NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF ENGINEERING CYBERNETICS

Exam
TTK4150 Nonlinear Control Systems
Friday December 18, 2015

SOLUTION

Problem 1 (13%)

a To show that the origin is the only equilibrium point x^* we have that

$$\begin{aligned}\dot{x}_1^* &= 0 = -{x_2^*}^3 \\ \dot{x}_2^* &= 0 = -{x_1^*}^2 x_2^* + {x_1^*}^3 - x_2^*\end{aligned}$$

which results in

$$\begin{aligned}&0 = -{x_2^*}^3 \Rightarrow x_2^* = 0 \\ &0 = -{x_1^*}^2 x_2^* + {x_1^*}^3 - x_2^* \Rightarrow 0 = {x_1^*}^3 \Rightarrow x_1^* = 0\end{aligned}$$

which gives that the origin is the only equilibrium point of the system.

b First we find the Jacobian of the system:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & -3x_2^2 \\ -2x_1x_2 + 3x_1^2 & -1 - x_1^2 \end{bmatrix}$$

Inserting for $(x_1, x_2) = (0, 0)$ gives

$$A \mid_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

which gives $\lambda_1 = 0$ and $\lambda_2 = -1$. Since one of the eigenvalues is zero, nothing can be concluded about stability of origin $(0, 0)$ using the indirect Lyapunov method.

c $V(x)$ is a positive definite, continuously differentiable function. We now find that

$$\begin{aligned}\dot{V}(x) &= x_1^3\dot{x}_1 + x_2^3\dot{x}_2 \\ &= -x_1^3x_2^3 + x_2^3(-x_1^2x_2 + x_1^3 - x_2) \\ &= -x_1^2x_2^4 - x_2^4 \leq 0 \quad \forall\, x \in \mathbb{R}^2\end{aligned}$$

$\dot{V}(x)$ is negative semidefinite, and LaSalle's theorem needs to be applied to prove asymptotic stability. Let $S = \{x \in \mathbb{R}^2 \mid x_2 = 0\}$. Let $x(t)$ be a solution that belongs identically to $S : x_2 = 0 \Rightarrow \dot{x}_2 = 0 \Rightarrow x_1^3 = 0 \Rightarrow x_1 = 0$, which means that no solution can stay identically in S other than $(0, 0)$. By Corollary 4.2, $(0, 0)$ is asymptotically stable. In addition, since $V(x)$ is radially unbounded, the origin is globally asymptotically stable.

Problem 2 (20%)

a Consider the Lyapunov function candidate

$$V_1 = \frac{1}{2}x_1^2,$$

which is positive definite, decrescent and radially unbounded. Its derivative for $x_2 = 0$ becomes

$$\dot{V}_1 = x_1\dot{x}_1 = -|x|_1 x_1^2 \cos^2(t) - x_1^4 \leq -x^4$$

The right-hand side of the inequality is negative definite. Hence the origin is a uniformly globally asymptotically stable (UGAS) equilibrium point (see Theorem 4.9 in Khalil p. 152).

b Consider the Lyapunov function candidate

$$V_2 = \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2,$$

which is positive definite, decrescent and radially unbounded. Its derivative equals

$$\begin{aligned}\dot{V}_2 &= x_2\dot{x}_2 + x_3\dot{x}_3 \\ &= -x_2^2 - x_2^2\sin^2(t) + x_2^3 x_3 - x_2^3 x_3 - x_6 \leq -x_2^2 - x_3^6\end{aligned}$$

The right-hand side of the inequality is negative definite. Hence the origin is a UGAS equilibrium point.

c Consider the Lyapunov function candidate $V_1 = \frac{1}{2}x_1^2$. Then the derivative of V_1 with respect to time along the x_1-dynamics gives

$$\begin{aligned}\dot{V}_1 &= x_1\dot{x}_1 = -|x|_1 x_1^2\cos^2(t) - x_1^4 + x_1 x_2 \\ &\leq -x_1^4 + |x_1||x_2| = -(1-\theta)|x_1|^4 - \theta|x_1|^4 + |x_1||x_2| \quad \forall\, \theta \in (0, 1)\end{aligned}$$

Notice that

$$-\theta|x_1|^4 + |x_1||x_2| = -\theta|x_1|\left(|x_1|^3 - \frac{|x_2|}{\theta}\right) \leq 0 \text{ for } |x_1| \geq \sqrt[3]{\frac{|x_2|}{\theta}}$$

