NTNU – Trondheim
Norwegian University of
Science and Technology

TTT4120 Digital Signal Processing
Fall 2017

Lecture: Discrete Time Systems in Frequency Domain

Prof. Stefan Werner
stefan.werner@ntnu.no
Office B329

Department of Electronic Systems

Lecture in course book*

- Proakis, Manolakis Digital Signal Processing, 4th Ed.
  - 4.2.1 Fourier series for discrete-time periodic signals
  - 4.2.3 Fourier transform of discrete-time aperiodic signals
  - 4.3 Frequency-domain and time-domain signal properties
  - 5.1.1 Response to complex exponential and sinusoidal…
  - 5.1.4 Response to aperiodic input signals
  - 5.4.1 Ideal filter characteristics

*Level of detail is defined by lectures and problem sets

Contents and learning outcomes

- Fourier series for periodic signals
- Fourier transform for aperiodic signals
- Signal properties in time and frequency domains
- Properties of the Fourier transform
- Frequency domain representation of LTI systems – the frequency response function $H(\omega)$

Frequency analysis of DT signals

- The impulse response of a linear time-invariant system $h[n]$ allows us to compute the response to an arbitrary input $x[n]$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$$
$$= \sum_k h[k]x[n-k]$$

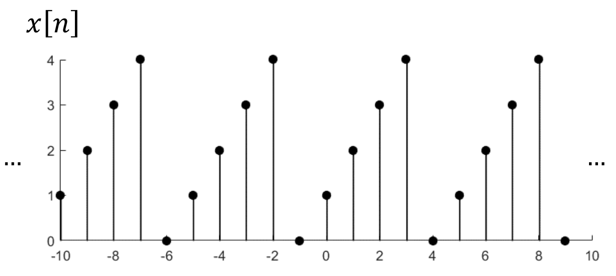
- Convolution sum is based on the fact that any input sequence can be decomposed as a linear combination of scaled and delayed unit impulse sequences, $x[n] = \sum_k x[k]\delta[n-k]$
- We can choose to represent the signal using a linear combination of some other basis signals

Frequency analysis of DT signals…

- Most signals of practical interest can be decomposed into a sum of sinusoidal components, or complex exponentials
- Using such a combination, a signal is said to be represented in frequency domain
  - Periodic signals $\Longrightarrow$ Fourier series
  - Finite-energy signals $\Longrightarrow$ Fourier transform
- We shall see that this decomposition is very important in the analysis of linear time-invariant systems
  - Response to a sinusoidal input signal is a sinusoid with the same frequency but different amplitude and phase
  - Linear combination sinusoids at input produces a similar linear combination of sinusoids at output

5

Discrete-time Fourier series (DTFS)



- Discrete-time signal $x[n]$ periodic with period N

$$x[n+N] = x[n], \forall n$$

6

Discrete-time Fourier series (DTFS)

- Fourier series representation for $x[n]$ consists of a weighted sum of N harmonically related exponentials $e^{j2\pi k/N}$

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \text{ (synthesis equation)}$$

- Fourier coefficients c_k provide frequency-domain information of $x[n]$ and are given by

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \text{ (analysis equation)}$$

- Spectrum of periodic sequence is periodic

$$c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi(k+N)n/N} = c_k$$

7

Discrete-time Fourier series (DTFS)…

- Only need to concentrate on a single period in frequency

$$0 \le \omega_k \le 2\pi \quad \text{or} \quad -\pi \le \omega_k \le \pi$$

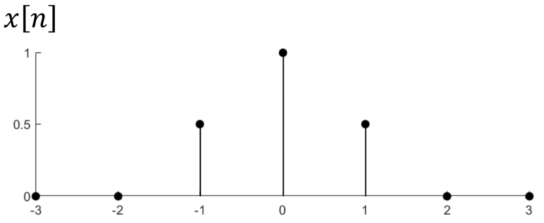
with $\omega_k = 2\pi k/N$

- Periodic signal in time-domain ⇒ discrete spectrum
- Example: $x_1[n] = \cos \pi^2 n$

$$x_2[n] = \cos\frac{\pi n}{4}$$

8

Discrete-time Fourier transform (DTFT)



