NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF ENGINEERING CYBERNETICS

Exam
TTK4150 Nonlinear Control Systems
Thursday December 11, 2014

SOLUTION

Problem 1 (14%)

For the equilibrium points x^* we have that

$$\dot{x}_1^* = 0 = (1 - x_1^*)x_1^* - \frac{2x_1^* x_2^*}{1 + x_1^*}$$
$$\dot{x}_2^* = 0 = \frac{(1 - x_2^*)x_2^*}{1 + x_1^*}$$

Which results in three equilibrium points:

$$x^*(1) = (0, 0)^T, \;\; x^*(2) = (0, 1)^T, \;\; x^*(3) = (1, 0)^T,$$

To find the qualitative behavior of the system around these equilibrium points we first have to linearize it around the equilibrium points. The Jacobian is:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 - 2x_1 - \frac{2x_2}{(1+x_1)^2} & -\frac{2x_1}{1+x_1} \\ -\frac{(1-x_2)x_2}{(1+x_1)^2} & \frac{1-2x_2}{1+x_1} \end{bmatrix}$$

For the first equilibrium point:

$$x^*(1) = (0, 0)^T \rightarrow A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow (\lambda - 1)^2 = 0 \rightarrow \lambda_{1,2} = 1$$

Thus the first equilibrium point is an unstable node, i.e. it is an unstable equilibrium point.

For the second equilibrium point:

$$x^*(2) = (0, 1)^T \rightarrow A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow (\lambda + 1)^2 = 0 \rightarrow \lambda_{1,2} = -1$$

Thus the second equilibrium point is a stable node, i.e. it is a stable equilibrium point.

For the third equilibrium point:

$$x^*(3) = (1,0)^T \rightarrow A_1 = \begin{bmatrix} -1 & -1 \\ 0 & \frac{1}{2} \end{bmatrix} \rightarrow (\lambda+1)(\lambda-0.5) = 0 \rightarrow \lambda_1 = -1,\ \lambda_2 = 0.5$$

Hence we have two real eigenvalues, where one is positive and one is negative. This means that this is a saddle point, and it is thus an unstable equilibrium point.

Problem 2 (10%)

$V(x)$ is continuously differentiable, radially unbounded and positive definite. So if we can choose u such that its time derivative is negative except at the origin, then Lyapunov theorem for global asymptotic stability is fulfilled. The time derivative of V is:

$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = 10x_1^2x_2 + 3x_1^7x_2 + 9x_2u$$

By selection of input as:

$$u = -\frac{1}{9x_2}(10x_1^2x_2 + 3x_1^7x_2 + kx_2^2) = -\frac{10}{9}x_1^2 - \frac{1}{3}x_1^7 - \frac{k}{9}x_2$$

which results in $\dot{V}(x) = -kx_2^2$ (Note that $u = -\frac{10}{9}x_1^2 - \frac{1}{3}x_1^7 - \frac{1}{9}\phi(x_2)x_2$ for any $\phi(x_2)$ that satisfies $x_2\phi(x_2) > 0, \forall x_2 \neq 0$ will make $\dot{V}$ negative definite in x_2, i.e., negative semidefinite in x). We thus have $\dot{V}(x) = 0$ when $x_2 = 0$, i.e. on the whole x_1-axis the derivative of the Lyapunov function candidate is zero.

So to prove global asymptotic stability, we need to use LaSalle's theorem. The condition of V being continuously differentiable is satisfied. Furthermore, since V is radially unbounded, we have that the level sets Ω_c are bounded for all values of c. Also $\dot{V} \leq 0$. What remains to show is that the origin is the only invariant set on the x_1-axis. On the x_1-axis, the system, including the controller, reduces to

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= -10x_1^2 \neq 0\ \forall x_1 \neq 0 \end{aligned}$$

hence the origin constitutes the only invariant set on it, i.e., no solution can stay in the set where $\dot{V} = 0$ other than the trivial solution $x = 0$. LaSalle's theorem thus guarantees global asymptotic stability.

