

## TTT4120 Digital Signal Processing Fall 2017

### Filtering of Discrete Random Signals

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## Lecture in course book\*

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- Proakis, Manolakis Digital Signal Processing, 4<sup>th</sup> Ed.
  - 12.1 Random signals, correlation functions, and power spectra
  - 5.3 Correlation functions and spectra at the output of LTI systems
- A comprehensive overview of topics treated in the lecture, see [“Introduksjon til statistisk signalbehandling”](#) on ItsLearning

\*Level of detail is defined by lectures and problem sets

## Contents and learning outcomes

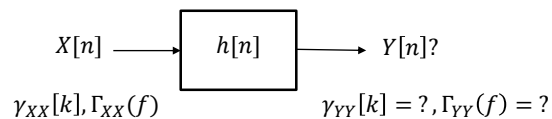
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- Filtering of stochastic signals in time-domain
- Frequency-domain interpretation
- Example: Power density spectrum of AR(1) process

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## Filtering of stochastic signals

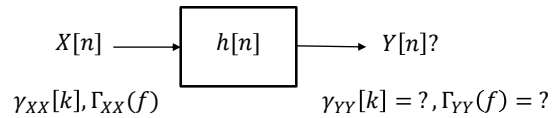
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- Let  $X[n]$  be a wide-sense stationary process
- Linear time-invariant filter described by  $h[n]$ ,  $H(z)$ , or  $H(f)$
- Can we relate output signal  $Y[n]$  to input signal  $X[n]$ ?

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## Filtering of stochastic signals...

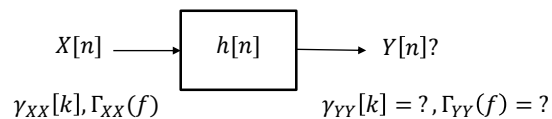


- Consider a single realization  $x[n]$  of process  $X[n]$
- Each input realization  $x[n]$  produces output realization  $y[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

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## Filtering of stochastic signals...



- Since  $x[n]$  is a realization of  $X[n]$ ,  $y[n]$  is a realization of the random process  $Y[n]$

$$Y[n] = \sum_{k=-\infty}^{\infty} h[k]X[n-k]$$

- We want to relate the statistical properties of output process  $Y[n]$  to the statistical properties of input process  $X[n]$

$$E\{Y[n]\} = ?, \gamma_{YY}[k] = ?, \Gamma_{YY}(f) = ?$$

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## Filtering of stochastic signals...

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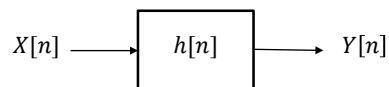
- Expected value of output process  $Y[n]$ :

$$\begin{aligned}
 m_Y &= E\{Y[n]\} = E\left\{\sum_{k=-\infty}^{\infty} h[k]X[n-k]\right\} \\
 &= \sum_{k=-\infty}^{\infty} h[k]E\{X[n-k]\} \\
 &= m_X \sum_{k=-\infty}^{\infty} h[k] \\
 &= m_X \sum_{k=-\infty}^{\infty} h[k]e^{j2\pi 0k} \\
 &= m_X H(0)
 \end{aligned}$$

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## Filtering of stochastic signals...

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- Example: WSS signal  $X[n]$  with mean  $m_X = 3$  is filtered by LTI system  $H(f) = \frac{1}{1-0.5e^{-j2\pi f}}$ . Compute the mean of output process  $Y[n]$ .

$$m_Y = E\{Y[n]\} = m_X H(0) = \frac{3}{1-0.5e^{-j2\pi 0}} = 6$$

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## Filtering of stochastic signals...

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- Autocorrelation sequence of output process  $Y[n]$ :

$$\begin{aligned}\gamma_{YY}[l] &= E\{Y[n]Y[n+l]\} = h[-l] * h[l] * \gamma_{XX}[l] \\ &= r_{hh}[l] * \gamma_{XX}[l]\end{aligned}$$

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## Filtering of stochastic signals...

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- Proof:

$$\begin{aligned}\gamma_{YY}[l] &= E\left\{\left(\sum_{i=-\infty}^{\infty} h[i]X[n-i]\right)\left(\sum_{j=-\infty}^{\infty} h[j]X[n+l-j]\right)\right\} \\ &= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h[i]h[j]E\{X[\textcolor{red}{n-i}]X[\textcolor{red}{n+l-j}]\} \\ &= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h[i]h[j]\gamma_{XX}[l-j+i] \\ &= \sum_{i=-\infty}^{\infty} h[i] \sum_{j=-\infty}^{\infty} \textcolor{red}{h[j]\gamma_{XX}[(l+i)-j]} \\ &= \sum_{i=-\infty}^{\infty} h[i] \textcolor{red}{g[l+i]} \text{ with } g[l] = h[l] * \gamma_{XX}[l] \\ &= \sum_{k=-\infty}^{\infty} h[-k]g[l-k] = h[-l] * g[l]\end{aligned}$$

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## Filtering of stochastic signals...

