NTNU – Trondheim
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TTT4120 Digital Signal Processing
Fall 2017

Modeling of Stochastic Processes: Parametric Spectral Estimation

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Lecture in course book*

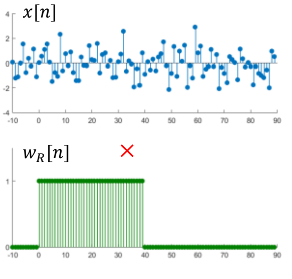
- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 12.2 Innovations representation of stationary random processes
  - 14.3 Parametric methods for spectral estimation
- A comprehensive overview of topics treated in the lecture, see "Introdukjon til statistisk signalbehandling" on Blackboard

*Level of detail is defined by lectures and problem sets

Contents and learning outcomes

- Non-parametric versus parametric models
- Innovations representation
- Rational power spectra: AR, MA, ARMA
- AR models and Yule-Walker equations

Nonparametric PSD estimation



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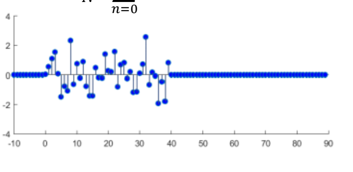
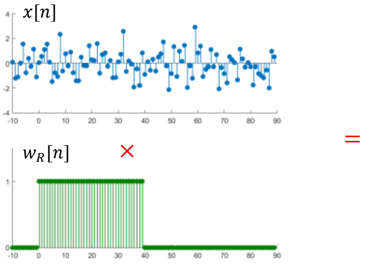
$$\hat{\gamma}_{XX}[l] = \frac{1}{N}\sum_{n=0}^{N-|l|-1} x[n]x[n+|l|] = w_B[l]\,\gamma_{XX}[l]$$



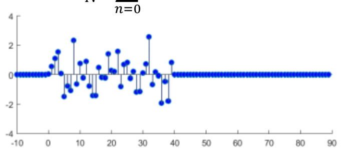
$$\hat{\Gamma}_{XX}(f) = \mathcal{F}\{\hat{\gamma}_{XX}[l]\} = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]e^{-j2\pi f n}\right|^2$$

- Nonparametric power spectrum estimation: Only assumptions about stochastic process $X[n]$ are wide-sense stationarity and ergodicity
  - + Relatively simple and easy to compute using the FFT_N
  - – Requires long data records for good frequency resolution
  - – Spectral leakage due to windowing ⇒ Can mask weak signals

Nonparametric PSD estimation…



$$\hat{\gamma}_{XX}[l] = \frac{1}{N}\sum_{n=0}^{N-|l|-1} x[n]x[n+|l|] = w_B[l]\,\gamma_{XX}[l]$$



$$\hat{\Gamma}_{XX}(f) = \mathcal{F}\{\hat{\gamma}_{XX}[l]\} = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]e^{-j2\pi fn}\right|^2$$

- Basic limitations of parametric spectrum estimation
  - Inherent assumption that $\hat{\gamma}_{XX}[m] = 0$ for some $m \geq N$
  - Inherent assumption that the data is periodic with N

5

Parametric PSD estimation

- Consider methods that relax the above assumptions and can extrapolate the values of the autocorrelation function for $m \geq N$

$$\hat{\gamma}_{XX}[m], |m| \leq N-1 \Longrightarrow \hat{\gamma}_{XX}[m], |m| \geq N$$

- Requires *a priori* information on how data signal is generated
- A parametric model for the signal generation is constructed
  - Sufficient to find the values of the model parameters
  - Often provides us with a better description of the process, whenever the model is close to reality
  - Eliminates need for windowing and assumption that $\hat{\gamma}_{XX}[m] = 0$ for some $m \geq N$