Problem 3 (12%)

a For the function $V(t,x)$ to be decrescent, a positive definite function $W_1(x)$ should be found such that:

$$V(t,x) = x_1^2 + \frac{1}{b+\cos t}x_2^2 \leq W_1(x)$$

From the structure of $V(t,x)$ we see that the function $W_1(x)$ needs to be in the form $W_1(x) = x_1^2 + px_2^2$, where p needs to be positive for the function to be positive definite. It can thus be concluded that V is decrescent when

$$0 < \frac{1}{b+\cos t} \leq p$$

Since $|\cos t| \leq 1$, by selection of $b > 1$ there always exists a p such that above condition is satisfied.

For the function $V(t,x)$ to be positive definite, a positive definite function $W_2(x)$ should be found such that:

$$V(t,x) = x_1^2 + \frac{1}{b+\cos t}x_2^2 \geq W_2(x)$$

From the structure of V we see that the structure of $W_2(x)$ needs to be of $W_2(x) = x_1^2 + qx_2^2$, where $q > 0$. It can thus be concluded that V is positive definite when

$$\frac{1}{b+\cos t} \geq q > 0$$

Since $|\cos t| \leq 1$, by selection of $b > 1$ there always exists a q such that above condition is satisfied.

As the result, V is decrescent and positive definite for $b > 1$.

b Furthermore, the derivative of the Lyapunov function should be negative semidefinite in order for the origin to be uniformly stable.:

$$\begin{aligned}\dot{V} &= 2x_1\dot{x}_1 + \frac{2x_2\dot{x}_2}{b+\cos t} + \frac{\sin t}{(b+\cos t)^2}x_2^2 \\ &= 2x_1x_2 - \frac{2x_2^2}{b+\cos t} - 2x_1x_2 + \frac{\sin t}{(b+\cos t)^2}x_2^2 \\ &= \frac{(-2b - 2\cos t + \sin t)x_2^2}{(b+\cos t)^2} \leq 0\end{aligned}$$

To make $\dot{V}$ negative semidefinite, we should have

$$-2b - 2\cos t + \sin t \leq 0 \rightarrow -2b - 2\cos t + \sin t \leq -2b + 2.24 \leq 0 \rightarrow -2b \leq -2.24 \rightarrow b \geq 1.12$$

It thus follows from this and the result in **a** that the origin is a uniformly stable equilibrium point for $b \geq 1.12$

Problem 4 (25%)

a The Lyapunov function candidate is continuously differentiable. Also, using that $\int \psi(z)dz \geq \int k_1 z dz = 1/2k_1x_1^2$, we see that $V(x) \geq 1/2kx_1^2 + x_2^2$, and is thus both positive definite and radially unbounded. Furthermore, when $\delta = 0$ we have

$$\begin{aligned}\dot{V}(x) &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = \psi(x_1)(-3x_1 + 2x_2) + x_2(-2\psi(x_1) - x_2) \\ &= -3x_1\psi(x_1) - x_2^2 \leq -3k_1x_1^2 - x_2^2\end{aligned}$$

for all x, i.e. $\dot{V}$ is negative definite in x. Since $V(x)$ satisfies all conditions of Theorem 4.2 in Khalil, it follows that the origin is GAS. (Furthermore, V(x) satisfies all conditions of Theorem 4.10, and the origin is thus GES).

b Using $V(x) = \int_0^{x_1} \psi(z)\, dz + \frac{1}{2}x_2^2$ we have

$$\begin{aligned}\dot{V}(x) &= \psi(x_1)(-3x_1 + 2x_2) + x_2(-2\psi(x_1) - x_2 + \delta) = -3x_1\psi(x_1) - x_2^2 + x_2\delta \\ &\le -3k_1x_1^2 - x_2^2 + x_2\delta = -\phi(x) + \delta y\end{aligned}$$

where $\phi(x) = 3k_1x_1^2 + x_2^2$ is positive definite. Hence it is (state) strictly passive.

