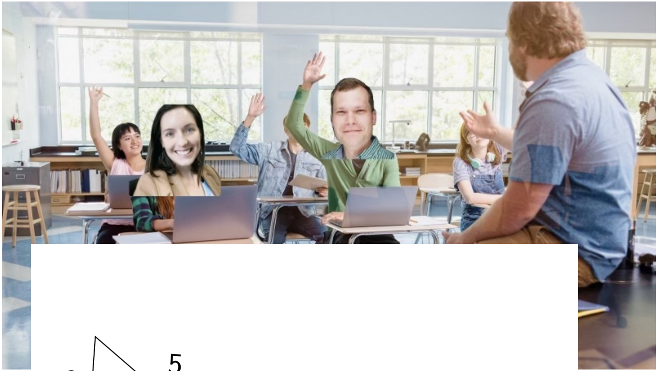
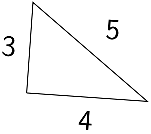
An Introduction to
Algebraic Number Theory

Alistair Pattison

November 2, 2023

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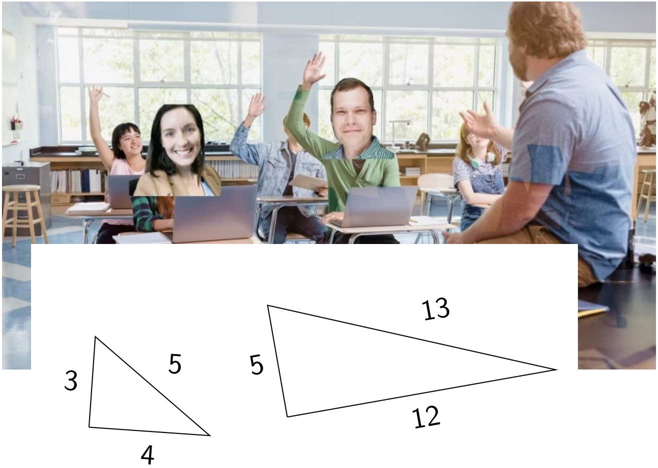
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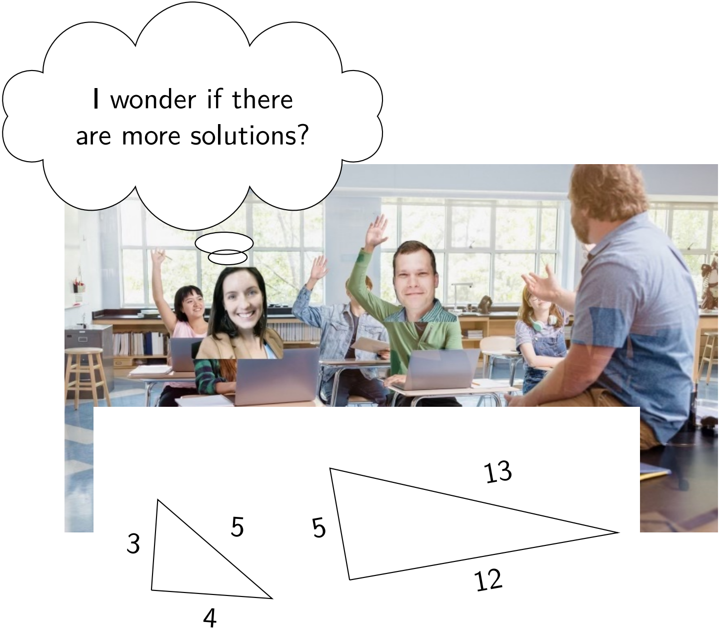
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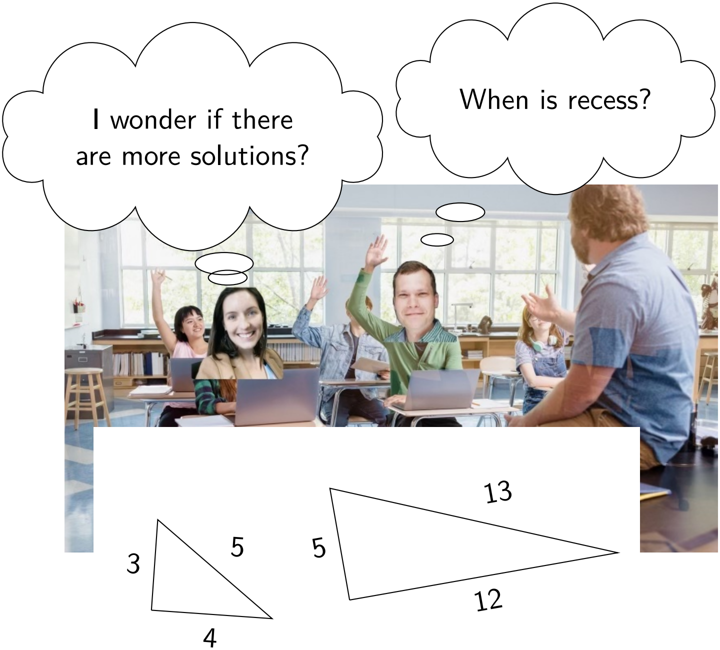
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Generating Primitive Pythagorean Triples