Hence

$$\begin{aligned}\alpha_1(|x_1|) \leq V_1(x_1) \leq \alpha_2(|x_1|) \\ \dot{V}_1 \leq -W(x_1), \quad \forall\, |x_1| \geq \rho(|x_2|),\end{aligned}$$

where $\alpha_1(|x_1|) = \alpha_2(|x_1|) = \frac{1}{2}x_1^2$ are class-$\mathcal{K}_\infty$ functions, the function $W(x_1) = (1 - \theta)|x_1|^4$ is positive definite and the function $\rho(|x_2|) = \sqrt[3]{\frac{|x_2|}{\theta}}$ is a class-$\mathcal{K}$function. Thus, by Theorem 4.19 (in Khalil p. 178) the system is input-to-state stable.

d By applying Lemma 4.7 it can be seen that the origin of the cascaded system is UGAS, since the subsystem Σ_1 with x_2 is ISS and the origin of the subsystem Σ_2 is UGAS.

Problem 3 (12%)

a The unperturbed system is

$$\dot{x} = -\alpha_1 x$$

It can easily be seen that the system is linear and has a Hurwitz state matrix since $\alpha_1 > 0$. Consequently, it can be concluded by Corollary 4.3 that the origin is exponentially stable.

Using the Lyapunov function

$$V(t,x) = \frac{1}{2}x^2$$

it can be shown that the derivative of $V(x)$ with respect to time along the x-dynamics gives

$$\dot{V}(t,x) = -\alpha_1 x^2 \quad \forall\, t \geq 0$$

This satisfy all the conditions in Theorem 4.10, which results in that origin is globally exponentially stable, since $V(t,x)$ is radially unbounded.

b We assume that $g(t,x)$ satisfy the linear growth bound

$$\begin{aligned} ||g(t,x)|| \leq \gamma ||x||, \quad \forall\, t \geq 0 \\ \gamma > 0. \end{aligned}$$

The conditions to the Lyapunov function are

$$\begin{aligned} &c_1||x||^2 \leq V(t,x) \leq c_2||x||^2 \\ &\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3||x||^2 \\ &\left\| \frac{\partial V}{\partial x} \right\| \leq c_4||x|| \end{aligned}$$

The first two conditions was satisfied in **a** with $c_1 = c_2 = \frac{1}{2}$ and $c_3 = \alpha_1$. The last condition becomes

$$\left\| \frac{\partial V}{\partial x} \right\| = x$$

which mean that $c_4 = 1$. Taking the derivative of $V(x)$ with respect to time along the trajectory of the perturbed system gives

$$\begin{aligned} \dot{V}(t,x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) + \frac{\partial V}{\partial x} g(t,x) \\ &\leq -\alpha_1|x|^2 + \left\| \frac{\partial V}{\partial x} \right\| \, ||g(t,x)|| \\ &\leq -\alpha_1|x|^2 + \gamma|x|^2 \end{aligned}$$

If $\gamma < \alpha_1$ then

$$\dot{V}(t,x) \leq -(\alpha_1 - \gamma)|x|^2, \ (\alpha_1 - \gamma) > 0,$$

$\dot{V}(t,x)$ is negative definite, thus we can conclude from Lemma 9.1 that origin of the perturbed system is exponentially stable if $\gamma < \alpha_1$. In addition, since all the assumptions hold globally then the origin is globally exponentially stable.

c The scalar system can be rewritten as

$$\dot{x} = \varepsilon\left(-(1+\sin^2(t))x + \frac{1}{2}x\right) = \varepsilon f(t,x) \tag{1}$$

This can be associated with an autonomous average system

$$\dot{x} = \varepsilon f_{av}(x)$$

where

$$f_{av}(x) = \frac{1}{T}\int_0^T f(\tau,x)d\tau$$

Taking the integral of the system (1) becomes

$$\begin{aligned}
f_{av}(x) &= \frac{1}{T}\int_0^T -(1+\sin^2(t))x + \frac{1}{2}x d\tau \\
&= \frac{1}{T}\int_0^T -\left(1+\frac{1}{2}(1-\cos(2\tau))\right)x + \frac{1}{2}x d\tau \\
&= \frac{1}{2T}\int_0^T -\big(3-\cos(2\tau)\big)\,x + x d\tau \\
&= \frac{1}{2T}\left[-\left(3T - \frac{1}{2}\sin(2T)\right)x + Tx\right]
\end{aligned}$$