c Using $V(x) = \int_0^{x_1} \psi(z)\, dz + \frac{1}{2}x_2^2$ we have

$$\begin{aligned}\dot{V}(x) &= \psi(x_1)(-3x_1 + 2x_2) + x_2(-2\psi(x_1) - x_2 + \delta) = -3x_1\psi(x_1) - x_2^2 + x_2\delta \\ &\le -3k_1x_1^2 - x_2^2 + x_2\delta \le -x_2^2 + x_2\delta = -y\rho(y) + \delta y\end{aligned}$$

where $\rho(y) = y$. Hence, it is output strictly passive.

d Using $V(x) = \int_0^{x_1} \psi(z)\, dz + \frac{1}{2}x_2^2$ we have that inequality (4.48) in Theorem 4.19 in Khalil is satisfied with $\alpha_1(\|x\|) = \frac{1}{2}k_1x_1^2 + \frac{1}{2}x_2^2$ and $\alpha_2(\|x\|) = \frac{1}{2}k_2x_1^2 + \frac{1}{2}x_2^2$. Furthermore,

$$\begin{aligned}\dot{V}(x) &\le -3k_1x_1^2 - x_2^2 + x_2u \\ &\le -\min(3k_1, 1)\|x\|^2 + \|x\||\delta| \\ &= -\big(\min(3k_1, 1) - \theta\big)\|x\|^2 - \theta\|x\|^2 + \|x\||\delta| \qquad \text{for } 0 < \theta < \min(3k_1, 1) \\ &\le -\big(\min(3k_1, 1) - \theta\big)\|x\|^2\end{aligned}$$

when $-\theta\|x\|^2 + \|x\||\delta| \le 0$, i.e. $\|x\| \ge \frac{|\delta|}{\theta}$. Then, by theorem 4.19 in Khalil, the system is ISS.

e The unforced system is given by

$$\begin{aligned}\dot{x}_1 &= -3x_1 + 2x_2 \\ \dot{x}_2 &= -2\psi(x_1) - x_2 \\ y &= x_2\end{aligned}$$

Consider

$$S = \{x \in \mathbb{R}^2 | y = 0\} = \{x \in \mathbb{R}^2 | x_2 = 0\}$$

then for every $(x_1, x_2) \in S$, i.e. $y(t) \equiv 0$, we have $x_2(t) \equiv 0 \implies \dot{x}_2 \equiv 0 \implies -2\psi(x_1) - 0 \equiv 0 \implies \psi(x_1) \equiv 0 \implies x_1(t) \equiv 0, \dot{x}_1 \equiv 0$. Hence, no other solution can stay identically in S other than the zero solution. Thus the system is zero state observable.

Problem 5 (27%)

a We differentiate the output $y = x_1$ to find the relative degree:

$$\begin{aligned}\dot{y} &= \dot{x}_1 = x_1^2 + x_2 \\ \ddot{y} &= 2x_1\dot{x}_1 + \dot{x}_2 = 2x_1^3 + 2x_1x_2 + x_3^2 + u\end{aligned}$$

The relative degree of the system is thus $\rho = 2$ in $\mathbb{R}^3$. It exists and is well defined, hence the system is input-output linearizable.

b The system can be written as

$$\dot{x} = f(x) + g(x)u$$

where

$$f(x) = \begin{bmatrix} x_1^2 + x_2 \\ x_3^2 \\ x_2 - kx_3 \end{bmatrix} \qquad g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- First, the external coordinates ξ are found. Since $\rho = 2$, ξ is of dimension 2.

$$\begin{aligned} \xi_1 &= y = x_1 \\ \xi_2 &= L_f y = \frac{\partial y}{\partial x} f(x) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} f(x) = x_1^2 + x_2 \end{aligned}$$

- Then the internal coordinates η are found. Since the system state has dimension 3, and ξ is of dimension 2, η is of dimension $3 - 2 = 1$. We will choose $\eta = \phi(x)$ such that the resulting coordinate transformation $T(x)$ is a diffeomorphism, $L_g\phi = 0$ and $T(0) = 0$. The Jacobian of the coordinate transformation T with the chosen external coordinates is

$$\frac{\partial T}{\partial x} = \begin{bmatrix} \frac{\partial \phi}{\partial x_1} & \frac{\partial \phi}{\partial x_2} & \frac{\partial \phi}{\partial x_3} \\ 1 & 0 & 0 \\ 2x_1 & 1 & 0 \end{bmatrix}$$

