NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF ENGINEERING CYBERNETICS

Exam
TTK4150 Nonlinear Control Systems
Monday December 12, 2011

SOLUTION

Problem 1 (15%)

a The Jacobian matrix is

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

$$= \begin{pmatrix} -2x_1x_2 & 13 - x_1^2 - 3x_2^2 \\ -(13 - 3x_1^2 - x_2^2) & 2x_1x_2 \end{pmatrix}$$

b The Jacobian matrix evaluated at $x = (1, 0)$ is given by:

$$A_1 := \left.\frac{\partial f}{\partial x}\right|_{x=(1,0)} = \begin{pmatrix} 0 & 12 \\ -10 & 0 \end{pmatrix}$$

Computing the eigenvalues of the A_1, we find that $\lambda(A_1) = \{\sqrt{120}i, -\sqrt{120}i\}$, and we can conclude that the equilibrium point is a center.

c To find the other equilibria, we look for the points x^*, such that $f(x^*) = (f_1(x^*), f_2(x^*)) = 0$. From $f_1 = 0$ we see that either $x_2 = 0$, or $(13 - x_1^2 - x_2^2) = 0$. From $f_2 = 0$ we see that $(13 - x_1^2 - x_2^2) = 0$ is not possible. Inserting $x_2 = 0$ into $f_2 = 0$ we get that x_1 must be a solution to $x_1^3 - 13x_1 + 12 = 0$. It was already given that $x = (1, 0)$ is an equilibrium point. Polynomial division of $x_1^3 - 13x_1 + 12 = 0$ by $x_1 - 1$ gives $x_1^2 + x_1 - 12 = 0$. This second-order equation has solutions $x_1 = -4$ and $x_1 = 3$. By computing the eigenvalues of

$$A_2 := \left.\frac{\partial f}{\partial x}\right|_{x=(-4,0)} = \begin{pmatrix} 0 & -3 \\ 35 & 0 \end{pmatrix}, \quad A_3 := \left.\frac{\partial f}{\partial x}\right|_{x=(3,0)} = \begin{pmatrix} 0 & 4 \\ 14 & 0 \end{pmatrix},$$

we find that $\lambda(A_2) = \{\sqrt{105}i, -\sqrt{105}i\}$ and $\lambda(A_3) = \{\sqrt{56}, -\sqrt{56}\}$. The equilibrium point $x = (-4, 0)$ is therefore a center, while the equilibrium point $x = (3, 0)$ is a saddle point.

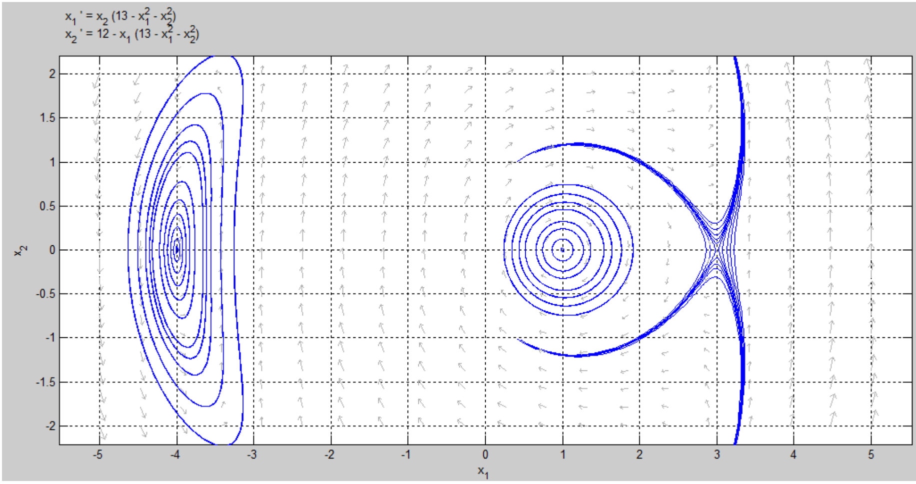

Figure 1: Equilibrium points

d Figure 1 shows the phase portrait of the equilibrium points of the system. Clearly there are centers at $(-4, 0)$ and $(1, 0)$ as well as a saddle point at $(3, 0)$.

e Bendixson's (negative) criterion cannot be used to prove that no periodic orbit is lying entirely within D. Although D is simply connected, the expression

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -2x_1x_2 + 2x_1x_2 = 0$$

and the test is inconclusive. However, the index theorem states that inside any periodic orbit, there must be at least one equilibrium point. As D does not contain any equilibrium point, there cannot be any periodic orbit lying entirely within D.

