TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2016 - Solution to Assignment 2

1. (a) By using the fact that the derivatives are zero in an equilibrium point, the following equations must be true

$$\begin{aligned} 0 &= x_2^* \\ 0 &= -\frac{f_3}{m}x_1^{*3} - \frac{f_1}{m}x_1^* - \frac{d}{m}x_2^* - g \\ &= -\frac{f_3}{m}x_1^{*3} - \frac{f_1}{m}x_1^* - g \end{aligned}$$

Inserting $(0,0)$ into the above equations leads to an illegal expression since g is not zero but $9.81 m/s^2$. Therefore $(0,0)$ is not an equilibrium point.
With $u = u_0$ we have the equations

$$0 = x_2^* \tag{1}$$

$$0 = -\frac{f_3}{m}x_1^{*3} - \frac{f_1}{m}x_1^* - g + \frac{u_0}{m} \tag{2}$$

Inserting $(x_1^*, x_2^*) = (x_{1d}, 0)$ into 2 gives

$$0 = -f_3 x_{1d}^3 - f_1 x_{1d} - mg + u_0$$

thus

$$u_0 = f_3 x_{1d}^3 + f_1 x_{1d} + mg \tag{3}$$

(b) In the original system equations we insert $x_1 = \tilde{x}_1 + x_{1d}$, $\tilde{x}_2 = x_2$ and $u = u_0 + \tilde{u}$. We have

$$\begin{aligned} \dot{\tilde{x}}_1 &= x_2 = \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= -\frac{f_3}{m}x_1^3 - \frac{f_1}{m}x_1 - \frac{d}{m}x_2 - g + \frac{u}{m} \\ &= -\frac{f_3}{m}\left(\tilde{x}_1 + x_{1d}\right)^3 - \frac{f_1}{m}\left(\tilde{x}_1 + x_{1d}\right) - \frac{d}{m}\tilde{x}_2 - g + \frac{\left(f_3 x_{1d}^3 + f_1 x_{1d} + mg + \tilde{u}\right)}{m} \end{aligned}$$

The resulting system equations are

$$\dot{\tilde{x}}_1 = \tilde{x}_2 \tag{4}$$

$$m\dot{\tilde{x}}_2 = -f_3\left[\left(\tilde{x}_1 + x_{1d}\right)^3 - x_{1d}^3\right] - f_1\tilde{x}_1 - d\tilde{x}_2 + \tilde{u} \tag{5}$$

In the equilibrium point for $\tilde{u} = 0$ we have

$$\begin{aligned} 0 &= \tilde{x}_2^* \\ 0 &= -f_3\left[\left(\tilde{x}_1^* + x_{1d}\right)^3 - x_{1d}^3\right] - f_1\tilde{x}_1^* - d\tilde{x}_2^* \\ &= -f_3\left[\left(\tilde{x}_1^* + x_{1d}\right)^3 - x_{1d}^3\right] - f_1\tilde{x}_1^* \end{aligned}$$

The equilibrium point is now in the origin.

(c) The Jacobian is calculated for (4)–(5).

$$A = \left[\begin{array}{cc} \frac{\partial f_1}{\partial \tilde{x}_1} & \frac{\partial f_1}{\partial \tilde{x}_2} \\ \frac{\partial f_2}{\partial \tilde{x}_1} & \frac{\partial f_2}{\partial \tilde{x}_2} \end{array} \right] \Bigg|_{x=(0,0)} = \left[\begin{array}{cc} 0 & 1 \\ -\frac{3f_3 x_{1d}^2 + f_1}{m} & -\frac{d}{m} \end{array} \right] \tag{6}$$

(Note that you may instead calculate the Jacobian for the original system, as long as you use the correct equilibrium point for this system.)
To find out whether A is Hurwitz or not, the eigenvalues of the matrix must be calculated

$$\lambda I - A = \left[\begin{array}{cc} \lambda & 0 \\ 0 & \lambda \end{array} \right] - \left[\begin{array}{cc} 0 & 1 \\ -\frac{3f_3 x_{1d}^2 + f_1}{m} & -\frac{d}{m} \end{array} \right] = \left[\begin{array}{cc} \lambda & -1 \\ \frac{3f_3 x_{1d}^2 + f_1}{m} & \lambda + \frac{d}{m} \end{array} \right] \tag{7}$$

$$|\lambda I - A| = \lambda(\lambda + \frac{d}{m}) + \frac{3f_3 x_{1d}^2 + f_1}{m} \tag{8}$$

$$= \lambda^2 + \frac{d}{m}\lambda + \frac{3f_3 x_{1d}^2 + f_1}{m} \tag{9}$$

The eigenvalues of A are thus given as

$$\lambda = \frac{1}{2}\left(-\frac{d}{m} \pm \sqrt{\left(\frac{d}{m}\right)^2 - \frac{4\left(3f_3 x_{1d}^2 + f_1\right)}{m}} \right) \tag{10}$$

Since $f_1, f_3, x_{1d}^2, m > 0$

$$\frac{d}{m} > \sqrt{\left(\frac{d}{m}\right)^2 - \frac{4\left(3f_3 x_{1d}^2 + f_1\right)}{m}} \tag{11}$$

and the eigenvalues will always lie in the left half plane which means that A is Hurwitz. This means that $(0,0)$ of (4)–(5) is locally asymptotically stable.