  We see that the Jacobian is nonsingular in the whole state space $\mathbb{R}^3$, something which implies that the transformation is a diffeomorphism, if $\frac{\partial \phi}{\partial x_3} = 1$ (or another constant value).
  Furthermore, the condition $L_g\phi = 0$ can be written as

$$\frac{\partial \phi(x)}{\partial x} g(x) = 0$$

  Inserting for $g(x)$, this implies

$$\frac{\partial \phi(x)}{\partial x_2} = 0$$

  A $\phi(x)$ which satisfies all three conditions on ϕ is

$$\phi(x) = x_3$$

- This gives the following diffeomorphism

$$z = T(x) = \begin{bmatrix} \eta \\ \cdots \\ \xi \end{bmatrix} = \begin{bmatrix} x_3 \\ \cdots\cdots \\ x_1 \\ x_1^2 + x_2 \end{bmatrix}$$

  Since this is a diffeomorphism, its inverse exists and is smooth, and can be found as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 - \xi_1^2 \\ \eta \end{bmatrix} = T^{-1}(z)$$

- Finally, the system is written in normal form

$$\begin{aligned}\dot{\eta} &= \xi_2 - \xi_1^2 - k\eta = f_0(\eta, \xi) \\ \dot{\xi} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \left[u + \left(2x_1^3 + 2x_1 x_2 + x_3^2 \right) \right] \\ &= A_c \xi + B_c \gamma(x) \left[u - \alpha(x) \right]\end{aligned}$$

Clearly, $\gamma(x) = L_g L_f h(x) = 1$ and $\alpha(x) = -L_f^2 h(x) / L_g L_f h(x) = -\left(2x_1^3 + 2x_1 x_2 + x_3^2\right)$ Since $T(z)$ is a global diffeomorphism, this transformation is valid in the entire state space $\mathbb{R}^3$. This can also be seen by noting that since $\alpha(x) = -\frac{L_f^2 y}{\gamma(x)}$, the transformation is valid when $\gamma(x) \neq 0$, and since $\gamma(x) = 1$, the transformation is valid for the entire $\mathbb{R}^3$ space.

c An input-output linearizing controller is given by

$$u = \alpha(x) + \beta(x) v$$

where $\beta(x) = \frac{1}{\gamma(x)} = 1$ and $\alpha(x) = -\left(2x_1^3 + 2x_1 x_2 + x_3^2\right)$. This gives

$$u = -\left(2x_1^3 + 2x_1 x_2 + x_3^2\right) + v$$

d The external dynamics ξ with the input-output linearizing controller inserted is given by:

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v\end{aligned}$$

$\xi = [\xi_1 \quad \xi_2]^T$ can be stabilized by $v = -k_1 \xi_1 - k_2 \xi_2$, which gives the closed loop dynamics

$$\dot{\xi} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \xi = A\xi$$

The eigenvalues of the closed loop system are

$$\lambda = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1}}{2}$$

which have negative real parts as long as $k_1, k_2 > 0$. This means that A is Hurwitz and $\dot{\xi} = A\xi$ is asymptotically stable at the origin with $v = -k_1 \xi_1 - k_2 \xi_2$, $k_1, k_2 > 0$.

e The system is minimum phase if the origin of the zero dynamics $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable. The zero dynamics is given by

$$\dot{\eta} = f_0(\eta, 0) = -k\eta$$

This is a linear differential equation with eigenvalue $\lambda = -k$, and the origin is thus GES. This can also be shown by using the Lyapunov function

$$V(\eta) = \frac{1}{2}\eta^2$$

Differentiating V along the trajectories of the zero dynamics gives

$$\dot{V}(\eta) = -k\eta^2$$

Since $V(\eta)$ is continuously differentiable and positive definite, and $\dot{V}(\eta)$ is negative definite, the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable. The system is therefore minimum phase.