Goal: find all relatively prime $x, y, z \in \mathbb{Z}$ such that

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$$z^2 = \alpha\beta \implies \alpha = \gamma^2 \quad \gamma \in \mathbb{Z}[i]$$

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$$\begin{aligned} x &= a^2 - b^2 \\ y &= 2ab \\ z &= a^2 + b^2 \end{aligned}$$

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$$z^2 = x^2 + y^2$$

a	b	
2	1	$3^2 + 4^2 = 5^2$
3	2	$5^2 + 12^2 = 13^2$
4	3	$7^2 + 24^2 = 25^2$
4	2	$12^2 + 16^2 = 20^2$
4	1	$15^2 + 8^2 = 17^2$

Algebraic Number Theory

ALGEBRAIC NUMBER THEORY

Using tools from algebra like rings and field extensions

Algebraic Number Theory

Using tools from algebra like rings and field extensions

Generating insight about the integers and the primes

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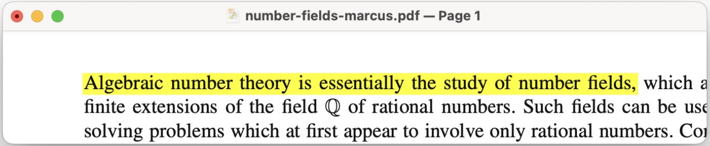
number-fields-marcus.pdf — Page 1

Algebraic number theory is essentially the study of number fields, which a
finite extensions of the field $\mathbb{Q}$ of rational numbers. Such fields can be use
solving problems which at first appear to involve only rational numbers. Co

Outline

1. MATH 342 in 3:42

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2. Number Fields and Number Rings

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3. The Ideal Class Group

MATH 342 in 3:42

Ring Hierarchy

Rings $\longrightarrow$ Integral Domains $\longrightarrow$ Integrally Closed Domains $\longrightarrow$ Unique Factorization Domains $\longrightarrow$ Principal Ideal Domains $\longrightarrow$ Fields

Definition (Commutative Ring)

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- Multiplication: $ab = ba \in R$
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- *Not* division: $a/b \notin R$

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An (additive) subgroup, I, such that $ra \in I$ for all $r \in R$, $a \in I$.

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- This is a generalization of Euclid's Lemma

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- **Non-example**: in $\mathbb{Z}/8\mathbb{Z}$, we have $4 \cdot 2 = 0$

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- **Example**: $\mathbb{Z}$
- **Non-example**: $\mathbb{Z}[i\sqrt{5}]$:

$$6 = 2 \cdot 3 = (1 + i\sqrt{5})(1 - i\sqrt{5})$$

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- **Non-example**: $\mathbb{Z}[x]$ because of $(2, x)$

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"Things like the rationals" or "rings where you can divide".

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$$f(x) = x^2 + 5$$

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$$e \qquad \pi \qquad \sqrt{7} \qquad 1 + 2\sqrt{17}$$

- But f is equally happy living in

$$\mathbb{Q}(i\sqrt{5}) = \{a + bi\sqrt{5} : a, b \in \mathbb{Q}\}$$

Number Fields and Number Rings

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Non-examples

- $\mathbb{Q}[\pi]$ because π is transcendental

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- $\mathbb{A}$ is a subring of $\mathbb{C}$

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- $\mathcal{O}_{\mathbb{Q}[\sqrt{5}]} = \mathbb{Z}[1, \frac{1}{2} + \frac{1}{2}\sqrt{5}]$

Dedekind Domains

Theorem

Number rings are Dedekind domains.

Definition (Dedekind domain)

A *Dedekind domain* is an integrally closed domain R such that

1. every ideal is finitely generated and
2. every nonzero prime ideal is maximal.

Dedekind Domains

Factoring ideals

Theorem

Every ideal of a Dedekind domain R uniquely factors into prime ideals.

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A Dedekind domain has unique factorization iff it's a PID.

Dedekind Domains

In the class hierarchy

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$$\begin{tikzcd}
 & \text{Integrally Closed Domains} \arrow[dl] \arrow[dr] & \\
\text{UFDs} \arrow[dr] & & \text{Dedekind Domains} \arrow[dl] \\
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Dedekind Domains

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- **UFD, not DD**
  $\mathbb{R}[x_1, x_2, \ldots]$, $\mathbb{Q}[x, y]$

The Ideal Class Group

The Ideal Class Group

Definition (Ideal Class Group)

Let $K = \mathbb{Q}[\alpha]$ be a number field. The *class group* of K is the set of ideals of $\mathcal{O}_K$, modulo the equivilence relation

$$I \sim J \text{ iff } \alpha I = \beta J \text{ for some nonzero } \alpha, \beta \in R.$$

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- The class group of $\mathbb{Q}[i\sqrt{5}]$ is $\mathbb{Z}/2\mathbb{Z}$.

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Theorem

Class numbers are always finite.

Thank you!

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Slides/paper