Problem 2 (15%)

a The derivative of $f(x)$ is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -e^{x_1x_2}\left(1 + x_1x_2\right) & x_1^2 e^{x_1x_2} \end{bmatrix} \tag{1}$$

Clearly $\frac{\partial f}{\partial x}$ exits and it is continuous, hence $f(x)$ is continuously differentiable. Since both $f(x)$ and $\frac{\partial f}{\partial x}$ are continuous in both time and $\mathbb{R}^2$, the function is by Lemma 3.2 in Khalil locally Lipschitz. $\frac{\partial f}{\partial x}$ is however not uniformly bounded, and the function is therefore not globally Lipschitz.

b Linearizing the system about the equilibrium point gives

$$A := \left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

The eigenvalues of the Jacobian matrix are $\lambda(A) = \{i, -i\}$. In this case linearization fails to determine stability of the equilibrium point. However, since A is not Hurwitz, we know that the system in *not* exponentially stable (Corollary 4.3 of Khalil).

c The Lyapunov function candidate is continuously differentiable, and positive definite. The time derivative is

$$\begin{aligned}\dot{V}(x) =& x_1x_2 - x_2x_1e^{x_1x_2} \\ =& x_1x_2(1 - e^{x_1x_2}) \,.\end{aligned}$$

Note that $\dot{V}(x) < 0$ is $x_1x_2 \neq 0$ and $\dot{V}(x) = 0$ if $x_1x_2 = 0$; therefore $\dot{V}(x)$ is negative semidefinite on $\mathbb{R}^2$, and the origin is stable. Let $S = \{x \in \mathbb{R}^2 \,|\, x_1x_2 = 0\}$ and let $x(t)$ be a solution that belongs identically to S. Then either $x_1(t) = 0\,\forall t$ or $x_2(t) = 0\,\forall t$. However,

$$x_1(t) \equiv 0 \Rightarrow \dot{x}_1(t) \equiv 0 \Rightarrow x_2 \equiv 0,$$

and

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow x_1 \equiv 0.$$

Therefore, the only solution that can stay identically in S is the trivial solution $x(t) \equiv 0$. Since V is radially unbounded, by Corollary 4.2 it follows that the origin in globally asymptotically stable.

Problem 3 (15%)

a We see that

$$\begin{aligned}y(t)u(t) =& f(u(t))u(t) \\ \geq& \epsilon f^2(u(t)) \\ =& \epsilon y^2(t) \,,\end{aligned}$$

and hence the system $y = f(u)$ is output strictly passive according to Definition 6.1 in Khalil.

b We have that

$$f(u)u = (1 + u^2)f^2(u) \geq \epsilon f^2(u) \,, \tag{2}$$

for all $\epsilon \in (0, 1]$, and hence the $y = f(u)$ is output strictly passive.

c Let $\tilde{f}(u) = Kf(u)$.

$$\left\|\tilde{f}(u)\right\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty \frac{K^2u^2}{(1+u^2)^2}dt} \leq K\sqrt{\int_0^\infty u^2dt} = K\|u\|_{\mathcal{L}_2}$$

And the $\mathcal{L}_2$-gain of $\tilde{f}(u)$ is $\gamma_1 = K$. The gain γ_2 of $g(s)$ is

$$
\begin{aligned}
\gamma_2 =& \sup_{\omega \in \mathbb{R}} \sqrt{g(-j\omega)g(j\omega)} \\
=& \sup_{\omega \in \mathbb{R}} \sqrt{\frac{1}{(\omega^2+1)(\omega^2+4)}} \\
=& \frac{1}{2},
\end{aligned}
$$

since we see that the supremum is attained for $\omega = 0$. The system can written on the form of Figure 7.1 i Khalil, with $r = 0$, $G(s) = g(s)$ and $\psi(\cdot) = Kf(u)$. According to the small-gain theorem, the system will be finite-gain $\mathcal{L}_2$-stable if $\gamma_1\gamma_2 < 1$ which means that

$$\gamma_1\gamma_2 = K\frac{1}{2} < 1 \tag{3}$$

$$\downarrow \tag{4}$$

$$K < 2 \tag{5}$$

and the gain K must be less than 2.