- Discrete-time signal $x[n]$ is *aperiodic* but has *finite energy*

9

Discrete-time Fourier transform (DTFT)

- Discrete-time Fourier transform (DFTF) of $x[n]$:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \text{ (analysis equation)}$$

- Represents the frequency content of $x[n]$ and is 2π-periodic

$$X(\omega + 2\pi k) = X(\omega)$$

- Frequency range for any discrete-time signal $x[n]$ is limited to $(-\pi, \pi)$ or $(0, 2\pi)$
- We may obtain $x[n]$ from $X(\omega)$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega \text{ (synthesis equation)}$$

- Notation: $x[n] \overset{\mathcal{F}}{\leftrightarrow} X(\omega)$

10

Discrete-time Fourier transform (DTFT)…

- Examples: $x_1[n] \;= \delta[n] \overset{\mathcal{F}}{\leftrightarrow} X_1(\omega) = ?$

$$x_2[n] = ? \overset{\mathcal{F}}{\leftrightarrow} X_2(\omega) \;= \delta(\omega)$$

$$x_3[n] \;= a^n u[n] \overset{\mathcal{F}}{\leftrightarrow} X_3(\omega) = ?$$

$$x_4[n] = ? \overset{\mathcal{F}}{\leftrightarrow} X_4(\omega) \;= \begin{cases} 1, |\omega| \le \omega_c < \pi \\ 0, \quad \text{otherwise} \end{cases}$$

Discrete-time Fourier transform (DTFT)…

- Answers:

$$X_1(\omega) = \sum_{n=-\infty}^{\infty} x_1[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$$

$$x_2[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\omega) e^{j\omega n} d\omega = \frac{1}{2\pi}$$

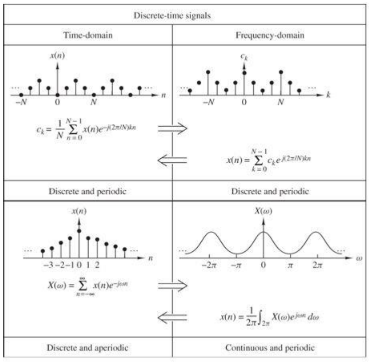
$$X_3(\omega) = \sum_{n=-\infty}^{\infty} x_3[n] e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n = \frac{1}{1 - a e^{-j\omega}}, |a| < 1$$

$$x_4[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} X(\omega) e^{j\omega n} d\omega = \frac{1}{2\pi} \cdot \frac{1}{jn} \left\{ e^{j\omega_c n} - e^{-j\omega_c n} \right\} =$$

$$= \frac{1}{\pi n} \sin(\omega_c n)$$

Summary DTFS and DTFT



- Discrete-time signals have periodic spectra
- Periodic signals ⇒ discrete spectra $\omega_k = \frac{2\pi k}{N}, \ \Delta f = 1/N$
- Aperiodic signals have continuous spectra

14

Properties of the DTFT

- Symmetry
- Time-shift
- Time-reversal
- Convolution theorem
- Frequency shifting
- Modulation theorem
- Parseval
- Window theorem

15

Properties of the DTFT…

- Symmetry:
- By expressing $x[n]$ in its real and imaginary parts, i.e.,

$$x[n] = x_R[n] + jx_I[n] \overset{\mathcal{F}}{\leftrightarrow} X_R(\omega) + jX_I(\omega)$$

we can derive a number of symmetry properties

- Example: Real and even signals have real-valued even spectra

$$X(\omega) = X(-\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X_R(\omega) + j \cdot 0$$

- Check all possibilities: real/imag and even/odd
- Example: $x[n]$ imaginary and odd $\Longrightarrow X(\omega)$?

