NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF ENGINEERING CYBERNETICS

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Exam
TTK4150 Nonlinear Control Systems

Wednesday December 19, 2012

Hours: 09.00 – 13.00

Aids:	D - No printed or written materials allowed. NTNU type approved calculator with an empty memory allowed.
Language:	English
No. of pages:	6
Grades available:	January 23, 2012

This exam counts for 100% of the final grade.

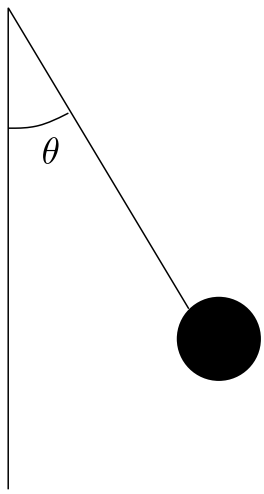

Figure 1: Pendulum

Problem 1 (15%)

The nonlinear dynamics for a pendulum is given by

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

where $l = 1$m is the length of the pendulum, $m = 10$g the mass, $k = 25$ a constant friction parameter and θ the angle subtended by the rod and the vertical axis through the pivot point, see Figure 1. The acceleration of gravity g is 9.81m/s^2.

a **[3%]** Choose appropriate state variables and write down the state equations $\dot{x} = f(x)$, where $x = (x_1, x_2)^T$. Find the Jacobian matrix $\partial f/\partial x$.

b **[5%]** Find all equilibrium points of the system and classify the qualitative behavior of each of them. Sketch the phase portraits.

c **[4%]** What can be said about the stability in these equilibrium points? Comment on the physics behind.

d **[3%]** Proove that the system has no periodic orbits in $\mathbb{R}^2$.

Problem 2 (15%)

Consider the second order system

$$\begin{aligned}\dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + \left(-x_1^2 + a\right) x_2\end{aligned}$$

where a is a constant. First, let $a = -1$ in **a**, **b** and **c**.

a **[3%]** Show that the origin is the only equilibrium point.

b [4%] Using Lyapunov's indirect method, what is the strongest conclusion you can make about the stability properties of the origin?

c [5%] Use the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

to show that the origin is globally asymptotically stable.

d [3%] Letting $a = 1$, what conclusions can now be drawn about the stability of the origin?

Problem 3 (10%)

For each of the functions $f(x)$ in **a** and **b**, determine whether f is (1) locally Lipschitz and (2) globally Lipschitz.

a [3%] $f(x) = -x + a\sin x$

b [3%] $f(x) = \tan x$

Hint: $\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$

c [4%] Describe the difference between a function $f(x)$ being (1) locally Lipschitz, (2) Lipschitz and (3) globally Lipschitz. Below the Lipschitz condition is given:

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

Problem 4 (26%)

Consider the system

$$\dot{x}_1 = x_2 \tag{1a}$$
$$\dot{x}_2 = -\left(1 + x_1^4\right)x_2^3 - (2x_1 + \sin x_1) + u \tag{1b}$$
$$y = x_2 \tag{1c}$$

a [2%] Show that

$$V(x) = \int_0^{x_1} 2\theta + \sin\theta d\theta + \frac{1}{2}x_2^2$$

can be used as a storage function.

Hint: You may use the fact that the function $\psi(\theta) = 2\theta + \sin\theta$ satisfies $\psi(\theta)\theta > 0 \; \forall\, \theta \neq 0$ and $\psi(0) = 0$.

b [4%] Show that for the output $y = x_2$, the system is output strictly passive.

c [4%] Show that for the output $y = x_2$, the system is zero-state observable.

d **[5%]** Show that the closed loop system with $u = -2x_2 - \sin x_2$ is globally asymptotically stable at the origin.

e **[4%]** Consider the system

$$\dot{z}_1 = z_2 \tag{2a}$$
$$\dot{z}_2 = \left(-z_1^2 + v\right) z_2^3 - (2z_1 + \sin z_1) + v \tag{2b}$$

Find an output $\bar{y} = h(z)$ such that this system is passive with input v and output $\bar{y}$. Use the storage function $V(z)$ where $V(\cdot)$ was defined in **a**.

f **[7%]** Consider system (1) and (2) interconnected through

$$u = \alpha(z) + \tilde{u} \tag{3a}$$
$$v = \beta(x) + \tilde{v} \tag{3b}$$

where $u_c = (\tilde{u}, \tilde{v})^T \in \mathbb{R}^2$ is a new input. See Figure 2.

