

## TTT4120 Digital Signal Processing Fall 2017

### Lecture: The Sampling Theorem

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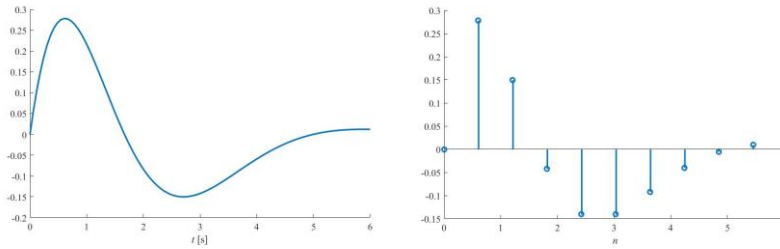
## Lecture in course book\*

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- Proakis, Manolakis Digital Signal Processing, 4<sup>th</sup> Ed.
  - 1.4.2 The sampling theorem
  - 1.4.6 Digital to analog conversion
  - 6.1 Ideal sampling and reconstruction of continuous-time signals

\*Level of detail is defined by lectures and problem sets

## Preliminary questions



- How fast must we sample the continuous-time signal (left) without losing information?
- What continuous-time signal corresponds to the discrete-time signal (right)?

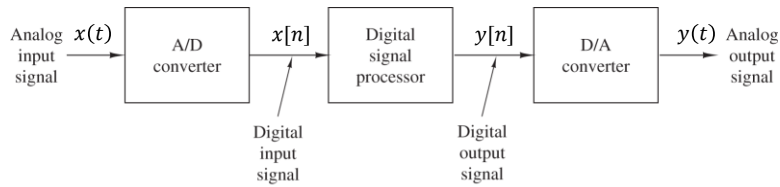
3

## Contents and learning outcomes

- Sampling of sinusoids and aliasing (partially covered Lect.1)
- Sampling theorem:
  - Ideal reconstruction of continuous-time signals
- Wagon wheel effect

4

## Periodic sampling



- Sampling – Processing – Reconstruction
- A signal is read (sampled) at a regular interval

$$x[n] = x(t_n) = x(nT) = x\left(\frac{n}{F_s}\right)$$

- Sampling interval  $T = \frac{1}{F_s}$ ,  $F_s$  being the sampling frequency

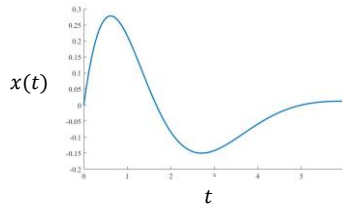
5

## Periodic sampling...

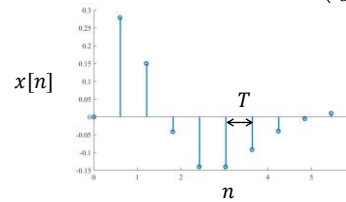
- Examples of sampling rate standards:
  - CD audio:  $F_s = 44.1$  kHz
  - TV frame rate:  $F_s = 100, 200, 400$  fr/s

6

## Periodic sampling...



$$x[n] = x(t_n) = x(nT) = x\left(\frac{n}{F_s}\right)$$



- Under what conditions is  $x[n]$  a good representation of  $x(t)$ ?
  - Appropriate choice of  $T$  or  $F_s$
- Under what conditions can  $x(t)$  be recovered from  $x[n]$ ?
  - Interpolation formula is needed
- Conditions are provided by the [sampling theorem](#)

7

## Sampling of sinusoids and aliasing

- Why considering sinusoidal signals?
- Many practical signals can be represented by the Fourier transform (or Fourier series)

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF \\ &= \int_{-\infty}^{\infty} X(F) (\cos 2\pi Ft + j \sin 2\pi Ft) dF \end{aligned}$$

- The concepts of sampling a single sinusoidal signal carry over to the case of more complicated signals.