16

Properties of the DTFT…

- Answer ($x[n]$ imaginary and odd $\Longrightarrow X(\omega)$?):

$x[n] = x_R[n] + jx_I[n] = jx_I[n]$ (imaginary)

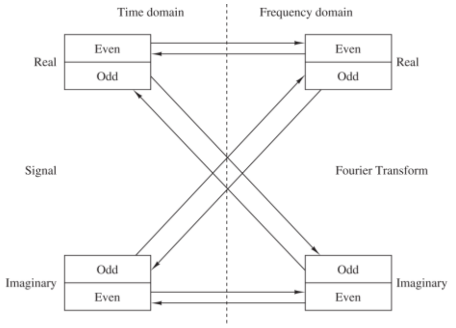
$x[-n] = -x[n]$ (odd)

$$X(\omega) = \sum_{n=-\infty}^{\infty} jx_I[n]e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{\infty} jx_I[n](\cos\omega n - j\sin\omega n)$$
$$= \sum_{n=-\infty}^{\infty} (jx_I[n]\cos\omega n - j^2 \sin\omega n)$$
$$= 2\sum_{n=0}^{\infty} x_I[n]\sin\omega n \quad \text{(Real-valued)}$$

$$X(-\omega) = 2\sum_{n=0}^{\infty} x_I[n]\sin[-\omega n]$$
$$= -2\sum_{n=0}^{\infty} x_I[n]\sin[\omega n] = -X(\omega) \quad \text{(Odd)}$$

17

Properties of the DTFT…



- Rewrite signals in terms of odd and even parts

$$x[n] = (x_R^e[n] + jx_I^e[n]) + (x_R^o[n] + jx_I^o[n])$$
$$X(\omega) = (X_R^e(\omega) + jX_I^e(\omega)) + (X_R^o(\omega) + jX_I^o(\omega))$$

18

Properties of the DTFT…

- Time-shift: $x[n-k] \overset{\mathcal{F}}{\leftrightarrow} e^{-j\omega k}X(\omega) = |X(\omega)|e^{j(\angle X(\omega)-\omega k)}$
- Time-reversal: $x[-n] \overset{\mathcal{F}}{\leftrightarrow} X(-\omega)$
- Convolution: $x_1[n] * x_2[n] \overset{\mathcal{F}}{\leftrightarrow} X_1(\omega)X_2(\omega)$
- Frequency shifting: $e^{j\omega_0 n}x[n] \overset{\mathcal{F}}{\leftrightarrow} X(\omega-\omega_0)$
- Modulation: $x[n]\cos\omega_0 n \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}[X(\omega-\omega_0) + X(\omega+\omega_0)]$
- Parseval: $\sum_n |x[n]|^2 \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi}|X(\omega)|^2 d\omega$
- Windowing: $x_1[n]x_2[n] \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$

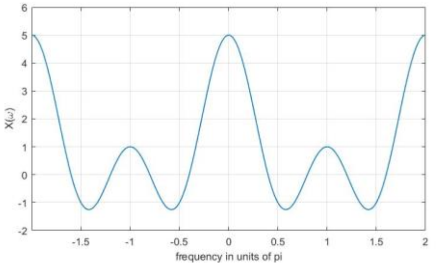
19

Properties of the DTFT…

- Example (symmetry): Pulse in time domain

$$x[n] = \{1,1,\underline{1},1,1\}$$

Sequence $x[n]$ is real and even $\Rightarrow$ $X(\omega)$ is real and even



Matlab
```
n = -2:2; x = ones(1,5);
k = -200:200; w = (pi/100)*k;
X = x * (exp(-j*pi/100)).^(n'*k);
plot(w/pi,real(X));grid
```


Properties of the DTFT…

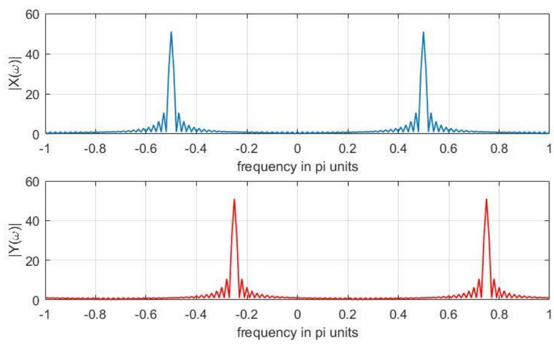
- Example (frequency shift):

$$x[n] = \cos\frac{\pi}{2}n\,,\ 0 \le n \le 100$$

$$y[n] = e^{j\frac{\pi}{4}n}x[n]$$

- Can you guess the shape of the spectra?