2. (a) The phase portrait can be seen in Figure 1.

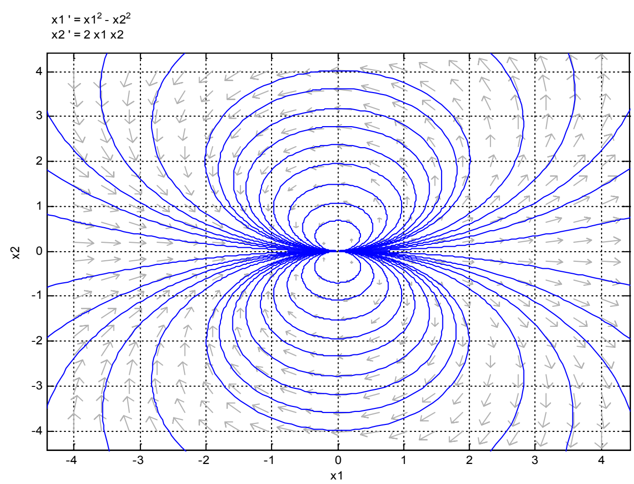

Figure 1: Phase portrait

The origin is not stable in the sense of Lyapunov. Given any $\varepsilon > 0$, no matter how small a δ we choose for the region of initial condition there always some initial conditions close to the x_1-axis which will exit the ε-region before converging to the origin.

(b) We need to show two conditions for asymptotic stability. First we need to show that for any given $\varepsilon > 0$ we could always find a $\delta > 0$ such that $\|x(0)\| < \delta \Longrightarrow \|x(t)\| < \varepsilon,\ \forall t \geq 0$. Furthermore we need to show that when the initial condition is on some domain every trajectory converges to the origin.

The solution is given by $x(t) = e^{\alpha t}x(0)$. We then have $|x(t)| \leq |x(0)|$ for all $t \geq 0$ since $\alpha < 0$. Given any $\varepsilon > 0$, choose $\delta = \varepsilon$ to show that for all $|x(0)| < \delta = \varepsilon$ it follows that $|x(t)| < \varepsilon,\ \forall t \geq 0$. Thus the origin is stable. From the solution it is easy to see that for any δ we have

$$|x(0)| < \delta \implies \lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{\alpha t}x(0) = 0.$$

as $\alpha < 0$.

3. Remark: the answer to this exercise may vary a lot depending on the chosen parameters of Lyapunov function and the domain D. One possible solution set is presented below.

(a) The scalar system is given by $\dot{x} = -x^3 - x^5$ with equilibrium point at the origin. Suppose $V(x) = px^2$ where $p > 0$. Then V is positive definite. Taking the derivative along the trajectory we have $\dot{V} = 2px\dot{x} = 2px(-x^3 - x^5) = -2px^4 - 2px^6$ which is negative definite. Hence the origin is asymptotically stable. Further V is radially unbounded which implies that the origin is globally asymptotically stable.

(b) The system is given by

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_2\\ \dot{x}_2 &= x_1 - x_2^3\end{aligned}$$

where it can be seen that the equilibrium point is given by

$$(x_1^*, x_2^*) = (0, 0)$$

A general quadratic Lyapunov function candidate is given by

$$V(x) = \frac{1}{2}x^T P x, \quad P = P^T$$

which is positive definite if and only if all the leading principal minors of P are positive, that is

$$\begin{aligned}p_{11} &> 0\\ p_{11}p_{22} - p_{12}^2 &> 0\end{aligned}$$

(and it follows that $p_{22} > 0$). The derivative of the Lyapunov function candidate along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P x\\ &= \begin{bmatrix} -x_1 - x_2 \\ x_1 - x_2^3 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\\ &= -p_{11}x_1x_2 - p_{12}x_1x_2 + p_{22}x_1x_2 - p_{11}x_1^2 + p_{12}x_1^2 - p_{12}x_2^2 - p_{22}x_2^4 - p_{12}x_1x_2^3\\ &= -(p_{11} - p_{12})x_1^2 - (p_{11} + p_{12} - p_{22})x_1x_2 - p_{12}x_2^2 - p_{22}x_2^4 - p_{12}x_1x_2^3\end{aligned}$$

In order to eliminate the undesirable terms, p_i is chosen according to

$$\begin{aligned} p_{11}+p_{12}-p_{22} &= 0 \\ p_{12} &= 0 \\ &\Rightarrow \ p_{11}=p_{22} \end{aligned}$$

which fulfill the requirements imposed in order to guarantee $V(x)$ to be positive definite. The derivative of $V(x)$ with respect to time is

$$\begin{aligned} \dot{V}(x) &= -p_{11}x_1^2 - p_{11}x_2^4 \\ &< 0 \quad \forall x \in R^2 - \{0\} \end{aligned}$$

Since $V(x)$ is radially unbounded it follows that the origin is globally asymptotically stable.