8

## Sampling of sinusoids and aliasing...

- Consider the continuous-time signal

$$x(t) = \cos \Omega t = \cos 2\pi F t$$

with angular frequency  $\Omega$  [rad/s], or frequency  $F$  [Hz]

- Periodic sampling at regular time intervals  $t_n = nT = 1/F_s$

$$x[n] \equiv x(t_n) = \cos 2\pi F n T = \cos 2\pi \frac{F}{F_s} n = \cos \underbrace{2\pi f}_{\omega} n$$

- Spectrum of digital signal is periodic with period  $\omega = 2\pi$  (or  $f = 1$ ), where  $f = 1/2$  represents the highest frequency

$$\therefore f = \frac{F}{F_s} \leq \frac{1}{2}$$

9

## Sampling of sinusoids and aliasing...

- Example 1: Sample signal  $x(t) = \cos 2\pi 400t$  at  $F_s=1000$

$$\begin{aligned} x[n] &= \cos 2\pi 400 n T \\ &= \cos 2\pi \frac{400}{1000} n = \cos 2\pi (0.4 + k) n \end{aligned}$$

- Spectrum of sampled signal  $X(f)$  obtained directly from

$$x[n] = \frac{1}{2} (e^{j2\pi(0.4+k)n} + e^{-j2\pi(0.4+k)n})$$

10

## Sampling of sinusoids and aliasing...

- Example 2: Sample signal  $x(t) = \cos 2\pi 400t + \cos 2\pi 800t$  at  $F_s=1000$

$$\begin{aligned} x[n] &= \cos 2\pi 400nT + \cos 2\pi 800nT \\ &= \cos 2\pi(0.4 + k)n + \cos 2\pi(\underbrace{0.8}_{1-0.2} + k)n \\ &= \cos 2\pi(0.4 + k)n + \cos 2\pi(-0.2 + k)n \end{aligned}$$

- Spectrum of sampled signal  $X(f)$  obtained directly from:

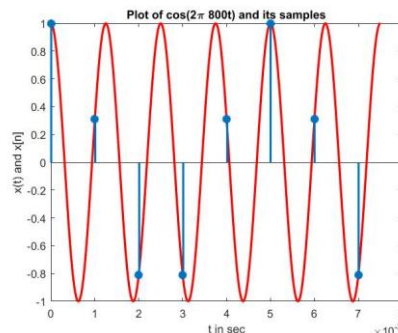
$$x[n] = \frac{1}{2}(e^{j2\pi 0.4n} + e^{-j2\pi 0.4n} + e^{j2\pi 0.2n} + e^{-j2\pi 0.2n})$$

- Distortion: highest analog frequency (800 Hz) appears as low-frequency component in digital spectrum (200 Hz)

11

## Sampling of sinusoids and aliasing...

- Example 2 (cont.): Simultaneous plot of  $\cos 2\pi 800t$  and its samples when  $F_s = 1000$



- Samples appear to be from  $\cos 2\pi 200t$

12

## Sampling of sinusoids and aliasing...

- To avoid aliasing (folding of high frequency components around  $f = 1/2$ , the following condition must be satisfied (Lecture 1)

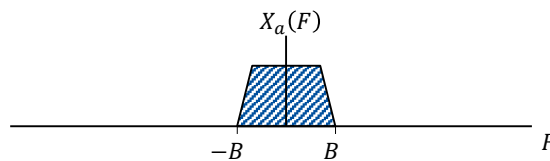
$$\frac{F}{F_s} \leq \frac{1}{2}, \forall F \Rightarrow F_s \geq 2F_{\max}$$

- We shall see that any **bandlimited** continuous-time signal can be reconstructed if sampled above the Nyquist rate  $2F_{\max}$

13

## Ideal reconstruction of continuous-time signals

- Bandlimited signal:



*A signal is bandlimited if there exists a finite frequency  $B$  (or  $\omega_B$ ) such that  $X_a(F)$  (or  $X_a(\omega)$ ) is zero for  $F > B$  (or  $\omega > \omega_B$ ). The frequency  $B = \omega_B/2\pi$  is called the signal bandwidth in Hz.*

14

## Ideal reconstruction of continuous-time ...

- Sampling Theorem:

*A bandlimited analog signal  $x_a(t)$  can be reconstructed from its sample values  $x[n] = x_a(nT)$  if the signal is sampled at rate*

$$F_s = \frac{1}{T} \geq 2F_{\max} = 2B,$$

*where  $F_{\max} = B$  is the highest frequency contained in  $x_a(t)$ .  
Otherwise aliasing would result in  $x[n]$ .*

- Sampling rate  $F_N = 2F_{\max}$  is called the **Nyquist rate**
- Highest analog frequency represented in  $x[n]$  is  $\frac{F_s}{2}$

15

## Ideal reconstruction of continuous-time ...

- Example: What is the Nyquist rate for the following signals?

$$x_1(t) = \cos 2\pi 400t + \cos 2\pi 800t$$

$$x_2(t) = \cos 100\pi t + 3 \cos 200\pi t$$

$$x_3(t) = \cos 150\pi t + 10 \sin(600\pi t + \theta)$$

- Can the signals be sampled at rate  $F_N$  without problems?