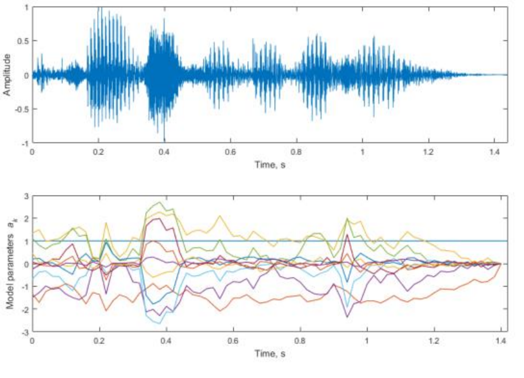
6

Parametric PSD estimation…

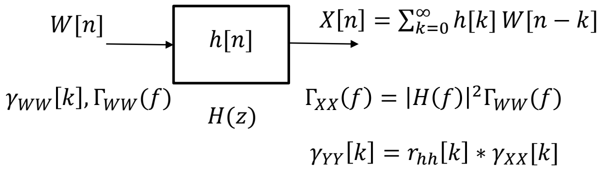
- Parametric modeling does not provide an exact representation
  - Approximation characterized by few parameters
  - Enhanced spectral resolution: especially for finite data records, e.g., due to time-variant or transient phenomena
  - Efficient signal compression (e.g., LPC of speech)
- Different parametric models
  - Rational models
  - All-pole models lead to linear equation systems

Parametric PSD estimation…

- Example: Model a speech signal (parameters change every 20ms)



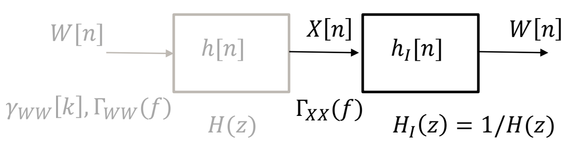
Innovations representations



- Wide-sense stationary random processes can be represented as the output of a causal and causually invertible system excited by a white noise process
- This representation is called the Wold representation

9

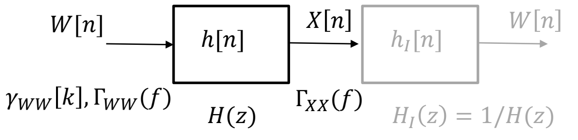
Innovations representations…



- Consequently, a WSS random process can be represented by the output of the inverse system, which is a white process
  - A random process can be transformed into a white process by passing $X[n]$ through a linear filter
  - $H_I(z)$ is a called a whitening filter

10

Innovations representations…



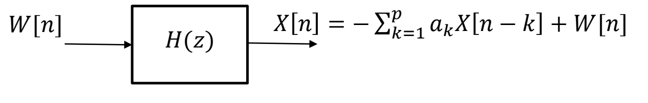
- Restrict our attention to cases where the PSD of $X[n]$ is rational

$$\Gamma_{XX}(f) = \frac{\sigma_W^2 B(z) B(z^{-1})}{A(z)A(z^{-1})}\bigg|_{z=e^{j\omega}} \text{ or } H(z) = \frac{B(z)}{A(z)}$$

- $H(z)$ is causal stable and minimum-phase $\Rightarrow$ $H_I(z)$ is also causal stable and minimum phase
- By knowing $H(z)$, described by a few parameters, we can go from the statistical properties of $W[n]$ to $X[n]$, and vice versa

11

Model types: AR process



- For an autoregressive (AR) process, filter $H(z)$ has only poles,

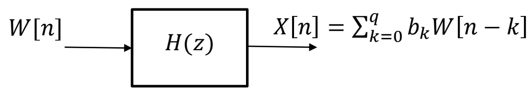
$$H(z) = \frac{1}{A(z)} = \frac{1}{1+\sum_{k=1}^{p} a_k z^{-k}}$$

and is described in time domain as

$$X[n] = -\sum_{k=1}^{p} a_k X[n-k] + W[n]$$

12

Model types: MA process



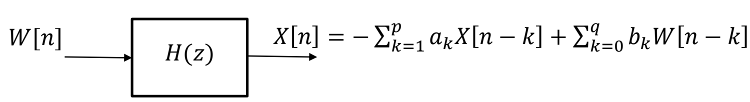
- For a moving average (MA) process, filter $H(z)$ has only zeros,

$$H(z) = B(z) = \sum_{k=0}^{q} b_k z^{-k}$$

and is described in time domain as

$$X[n] = \sum_{k=0}^{q} b_k W[n-k]$$

Model types: ARMA process



- For an autoregressive moving average (ARMA) process, filter $H(z)$ has both zeros and poles,