Find the functions $\alpha(z), \beta(x)$ and the new output $y_c = \left(y_{c1}(x), y_{c2}(z)\right)$ such that the closed-loop system (1), (2), (3) with input u_c and output y_c is passive.

Problem 5 (26%)

Consider the system

$$\begin{aligned}
\dot{x}_1 &= x_3 - x_2 \\
\dot{x}_2 &= -x_2 - u \\
\dot{x}_3 &= x_1^2 - x_3 + u \\
y &= x_1
\end{aligned}$$

a **[3%]** Find the relative degree ρ of the system. Is the system input-output linearizable?

b **[10%]** Transform the system into normal form

$$\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi} &= A_c \xi + B_c \gamma(x) \left[u - \alpha(x)\right]
\end{aligned}$$

Specify the diffeomorphism $z = T(x) = \begin{bmatrix} \eta & \xi \end{bmatrix}^T$, the functions $\gamma(x)$, $\alpha(x)$ and $f_0(\eta, \xi)$ and the matrices A_c and B_c. In which domain is the transformation valid?

c **[3%]** Find an input-output linearizing controller on the form $u = \alpha(x) + \beta(x) v$.

d **[3%]** Find a controller v such that the external dynamics ξ is asymptotically stable at the origin.

e **[4%]** Is the system minimum phase?

Hint: If you were not able to solve **b** you may use the following equations for the internal dynamics:

$$\dot{\eta} = -\eta + \xi_1^2$$

f [3%] Is the closed-loop system $\begin{bmatrix} \eta & \xi \end{bmatrix}^T$ asymptotically stable at the origin?

Problem 6 (8%)

Investigate the input-to-state stability for the following system.

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= -x_1 - x_2 + u \end{aligned}$$

Appendix: Formulae

$$xy \leq \frac{\varepsilon^p}{p}|x|^p + \frac{1}{q\varepsilon^q}|y|^q$$

$$y(t) \approx a \sin\theta$$
$$\theta = \omega t$$

$$\psi(y(t)) \approx b + c_c \cos\theta + c_s \sin\theta = b + c\sin(\theta + \phi)$$

$$b = \frac{1}{2\pi}\int_0^{2\pi} \psi(a\sin\theta)\mathrm{d}\theta$$
$$c_c = \frac{1}{\pi}\int_0^{2\pi} \psi(a\sin\theta)\cos\theta\mathrm{d}\theta$$
$$c_s = \frac{1}{\pi}\int_0^{2\pi} \psi(a\sin\theta)\sin\theta\mathrm{d}\theta$$

$$c = \sqrt{{c_s}^2 + {c_c}^2}$$
$$\phi = \arctan\left(\frac{c_c}{c_s}\right)$$

$$\Psi(a,\omega) = \frac{c\,\mathrm{e}^{j(\theta+\varphi)}}{a\,\mathrm{e}^{j\theta}} = \frac{c\,\mathrm{e}^{j\varphi}}{a} = \frac{c_s + jc_c}{a}$$
$$|\Psi(a,\omega)| = \frac{c}{a}$$
$$\angle\Psi(a,\omega)| = \phi$$

$$G(j\omega)\Psi(a,\omega) + 1 = 0$$

$$\mathrm{e}^{\alpha+j\beta} = \mathrm{e}^{\alpha}(\cos\beta + j\sin\beta)$$
$$\cos^2\alpha + \sin^2\alpha = 1$$
$$\sin\alpha = \pm\sqrt{1-\cos^2\alpha}$$
$$\cos\alpha = \pm\sqrt{1-\sin^2\alpha}$$