(c) The system is given by

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

where it can be seen that the equilibrium point is $(x_1^*, x_2^*) = (0,0)$. A general quadratic Lyapunov function candidate is given by

$$V(x) = \frac{1}{2}x^T P x, \quad P = P^T$$

which is positive definite if and only if all the leading principal minors of P are positive

$$\begin{aligned} p_{11} &> 0 \\ p_{11}p_{22} - p_{12}^2 &> 0 \end{aligned}$$

(and it follows that $p_{22} > 0$). The derivative of the Lyapunov function candidate along the trajectories of the system is given by

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x \\ &= \begin{bmatrix} -x_1 + x_2^2 \\ -x_2 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} -x_1 + x_2^2 \\ -x_2 \end{bmatrix}^T \begin{bmatrix} p_{11}x_1 + p_{12}x_2 \\ p_{12}x_1 + p_{22}x_2 \end{bmatrix} \\ &= \left(-x_1 + x_2^2\right)(p_{11}x_1 + p_{12}x_2) - x_2(p_{12}x_1 + p_{22}x_2) \\ &= p_{12}x_2^3 - p_{11}x_1^2 - p_{22}x_2^2 - 2p_{12}x_1x_2 + p_{11}x_1x_2^2 \end{aligned}$$

By choosing $p_{12} = 0$, the term x_2^3 and x_1x_2 vanishes and the derivative is rewritten as

$$\begin{aligned} \dot{V}(x) &= -p_{11}x_1^2 - p_{22}x_2^2 + p_{11}x_1x_2^2 \\ &= -p_{11}x_1^2 - (p_{22} - p_{11}x_1)\,x_2^2 \\ &= -p_{11}x_1^2 - p_{11}\left(\frac{p_{22}}{p_{11}} - x_1\right)x_2^2 \\ &< 0, \quad \forall \frac{p_{22}}{p_{11}} > x_1 \end{aligned}$$

By taking $D = \left\{x \in R^n | x_1 < \frac{p_{22}}{p_{11}}\right\}$, where $\frac{p_{22}}{p_{11}}$ may be chosen arbitrary large, it follows that the equilibrium point is asymptotically stable.

(d) The system is given by

$$\begin{aligned}\dot{x}_1 &= (x_1 - x_2)\left(x_1^2 + x_2^2 - 1\right)\\ \dot{x}_2 &= (x_1 + x_2)\left(x_1^2 + x_2^2 - 1\right)\end{aligned}$$

where it can be seen that the equilibrium points are given by

$$(x_1^*, x_2^*) = (0, 0)$$

and the set

$$x_1^{*2} + x_2^{*2} = 1$$

This implies that the origin can not be globally asymptotically stable, since any initial condition in $\{x \in R^2 | x_1^2 + x_2^2 = 1\}$ implies that the solution stays in that set. A general quadratic Lyapunov function candidate is given by

$$V(x) = \frac{1}{2} x^T P x, \quad P = P^T$$

which is positive definite if and only if all leading principal minors of P are positive, that is

$$\begin{aligned}p_{11} &> 0\\ p_{11}p_{22} - p_{12}^2 &> 0\end{aligned}$$

(and it follows that $p_{22} > 0$). The derivative of the Lyapunov function candidate along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P x\\ &= \begin{bmatrix} (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\ (x_1 + x_2)(x_1^2 + x_2^2 - 1) \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\\ &= \left(2x_1x_2p_{12} - x_1x_2p_{11} + x_1x_2p_{22} + x_1^2p_{11} + x_1^2p_{12} - x_2^2p_{12} + x_2^2p_{22}\right)\left(x_1^2 + x_2^2 - 1\right)\\ &= x^T \begin{bmatrix} p_{11} + p_{12} & p_{12} - \frac{1}{2}p_{11} + \frac{1}{2}p_{22} \\ p_{12} - \frac{1}{2}p_{11} + \frac{1}{2}p_{22} & p_{22} - p_{12} \end{bmatrix} x \left(x_1^2 + x_2^2 - 1\right)\\ &= x^T Q x \left(x_1^2 + x_2^2 - 1\right)\end{aligned}$$