Since the function (1) is π-periodic in t, the average function is given by

$$\begin{aligned}
f_{av}(x) &= \frac{1}{2\pi}\left[-\left(3\pi - \frac{1}{2}\sin(2\pi)\right)x + \pi x\right] \\
&= \frac{1}{2\pi}\left[-3\pi x + \pi x\right] \\
&= -x
\end{aligned}$$

The average system $\dot{x} = -\varepsilon x$ has an equilibrium point at $x^* = 0$. The Jacobian function at this point is -1. Thus the equilibrium point $x^* = 0$ is exponential stable. By Theorem 10.4, we can conclude that, for sufficient small ε, the system has an exponentially stable periodic solution of period π in an $O(\varepsilon)$ neighbourhood of $x^* = 0$. Moreover, for initial states sufficiently near $x = 0$, $x(t,\varepsilon) = x_{av}(t,\varepsilon) + O(\varepsilon)$ for all $t \geq 0$.

Problem 4 (15%)

a Using

$$V(x) = \frac{k_1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{k_2}{2k_5}x_3^2$$

we have

$$\begin{aligned}\dot{V}(x) =& k_1(x_2 - k_1x_1)x_1 + \left(-k_1x_1 - k_2x_3 - \frac{k_3}{\sqrt{k_4 + x_1^2}}x_2\right)x_2 \\ &+ \frac{k_2}{k_5}\left(k_5x_2 - k_5x_3 + \frac{k_5}{k_2}u\right)x_3 \\ =& -k_1^2x_1^2 - \frac{k_3}{\sqrt{k_4 + x_1^2}}x_2^2 - k_2x_3^2 + ux_3 \\ =& -\psi(x) + uy\end{aligned}$$

where $\psi(x) = k_1^2x_1^2 + \frac{k_3}{\sqrt{k_4+x_1^2}}x_2^2 + k_2x_3^2$ is positive definite. Hence it is (state) strictly passive.

b Using

$$V(x) = \frac{k_1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{k_2}{2k_5}x_3^2$$

we have

$$\begin{aligned}\dot{V}(x) =& k_1(x_2 - k_1x_1)x_1 + \left(-k_1x_1 - k_2x_3 - \frac{k_3}{\sqrt{k_4 + x_1^2}}x_2\right)x_2 \\ &+ \frac{k_2}{k_5}\left(k_5x_2 - k_5x_3 + \frac{k_5}{k_2}u\right)x_3 \\ =& -k_1^2x_1^2 - \frac{k_3}{\sqrt{k_4 + x_1^2}}x_2^2 - k_2x_3^2 + ux_3 \\ \leq& -k_2x_3^2 + uy \\ =& -y\rho(y) + uy\end{aligned}$$

where $\rho(y) = k_2y$. Hence, it is output strictly passive.

c The unforced system is given by

$$\begin{aligned}\dot{x}_1 &= -k_1x_1 + x_2 \\ \dot{x}_2 &= -k_1x_1 - k_2x_3 - \frac{k_3}{\sqrt{k_4 + x_1^2}}x_2 \\ \dot{x}_3 &= k_5x_2 - k_5x_3 \\ y &= x_3\end{aligned}$$

Consider

$$S = \{x \in \mathbb{R}^3 | y = 0\} = \{x \in \mathbb{R}^3 | x_3 = 0\}$$

then for every $(x_1, x_2, x_3) \in S$, i.e. $y(t) \equiv 0$, we have $x_3(t) \equiv 0 \implies \dot{x}_3 \equiv 0 \implies k_5 x_2 - 0 \equiv 0 \implies x_2 \equiv 0 \implies \dot{x}_2(t) \equiv 0 \implies -k_1 x_1 - 0 \equiv 0 \implies x_1 \equiv 0, \dot{x}_1 \equiv 0$. Hence, no other solution can stay identically in S other than the zero solution. Thus the system is zero state observable.

d It has been concluded that the system is strictly passive, output strictly passive and zero-state observable. From these results it can be said by applying Lemma 6.7 that the origin of the unforced system $\dot{x} = f(x, 0)$ is asymptotically stable. Furthermore, since the storage function $V(x)$ is radially unbounded, it can be said that the origin is globally asymptotically stable.

