TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2016 - Assignment 5
Due date: Friday 18 November at 16.00.

1. Consider the system

$$a\dot{x} = -x + \frac{1}{k}h(x) + u$$
$$y = h(x)$$

where a and k are positive constants and $h \in [0, k]$. Show that the system is passive with $V(x) = a\int_0^x h(\sigma)\,d\sigma$ as the storage function.

2. Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -h(x_1) - ax_2 + u$$
$$y = kx_2 + u$$

where $a > 0$, $k > 0$, $h \in [\alpha_1, \infty]$, and $\alpha_1 > 0$. Let $V(x) = k\int_0^x h(s)\,ds + x^T P x$, where $p_{11} = ap_{12}$, $p_{22} = k/2$, and $0 < p_{12} < \min\{2\alpha_1, ak/2\}$. Using $V(x)$ as a storage function show that the system is strictly passive.

3. Consider again the Duckmaze system from the previous assignments.

(a) Consider the transformed system from Assignment 2 (Exercise 1b):

$$\dot{\tilde{x}}_1 = \tilde{x}_2 \tag{1}$$
$$m\dot{\tilde{x}}_2 = -f_3\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] - f_1\tilde{x}_1 - d\tilde{x}_2 + \tilde{u} \tag{2}$$

Define the output

$$y = \tilde{x}_2 \tag{3}$$

As in Assignment 3 (Exercise 1a), use $V = \frac{1}{2}\left(\tilde{x}_1^2 + m\tilde{x}_2^2\right)$ as Lyapunov function candidate.
Outline a control law that makes the system passive from the new control input v to the output y (in Khalil this technique is described as feedback passivation).
Note: This topic (Chapter 14.4: Passivity-based Control) will be covered in the lectures later on - this problem is however an easy introduction which you will be able to solve.

(b) Is the system zero state observable?

(c) Explain why the origin can be globally stabilized. Derive a controller that globally stabilizes the origin.

(d) An unknown constant disturbance w is acting on the system, i.e. the system equations are changed to

$$\dot{x}_1 = x_2 \tag{4}$$

$$\dot{x}_2 = -\frac{f_3}{m}x_1^3 - \frac{f_1}{m}x_1 - \frac{d}{m}x_2 - g + \frac{u}{m} + \frac{w}{m} \tag{5}$$

Do we still have $x_1^* = \lim_{t\to\infty} x_1 = x_{1d}$?

When investigating passivity for interconnected systems, the first step is often to try a storage function as a sum of the storage functions for the interconnected systems. This is illustrated in Exercise 3.

4. Show that the parallel connection of two passive (respectively, input strict passive, output strict passive, strictly passive) dynamical system is passive (respectively, input strict passive, output strict passive, strictly passive).
(Note: In this problem, for output strictly passivity you can assume that $y_i^T \rho_i(y_i) \geq \delta_i y_i^T y_i$ for some positive δ_i)

5. Consider the system

$$\begin{aligned}
\dot{x}_1 &= -3x_1 + 2x_2 \\
\dot{x}_2 &= -2\psi(x_1) - x_2 + \delta \\
y &= x_2
\end{aligned}$$

with $x = [x_1, x_2]^T \in \mathbb{R}^2$, where $k_1 z^2 \leq z\psi(z) \leq k_2 z^2$ for all $z \in \mathbb{R}$ and for some $k_1, k_2 > 0$. The time varying $\delta(t)$ is the disturbance to the system.

(a) For zero disturbance (i.e. $\delta(t) = 0$ for all t) show that the origin is globally asymptotically stable using $V(x) = \int_0^{x_1} \psi(z)\,dz + \frac{1}{2}x_2^2$. Hint: $\frac{d}{dv}\int_0^v \psi(z)\,dz = \psi(v)$.

(b) Show that the system from δ to y is strictly passive (state strictly passive).

(c) Show that the system from δ to y is also output strictly passive.

(d) Show that the system is input to state stable when δ is viewed as the input.

(e) Show that the system is zero state observable when δ is viewed as the input.

6. Consider a PID controller

$$h(s) = K_p\beta\frac{(1 + T_i s)(1 + T_d s)}{(1 + \beta T_i s)(1 + \alpha T_d s)} \tag{6}$$

as a system with $K_p = 1$, $T_d = 1$, $T_i = 2$, $\beta = 1.5$ and $\alpha = 0.5$.

