Comparison Lemma Example

Using Comparison lemma.Show that the solution of the state equation

$$\dot{x}_1 = -x_1 + \frac{2x_2}{1+x_2^2}, \qquad \dot{x}_2 = -x_2 + \frac{2x_1}{1+x_1^2} \tag{1}$$

satisfies the inequality

$$||x(t)||_2 \leq e^{-t}||x(0)||_2 + \sqrt{2}(1-e^t). \tag{2}$$

First we define a new variable $V = ||x||_2 = x_1^2 + x_2^2$, which is positive define function. The derivative of V becomes

$$\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = -2x_1^2 - 2x_2^2 + \frac{4x_1x_2}{1+x_2^2} + \frac{4x_1x_2}{1+x_1^2} \tag{3}$$

$$\leq -2V + 4|x|_1\frac{|x|_2}{1+x_2^2} + 4|x|_2\frac{|x|_1}{1+x_1^2} \tag{4}$$

$$\leq -2V + 2|x|_1 + 2|x|_2 \quad \text{since } \frac{|y|}{1+y^2} \leq \frac{1}{2} \tag{5}$$

$$\leq -2V + 2\sqrt{2}\sqrt{V} \quad \text{since } ||x||_1 \leq \sqrt{n}||x||_2 \tag{6}$$

Let us define a new variable $W = \sqrt{V} = ||x||_2$. The derivative becomes

$$\dot{W} = \frac{\dot{V}}{2\sqrt{V}} \leq -W + \sqrt{2} \quad \forall\ V \neq 0 \tag{7}$$

At $V = 0$, we have

$$\frac{|W(t+h) - W(t)|}{h} = \frac{|W(t+h)|}{h} = \frac{1}{h}||x(t+h)||_2 \tag{8}$$

From example 3.9 in khalil it can be seen that

$$\lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} ||f(x(\tau))||_2 d\tau = 0 \tag{9}$$

Thus

$$D^+W(t) \leq -W(t) + \sqrt{2}\forall\ t \geq 0 \tag{10}$$

Let $u(t)$ be the solution to the differential equation

$$\dot{u} = -u + \sqrt{2}, \quad u(0) = ||x(0)||_2 \tag{11}$$

The comparison lemma then states that

$$||x(t)||_2 \leq u(t) = \exp^{-t}||x(0)||_2 + \sqrt{2}(1-\exp^{-t}) \tag{12}$$