16



## Ideal reconstruction of continuous-time ...

- Continuous-time signal:  $x_a(t) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi Ft} dF$   
 $X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi Ft} dt$
- Discrete-time signal:  $x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) e^{j2\pi fn} df$   
 $X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fn}$
- Relationship between  $f$  and  $F$   
 $x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) e^{j2\pi fn} df = x_a(nT) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi FnT} dF$

17

## Ideal reconstruction of continuous-time ...

- Make use following relations  

$$f = \frac{F}{F_s}, e^{j2\pi FnT} = e^{\frac{j2\pi F}{F_s} n} = e^{\frac{j2\pi F}{F_s} (n - kF_s)}$$

- Then, we can manipulate the former expression into

$$\frac{1}{F_s} \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} X\left(\frac{F}{F_s}\right) e^{\frac{j2\pi nF}{F_s}} dF = \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \sum_{k=-\infty}^{\infty} X_a(F - kF_s) e^{\frac{j2\pi F}{F_s} n} dF$$

- Relation between sampled and analog spectra

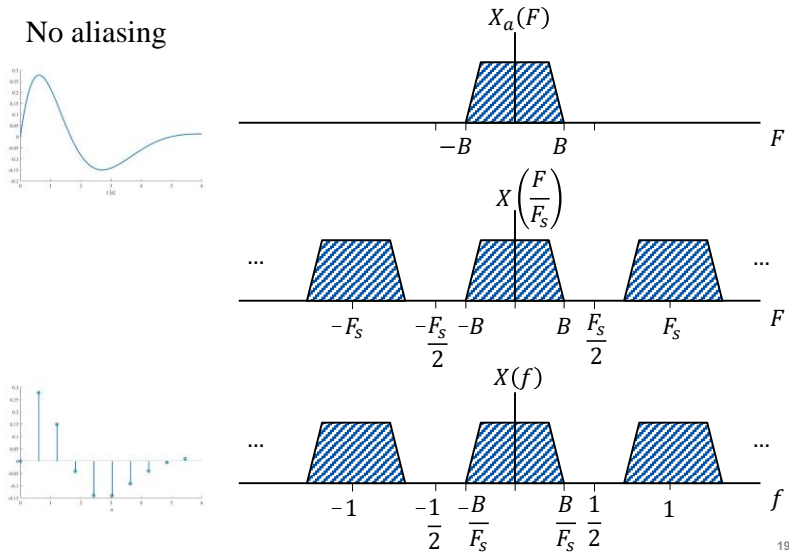
$$X(F/F_s) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s), \text{ or}$$

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a([f - k]F_s)$$

18

## Ideal reconstruction of continuous-time ...

- No aliasing



19

## Ideal reconstruction of continuous-time ...

- If discrete-time signal  $x[n]$  has no aliasing in spectrum  $X(F)$

$$X_a(F) = \begin{cases} \frac{1}{F_s} X\left(\frac{F}{F_s}\right), & |F| \leq \frac{F_s}{2} \\ 0, & |F| > \frac{F_s}{2} \end{cases}$$