d From the figure given in the exam it can be seen that nonlinearity is odd, that is $f(-u) = -f(u)$. Since the nonlinearity is odd, its describing function will be real. Furthermore, we see that

$$\Psi(a) = \frac{2}{\pi}\int_0^{\pi} \frac{(\sin\theta)^2}{1 + a^2(\sin\theta)^2}\,\mathrm{d}\theta > 0$$

and hence the locus of $-1/\Psi(a)$ in the complex plane will be confined to the negative real axis. The Nyquist plot $g(j\omega)$ for $\omega > 0$ does not intersect the negative real axis, and the describing function does not predict any sustained oscillations.

Problem 4 (9%)

a Since

$$0 \le e^{-3t} \le 1 \quad \rightarrow \quad 1 \le 1 + e^{-3t} \le 2 \tag{6}$$

we can write

$$W_1(x) = x_1^2 + x_2^2 \le V(t,x) \le x_1^2 + 2x_2^2 = W_2(x) \tag{7}$$

Since both $W_1(x)$ and $W_2(x)$ are continuous positive definite functions ($W_1(0) = W_2(0) = 0$ and $W_1(x) > 0, W_2(x) > 0 \forall x \neq 0$), $V(t,x)$ is positive definite and decrescent.

We also have that $V(t,x) \to \infty$ as $\|x\|_2 \to \infty$, and hence $V(t,x)$ is radially unbounded.

b

$$\dot{V}(t,x) = -2x_1^2 + 2x_1x_2 - \left(2 + 5e^{-3t}\right)x_2^2 \le -2x_1^2 + 2x_1x_2 - 2x_2^2 \tag{8}$$

$$= -x^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x = -x^T Q x \tag{9}$$

Q is positive definite (it is symmetric and its eigenvalues $\lambda_1 = 1, \lambda_2 = 3$ are both positive), hence $\dot{V}(t,x)$ is negative definite.

By recalling that a postive definite function $x^T Hx$ satisfies

$$\lambda_{min}(H)x^T x \leq x^T Hx \leq \lambda_{max}(H)x^T x \tag{10}$$

we see that

$$x_1^2 + x_2^2 = \|x\|_2^2 \leq V(t,x) \leq 2(x_1^2 + x_2^2) = 2\|x\|_2^2 \tag{11}$$

$$\dot{V}(t,x) \leq \lambda_{max}(Q)x^T x = \lambda_{max}(Q)\|x\|_2^2 \tag{12}$$

where $\lambda_{max}(Q)$ is a positive constant. Since this holds globally, and $V(t,x)$ is continuously differentiable the system is, by Theorem 4.10 in Khalil, globally exponentially stable.

Problem 5 (30%)

a

$$\begin{aligned} y &= x_4 \\ \dot{y} &= \dot{x}_4 = x_3 \\ \ddot{y} &= \dot{x}_3 = 5x_2x_3^2 + u \end{aligned}$$

The relative degree of the system is $\rho = 2$. It exists and is well defined, hence the system is input-output linearizable.

b The system can be written as

$$\dot{x} = f(x) + g(x)u \tag{13}$$

where

$$f(x) = \begin{bmatrix} x_2 - x_1 \\ -x_2 \\ 5x_2x_3^2 \\ x_3 \end{bmatrix} \qquad g(x) = \begin{bmatrix} 0 \\ -x_3 \\ 1 \\ 0 \end{bmatrix} \tag{14}$$

- First, the external dynamics ξ are found. Since $\rho = 2$, ξ is of dimension 2.

$$\begin{aligned} \xi_1 &= y = x_4 \\ \xi_2 &= L_f y = \frac{\partial y}{\partial x} f(x) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} f(x) = x_3 \end{aligned}$$

- Then the internal dynamics η are found. Since ξ is of dimension 2, η is of dimension $4 - 2 = 2$. With $\eta_1 = \phi_1(x)$ and $\eta_2 = \phi_2(x)$, we have that

$$\frac{\partial \phi_1(x)}{\partial x} g(x) = \frac{\partial \phi_2(x)}{\partial x} g(x) = 0 \tag{15}$$

Inserting for $g(x)$ we have that

$$-\frac{\partial \phi_1(x)}{\partial x_2} x_3 + \frac{\partial \phi_1(x)}{\partial x_3} = 0 \tag{16}$$

$$-\frac{\partial \phi_2(x)}{\partial x_2} x_3 + \frac{\partial \phi_2(x)}{\partial x_3} = 0 \tag{17}$$