In addition, by applying Lemma 6.5 it can be concluded that the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain $\gamma \leq \frac{1}{k_2}$.

Problem 5 (30%)

a We differentiate the output $y = x_2$ to find the relative degree:

$$\dot{y} = \dot{x}_2 = x_3 - x_1$$
$$\ddot{y} = \dot{x}_3 - \dot{x}_1 = -x_3 - 2x_1 + u - x_1 - x_2 + u = -3x_1 - x_2 - x_3 + 2u$$

The relative degree of the system is thus $\rho = 2$ in $\mathbb{R}^3$. It exists and is well defined, hence the system is input-output linearizable.

b The system can be written as

$$\dot{x} = f(x) + g(x)u$$

where

$$f(x) = \begin{bmatrix} x_1 + x_2 \\ x_3 - x_1 \\ -2x_1 - x_3 \end{bmatrix} \qquad g(x) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- First, the external coordinates ξ are found. Since $\rho = 2$, ξ is of dimension 2.

$$\xi_1 = y = x_2$$
$$\xi_2 = L_f y = \frac{\partial y}{\partial x} f(x) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} f(x) = x_3 - x_1$$

- Then the internal coordinates η are found. Since the system state has dimension 3, and ξ is of dimension 2, η is of dimension $3 - 2 = 1$. With $\eta = \phi(x)$, we have that

$$\frac{\partial \phi(x)}{\partial x} g(x) = 0$$

Inserting for $g(x)$, this implies

$$-\frac{\partial \phi(x)}{\partial x_1} + \frac{\partial \phi(x)}{\partial x_3} = 0$$

A $\phi(x)$ which satisfies this is

$$\phi(x) = x_1 + x_3$$

- This gives the following diffeomorphism

$$z = T(x) = \begin{bmatrix} \eta \\ \cdots \\ \xi \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ \cdots\cdots \\ x_2 \\ -x_1 + x_3 \end{bmatrix}$$

Since this is a diffeomorphism, its inverse exists and is smooth, and can be found as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\eta - \xi_2) \\ \xi_1 \\ \frac{1}{2}(\eta + \xi_2) \end{bmatrix} = T^{-1}(z)$$

- Next, $\gamma(x)$ and $\alpha(x)$ are found

$$\begin{aligned} \gamma(x) &= L_g L_f y = 2 \\ \alpha(x) &= -\frac{L_f^2 y}{\gamma(x)} = -\frac{1}{2}(-(x_1 + x_2) - x_3 - 2x_1) = -\frac{1}{2}(-3x_1 - x_2 - x_3) \end{aligned}$$

- Finally, the system is written in normal form

$$\begin{aligned} \dot{\eta} &= \dot{x}_1 + \dot{x}_3 = -x_1 + x_2 - x_3 \\ &= -\frac{1}{2}(\eta - \xi_2) + \xi_1 - \frac{1}{2}(\eta - \xi_2) = \xi_1 - \eta = f_0(\eta, \xi) \\ \dot{\xi} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 2 \left[u + \frac{1}{2}(-3x_1 - x_2 - x_3) \right] \\ &= A_c \xi + B_c \gamma(x) \left[u - \alpha(x) \right] \end{aligned}$$

This can be seen by noting that since $\alpha(x) = -\frac{L_f^2 y}{\gamma(x)}$, the transformation is valid when $\gamma(x) \neq 0$, and since $\gamma(x) = 2$, the transformation is valid for the entire $\mathbb{R}^3$ space.

c An input-output linearizing controller is given by

$$u = \alpha(x) + \beta(x) v$$

where $\beta(x) = \frac{1}{\gamma(x)} = \frac{1}{2}$ and $\alpha(x) = -\frac{1}{2}(-3x_1 - x_2 - x_3)$. This gives

$$u = -\frac{1}{2}(-3x_1 - x_2 - x_3) + \frac{1}{2} v$$

d The external dynamics ξ with the input-output linearizing controller inserted is given by:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v \end{aligned}$$

