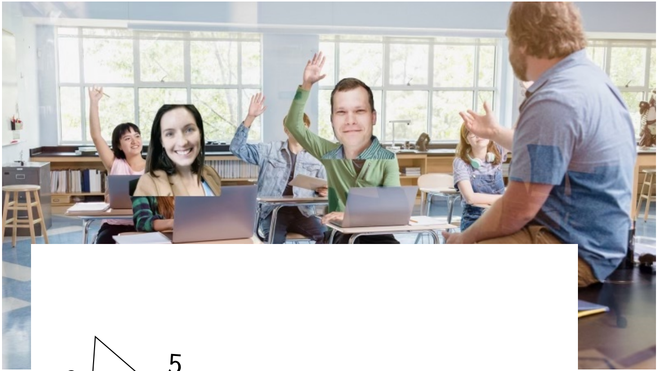
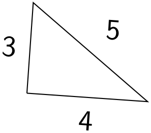
An Introduction to Algebraic Number Theory

Alistair Pattison

November 2, 2023

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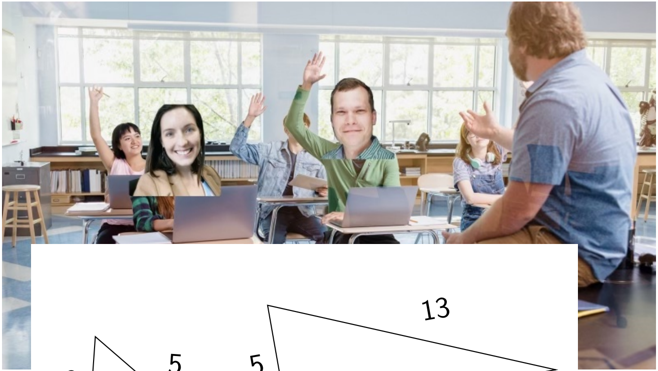

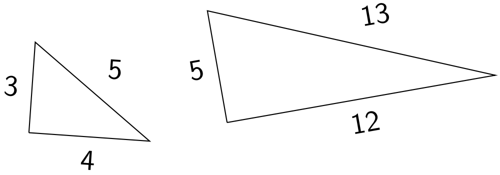
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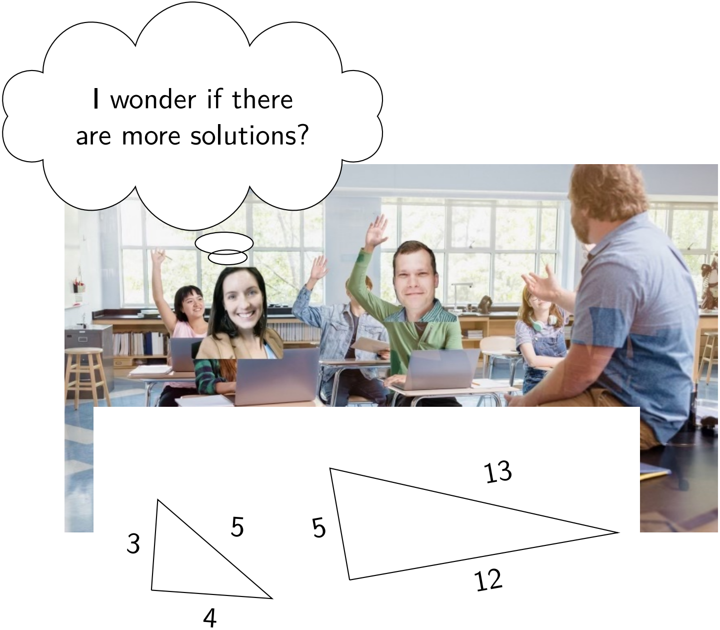
I wonder if there
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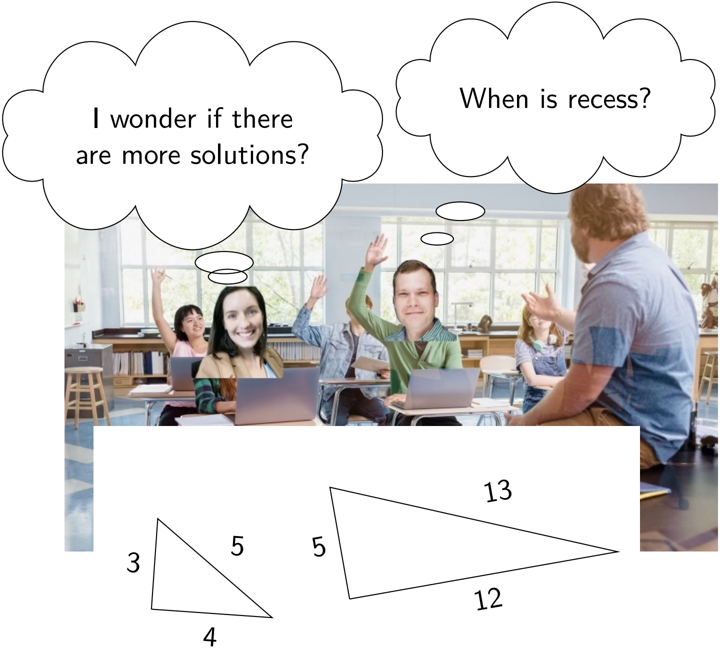
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Generating Primitive Pythagorean Triples