By choosing Q such that $x^T Q x > 0 \quad \forall x \neq 0$ and taking $D = \{x \in R^2 | x_1^2 + x_2^2 < 1\}$, it can be seen that

$$\dot{V}(x) < 0 \quad \forall x \in D$$

Choosing $p_{12} = 0$, the matrix P is positive definite if and only if

$$\begin{aligned}p_{11} &> 0\\ p_{22} &> 0\end{aligned}$$

and the matrix Q is positive definite if and only if

$$\begin{aligned}p_{11} &> 0\\ p_{22} &> 0\\ p_{11} &< (3 + 2\sqrt{2})p_{22}\end{aligned}$$

The latter equation is found by solving the determinate of Q. Thus by choosing $p_{11}, p_{22} > 0, p_{11} < (3 + 2\sqrt{2})p_{22}$ and $p_{12} = 0$ we have shown that the origin of the system is asymptotically stable.

4. We first introduce a change of variables

$$\begin{aligned} z_1 &= x_1 - x_1^* \\ z_2 &= x_2 - x_2^* \end{aligned}$$

where $(x_1^*, x_2^*) = (-1, 1)$. We could write our differential equations as

$$\begin{aligned} \dot{z}_1 &= \dot{x}_1 = -x_1^2 x_2 - 2x_1 x_2 + x_1^2 + 2x_1 \\ &= -(z_1 - 1)^2 (z_2 + 1) - 2(z_1 - 1)(z_2 + 1) + (z_1 - 1)^2 + 2(z_1 - 1) \\ &= z_2 \left(1 - z_1^2\right) \end{aligned}$$

$$\begin{aligned} \dot{z}_2 &= \dot{x}_2 = x_1^3 + 2x_1^2 + x_1^2 x_2 + 2x_1 x_2 \\ &= (z_1 - 1)^3 + 2(z_1 - 1)^2 + (z_1 - 1)^2 (z_2 + 1) + 2(z_1 - 1)(z_2 + 1) \\ &= -\left(z_1 + z_2 - z_1^3 - z_1^2 z_2\right) \\ &= -(z_1 + z_2)\left(1 - z_1^2\right) \end{aligned}$$

and the new system becomes

$$\dot{z} = \begin{bmatrix} z_2 \\ -(z_1 + z_2) \end{bmatrix} \left(1 - z_1^2\right) \tag{12}$$

Let our Lyapunov function candidate be in the form $V(z) = z^T P z$ where $z^T = \left[\begin{array}{cc} z_1 & z_2 \end{array}\right]$ and

$$P = \left[\begin{array}{cc} p_{11} & p_{12} \\ p_{12} & p_{22} \end{array}\right]$$

is a positive definite symmetric matrix. Each component on the matrix P will be assigned later to a specific value such that our function candidate V becomes a Lyapunov function (in this case P does NOT have to be UNIQUE). Expanding the Lyapunov function candidate we have $V(z) = p_{11} z_1^2 + 2p_{12} z_1 z_2 + p_{22} z_2^2$. Taking the derivative of the function along the trajectory we have

$$\begin{aligned} \dot{V} &= 2p_{11} z_1 \dot{z}_1 + 2p_{12} \dot{z}_1 z_2 + 2p_{12} z_1 \dot{z}_2 + 2p_{22} z_2 \dot{z}_2 \\ &= 2p_{11} z_1 z_2 \left(1 - z_1^2\right) + 2p_{12} z_2^2 \left(1 - z_1^2\right) - 2p_{12}\left(z_1^2 + z_1 z_2\right)\left(1 - z_1^2\right) - 2p_{22}\left(z_1 z_2 + z_2^2\right)\left(1 - z_1^2\right) \\ &= 2p_{11} z_1 z_2 - 2p_{11} z_1^3 z_2 + 2p_{12} z_2^2 - 2p_{12} z_2^2 z_1^2 - 2p_{12} z_1^2 + 2p_{12} z_1^4 - 2p_{12} z_1 z_2 \\ &\quad +2p_{12} z_1^3 z_2 - 2p_{22} z_1 z_2 + 2p_{22} z_1^3 z_2 - 2p_{22} z_2^2 + 2p_{22} z_2^2 z_1^2 \\ &= -2p_{12} z_1^2 + 2(p_{11} - p_{12} - p_{22}) z_1 z_2 - 2(p_{22} - p_{12}) z_2^2 + H.O.T. \end{aligned}$$

where the higher order term is given by

$$H.O.T. = -2p_{11} z_1^3 z_2 - 2p_{12} z_2^2 z_1^2 + 2p_{12} z_1^4 + 2p_{12} z_1^3 z_2 + 2p_{22} z_1^3 z_2 + 2p_{22} z_2^2 z_1^2$$

We could rewrite the derivative as

$$\dot{V} = -z^T Q z + H.O.T.$$

where

$$Q = \left[\begin{array}{cc} 2p_{12} & -p_{11} + p_{12} + p_{22} \\ -p_{11} + p_{12} + p_{22} & 2(p_{22} - p_{12}) \end{array}\right]$$