Problem 6 (12%)

- Using the Lyapunov function

$$\begin{aligned} V_1 &= \frac{1}{2}x_1^2 \\ \dot{V}_1 &= x_1\dot{x}_1 = 5x_1^2x_2 + x_1^3 = x_1^2\left(5x_2 + x_1\right) \end{aligned}$$

  With $x_2 = \varphi(x_1) = -\frac{x_1}{5} - \frac{k_1}{5}x_1^2$, x_1 is asymptotically stable:

$$\dot{V}_1 = -k_1x_1^4 < 0 \quad \forall \quad x_1 \neq 0$$

- A new variable z is defined as $z = x_2 - \varphi(x_1)$. The dynamics for $\dot{x}_1$ in the new coordinates is then

$$\dot{x}_1 = 5x_1x_2 + x_1^2 = 5x_1(z + \varphi(x_1)) + x_1^2 = 5x_1z - x_1^2 - k_1x_1^3 + x_1^2 \quad = -k_1x_1^3 + 5x_1z$$

- Next, the expression for $\dot{z}$ is found

$$\begin{aligned} \dot{z} &= \dot{x}_2 - \frac{\partial\varphi(x)}{\partial t} \\ &= -4x_2^2 + u + \frac{1}{5}\dot{x}_1 + \frac{k_1}{5}2x_1\dot{x}_1 \\ &= -4x_2^2 + u + \dot{x}_1\left(\frac{1}{5} + \frac{2k_1}{5}x_1\right) \\ &= -4x_2^2 + u + \left(5x_1x_2 + x_1^2\right)\left(\frac{1}{5} + \frac{2k_1}{5}x_1\right) \end{aligned}$$

- Finally, the overall stabilizing input u is found using the following continuously differentiable and positive definite Lyapunov function

$$\begin{aligned} V_2 &= \frac{1}{2}x_1^2 + \frac{1}{2}z^2 \\ \dot{V}_2 &= x_1\dot{x}_1 + z\dot{z} = -k_1x_1^4 + 5x_1^2z + z\left[-4x_2^2 + u + \left(5x_1x_2 + x_1^2\right)\left(\frac{1}{5} + \frac{2k_1}{5}x_1\right)\right] \\ &= -k_1x_1^4 + z\left[5x_1^2 - 4x_2^2 + u + \left(5x_1x_2 + x_1^2\right)\left(\frac{1}{5} + \frac{2k_1}{5}x_1\right)\right] \end{aligned}$$

u is chosen such that

$$5x_1^2 - 4x_2^2 + u + \left(5x_1x_2 + x_1^2\right)\left(\frac{1}{5} + \frac{2k_1}{5}x_1\right) = -k_2 z$$

Which means that

$$u = -5x_1^2 + 4x_2^2 - \left(5x_1x_2 + x_1^2\right)\left(\frac{1}{5} + \frac{2k_1}{5}x_1\right) - k_2 z$$

and

$$\dot{V}_2 = -k_1x_1^4 - k_2z^2 < 0 \quad \forall \quad (x_1, z) \neq (0,0)$$

Since $V_2(x_1, z)$ is continuously differentiable and positive definite, and $\dot{V}_2(x_1, z)$ is negative definite, u asymptotically stabilizes $(x_1, z) = (0, 0)$. Since the transformation $(x_1, x_2) \to (x_1, z)$ is a global diffeomorphism, this also means that the origin is asymptotically stabilized at the origin. In addition, since $V_2(x_1, z)$ is radially unbounded and there are no singularities in u, $x = 0$ is globally asymptotically stable.

The globally stabilizing controller in original coordinates is as follows

$$u = -5x_1^2 + 4x_2^2 - \left(5x_1x_2 + x_1^2\right)\left(\frac{1}{5} + \frac{2k_1}{5}x_1\right) - k_2\left(x_2 + \frac{1}{5}x_1 + \frac{k_1}{5}x_1^2\right)$$