A $\phi_1(x)$ and a $\phi_2(x)$ which satisfies this are

$$\phi_1(x) = x_1 \tag{18}$$

$$\phi_2(x) = x_2 + \frac{1}{2} x_3^2 \tag{19}$$

- This gives the following diffeomorphism candidate

$$z = T(x) = \begin{bmatrix} \eta \\ \cdots \\ \xi \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 + \frac{1}{2}x_3^2 \\ \cdots\cdots \\ x_4 \\ x_3 \end{bmatrix} \tag{20}$$

Clearly $T(x)$ is continuously differentiable, but we also need to check that its inverse exists and that it is also continuously differentiable.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \eta_2 - \frac{1}{2}\xi_2^2 \\ \xi_2 \\ \xi_1 \end{bmatrix} = T^{-1}(z)$$

$T^{-1}(z)$ exists, and it is also continuously differentiable, hence $T(x)$ is a diffeomorphism.

- Next, $\gamma(x)$ and $\alpha(x)$ are found

$$\gamma(x) = L_g L_f y = 1$$

$$\alpha(x) = -\frac{L_f^2 y}{\gamma(x)} = -5x_2x_3^2$$

- And finally, the system is written in normal form

$$\dot{\eta} = \begin{bmatrix} \eta_2 - \frac{1}{2}\xi_2^2 - \eta_1 \\ \left(\eta_2 - \frac{1}{2}\xi_2^2\right)\left(5\xi_2^3 - 1\right) \end{bmatrix} = f_0\left(\eta, \xi\right) \tag{21}$$

$$\dot{\xi} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 1 \left[u + 5x_2x_3^2\right] \tag{22}$$

$$= A_c\xi + B_c\gamma(x)\left[u - \alpha(x)\right] \tag{23}$$

Since $\alpha(x) = -\frac{L_f^2 y}{\gamma(x)}$, the transformation is valid when $\gamma(x) \neq 0$. Since $\gamma(x) = 1$, this transformation is valid for the entire $\mathbb{R}^4$ space.

c

$$u = \alpha(x) + \beta(x)v \tag{24}$$

where $\beta(x) = \frac{1}{\gamma(x)} = 1$ and $\alpha(x) = -5x_2x_3^2$. This gives

$$u = -5x_2x_3^2 + v \tag{25}$$

d The external dynamics ξ with (25) inserted:

$$\dot{\xi}_1 = \xi_2 \tag{26}$$
$$\dot{\xi}_2 = v \tag{27}$$

$\xi = [\xi_1 \;\; \xi_2]^T$ can be stabilized by $v = -k_1\xi_1 - k_2\xi_2$, which gives the closed loop dynamics

$$\dot{\xi} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \xi = A\xi \tag{28}$$

The eigenvalues of the closed loop system are

$$\lambda = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1}}{2} \tag{29}$$

which have negative real parts as long as $k_1, k_2 > 0$. This means that A is Hurwitz and $\dot{\xi} = A\xi$ is asymptotically stable at the origin with $v = -k_1\xi_1 - k_2\xi_2$, $k_1, k_2 > 0$.

e The system is minimum phase if $T(0) = 0$ (the origin is kept as an equilibrium point for $[\eta \;\; \xi]^T$) and $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable at the origin.

$$\dot{\eta} = f_0(\eta, 0) = \begin{bmatrix} \eta_2 - \eta_1 \\ -\eta_2 \end{bmatrix} \tag{30}$$

Applying the Lyapunov function

$$V(\eta) = \eta^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \eta = \eta_1^2 + \eta_1\eta_2 + \eta_2^2 \tag{31}$$

gives

$$\begin{aligned} \dot{V}(x) &= 2\eta_1(\eta_2 - \eta_1) - \eta_1\eta_2 + (\eta_2 - \eta_1)\eta_2 - \eta_2^2 && (32) \\ &= -2\eta_1^2 - \eta_2^2 && (33) \end{aligned}$$

Since $V(x)$ is continuously differentiable and positive definite, and $\dot{V}(x)$ is negative definite, $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable at the origin. Clearly $T(0) = 0$, and the system as a whole is therefore minimum phase.

f Since the system is minimum phase, and the closed loop external dynamics are asymptotically stable at the origin, the closed loop system $[\eta \quad \xi]^T$ is asymptotically stable at the origin.

g Since $\dot{\eta} = f_0(\eta, 0)$ is min.phase, η will be bounded for sufficiently small $e(0)$, $\eta(0)$ and $R(t)$, and the overall control law

$$u = -5x_2x_3^2 + \ddot{r} - Ke \tag{34}$$

where $e = \xi - R$, $R = [r \quad \dot{r}]^T$and $K = [k_1 \quad k_2]^T$, will solve the tracking problem.