$\xi = [\xi_1 \ \ \xi_2]^T$ can be stabilized by $v = -k_1\xi_1 - k_2\xi_2$, which gives the closed loop dynamics

$$\dot{\xi} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \xi = A\xi$$

The eigenvalues of the closed loop system are

$$\lambda = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1}}{2}$$

which have negative real parts as long as $k_1, k_2 > 0$. This means that A is Hurwitz and $\dot{\xi} = A\xi$ is asymptotically stable at the origin with $v = -k_1\xi_1 - k_2\xi_2$, $k_1, k_2 > 0$.

e The system is minimum phase if the origin of the zero dynamics $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable. The zero dynamics is given by

$$\dot{\eta} = f_0(\eta, 0) = -\eta$$

This is a linear differential equation with eigenvalue $\lambda = -1$, and the origin is thus GES. This can also be shown by using the Lyapunov function

$$V(\eta) = \frac{1}{2}\eta^2$$

Differentiating V along the trajectories of the zero dynamics gives

$$\dot{V}(\eta) = -\eta^2$$

Since $V(\eta)$ is continuously differentiable and positive definite, and $\dot{V}(\eta)$ is negative definite, the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable. The system is therefore minimum phase.

f Since the system is minimum phase, and the closed-loop external dynamics is asymptotically stable at the origin, the closed-loop system $[\eta, \xi]^\top$ is asymptotically stable at the origin.

Problem 6 (10%) Consider the surge-motion model of a ship,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ m\dot{x}_2 + d(x_2)x_2 &= u, \end{aligned}$$

The control objective is to globally stabilize the origin of the system. The design approach of a backstepping controller is divided into several stages, including the definition of new state variables and finding the control law through control Lyapunov functions (CLF).

We start by choosing a positive definite (CLF)

$$V_1 = \frac{1}{2}x_1^2$$

the derivative of V_1 with respect to time along the x_1-dynamics gives

$$\dot{V}_1 = x_1\dot{x}_1 = x_1 x_2$$

With $x_2 = \varphi(x_1) = -k_1x_1$, the origin of the x_1-dynamic is uniform exponentially stable. Since

$$\dot{V}_1 = -k_1x_1^2 < 0 \quad \forall\, x_1 \neq 0$$

satisfy Theorem 4.10. A new variable z is defined as $z = x_2 - \varphi(x_1)$. The dynamics for $\dot{x}_1$ in the new coordinates is then

$$\dot{x}_1 = x_2 = z + \varphi(x_1) = z - k_1x_1$$

Next, the expression for $\dot{z}$ is found

$$\begin{aligned} m\dot{z} &= m\dot{x}_2 - m\frac{\partial\varphi(x_1)}{\partial t} \\ &= u - d(x_2)x_2 - m(-k_1\dot{x}_1) \\ &= u - d(x_2)x_2 - m(-k_1(z - k_1x_1)) \\ &= u - d(x_2)x_2 - m\dot{\varphi}, \end{aligned}$$

where $\dot{\varphi} = -k_1(z - k_1x_1)$. Finally, the overall stabilizing input u is found using the following continuously differentiable and positive definite Lyapunov function

$$\begin{aligned} V_2 &= \frac{1}{2}x_1^2 + \frac{1}{2}mz^2 \\ \dot{V}_2 &= x_1\dot{x}_1 + z\dot{z} = x_1(z - k_1x_1) + z(u - d(x_2)x_2 - m\dot{\varphi}) \\ &= -k_1x_1^2 + z(x_1 + u - d(x_2)x_2 - m\dot{\varphi}) \end{aligned}$$

u is chosen such that

$$x_1 + u - d(x_2)x_2 - m\dot{\varphi} = -k_2z$$

which means that

$$u = -x_1 + d(x_2)x_2 + m\dot{\varphi} - k_2z$$

and

$$\dot{V}_2 = -k_1x_1^2 - k_2z^2 < 0 \quad \forall\, (x_1, z) \neq (0,0)$$

Since $V_2(x_1, z)$ is continuously differentiable and positive definite, and $\dot{V}_2(x_1, z)$ is negative definite, u exponentially stabilizes $(x_1, z) = (0, 0)$. Since the transformation $(x_1, x_2) \to (x_1, z)$ is a global diffeomorphism, this also means that the origin is exponentially stabilized at the origin. In addition, since $V_2(x_1, z)$ is radially unbounded and there are no singularities in u, the origin is uniform global exponentially stable.