Goal: find all relatively prime $x, y, z \in \mathbb{Z}$ such that

$$z^2 = x^2 + y^2$$

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$$x \text{ and } y \text{ relatively prime} \implies \alpha \text{ and } \beta \text{ relatively prime}$$

$$z^2 = \alpha\beta \implies \alpha = u\gamma^2$$

$$\gamma \in \mathbb{Z}[i], u \in \{\pm 1, \pm i\}$$

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$$\begin{aligned} x &= a^2 - b^2 \\ y &= 2ab \\ z &= a^2 + b^2 \end{aligned}$$

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$$z^2 = x^2 + y^2$$

a	b	
2	1	$3^2 + 4^2 = 5^2$
3	2	$5^2 + 12^2 = 13^2$
4	3	$7^2 + 24^2 = 25^2$
4	2	$12^2 + 16^2 = 20^2$
4	1	$15^2 + 8^2 = 17^2$

Algebraic Number Theory

Algebraic Number Theory

Using tools from algebra like
rings and field extensions

ALGEBRAIC NUMBER THEORY

Using tools from algebra like rings and field extensions

Generating insight about the integers and the primes

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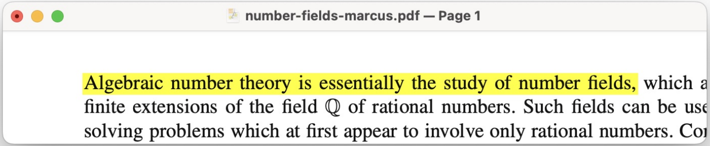

number-fields-marcus.pdf — Page 1

Algebraic number theory is essentially the study of number fields, which a
finite extensions of the field $\mathbb{Q}$ of rational numbers. Such fields can be use
solving problems which at first appear to involve only rational numbers. Co

Outline

1. MATH 342 in 3:42

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2. Number Fields and Number Rings

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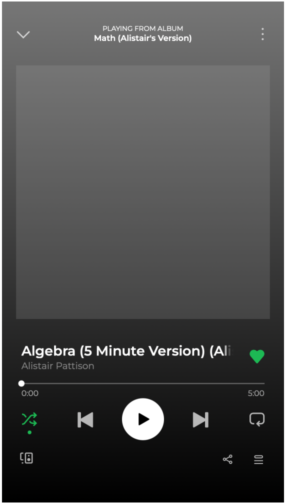
3. The Ideal Class Group

MATH 342 in 3:42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Math (Alistair's Version)
Algebra (5 Minute Version) (Ali
Alistair Pattison
0:00
5:00

Ring Hierarchy

Rings $\longrightarrow$ Integral Domains $\longrightarrow$ Integrally Closed Domains $\longrightarrow$ Unique Factorization Domains $\longrightarrow$ Principal Ideal Domains $\longrightarrow$ Fields

Definition (Commutative Ring)

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- Distributive property: $a(b + c) = ab + bc$
- *Not* division: $a/b \notin R$

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Definition (Ideal)

An (additive) subgroup, I, such that $ra \in I$ for all $r \in R$, $a \in I$.

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- **Example**: Even integers ($8 \cdot 7$ is even)

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- This is a generalization of Euclid's Lemma

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- **Example**: $\mathbb{Z}$, $\mathbb{Z}[i]$
- **Non-example**: in $\mathbb{Z}/8\mathbb{Z}$, we have $4 \cdot 2 = 0$

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A ring R is *integrally closed* if for all $\alpha/\beta \in \operatorname{Frac} R$ that are integral over R, then $\beta \mid \alpha$, i.e., $\alpha/\beta \in R$.

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Definition (UFD)

A commutative ring R is a *unique factorization domain* if every element factors uniquely into irreducible elements.

- **Example**: $\mathbb{Z}$
- **Non-example**: $\mathbb{Z}[i\sqrt{5}]$:

$$6 = 2 \cdot 3 = (1 + i\sqrt{5})(1 - i\sqrt{5})$$

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- **Example**: $\mathbb{Z}$, $\mathbb{Q}[x]$, $\mathbb{Z}[i]$
- **Non-example**: $\mathbb{Z}[x]$ because of $(2, x)$

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"Things like the rationals" or "rings where you can divide".

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- **Example**: $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{R}$

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$$f(x) = x^2 + 5$$

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&= (x - i\sqrt{5})(x + i\sqrt{5})
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- But f is equally happy living in

$$Q(i\sqrt{5}) = \{a + bi\sqrt{5} : a, b \in \mathbb{Q}\}$$

Number Fields and Number Rings

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Theorem

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where n is the degree of the minimal polynomial of α.

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Non-examples

- $\mathbb{Q}[\pi]$ because π is transcendental

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- $\mathbb{A}$ is a subring of $\mathbb{C}$

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- $\mathcal{O}_{\mathbb{Q}[\zeta]} = \mathbb{Z}[\zeta]$ for primitive roots ζ
- $\mathcal{O}_{\mathbb{Q}[\sqrt{5}]} = \mathsf{Z}[1, \, \frac{1}{2} + \frac{1}{2}\sqrt{5}]$

Dedekind Domains

Theorem

Number rings are Dedekind domains.

Definition (Dedekind domain)

A *Dedekind domain* is an integrally closed domain R such that

1. every ideal is finitely generated and
2. every nonzero prime ideal is maximal.

Theorem

Every ideal of a Dedekind domain R uniquely factors into prime ideals.

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- Allows replacing unique factorization of elements with unique factorization of ideals.

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Dedekind Domains

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The Ideal Class Group

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Definition (Ideal Class Group)

Let $K = \mathbb{Q}[\alpha]$ be a number field. The *class group* of K is the set of ideals of $\mathcal{O}_K$, modulo the equivilence relation

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Class numbers are always finite.

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Slides