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1+\sum_{k=1}^{p} a_k z^{-k}}$$

and is described in time domain as

$$X[n] = -\sum_{k=1}^{p} a_k X[n-k] + \sum_{k=0}^{q} b_k W[n-k]$$

Parameter estimation

$$W[n] \longrightarrow \boxed{H(z)} \longrightarrow X[n] = -\sum_{k=1}^{p} a_k X[n-k] + \sum_{k=0}^{q} b_k W[n-k]$$

- The preceeding models are described by a few parameters: $\{a_k\}$, $\{b_k\}$, and σ_W^2
- Parameter values are unknown $\Rightarrow$ need to estimate them from $X[n]$
- We need to find the parameters such that the model "resembles," or is "close" to, the true process
- Restrict our study to AR models

Parameter estimation…

$$W[n] \longrightarrow \boxed{H(z)} \longrightarrow X[n] = -\sum_{k=1}^{p} a_k X[n-k] + W[n]$$

- Restrict our study to AR models
  - Suitable for modelling processes characterized by sharp peaks in the spectrum
  - Other types of spectra can be modeled by increasing the model order, i.e., the number of filter coefficients
  - Suitable for many practical physical processes, e.g., speech, images, etc.

Statistical description of AR(*p*) process

- Model the problem with linear system

$$W[n] \longrightarrow \boxed{h[n]} \longrightarrow X[n] = -\sum_{k=1}^{p} a_k X[n-k] + W[n]$$

- WGN process $W[n]$: $E\{W[n]W[n-m]\} = \sigma_W^2 \delta[m]$
- The AR(p) process has p filter coefficients $\{a_k\}_{k=1}^{p}$
- Let us try to find a relation between the autocorrelation function $\gamma_{XX}[l]$ and filter coefficients a_k and σ_W^2

17

Statistical description of AR(*p*) process

- Autocorrelation function $\gamma_{XX}[l]$:

$$\begin{aligned}
\gamma_{XX}[l] &= E\{X[n]X[n-l]\} \\
&= E\left\{\left(-\sum_{k=1}^{p} a_k X[n-k] + W[n]\right)X[n-l]\right\} \\
&= E\left\{-\sum_{k=1}^{p} a_k X[n-k]X[n-l] + W[n]X[n-l]\right\} \\
&= -\sum_{k=1}^{p} a_k E\{X[n-k]X[n-l]\} + E\{W[n]X[n-l]\} \\
&= -\sum_{k=1}^{p} a_k \gamma_{XX}[l-k] + \gamma_{WX}[l]
\end{aligned}$$

- Let us take look at the crosscorrelation term $\gamma_{WX}[l]$

18

Statistical description of AR(*p*) process…

- Crosscorrelation term $\gamma_{WX}[l]$:

$$\gamma_{WX}[l] = E\{W[n]X[n-l]\} = E\{W[n+l]X[n]\}$$
$$= E\left\{W[n+l]\left(-\sum_{k=1}^{p} a_k X[n-k] + W[n]\right)\right\}$$
$$= E\left\{-\sum_{k=1}^{p} a_k W[n+l]X[n-k] + W[n+l]W[n]\right\}$$
$$= -\sum_{k=1}^{p} a_k E\{W[n+l]X[n-l]\} + E\{W[n+l]W[n]\}$$
$$= 0 + \sigma_W^2 \delta[l]$$

- Crosscorrelation term only take non-zero value lag $l = 0$

Statistical description of AR(*p*) process…

- Autocorrelation function $\gamma_{XX}[l]$ for an AR(p) process:

$$\boxed{\gamma_{XX}[|l|] = -\sum_{k=1}^{p} a_k \gamma_{XX}[|l|-k] + \sigma_W^2 \delta[l]}$$

- Discussion: How can we use the above equation to find $\gamma_{XX}[l]$ for all l, and what knowledge is required?