- Analog signal can be reconstructed from samples  $x[n]$

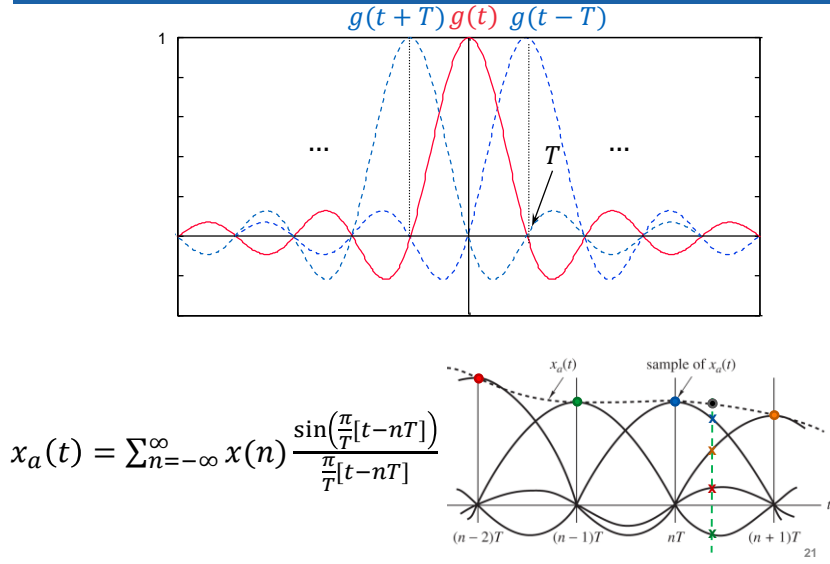
$$x_a(t) = \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} X_a(f) e^{j2\pi ft} df = \frac{1}{F_s} \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} X\left(\frac{f}{F_s}\right) e^{j2\pi ft} df = \dots$$

$$= \sum_{n=-\infty}^{\infty} x[n] \frac{\sin\left(\frac{\pi}{T}[t-nT]\right)}{\frac{\pi}{T}[t-nT]} = \sum_{n=-\infty}^{\infty} x[n] g[t-nT]$$

- Interpolation function is a sinc function

20

# Ideal reconstruction of continuous-time ...



# Ideal reconstruction of continuous-time ...

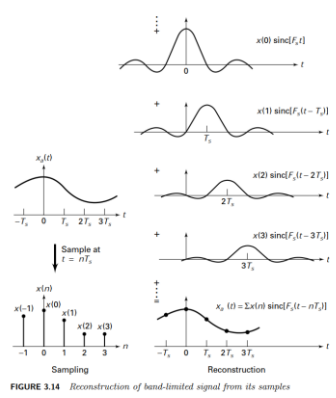
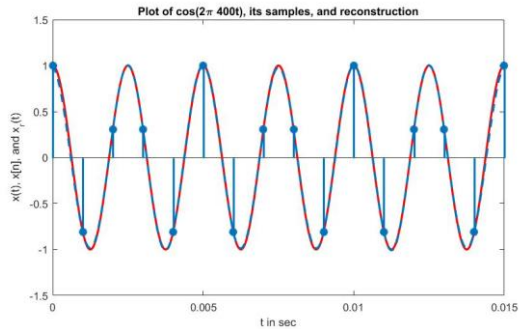


FIGURE 3.14 Reconstruction of band-limited signal from its samples

$$x_a(t) = \sum_{n=-\infty}^{\infty} x[n]g[t - nT]$$

## Ideal reconstruction of continuous-time ...

- Revisiting Example 1: Simultaneous plot of  $\cos 2\pi 400t$ , its samples, and reconstruction when  $F_s = 1000$

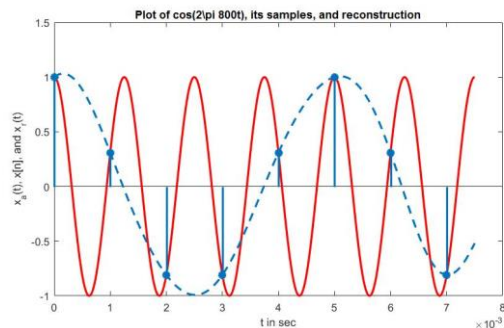


- Perfect reconstruction is possible

23

## Ideal reconstruction of continuous-time ...

- Revisiting Example 2: Simultaneous plot of  $\cos 2\pi 800t$ , its samples, and reconstruction when  $F_s = 1000$

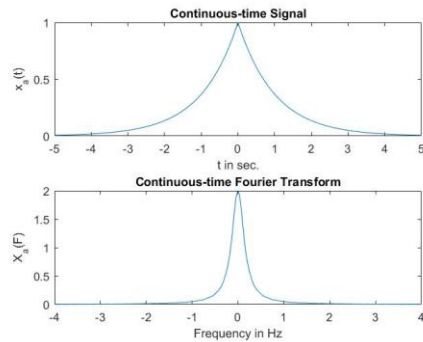


- Reconstruction of folded signal component  $\cos 2\pi 200t$

24

# Ideal reconstruction of continuous-time ...

- Example 3: Sample  $x_a(t) = e^{-|t|}$  at rates  $F_{s_1} = 5$  and  $F_{s_2} = 1$ .