To prove asymptotic stability it is enough to require our function candidate V to be a Lyapunov function on a neighborhood of the origin (does not have to be in the global region since we are not talking about global asymptotic stability). This means that it is enough to show $\dot{V}$ to be negative definite on a neighborhood of the origin. From this point of view we could see that near the origin, the quadratic term $z^T Q z$ dominates the higher order term $H.O.T.$ Hence, $\dot{V}$ will be negative definite in the neighborhood of the origin if the term $-z^T Q z$ is negative definite. In this case it is enough to show that Q is a positive definite matrix. Thus a sufficient condition for asymptotic stability is to have P and Q to be positive definite. By checking all the leading priciple minors of P and Q to be positive we might end up having nonunique solution of P and Q. One choice is by setting

$$p_{11} = 3, \quad p_{12} = 1, \quad p_{22} = 2$$

where we have

$$P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Since P and Q are positive definite matrices then $(z_1, z_2) = (0, 0)$ is asymptotically stable, or equivalently $(x_1, x_2) = (-1, 1)$ is asymptotically stable.

5. We have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(x_1 + x_2) - h(x_1 + x_2) \end{aligned}$$

We want to determine the gradient, $g(x)$, of the Lyapunov function so that

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1} \tag{13}$$

and

$$\dot{V}(x) = g^T(x) f(x) < 0 \quad \forall \quad x \neq 0 \tag{14}$$

$$V(x) = \int_0^x g^T(y) dy > 0 \quad \forall \quad x \neq 0 \tag{15}$$

Let

$$g(x) = \begin{bmatrix} \alpha x_1 + \beta x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}$$

where α, β, γ, $\delta > 0$. The symmetry requirement (13) gives

$$\beta = \gamma$$

The derivative of V along the trajectories of the system is now given by

$$\begin{aligned} \dot{V}(x) &= g(x)^T f(x) \\ &= \begin{bmatrix} \alpha x_1 + \beta x_2 \\ \beta x_1 + \delta x_2 \end{bmatrix}^T \begin{bmatrix} x_2 \\ -(x_1 + x_2) - h(x_1 + x_2) \end{bmatrix} \\ &= (\alpha x_1 + \beta x_2) x_2 + (\beta x_1 + \delta x_2)(-(x_1 + x_2) - h(x_1 + x_2)) \end{aligned}$$

Taking $\beta = \delta$

$$\begin{aligned} \dot{V}(x) &= (\alpha x_1 + \beta x_2) x_2 + \beta (x_1 + x_2)(-(x_1 + x_2) - h(x_1 + x_2)) \\ &= (\alpha x_1 + \beta x_2) x_2 - \beta (x_1 + x_2)^2 - \beta (x_1 + x_2) h(x_1 + x_2) \\ &= \alpha x_1 x_2 + \beta x_2^2 - \beta \left(x_1^2 + 2x_1 x_2 + x_2^2\right) - \beta (x_1 + x_2) h(x_1 + x_2) \\ &= \alpha x_1 x_2 + \beta x_2^2 - \beta x_1^2 - \beta 2 x_1 x_2 - \beta x_2^2 - \beta (x_1 + x_2) h(x_1 + x_2) \\ &= -\beta x_1^2 - (2\beta - \alpha) x_1 x_2 - \beta (x_1 + x_2) h(x_1 + x_2) \end{aligned}$$

Taking $\beta = \frac{1}{2}\alpha$ in order to get rid of the x_1x_2-term

$$\begin{aligned}\dot{V}(x) &= -\beta x_1^2 - \beta(x_1+x_2)h(x_1+x_2) \\ &< 0 \quad \forall x \in R^2\end{aligned}$$

since $zh(z) > 0 \ \forall \ z \neq 0$ and $\beta > 0$. The function V in constructed by

$$\begin{aligned}V(x) &= \int_0^{x_1} g_1(y_1, 0)\,dy_1 + \int_0^{x_2} g_2(x_1, y_2)dy_2 \\ V(x) &= \int_0^{x_1} \alpha y_1 dy_1 + \int_0^{x_2} (\gamma x_1 + \delta y_2)\,dy_2 \\ &= \alpha\left[\frac{1}{2}y_1^2\right]_0^{x_1} + \gamma x_1 [y_2]_0^{x_2} + \delta\left[\frac{1}{2}y_2^2\right]_0^{x_2} \\ &= \frac{1}{2}\alpha x_1^2 + \gamma x_1 x_2 + \frac{1}{2}\delta x_2^2 \\ &= \beta x_1^2 + \beta x_1 x_2 + \frac{\beta}{2}x_2^2 \\ &= x^T P x\end{aligned}$$

where

$$P = \begin{bmatrix} \beta & \frac{\beta}{2} \\ \frac{\beta}{2} & \frac{\beta}{2} \end{bmatrix}$$

and

$$\begin{aligned}\beta &> 0 \\ \frac{\beta^2}{2} - \frac{\beta^2}{4} = \frac{\beta^2}{4} &> 0\end{aligned}$$

which implies that $P > 0$ (and $V(x)$ is positive definite on R^2 and radially unbounded). By Theorem 4.2 it is concluded that the origin is globally asymptotically stable.

