NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF ENGINEERING CYBERNETICS

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Exam
TTK4150 Nonlinear Control Systems

Monday December 12, 2011

Hours: 09.00 – 13.00

Aids:	D - No printed or written materials allowed. NTNU type approved calculator with an empty memory allowed.
Language:	English
No. of pages:	6
Grades available:	January 12, 2012

This exam counts for 100% of the final grade.

Problem 1 (15%)

Consider the system:

$$\begin{aligned}\dot{x}_1 &= x_2(13 - x_1^2 - x_2^2)\\ \dot{x}_2 &= 12 - x_1(13 - x_1^2 - x_2^2)\end{aligned}$$

- **a [2%]** Let $\dot{x} = f(x)$ where $x = (x_1, x_2)^\top$. Find the Jacobian matrix $\partial f/\partial x$.
- **b [2%]** The system has an equilibrium point at $x = (1, 0)^\top$. Classify the *qualitative behavior* of the equilibrium point.
- **c [4%]** Find all other equilibrium points and classify the *qualitative behavior* of each of them.

  Hint: You may want to use polynomial division and the fact that one of the equilibrium points are located at $x = (1, 0)^\top$ to find the roots of the polynomial $x_1^3 - 13x_1 + 12$.
- **d [3%]** Make a sketch of the phase portrait around each of the above systems equilibrium points in the phase plane.

  Hint: Mark the equilibrium points in the phase plane, and consider the location and the qualitative behavior of each equilibrium. You might try to plot some of the vectors of the vector field $\dot{x} = f(x)$ for some points around each equilibrium.
- **e [4%]** Use the index method to prove that no periodic orbit lie entirely within $D := \{x \in \mathbb{R}^2 \,|\, ||x||_2^2 < 1\}$. Can the Bendixson Criterion be used to rule out the existence of a periodic orbit in D?

Problem 2 (15%)

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 e^{x_1 x_2}\end{aligned}$$

- **a [5%]** Find whether the right hand side of $\dot{x} = f(x)$, where $x = (x_1, x_2)^\top$, is continuously differentiable, locally Lipschitz and/or globally Lipschitz?
- **b [5%]** What conclusions can be drawn about stability, asymptotically stability and exponentially stability of the origin, using Lyapunov's indirect method (linearization)?
- **c [5%]** Determine by using the Lyapunov function candidate $V(x) = 1/2(x_1^2 + x_2^2)$ whether the equilibrium point is unstable, stable, asymptotically stable of globally asymptotically stable. State the strongest conclusion that you can verify.

Problem 3 (15%)

a [2%] Consider a nonlinear static system $\tilde{y} = f(u)$. Show that this system is output strictly passive if

$$uf(u) \geq \epsilon f^2(u) \,.$$

where ϵ is a positive constant.

b [3%] Use the statement in **a** to prove that the system $\tilde{y} = f(u)$, with

$$f(u) = \frac{u}{1+u^2} \,,$$

is output strictly passive.

c [5%] Consider the system in Figure 1 below, where

$$g(s) = \frac{1}{(s+1)(s+2)} \quad \text{and} \quad f(u) = \frac{u}{(1+u^2)} .$$

and view the function $f : \mathbb{R} \to \mathbb{R}$ as an operator that assigns to every input signal $u(t)$ the output signal $\tilde{y}(t) = f(u(t))$. Show that this operator is finite-gain $\mathcal{L}_2$ stable, and use the small gain theorem to determine for which values of the gain K the closed loop system is stable.

Hint: The $\mathcal{L}_2$ gain γ of a transfer function $g(s)$ is given by

$$\gamma = \sup_{\omega \in \mathbb{R}} |g(j\omega)| \,.$$

d [5%] Consider again the setup in Figure 1, with $g(j\omega)$ and $f(u)$ as in **c**. The nonlinear function $f(u)$ is shown in Figure 2, where as the Nyquist plot of the transfer function $g(j\omega)$ is shown in Figure 3. Based on the given information, does the describing function method predict sustained oscillations in the closed-loop system, and if so, for which values of K?

Hint: It is not necessary to calculate the describing function in order to answer the problem.

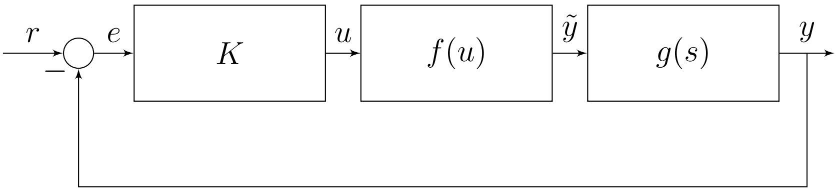

Figure 1: System overview. Notice that $\tilde{y}$ is the output of the nonlinearity.

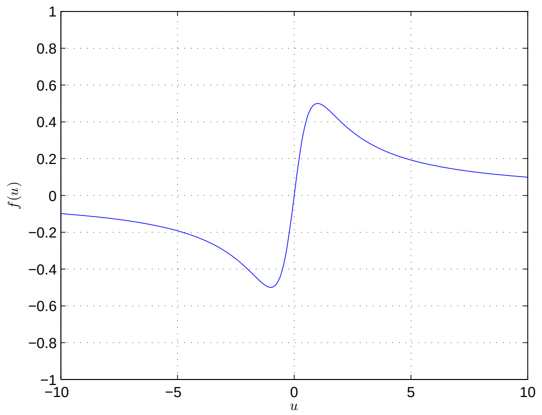

Figure 2: The function $f(u)$.