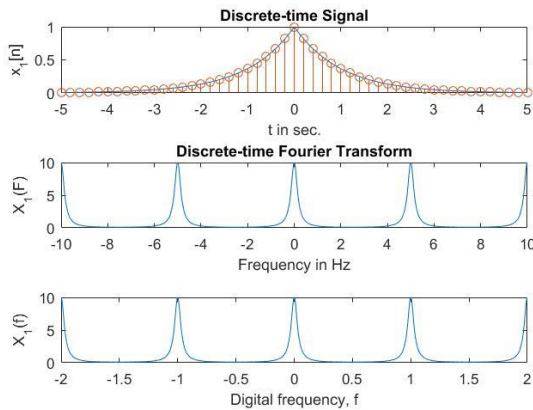
$$X(F) = \frac{2}{1 + (2\pi F)^2}$$

- How about spectra  $X(F)$  and  $X(f)$  for the two sampling rates? Sketch and draw conclusions about the reconstructed signals?

25

# Ideal reconstruction of continuous-time ...

- Example 3 (cont.):  $x_1[n] = e^{-|n|T_1} = e^{-\frac{|n|}{F_{s_1}}}$ ,  $F_{s_1} = 5$

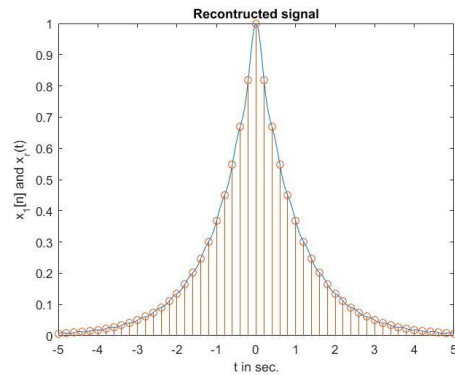


$$X\left(\frac{F}{F_s}\right) = \frac{1 - e^{-\frac{2}{F_s}}}{\left(1 - 2e^{-\frac{n}{F_s}} \cos \frac{2\pi F}{F_s} + e^{-\frac{2}{F_s}}\right)}$$

26

## Ideal reconstruction of continuous-time ...

- Example 3 (cont.):  $x_1[n] = e^{-nT_1} = e^{-\frac{n}{F_{s1}}}, F_{s1} = 5$

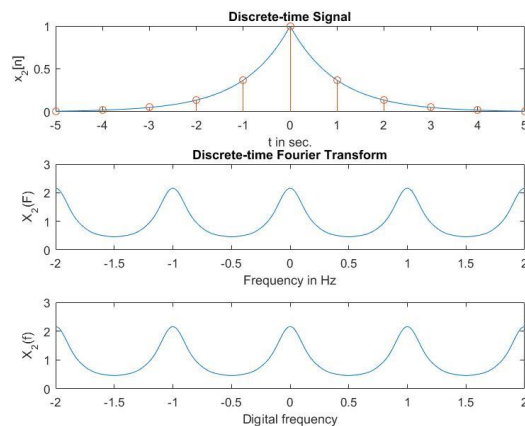


- Excellent reconstruction.

27

## Ideal reconstruction of continuous-time ...

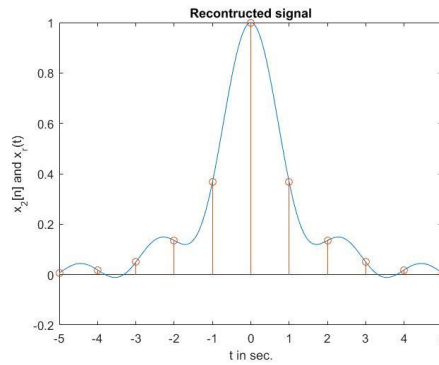
- Example 3 (cont.):  $x_2[n] = e^{-nT_2} = e^{-\frac{n}{F_{s2}}}, F_{s2} = 1$