6. The pendulum system is given by

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2\end{aligned}$$

A general quadratic Lyapunov function candidate is given by

$$\begin{aligned}V(x) &= \frac{1}{2}x^T P x + \frac{g}{l}(1 - \cos x_1) \\ &= \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{g}{l}(1 - \cos x_1)\end{aligned}$$

The quadratic form $\frac{1}{2}x^T P x$ is positive definite if and only if all the leading principal minors of P are positive

$$\begin{gathered}p_{11} > 0 \\ p_{11}p_{22} - p_{12}^2 > 0\end{gathered} \tag{16}$$

The derivative of the Lyapunov function candidate along the trajectories of the system is given by

$$\begin{aligned}
\dot{V}(x) &= x^T P \dot{x} + \frac{g}{l}\dot{x}_1 \sin x_1 \\
&= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \frac{g}{l}\dot{x}_1 \sin x_1 \\
&= (x_1 p_{11} + x_2 p_{12})\dot{x}_1 + (x_1 p_{12} + x_2 p_{22})\dot{x}_2 + \frac{g}{l}\sin x_1 \cdot \dot{x}_1 \\
&= \left[p_{11}x_1 + p_{12}x_2 + \frac{g}{l}\sin x_1\right] x_2 + (p_{12}x_1 + p_{22}x_2)\left[-\frac{g}{l}\sin x_1 - \frac{k}{m}x_2\right] \\
&= \frac{g}{l}(1 - p_{22})x_2 \sin x_1 - \frac{g}{l}p_{12}x_1 \sin x_1 + \left[p_{11} - p_{12}\frac{k}{m}\right] x_1 x_2 + \left[p_{12} - p_{22}\frac{k}{m}\right] x_2^2
\end{aligned}$$

The elements p_{11}, p_{12}, and p_{22} should be selected such that $\dot{V}(x)$ becomes negative definite. The signs of $x_2 sin x_1$ and $x_1 x_2$ change based on the quadrant of x_1 and x_2 and therefore, these two elements should be eliminated. This happens by choosing $p_{22} = 1$ and $p_{11} = (k/m)p_{12}$. Then, from (16), p_{12} should satisfy $0 < p_{12} < (k/m)$ to have a positive definite V. By choosing $p_{12} = \frac{1}{2}(k/m)$

$$\dot{V}(x) = -\frac{1}{2}\frac{g}{l}\frac{k}{m}x_1 \sin x_1 - \frac{1}{2}\frac{k}{m}x_2^2$$

Obviously, $x_1 sin x_1 > 0$ for all $-\pi < x_1 < \pi$. Therefor, by selecting

$$B_r = \{x \in R^2 \mid |x_1| < \pi\},$$

it concludes that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite in B_r. Thus, we can conclude that the origin is locally asymptotically stable.

7. The function $V(x) = 0.5\left(x_1^2 + x_2^2\right)$ is positive definite at all points which are not in the origin. Then

$$\begin{aligned}
\dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 \\
&= x_1\left(x_2 + \alpha x_1\left(\beta^2 - x_1^2 - x_2^2\right)\right) + x_2\left(-x_1 + \alpha x_2\left(\beta^2 - x_1^2 - x_2^2\right)\right) \\
&= \alpha\left(x_1^2 + x_2^2\right)\left(\beta^2 - x_1^2 - x_2^2\right)
\end{aligned}$$

Defining

$$U \triangleq \left\{x \in R^2 | \left\|x\right\|_2 \le r, 0 < r < \beta\right\}$$

which is nonempty, it follows that V and $\dot{V}$ are positive definite in U. By the Chetaev's theorem the origin is unstable.

8. (a) From the figure it can be seen that

$$\begin{aligned}
\dot{x}_1 &= -g(e) + 2x_2 - x_1 \\
\dot{x}_2 &= g(e) - x_2 \\
e &= -x_1
\end{aligned}$$

and the system is given by

$$\begin{aligned}
\dot{x}_1 &= x_1^3 + 2x_2 - x_1 \\
\dot{x}_2 &= -x_1^3 - x_2
\end{aligned}$$