Statistical description of AR(*p*) process…

- Autocorrelation function $\gamma_{XX}[l]$ for an AR(p) process:

$$\boxed{\gamma_{XX}[|l|] = -\sum_{k=1}^{p} a_k \gamma_{XX}[|l|-k] + \sigma_W^2 \delta[l]}$$

- Discussion: How can we use the above equation to find…
  - Crosscorrelation function $\gamma_{XX}[l]$ is specified for all l when
    1) the p filter coefficients $\{a_k\}_{k=1}^{p}$, and;
    2) the p+1 first values of $\gamma_{XX}[l]$, i.e., $\gamma_{XX}[0], \gamma_{XX}[1], \ldots, \gamma_{XX}[p]$
- How to find model parameters to the model $\{a_k\}_{k=1}^{p}$ given $\gamma_{XX}[l]$

21

Statistical description of AR(*p*) process…

$$\boxed{\gamma_{XX}[|l|] = -\sum_{k=1}^{p} a_k \gamma_{XX}[|l|-k] + \sigma_W^2 \delta[l]}$$

- Linear equation system ($\gamma_{XX}[l] = \gamma_{XX}[-l]$):

$l = 1$: $-\gamma_{XX}[1] = a_1\gamma_{XX}[0] + a_2\gamma_{XX}[1] + \cdots + a_p\gamma_{XX}[p-1]$

$l = 2$: $-\gamma_{XX}[2] = a_1\gamma_{XX}[1] + a_2\gamma_{XX}[0] + \cdots + a_p\gamma_{XX}[p-2]$

$l = 3$: $-\gamma_{XX}[3] = a_1\gamma_{XX}[2] + a_2\gamma_{XX}[1] + \cdots + a_p\gamma_{XX}[p-3]$

$\vdots$

$l = p$: $-\gamma_{XX}[p] = a_1\gamma_{XX}[p-1] + a_2\gamma_{XX}[p-2] + \cdots + a_p\gamma_{XX}[0]$

22

Statistical description of AR(*p*) process…

- Linear equation system in matrix form (Yule-Walker equations):

$$\underbrace{\begin{bmatrix} \gamma_{XX}[0] & \gamma_{XX}[1] & \cdots & \gamma_{XX}[p-1] \\ \gamma_{XX}[1] & \gamma_{XX}[0] & \cdots & \gamma_{XX}[p-2] \\ \vdots & \ddots & \ddots & \vdots \\ \gamma_{XX}[p-1] & \gamma_{XX}[p-2] & \cdots & \gamma_{XX}[0] \end{bmatrix}}_{\boldsymbol{\Gamma}_{XX}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}}_{\boldsymbol{a}} = - \underbrace{\begin{bmatrix} \gamma_{XX}[1] \\ \gamma_{XX}[2] \\ \vdots \\ \gamma_{XX}[p] \end{bmatrix}}_{\boldsymbol{\gamma}_{XX}}$$

and

$$\sigma_W^2 = \gamma_{XX}[0] + \sum_{k=1}^{p} a_k \gamma_{XX}[k]$$

- Solve for coefficients

$$\boldsymbol{a} = -\boldsymbol{\Gamma}_{XX}^{-1} \boldsymbol{\gamma}_{XX}$$

Statistical description of AR(*p*) process…

- Santy check: Model process $X[n] = -aX[n-1] + W[n]$ using an AR(2) process, and find parameters a_1 and a_2

AR(1) process: $\gamma_{XX}[l] = \sigma_W^2 \frac{(-a)^{|l|}}{1-a^2}, |l| \geq 0$

Yule-Walker equations:

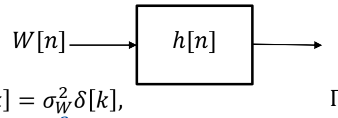
$$\underbrace{\begin{bmatrix} \gamma_{XX}[0] & \gamma_{XX}[1] \\ \gamma_{XX}[1] & \gamma_{XX}[0] \end{bmatrix}}_{\boldsymbol{\Gamma}_{XX}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}_{\boldsymbol{a}} = - \underbrace{\begin{bmatrix} \gamma_{XX}[1] \\ \gamma_{XX}[2] \end{bmatrix}}_{\boldsymbol{\gamma}_{XX}}$$

$$\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} -a \\ a^2 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix}^{-1} \begin{bmatrix} -a \\ a^2 \end{bmatrix}$$