Properties of the DTFT…



Matlab

```
n = 0:100; x = cos(pi*n/2);
k = -100:100; w = (pi/100)*k;
X = x * (exp(-j*pi/100)).^(n'*k);
y = exp(j*pi*n/4).*x;
Y = y * (exp(-j*pi/100)).^(n'*k);
subplot(2,1,1); plot(w/pi,abs(X));
subplot(2,1,2); plot(w/pi,abs(Y));
```

23

LTI systems in frequency domain

- Output of linear time-invariant system

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$$

- What is the output if the input is a complex exponential?

$$x[n] = Ae^{j\omega n}, \ -\infty < n < \infty \text{ and } \omega \in [-\pi, \pi]$$

- Compute the convolution

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$= AH(\omega)e^{j\omega n}$$

with $H(\omega) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega n}$ (frequency response)

24

LTI systems in frequency domain…

- Using the linearity of LTI systems

$$\sum_k A_k e^{j\omega_k n} \longrightarrow \boxed{h[n]} \longrightarrow \sum_k A_k H(\omega_k) e^{j\omega_k n}$$

- Frequency response is in general complex-valued

$$H(\omega) = |H(\omega)| e^{j\angle H(\omega)}$$

- Magnitude response: $|H(\omega)|$
- Phase response: $\angle H(\omega)$

LTI systems in frequency domain…

- Response to arbitrary input

$$\begin{matrix} x[n] \\ X(\omega) \end{matrix} \longrightarrow \boxed{h[n]} \longrightarrow \begin{matrix} y[n] = h[n] * x[n] \\ Y(\omega) \ = H(\omega)\, X(\omega) \end{matrix}$$

- Frequency response is in general complex-valued

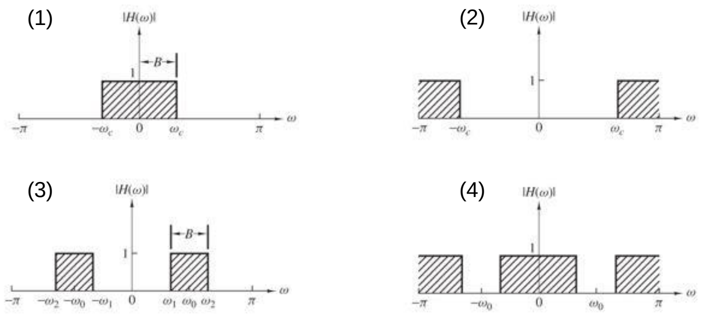
$$H(\omega) = |H(\omega)| e^{j\angle H(\omega)}$$

  with magnitude response $|H(\omega)|$ and phase response $\angle H(\omega)$

- Frequency response acts like a spectral shaping function
- LTI system that performs spectral shaping is referred to as filter

LTI systems in frequency domain…

- Ideal filter characteristics



- Practical to implement? What is the time-domain impulse response corresponding to (1)?

27

LTI systems in frequency domain…

- Example: LTI system $h[n] = 0.5^n u[n]$ excited by $x[n] = e^{j\frac{\pi}{2}n}$

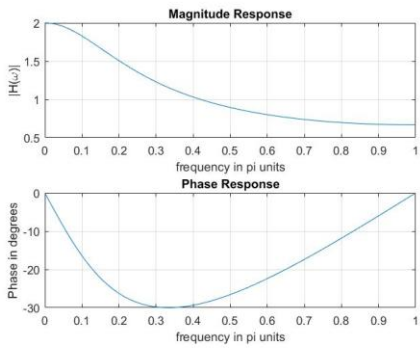
$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

1) Compute $y[n]$

2) Characterize the type of filter that $h[n]$ represents

28

LTI systems in frequency domain…



Matlab

```
w = [0:1:500]*pi/500
H = exp(j*w) ./ (exp(j*w) - 0.5*ones(1,501));
magH = abs(H); angH = angle(H);
subplot(2,1,1); plot(w/pi,magH); grid;
subplot(2,1,2); plot(w/pi,angH*180/pi); grid
```


Summary

Today:

- Signals and systems in frequency-domain
- Discrete-time Fourier series and transform (DTFS & DTFT)
- Filtering using LTI systems and ideal filters

Next:

- Start the journey of z-transforms