(b) Clearly the function $V(x)$ is positive definite and radially unbounded. The derivative of $V(x)$ along the trajectories of the system is given by

$$\begin{aligned}
\dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} = 2x^T P \dot{x} \\
&= -x_1^2 - x_2^2 - 2x_1^3 x_2 \\
&= -x_1^2 - x_2^2 - 2x^T \begin{bmatrix} 0 & \frac{1}{2}x_1^2 \\ \frac{1}{2}x_1^2 & 0 \end{bmatrix} x \\
&= -x^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x - x^T \begin{bmatrix} 0 & x_1^2 \\ x_1^2 & 0 \end{bmatrix} x \\
&= -x^T \begin{bmatrix} 1 & x_1^2 \\ x_1^2 & 1 \end{bmatrix} x \\
&= -x^T Q(x) x
\end{aligned}$$

where positive definiteness of $Q(x)$ implies that the origin is asymptotically stable. In order for $Q(x)$ to be positive definite, is is required that all its leading principal minors are positive. This imposes the requirements

$$\begin{aligned}
1 &> 0 \\
1 - x_1^4 &> 0
\end{aligned}$$

Taking $D = \{x \in R^2 |\, |x_1| < 1\}$ and applying Theorem 4.1, the origin is asymptotically stable.

(c) Since $V(x)$ is radially unbounded it is known that the set $\Omega_c = \{x \in R^2 | V(x) \leq c\}$, where c is chosen such that $|x_1| < 1 \quad \forall x \in \Omega_c$, is positively invariant. The constant c is obtained by

$$\begin{aligned}
c &= \min_{|x_1|=1} V(x) = \min_{|x_1|=1} x^T P x \\
&= \min_{|x_1|=1} \left(\frac{1}{2}x_1^2 + x_1 x_2 + \frac{3}{2}x_2^2 \right) \\
&= \min \begin{cases} \frac{1}{2} + x_2 + \frac{3}{2}x_2^2, & \text{if } x_1 = 1 \\ \frac{1}{2} - x_2 + \frac{3}{2}x_2^2, & \text{if } x_1 = -1 \end{cases}
\end{aligned}$$

The minimum of $V(x)$ along $x_1 = 1$ and $x_1 = -1$ is found through

$$\begin{aligned}
\frac{\partial}{\partial x_2}\left(\frac{1}{2} + x_2 + \frac{3}{2}x_2^2 \right) &= 1 + 3x_2 = 0 \\
\frac{\partial}{\partial x_2}\left(\frac{1}{2} - x_2 + \frac{3}{2}x_2^2 \right) &= -1 + 3x_2 = 0
\end{aligned}$$

which implies that $\left(-1, \frac{1}{3}\right)$ and $\left(-1, -\frac{1}{3}\right)$ are candidates for minimum:

$$\begin{aligned}
c &= \min_{|x_1|=1} V(x) \\
&= \min V(x)\big|_{x \in \left\{\left(-1,\frac{1}{3}\right),\left(1,-\frac{1}{3}\right)\right\}} \\
&= \min \left\{ V\left(-1, \frac{1}{3}\right), V\left(1, -\frac{1}{3}\right) \right\} \\
&= \frac{1}{3}
\end{aligned}$$

Take $\Omega = \Omega_{\frac{1}{3}}$, $E = \left\{ x \in \Omega | \dot{V}\left(x\right) = 0 \right\} = (0,0) = M$. This means that $(0,0)$ is the largest invariant set in E, and with respect to Theorem 4.4 Ω may be taken as an estimate of the region of attraction. Figure 2 shows a plot of the region of attraction.

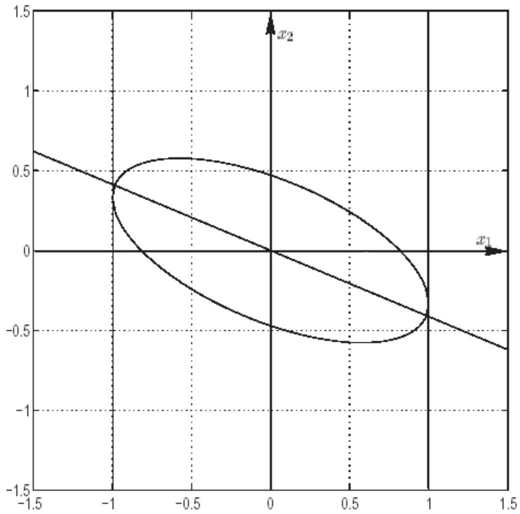

Figure 2: Region of attraction

9. The system is given by

$$\begin{aligned}\dot{x}_1 &= 4x_1^2x_2 - f_1(x_1)\left(x_1^2+2x_2^2-4\right)\\ \dot{x}_2 &= -2x_1^3 - f_2(x_2)\left(x_1^2+2x_2^2-4\right)\end{aligned}$$

