3.3 A simplified dynamic model

In this section we consider the dynamic extension of (97) as studied in [16]:

$$\begin{array}{rcl} \dot{x}_e &=& \omega y_e - v + v_r \cos\theta_e \\ \dot{y}_e &=& -\omega x_e + v_r \sin\theta_e \\ \dot{\theta}_e &=& \omega_r - \omega \\ \dot{v} &=& u_1 \\ \dot{\omega} &=& u_2 \end{array} \tag{108}$$

where u_1 and u_2 are regarded as torques or generalized force variables for the two-degree-of-freedom mobile robot.

Our aim is to find a control law $u = (u_1, u_2)^T$ of the form

$$\begin{array}{rcl} u_1 &=& u_1(t, x_e, y_e, \theta_e, v, \omega) \\ u_2 &=& u_2(t, x_e, y_e, \theta_e, v, \omega) \end{array} \tag{109}$$

such that the closed-loop trajectories of (108,109) are exponentially stable in any ball.

To solve this problem we first define

$$\begin{array}{rcl} v_e &=& v - v_r \\ \omega_e &=& \omega - \omega_r \end{array}$$

which leads to

$$\begin{bmatrix} \dot{x}_e \\ \dot{v}_e \\ \dot{y}_e \end{bmatrix} = \begin{bmatrix} 0 & -1 & \omega_r(t) \\ 0 & 0 & 0 \\ -\omega_r(t) & 0 & 0 \end{bmatrix} \begin{bmatrix} x_e \\ v_e \\ y_e \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} (u_1 - \dot{v}_r) + \begin{bmatrix} v_r \int_0^1 \sin(s\theta_e) ds & y_e \\ 0 & 0 \\ v_r \int_0^1 \cos(s\theta_e) ds & -x_e \end{bmatrix} \begin{bmatrix} \theta_e \\ \omega_e \end{bmatrix} \tag{110}$$

$$\begin{bmatrix} \dot{\theta}_e \\ \dot{\omega}_e \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_e \\ \omega_e \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u_2 - \dot{\omega}_r) \tag{111}$$

in which we again recognize a cascaded structure similar to the one in the previous section. We only need to find u_1 and u_2 such that the systems

$$\begin{bmatrix} \dot{x}_e \\ \dot{v}_e \\ \dot{y}_e \end{bmatrix} = \begin{bmatrix} 0 & -1 & \omega_r(t) \\ 0 & 0 & 0 \\ -\omega_r(t) & 0 & 0 \end{bmatrix} \begin{bmatrix} x_e \\ v_e \\ y_e \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_1$$

and

$$\begin{bmatrix} \dot{\theta}_e \\ \dot{\omega}_e \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_e \\ \omega_e \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2$$

are exponentially stable in any ball. In light of the previous section, that is not too difficult.

Proposition 6 Consider the system (108) in closed-loop with the controller

$$u_1 = \dot{v}_r + c_3 x_e - c_4 v_e \quad (112a)$$

$$u_2 = \dot{\omega}_r + c_5 \theta_e - c_6 \omega_e \quad (112b)$$

where $c_3 > 0$, $c_4 > 0$, $c_5 > 0$, $c_6 > 0$. If $\omega_r(t)$, $\dot{\omega}_r(t)$ and $v_r(t)$ are bounded and ω_r is PE then, the closed-loop system (108, 112) is exponentially stable in any ball. □

Proof. The closed-loop system (108, 112) can be written as

$$\begin{bmatrix} \dot{x}_e \\ \dot{v}_e \\ \dot{y}_e \end{bmatrix} = \begin{bmatrix} 0 & -1 & \omega_r(t) \\ c_3 & -c_4 & 0 \\ -\omega_r(t) & 0 & 0 \end{bmatrix} \begin{bmatrix} x_e \\ v_e \\ y_e \end{bmatrix} + \begin{bmatrix} v_r \int_0^1 \sin(s\theta_e) ds & y_e \\ 0 & 0 \\ v_r \int_0^1 \cos(s\theta_e) ds & -x_e \end{bmatrix} \begin{bmatrix} \theta_e \\ \omega_e \end{bmatrix}$$

$$\begin{bmatrix} \dot{\theta}_e \\ \dot{\omega}_e \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ c_5 & -c_6 \end{bmatrix} \begin{bmatrix} \theta_e \\ \omega_e \end{bmatrix}$$

which is of the form (34). We again have to verify the three assumptions of Theorem 1:

- Assumption on Σ_1: This system is GES (and therefore UGAS) under the assumptions imposed on $\omega_r(t)$ and c_2. The existence of a suitable V again follows from converse Lyapunov theory.
- Assumption on the interconnection term: Since $|v_r(t)| \le v_r^{max}$ for all $t \ge 0$ we have:

$$\|g(t,x,y)\| \le v_r^{max}\sqrt{2} + \|x\|.$$

- Assumption on Σ_2: follows from GES of Σ_2.

Therefore we can conclude UGAS from Theorem 1. Since both Σ_1 and Σ_2 are GES, Proposition 3 gives the desired result. ■

5 Conclusions

Motivated by practical problems such as global tracking of time-varying trajectories we have studied the stability analysis problem of cascaded nonlinear non-autonomous systems. Our contributions establish relations between sufficient conditions to ensure global uniform asymptotic stability of cascaded nonlinear systems.

We have illustrated the technique of cascaded-based control through different applications.

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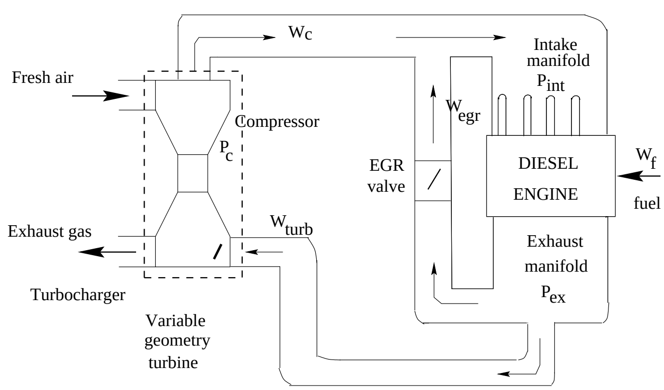

Figure 3: Turbo charged VGT-EGR diesel engine.

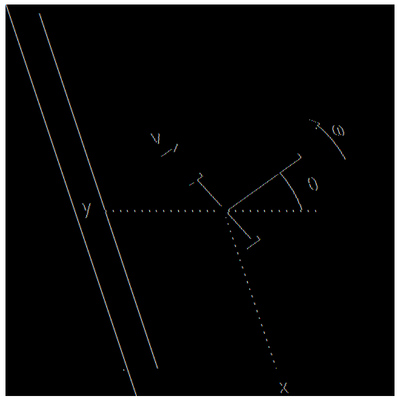

Figure 4: The mobile car.

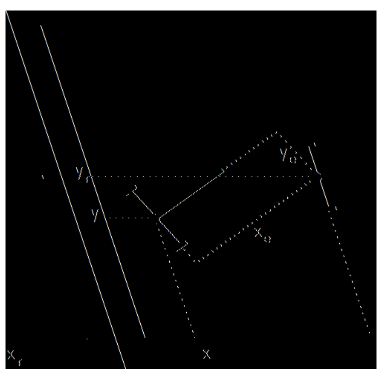

Figure 5: The error dynamics.