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- Power density spectrum of  $Y[n]$ :

$$\Gamma_{YY}(f) = \mathcal{F}\{\gamma_{YY}[k]\} = \mathcal{F}\{r_{hh}[k] * \gamma_{XX}[k]\}$$

$$= \mathcal{F}\{r_{hh}[k]\} \mathcal{F}\{\gamma_{XX}[k]\}$$

$$= S_{hh}(f) \Gamma_{XX}(f) = |H(f)|^2 \Gamma_{XX}(f)$$

- The output PDS is the input PDS multiplied by the magnitude-squared of the frequency response!

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## Filtering of stochastic signals...

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- Example: What is the output power  $E\{Y^2[n]\}$  of a linear system  $H(f)$  when the input  $X[n]$  is WGN?

$$\Gamma_{YY}(f) = |H(f)|^2 \Gamma_{XX}(f) = |H(f)|^2 \mathcal{F}\{\sigma_X^2 \delta[n]\}$$

$$= \sigma_X^2 |H(f)|^2$$

$$\sigma_Y^2 = E\{Y^2[n]\} = \gamma_{YY}[0]$$

$$= \int_{-0.5}^{0.5} \Gamma_{YY}(f) e^{j2\pi f \cdot 0} df = \sigma_X^2 \int_{-0.5}^{0.5} |H(f)|^2 df$$

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## Filtering of stochastic signals...

- Crosscorrelation sequence of processes  $Y[n]$  and  $X[n]$ :

$$\begin{aligned}\gamma_{YX}[l] &= E\{(\sum_{k=-\infty}^{\infty} h[k]X[n-k])X[n-l]\} \\ &= \sum_{k=-\infty}^{\infty} h[k]E\{X[n-k]X[n-l]\} \\ &= \sum_{k=-\infty}^{\infty} h[k]\gamma_{XX}[l-k] = h[l] * \gamma_{XX}[l]\end{aligned}$$

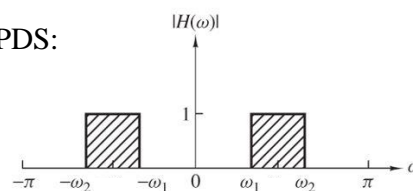
- Crosscorrelation spectrum of processes  $Y[n]$  and  $X[n]$ :

$$\begin{aligned}\Gamma_{YX}(f) &= \mathcal{F}\{h[k] * \gamma_{XX}[l]\} = \mathcal{F}\{h[k]\}\mathcal{F}\{\gamma_{XX}[k]\} \\ &= H(f)\Gamma_{XX}(f)\end{aligned}$$

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## Frequency domain interpretation

Interpretation PDS:



- Filter with narrow frequency band  $\Rightarrow \Gamma_{YY}(f) = |H(f)|^2 \Gamma_{XX}(f)$
- Average output power

$$\begin{aligned}E\{Y^2[n]\} &= \gamma_{YY}[0] = \int_{-0.5}^{0.5} \Gamma_{YY}(f) df \\ &= \int_{\frac{-\omega_1}{2\pi}}^{\frac{-\omega_2}{2\pi}} |H(f)|^2 \Gamma_{XX}(f) df + \int_{\frac{\omega_1}{2\pi}}^{\frac{\omega_2}{2\pi}} |H(f)|^2 \Gamma_{XX}(f) df\end{aligned}$$

- Area under  $\Gamma_{XX}(f)$  for  $\omega_1 \leq |\omega| \leq \omega_2$  is the average power for that frequency band  $\Rightarrow \Gamma_{XX}(f)$  can be viewed as density function for power in frequency domain

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## Example: PDS of AR(1) process

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- Example: Calculate  $\Gamma_{XX}(f)$  for the random process

$$X[n] = aX[n-1] + W[n], W[n] \sim N(0, \sigma_W^2)$$

$$\text{WGN: } E\{W[n]W[n+l]\} = \sigma_W^2 \delta[l]$$

- Approach 1: Calculate  $\gamma_{XX}[l]$  and take its Fourier transform
- Approach 2: Use the idea of LTI systems

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## Example: PDS of AR(1) process...

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- Approach 1: Calculate  $\gamma_{XX}[l]$  and take its Fourier transform
- Consider lag  $l = 0$ :

$$\begin{aligned} E\{X[n]X[n]\} &= \gamma_{XX}[0] = E\{(aX[n-1] + W[n])^2\} \\ &= E\{a^2X^2[n-1] + 2aX[n-1]W[n] + W^2[n]\} \\ &= a^2E\{X^2[n-1]\} + 2aE\{X[n-1]W[n]\} + E\{W^2[n]\} \\ &= a^2\gamma_{XX}[0] + 0 + \sigma_W^2 \end{aligned}$$

$$\Rightarrow \gamma_{XX}[0] = \frac{1}{1-a^2} \sigma_W^2$$

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## Example: PDS of AR(1) process...