As the functions f_1 and f_2 have the same sign as their arguments it follows that $x_1 f_1(x_1) \geq 0$, $x_2 f_2(x_2) \geq 0$, $f_1(0) = f_2(0) = 0$. Then it is easy to see that there are three equilibrium points, i.e. $(0,0)$, $\left(0, \pm\sqrt{2}\right)$. Next we want to show that $\{x \in R^2 | x_1^2 + 2x_2^2 - 4 = 0\}$ is a invariant set. We define a new variable $z \triangleq x_1^2 + 2x_2^2 - 4$. The derivative of z with respect to time is

$$\begin{aligned}\dot{z} &= 2x_1\dot{x}_1 + 4x_2\dot{x}_2\\ &= 2x_1\left[4x_1^2x_2 - f_1(x_1)\left(x_1^2+2x_2^2-4\right)\right]\\ &\quad +4x_2\left[-2x_1^3 - f_2(x_2)\left(x_1^2+2x_2^2-4\right)\right]\\ &= -2x_1f_1(x_1)\left(x_1^2+2x_2^2-4\right) - 4x_2f_2(x_2)\left(x_1^2+2x_2^2-4\right)\\ &= -\left(2x_1f_1(x_1) + 4x_2f_2(x_2)\right)\left(x_1^2+2x_2^2-4\right)\\ &= -2\left(x_1f_1(x_1) + 2x_2f_2(x_2)\right)z\end{aligned}$$

where it can be seen that $z = 0$ is an equilibrium point for this differential equation. This means that the relationship between x_1 and x_2 as given by z stays constant if $z = 0$ initially, and $\{x \in R^2 | x_1^2 + 2x_2^2 - 4 = 0\}$ is an invariant set for the system. Consider the function

$$V(x) = \left(x_1^2 + 2x_2^2 - 4\right)^2$$

which is radially unbounded. The derivative of V with respect to time is

$$\begin{aligned}\dot{V} &= 2\left(x_1^2+2x_2^2-4\right)\left(2x_1\dot{x}_1 + 4x_2\dot{x}_2\right)\\ &= -4\left(x_1f_1(x_1) + 2x_2f_2(x_2)\right)\left(x_1^2+2x_2^2-4\right)^2\end{aligned}$$

which is negative semidefinite. Let $D = R^2$. The set $\Omega_c = \left\{x \in R^2 | V(x) \leq c, \dot{V}(x) \leq 0\right\} = \{x \in R^2 | V(x) \leq c\}$ is a compact positively invariant set for any finite $c > 0$ due to the radially unboundedness of $V(x)$. Let $\Omega = \Omega_c$, the set E is given by

$$\begin{aligned}E &= \left\{x \in \Omega | \dot{V}(x) = 0\right\}\\ &= \left\{x \in \Omega | x_1^2 + 2x_2^2 - 4 = 0\ \right\} \cup \left\{x \in \Omega |\ x_1f_1(x_1) + 2x_2f_2(x_2) = 0\right\}\\ &= \left\{x \in \Omega | x_1^2 + 2x_2^2 - 4 = 0\ \right\} \cup (0,0)\end{aligned}$$

Since both $\{x \in \Omega | x_1^2 + 2x_2^2 - 4 = 0\ \}$ and $(0,0)$ are invariant sets (recall that the origin is an equilibrium point), the largest invariant set in E is given by $M = E$, and by Theorem 4.4 it can be concluded that every solution starting in Ω approaches $\{x \in \Omega | x_1^2 + 2x_2^2 - 4 = 0\}$ or the origin as $t \to \infty$. However, the set $\{x \in \Omega | x_1^2 + 2x_2^2 - 4 = 0\}$ is not a limit cycle since it contains equilibrium points $\left(0, \pm\sqrt{2}\right)$.

10. From $V(x) = \alpha x_1^2 + x_2^2$ and

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2\end{aligned}$$

we have

$$\begin{aligned}
\dot{V} &= 2\alpha x_1 \dot{x}_1 + 2x_2 \dot{x}_2 \\
&= 2\alpha x_1 x_2 + 2x_2 \left[-x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2\right] \\
&= -2x_2^2 - 2x_2^2 x_1^2 - 4x_1 x_2^3 - 2x_2^4 \\
&= -2x_2^2 \left[1 + x_1^2 + 2x_1 x_2 + x_2^2\right] \\
&= -2x_2^2 \left[1 + (x_1 + x_2)^2\right]
\end{aligned}$$

Thus $\dot{V}$ is negative semidefinite since $\dot{V}(x) = 0$ for $x = (x_1, 0)$ where $x_1 \in R$. When $\dot{V} = 0$ we should have $x_2 = 0$ and also $\dot{x}_2 = 0$. From $\dot{x}_2 = 0$ we should have $-x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2 = 0$ or equivalently $\alpha x_1 = 0$ since $x_2 = 0$. It follows that $\dot{V}(x) = 0$ identically on $x = (0, 0)$. Hence, by Corollary 4.1, the origin is asymptotically stable. Furthermore since V is radially unbounded, by Corollary 4.2 we conclude that the origin is globally asymptotically stable.