28

## Ideal reconstruction of continuous-time ...

- Example 3 (cont.):  $x_2[n] = e^{-nT_2} = e^{-\frac{n}{F_{s_2}}}$ ,  $F_{s_2} = 1$

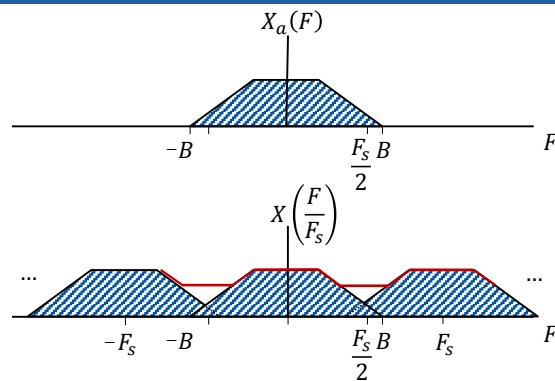


- Reconstructed signal quite different from actual one (aliasing).

29

## Ideal reconstruction of continuous-time ...

- Aliasing

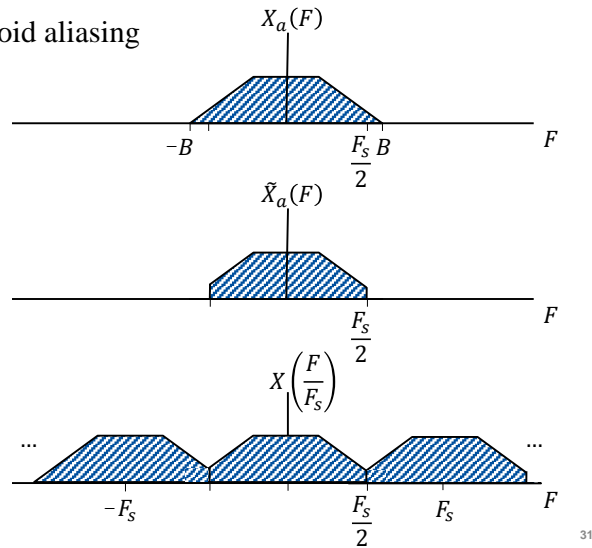


- Interpolation will produce  $\hat{x}_a(t)$  corresponding to aliased spectrum
- Prefilter  $x_a(t)$  to limit bandwidth before sampling

30

## Ideal reconstruction of continuous-time ...

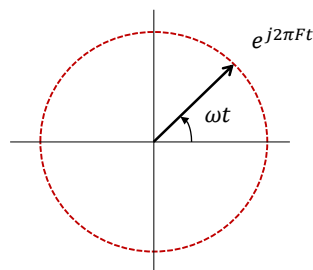
- Prefilter to avoid aliasing



31

## Example: Wagon wheel effect

- [Illusion of a wheel spinning in wrong direction](#)
- Imagine phasor rotating at angular speed  $\omega = 2\pi F$  rad/sec



- Starting at  $t = 0$ , take a snapshot every  $T$  seconds, i.e.,  $nT = \frac{n}{F_s}$
- Find values of  $T$  such that the sampled phasor appears to rotate in clockwise direction rather than counter-clockwise?

32



## Example: Wagon wheel effect...

- Demo on ItsLearning:

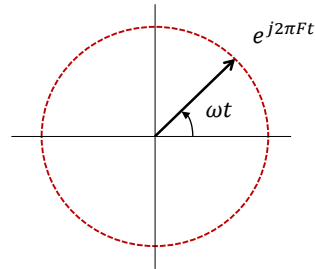
sampling\_rotating\_phasor.m

### Matlab

```
F = 1; Fs = 5; N = 3;
[f]=sampling_rotating_phasor(F,Fs,N);

F = 1; Fs = 4; N = 3;
[f]=sampling_rotating_phasor(F,Fs,N);

F = 1; Fs = 1.3; N = 3;
[f]=sampling_rotating_phasor(F,Fs,N);
```



33

## Summary

Today:

- Sampling of analog and aliasing
- Sampling theorem
- Ideal reconstruction of analog signals

Next:

- Sampling in frequency domain: Discrete Fourier Transform

34