- Consider lag  $l \geq 1$ :

$$\begin{aligned}
 \gamma_{XX}[l] &= E\{X[n]X[n+l]\} \\
 &= E\{X[n](aX[n+l-1] + W[n+l])\} \\
 &= E\{X[n](a^l X[n] + \sum_{j=0}^{l-1} a^j W[n+l-j])\} \\
 &= a^l E\{X^2[n]\} + \sum_{j=0}^{l-1} a^j E\{W[n+l-j]\} \\
 &= a^l \gamma_{XX}[0] + 0 = \frac{a^l}{1-a^2} \sigma_W^2
 \end{aligned}$$

- Symmetry  $\gamma_{XX}[l] = \gamma_{XX}[-l]$  provides the final answer

$$\gamma_{XX}[l] = \frac{a^{|l|}}{1-a^2} \sigma_W^2$$

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## Example: PDS of AR(1) process...

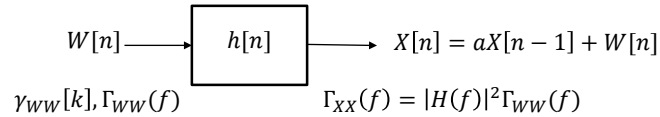
- Take the Fourier transform:

$$\begin{aligned}
 \Gamma_{XX}(f) &= \mathcal{F}\{\gamma_{XX}[l]\} \\
 &= \sum_{l=-\infty}^{\infty} \frac{a^{|l|} \sigma_W^2}{1-a^2} e^{-j2\pi f l} \\
 &= \frac{\sigma_W^2}{1-a^2} \sum_{l=-\infty}^{\infty} a^{|l|} e^{-j2\pi f l} \\
 &= \dots = \frac{\sigma_W^2}{1-a^2} \frac{1-a^2}{|1-a e^{-j2\pi f}|^2} \\
 &= \frac{\sigma_W^2}{|1-a e^{-j2\pi f}|^2}
 \end{aligned}$$

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## Example: PDS of AR(1) process...

- Approach 2: Model the problem with a linear system

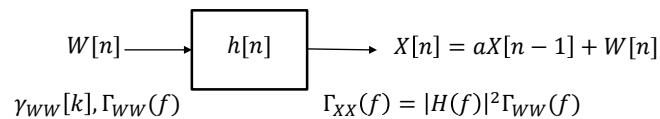


- WGN process  $W[n] \Rightarrow \gamma_{WW}[k] = ?, \Gamma_{WW}(f) = ?$
- What is the system frequency response  $H(f)$ ?

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## Example: PDS of AR(1) process...

- Find the frequency response  $H(f)$ :



- For any realization  $x[n]$ :  $X(z) = az^{-1}X(z) + W(z)$

$$\Rightarrow H(z) = \frac{1}{1-az^{-1}}$$

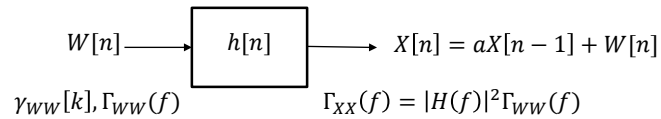
- Consequently we obtain the frequency response

$$H(f) = \frac{1}{1 - ae^{-j2\pi f}}$$

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## Example: PDS of AR(1) process...

- Power density spectrum of  $X[n]$ :



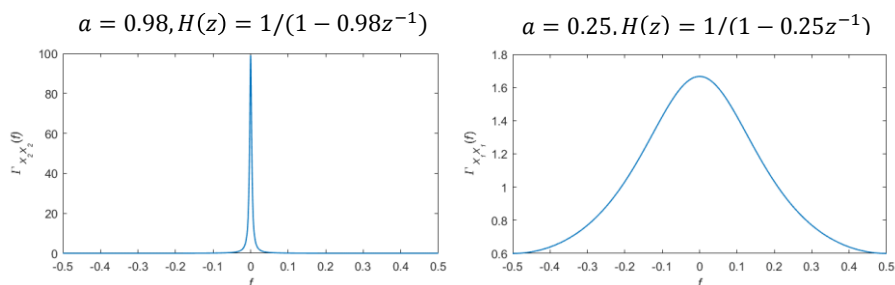
$$\Gamma_{XX}(f) = |H(f)|^2 \Gamma_{WW}(f)$$

$$= \frac{1}{|1 - ae^{-j2\pi f}|^2} \sigma_W^2$$

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## Example: PDS of AR(1) process...

- Power density spectrum of  $X[n]$ :



### Matlab

```
[H,W]=freqz(1,[1 -0.98],(-pi:pi/500:pi))
plot(W/pi,(1-0.98^2)*abs(H).^2)
```

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## Summary

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- Today we discussed:
  - Linear filtering of stochastic processes
  - Power density and cross-spectra
- Next:
  - Basics of parameter estimation