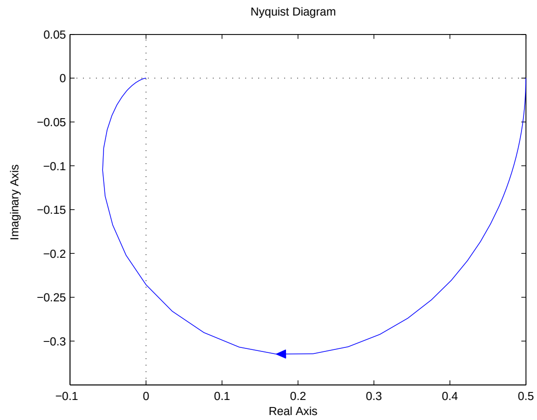

Figure 3: Nyquist plot for $g(j\omega)$ for $\omega > 0$.

Problem 4 (9%)

Consider the nonautonomous system

$$\begin{aligned}\dot{x}_1 =& -x_1 - e^{-3t}x_2\\ \dot{x}_2 =& x_1 - x_2\,.\end{aligned}$$

a [4%] To determine stability of the system consider the Lyapunov function candidate

$$V(t,x) = x_1^2 + (1+e^{-3t})x_2^2\,.$$

Determine whether $V(t,x)$ is positive definite, decrescent and radially unbounded.

b [5%] State the strongest stability property (of the origin) you can, using the Lyapunov function candidate from **a**.

Problem 5 (30%)

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1\\ \dot{x}_2 &= -x_2 - x_3u\\ \dot{x}_3 &= 5x_2x_3^2 + u\\ \dot{x}_4 &= x_3\\ y &= x_4\end{aligned}$$

a [3%] Find the relative degree ρ of the system. Is the system input-output linearizable?

b [10%] Transform the system into normal form

$$\begin{aligned}\dot{\eta} &= f_0(\eta,\xi)\\ \dot{\xi} &= A_c\xi + B_c\gamma(x)\left[u - \alpha(x)\right]\end{aligned}$$

Specify the diffeomorphism $z = T(x) = \begin{bmatrix}\eta & \xi\end{bmatrix}^T$, the functions $\gamma(x)$, $\alpha(x)$ and $f_0(\eta, \xi)$ and the matrices A_c and B_c. In which domain is the transformation valid?

c **[3%]** Find an input-output linearizing controller on the form $u = \alpha(x) + \beta(x)v$.

d **[3%]** Find a controller v such that the external dynamics ξ is asymptotically stable at the origin.

e **[6%]** Is the system minimum phase?

Hint: If you were not able to solve **b** you may use the following equations for the internal dynamics:

$$\begin{aligned}\dot{\eta}_1 &= \eta_2 - \frac{1}{2}\xi_2^2 - \eta_1 \\ \dot{\eta}_2 &= (\eta_2 - \frac{1}{2}\xi_2^2)(5\xi_2^3 - 1)\,.\end{aligned}$$

f **[2%]** Is the closed-loop system $\begin{bmatrix}\eta & \xi\end{bmatrix}^T$ asymptotically stable at the origin?

g **[3%]** Design a control law which ensures that the output $y(t) = \xi(t)$ follows an arbitrary reference trajectory $r_d(t)$, where $r_d(t)$ and its derivates up to $r_d^{(\rho)}$ are bounded for all $t \geq 0$ and the ρ-th derivative $r^{(\rho)}(t)$ is a piecewise continuous function of t.

Problem 6 (16%)

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1 x_2 - x_1^4 \\ \dot{x}_2 &= x_1 + u\end{aligned}$$

a **[10%]** Use backstepping to design a nonlinear control law that makes the orgin globally asymptotically stable.

b **[3%]** Given this control law, are there still any aspects of the system that might jeopardize the **global** asymptotic stability obtained by this control law?

c **[3%]** In general, for which systems is backstepping a better method than feedback linearization for designing a nonlinear control law?

Appendix: Formulae

$$xy \leq \frac{\varepsilon^p}{p}|x|^p + \frac{1}{q\varepsilon^q}|y|^q$$

$$y(t) \approx a \sin\theta$$
$$\theta = \omega t$$

$$\psi(y(t)) \approx b + c_c \cos\theta + c_s \sin\theta = b + c\sin(\theta + \phi)$$

$$b = \frac{1}{2\pi}\int_0^{2\pi} \psi(a\sin\theta)\mathrm{d}\theta$$
$$c_c = \frac{1}{\pi}\int_0^{2\pi} \psi(a\sin\theta)\cos\theta\mathrm{d}\theta$$
$$c_s = \frac{1}{\pi}\int_0^{2\pi} \psi(a\sin\theta)\sin\theta\mathrm{d}\theta$$

$$c = \sqrt{{c_s}^2 + {c_c}^2}$$
$$\phi = \arctan\left(\frac{c_c}{c_s}\right)$$

$$\Psi(a,\omega) = \frac{c\,\mathrm{e}^{j(\theta+\varphi)}}{a\,\mathrm{e}^{j\theta}} = \frac{c\,\mathrm{e}^{j\varphi}}{a} = \frac{c_s + jc_c}{a}$$
$$|\Psi(a,\omega)| = \frac{c}{a}$$
$$\angle\Psi(a,\omega)| = \phi$$

$$G(j\omega)\Psi(a,\omega) + 1 = 0$$

$$\mathrm{e}^{\alpha+j\beta} = \mathrm{e}^{\alpha}(\cos\beta + j\sin\beta)$$
$$\cos^2\alpha + \sin^2\alpha = 1$$
$$\sin\alpha = \pm\sqrt{1-\cos^2\alpha}$$
$$\cos\alpha = \pm\sqrt{1-\sin^2\alpha}$$