Problem 6 (16%)

a

- First the stabilizing virtual input is found using the following continuously differentiable and positive definite Lyapunov function

$$V_1 = \frac{1}{2}x_1^2 \tag{35}$$
$$\dot{V}_1 = x_1\left(x_1x_2 - x_1^4\right) = x_1^2\left(x_2 - x_1^3\right) \tag{36}$$

With $x_2 = x_1^3 - x_1^2$, x_1 is asymptotically stable:

$$\dot{V}_1 = x_1^2\left(x_1^3 - x_1^2 - x_1^3\right) = -x_1^4 < 0 \quad \forall \quad x_1 \neq 0 \tag{37}$$

- Then $x_1\phi\left(x\right) = x_1\left(x_1^3 - x_1^2\right)$ is added and subtracted from $\dot{x}_1$

$$\dot{x}_1 = x_1x_2 - x_1^4 + x_1\phi(x) - x_1\phi(x) \tag{38}$$
$$= -x_1^3 + x_1\left(x_2 - \phi(x)\right) \tag{39}$$

And the new variable z is defined as $z = x_2 - \phi(x)$.

- Next, the expression for $\dot{z}$ is found

$$\dot{z} = \dot{x}_2 - \frac{\partial\phi(x)}{\partial t} \tag{40}$$
$$= x_1 + u - \dot{x}_1\left(3x_1^2 + 2x_1\right) \tag{41}$$
$$= x_1 + u - \left(x_1x_2 - x_1^4\right)\left(3x_1^2 + 2x_1\right) \tag{42}$$

- Finally, the overall stabilizing input u is found using the following continuously differentiable and positive definite Lyapunov function

$$V_2 = \frac{1}{2}x_1^2 + \frac{1}{2}z^2 \tag{43}$$
$$\dot{V}_2 = -x_1^4 + z\left[x_1^2 + x_1 + u - \left(x_1x_2 - x_1^4\right)\left(3x_1^2 + 2x_1\right)\right] \tag{44}$$

u is chosen such that

$$x_1^2 + x_1 + u - \left(x_1x_2 - x_1^4\right)\left(3x_1^2 + 2x_1\right) = -z \tag{45}$$

Which means that

$$u = -z - x_1^2 - x_1 + \left(x_1 x_2 - x_1^4\right)\left(3x_1^2 + 2x_1\right) \tag{46}$$

and

$$\dot{V}_2 = -x_1^4 - z^2 < 0 \quad \forall \quad (x_1, z) \neq (0, 0) \tag{47}$$

Since $V_2(x_1, z)$ is continuously differentiable and positive definite, and $\dot{V}_2(x_1, z)$ is negative definite, u asymptotically stabilizes x_1 and z at the origin. Since $z = 0 \rightarrow x_2 = \phi(x)$ and $x_1 = 0 \rightarrow \phi(x) = 0$, this also means that x_2 is asymptotically stabilized at the origin. In addition, since $V_2(x_1, z)$ is radially unbounded and there are no singularities in u, x is globally asymptotically stable.

The globally stabilizing controller in original coordinates is as follows

$$u = -x_2 - x_1\left(1 + 2x_1 - x_1^2\right) + \left(x_1 x_2 - x_1^4\right)\left(3x_1^2 + 2x_1\right) \tag{48}$$

b Yes, there are no guaranties that the controller is physically able to produce the necessary output to stabilize the system. I.e., different (physical) limitations on the controller may jeopardize the global asymptotic stability.

c For systems which naturally include “good” nonlinearities (damping) it is better to apply backstepping than feedback linearization. The controller based on feedback linearization will always cancel all nonlinearities, also those who provides damping to the system, which is unnecessary and inefficient. With backstepping, however, one can decide to keep “good” nonlinearities through the control design so that these terms aid the controller in stabilizing the system.