Solve for coefficients: $a_1 =?$, $a_2 =?$

PSD of an AR(p) process

- Once we have filter coefficients $\{a_k\}_{k=1}^{p}$ we can compute the PSD



$X[n] = -\sum_{k=1}^{p} a_k X[n-k] + W[n]$

$\gamma_{WW}[k] = \sigma_W^2 \delta[k]$,
$\Gamma_{WW}(f) = \sigma_W^2$

$\Gamma_{XX}(f) = |H(f)|^2 \Gamma_{WW}(f) = |H(f)|^2 \sigma_W^2$

- Frequency response of filter

$$H(f) = \frac{1}{1+\tilde{A}(f)}, \text{ with } \tilde{A}(z) = \sum_{k=1}^{p} a_k z^{-k}$$

- Finally we obtain the PSD as

$$\Gamma_{XX}(f) = \frac{\sigma_W^2}{\left|1+\tilde{A}(f)\right|^2}$$

25

PSD of an AR(p) process…

- Example: Model $X[n] = W[n] - bW[n-1]$, $W[n] \sim N(0,1)$ using an AR(2) process, and find parameters a_1 and a_1

- Autocorrelation: $\gamma_{XX}[l] = (1+b^2)\delta[l] - b\delta[l-1] - b\delta[l+1]$
- Yule-Walker equations:

$$\begin{bmatrix} 1+b^2 & -b \\ -b & 1+b^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} -b \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1+b^2 & -b \\ -b & 1+b^2 \end{bmatrix}^{-1} \begin{bmatrix} b \\ 0 \end{bmatrix}$$

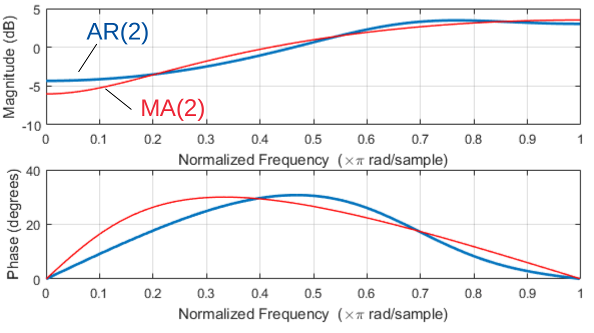
Solve for coefficients: $a_1 = \frac{b(1+b^2)}{b^2(1+b^2)+1}$, $a_2 = \frac{b^2}{b^2(1+b^2)+1}$

$$\sigma_W^2 = \gamma_{XX}[0] + a_1 \gamma_{XX}[1]$$

26

PSD of an AR(p) process…

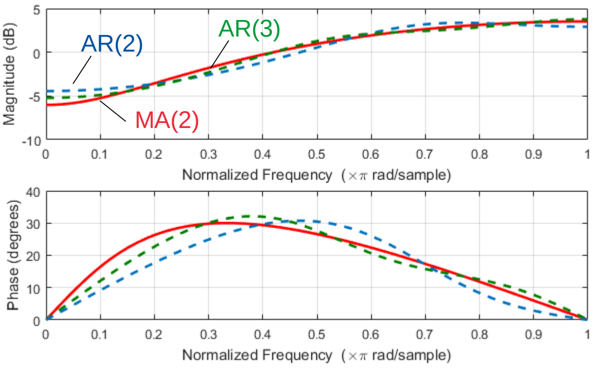
- Example: Model $X[n] = W[n] - 0.5W[n-1]$, $W[n] \sim N(0,1)$



27

PSD of an AR(p) process…

- Example: Model $X[n] = W[n] - 0.5W[n-1]$, $W[n] \sim N(0,1)$



28

PSD of an AR(p) process…

- Example: Power spectrum of an AR process is given by

$$\Gamma_{XX}(f) = \frac{\sigma_W^2}{|A(f)|^2} = \frac{25}{\left|1 - e^{-j2\pi f} + \frac{1}{2}e^{-j4\pi f}\right|^2}$$

where σ_W^2 is the variance of the input sequence.