(a) Show that

$$|h(j\omega)| \leq \frac{K_p\beta}{\alpha}\ \forall\omega \tag{7}$$

(b) Show that

$$\text{Re}\left[h\left(j\omega\right)\right] \geq K_p \; \forall \omega \tag{8}$$

For the rest of the exercise, assume that the conditions 7–8 hold for all cases where $K_p > 0$, $0 \leq T_d < T_i$, $1 \leq \beta < \infty$ and $0 < \alpha \leq 1$.

(c) Show that the system is passive (Hint: See Appendix A).

(d) Show that the system is input strictly passive (Hint: See Appendix A).

(e) Show that the system is output strictly passive (Hint: See Appendix A).

(f) Show that the system is zero-state observable.

7. Euler equations for a rotating rigid spacecraft are given by

$$\begin{aligned}
J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 + u_1 \\
J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 + u_2 \\
J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_1\omega_2 + u_3
\end{aligned}$$

where ω_1 to ω_3 are the components of the angular velocity vector along the principal axes, u_1 to u_3 are the torque inputs applied about the principal axes, and J_1 to J_3 are the principal moments of the inertia.

(a) Show that the map from $u = [u_1, u_2, u_3]^T$ to $\omega = [\omega_1, \omega_2, \omega_3]^T$ is lossless.

(b) Let $u = -K\omega + v$, where K is a positive definite symmetric matrix. Show that the map from v to ω is finite-gain $\mathcal{L}_2$ stable.

(c) Show that, when $v = 0$, the origin $\omega = 0$ is globally asymptotically stable.

(Hint: Use $V\left(\omega\right) = \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2$).

8. Consider the feedback connection of Figure 6.11 in Khalil with

$$H_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - h_1(x_2) + e_1 \\ y_1 = x_2 \end{cases}$$

$$H_2 : \begin{cases} \dot{x}_3 = -x_3 + e_2 \\ y_2 = h_2(x_3) \end{cases}$$

where h_1 and h_2 are locally Lipschitz function, which satisfy $h_1 \in (0, \infty]$, $h_2 \in (0, \infty]$ and $|h_2(z)| \geq |z|/(1 + z^2)$ for all z.

(a) Show that the feedback connection is passive.

(b) Show that the origin of the unforced system is globally asymptotically stable.

(Hint: Use $V_1\left(x_1, x_2\right) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$ and $V_2\left(x_3\right) = \int_0^{x_3} h_2\left(z\right) dz$).

9. Repeat the previous exercise for

$$H_1 : \begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = -x_1^3 - x_2 + e_1 \\ y_1 = x_2 \end{cases}$$

$$H_2 : \begin{cases} \dot{x}_3 = -x_3 + e_2 \\ y_2 = x_3^3 \end{cases}$$

(Hint: Use $V_1(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ and $V_2(x_3) = \frac{1}{4}x_3^4$).

10. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + \psi(u) \end{aligned}$$

where ψ is a locally Lipschitz function that satisfies $\psi(0) = 0$ and $u\psi(u) > 0$ for all $u \neq 0$. Design a globally stabilizing state feedback controller. (Hint: Theorem 14.4 in Khalil).

11. **Optional exercise:** Verify that a function in the sector $[K_1, K_2]$ can be transformed into a function in the sector $[0, \infty]$ by input feedforward followed by output feedback, as shown in Figure 6.7 in Khalil.

Appendix A: Passivity

A linear system given by the scalar transfer function $h(s)$ such that all poles p_i satisfy $\mathrm{Re}\,[p_i] \leq 0$ is

- passive if $\mathrm{Re}\,[h(j\omega)] \geq 0 \quad \forall\omega$ such that $j\omega$ is not a pole.
- input strictly passive if $\mathrm{Re}\,[h(j\omega)] \geq \delta > 0 \quad \forall\omega$ such that $j\omega$ is not a pole.
- output strictly passive if $\mathrm{Re}\,[h(j\omega)] \geq \epsilon\,|h(j\omega)|^2 > 0 \quad \forall\omega$ such that $j\omega$ is not a pole, and for some positive ϵ.