TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2016 - Assignment 2
Due date: Friday 30 September at 16.00.

1. Consider again the Duckmaze system from Assignment 1 (Exercise 4) given by

$$\dot{x}_1 = x_2 \tag{1}$$
$$\dot{x}_2 = -\frac{f_3}{m}x_1^3 - \frac{f_1}{m}x_1 - \frac{d}{m}x_2 - g + \frac{u}{m} \tag{2}$$

(Note: It is possible to do this exercise even if you did not do Assignment 1)

   (a) Is $(0, 0)$ an equilibrium point for $u = 0$?
   Since it is desirable to control the position to a desired position x_{1d}, the equilibrium point of (1)–(2) should be placed at $(x_{1d}, 0)$. What conditions must be satisfied for x_1^* and x_2^* and a constant input $u = u_0$ for $(x_{1d}, 0)$ to be an equilibrium point? Find u_0.

   (b) Split the states and input into stationary values (x_0, u_0) and difference terms $(\tilde{x}, \tilde{u})$. You will have $x = x_0 + \tilde{x}$ and $u = u_0 + \tilde{u}$.
   Derive the equations for $\tilde{x}$ with $\tilde{u}$ as an input. What is the equilibrium point for $\tilde{u} = 0$?

   (c) Calculate the Jacobian of the system and denote this A. Is A Hurwitz or not? What does this mean related to the stability of the equilibrium point?

2. (a) Consider

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_2^2\\ \dot{x}_2 &= 2x_1x_2\end{aligned}$$

   Construct the phase portrait of the system. Is the origin stable? Provide your argument with respect to Definition 4.1. on page 112 of Khalil (qualitative argument is enough).

   (b) Use Definition 4.1. (on page 112 of Khalil) to show that the origin of the following system

$$\dot{x} = \alpha x$$

   is asymptotically stable for $\alpha < 0$. (Note: in addition to convergence you also have to show quantitatively that for any given ε you could obtain a δ which depends on ε).

3. For the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable. Comment also on the possibility of a global result. (Hint: see Appendix, at the last page of this assignment)

   (a) The scalar system

$$\dot{x} = -x^3 - x^5, \qquad x \in R$$

   (b) The system

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_2 \\ \dot{x}_2 &= x_1 - x_2^3 \end{aligned}$$

   (c) The system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

   (Note: To comment on the possibility of a global result, use simulations.)

   (d) The system

$$\begin{aligned} \dot{x}_1 &= (x_1 - x_2)\left(x_1^2 + x_2^2 - 1\right) \\ \dot{x}_2 &= (x_1 + x_2)\left(x_1^2 + x_2^2 - 1\right) \end{aligned}$$

4. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1^2 x_2 - 2x_1x_2 + x_1^2 + 2x_1 \\ \dot{x}_2 &= x_1^3 + 2x_1^2 + x_1^2 x_2 + 2x_1x_2 \end{aligned}$$

   Do a change of variables

$$\begin{aligned} z_1 = x_1 - x_1^* \\ z_2 = x_2 - x_2^* \end{aligned}$$

   where $(x_1^*, x_2^*) = (-1, 1)$ to shift the equilibrium point to the origin.
   By using a quadratic Lyapunov function candidate, show that the equilibrium point is asymptotically stable.
   Hint I: The resultant system will be of the form $\dot{z} = f(z)(1 - z_1^2)$
   Hint II: Remember that close to the origin, the higher order terms will be dominated by lower order terms.

5. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(x_1 + x_2) - h(x_1 + x_2) \end{aligned}$$

   where h is continuously differentiable and $zh(z) > 0$ for all $z \neq 0$. Using the variable gradient method, find a Lyapunov function that shows that the origin is globally asymptotically stable.

6. Consider the Pendulum system with friction as

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2\end{aligned}$$

where the friction component is expressed by $(k/m)x_2$. Use the general Lyapunov function

$$V(x) = \frac{1}{2}x^T P x + \frac{g}{l}(1-\cos x_1)$$

where

$$P = \begin{bmatrix} p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}, \qquad P = P^T > 0$$

to examine the stability characteristic of the system. In particular, prove that the origin is locally asymptotically stable by an appropriate selection of matrix P?

7. Consider again

$$\begin{aligned}\dot{x}_1 &= x_2 + \alpha x_1\left(\beta^2 - x_1^2 - x_2^2\right)\\ \dot{x}_2 &= -x_1 + \alpha x_2\left(\beta^2 - x_1^2 - x_2^2\right)\end{aligned}$$

where $\alpha, \beta > 0$ are constants. Using Chetaev's theorem (theorem 4.3 in Khalil's book), show that the origin is unstable! Hint: $V = 0.5\left(x_1^2 + x_2^2\right)$.

8. Consider the system in Figure 1 where the nonlinear function is given by $g(e) = e^3$.

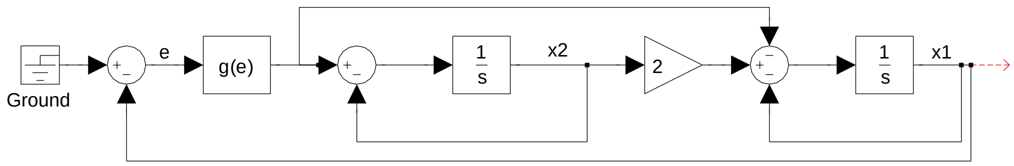

Figure 1: Block diagram of the system

(a) Find the state space model.

(b) Show that the origin is asymptotically stable using the Lyapunov function

$$V(x) = x^T P x$$

where

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1\\ 1 & 3\end{bmatrix}$$

(c) Sketch an estimate of the region of attraction in the (x_1, x_2)-plane.

9. Consider the system

$$\begin{aligned}\dot{x}_1 &= 4x_1^2 x_2 - f_1(x_1)\left(x_1^2 + 2x_2^2 - 4\right)\\ \dot{x}_2 &= -2x_1^3 - f_2(x_2)\left(x_1^2 + 2x_2^2 - 4\right)\end{aligned}$$

where the continuous functions f_1 and f_2 have the same sign as their arguments, i.e.

$$\begin{aligned} x_1 f_1(x_1) > 0 \quad \text{for} \quad x_1 \neq 0 \\ x_2 f_2(x_2) > 0 \quad \text{for} \quad x_2 \neq 0 \\ f_1(0) = f_2(0) = 0 \end{aligned}$$

Show that $\{x \in R^2 | x_1^2 + 2x_2^2 - 4 = 0\}$ and $(x_1, x_2) = (0, 0)$ are invariant sets, and that every trajectory approaches the sets when $t \to \infty$. Why do you think that the set $\{x \in R^2 | x_1^2 + 2x_2^2 - 4 = 0\}$ is not a limit cycle?
Hint: Apply Theorem 4.4 on page 128 of Khalil and use $V(x) = (x_1^2 + 2x_2^2 - 4)^2$.

10. Using $V(x) = \alpha x_1^2 + x_2^2$ where $\alpha > 0$ show that the origin of the following system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2 \end{aligned}$$

is globally asymptotically stable!

Appendix

A symmetric matrix $P = P^T$ is positive definite if:

- All eigenvalues of P are greater than zero.

or

- All leading principal minors of P are greater than zero.

Definition: Leading principal minors
Given an NxN matrix A, a leading principal submatrix of A is a submatrix formed by deleting all but the first n rows and columns. A leading principal minor is the determinant of a leading principal submatrix. Thus, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then the leading principal minors are

$$|a_{11}|, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$