a) Determine the difference equation for generating the AR process when the excitation is white noise

b) Determine the system function for the whitening filter

29

PSD of an AR(p) process…

- Only access to finite-length realization, $x[n]$, of process $X[n]$
  - True $\gamma_{XX}[l]$ must be estimated from $x[n] \Rightarrow \hat{\gamma}_{XX}[l]$
  - Parameter values computed using $\hat{\gamma}_{XX}[l]$ becomes parameter estimates $\{\hat{a}_k\} \Rightarrow$ Power spectrum estimate

$$\hat{\Gamma}_{XX}(f) = \frac{\hat{\sigma}_W^2}{\left|\hat{A}(f)\right|^2} = \frac{\hat{\sigma}_W^2}{\left|1 + \sum_{k=1}^{p} \hat{a}_k e^{-j2\pi f k}\right|^2}$$

$$\boxed{\begin{array}{l} \gamma_{XX}[l] \longrightarrow \{a_k\} \longrightarrow \Gamma_{XX}(f) \\ \qquad\qquad\quad \downarrow \text{(estimation)} \\ \hat{\gamma}_{XX}[l] \longrightarrow \{\hat{a}_k\} \longrightarrow \hat{\Gamma}_{XX}(f) \end{array}}$$

30

PSD of an AR(p) process…

- Example: Estimate $\Gamma_{XX}(f)$ from an N-point realization of
  $X[n] = aX[n-1] + W[n],\ W[n] \sim N(0, \sigma_w^2)$

1. Compute estimates of $\hat{\gamma}_{XX}[0]$ and $\hat{\gamma}_{XX}[1]$:

$$\hat{\gamma}_{XX}[l] = \frac{1}{N}\sum_{n=0}^{N-|l|-1} x[n]x[n+|l|]\,,\ l = 0{,}1$$

2. Estimate AR(1) parameter and noise variance (Yule-Walker):

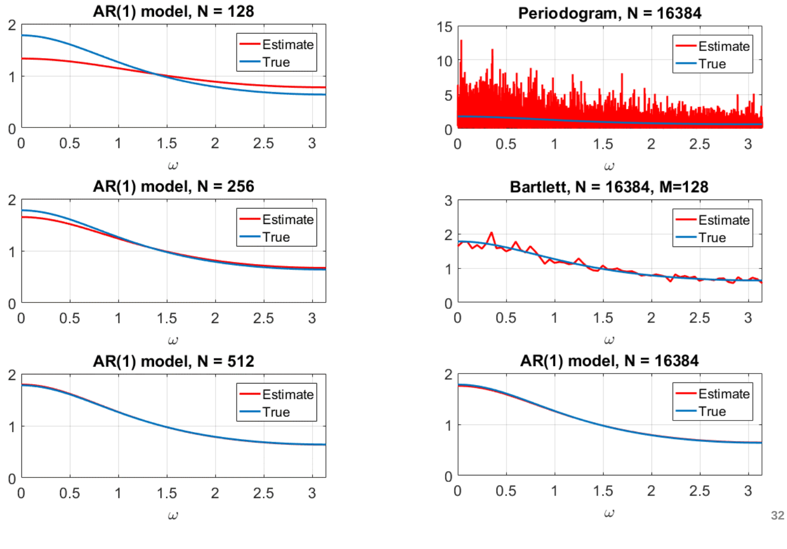
$$\hat{a} = -\frac{\hat{\gamma}_{XX}[1]}{\hat{\gamma}_{XX}[0]},\ \sigma_W^2 = \hat{\gamma}_{XX}[0] + \hat{a}\hat{\gamma}_{XX}[1]$$

3. Power spectrum estimate:

$$\hat{\Gamma}_{XX}(f) = \frac{\hat{\sigma}_W^2}{|\hat{A}(f)|^2} = \frac{\hat{\sigma}_W^2}{|1+\hat{a}e^{-j2\pi fk}|^2}$$

31

PSD of an AR(1) process…



32

PSD of an AR(1) process…

```
Matlab
% AR prosess
a = [1 -0.25];
[H,Omega] = freqz(1,a,1024); % True spectrum

% WGN
N = 2^14;
W = randn(N,1);

% Observed AR prosess
X = filter(1,a,W);

% Estimate AR process
[a_e, sigmaW2_e] = aryule(X,1); % Estimate
[He,W]=freqz(sigmaW2_e, a_e, 1024);

plot(W/pi,10*log10(abs(H))), hold on
plot(W/pi,10*log10(abs(He)))
```


Summary

- Today we discussed:
  - Parametric spectral estimation
- Next:
  - Linear prediction