Numerical Relativity Homework 1

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1 Question 1

An hyperbolic partial differential equation can always be rewritten in terms of the solution vector $\bar{u}$ with dimension n, as:

$$\partial_t \bar{u} + \bar{\bar{A}}^i \partial_i \bar{u} = \bar{S} \quad i = 1, 2, 3 \quad (3D)$$

where $\bar{S}$ is the source vector, while $\bar{\bar{A}}^i$ is a matrix with $n \times n$ components. For example, for the wave equation $\bar{\bar{A}}$ is the velocity matrix, while for conservation laws it's the Jacobian matrix.

If we now consider an arbitrary unit vector n^i, we define the characteristic matrix of the system as $\bar{\bar{P}} \equiv \bar{\bar{A}}^i \cdot n_i$. *Strongly hyperbolic systems* are the one whose matrix $\bar{\bar{P}}$ has real eigenvalues and a complete set of them for every unit vector n^i, which implies that $\bar{\bar{P}}$ can be diagonalized. On the contrary, for *weakly hyperbolic systems* the matrix $\bar{\bar{P}}$ has real eigenvalues but not a complete set of them and so $\bar{\bar{P}}$ can't be diagonalized.

From a numerical point of view we are interested in rewriting the partial differential equation to have a strongly hyperbolic system. This is due to the theorem that states that "strongly hyperbolic systems are well-posed problems, while weakly hyperbolic systems are not".
A *well-posed problem* means that the norm of the solution vector doesn't grow faster in time than an exponential:

$$\left\|\bar{u}(t, x^i)\right\| \leq k e^{\alpha t} \left\|\bar{u}(0, x^i)\right\|$$

This theorem underlines the importance of the choice of the functions to rewrite the hyperbolic second order equations as first order equations. From an analytical point of view every choice is the same and the analytical solution is always fine. On the other hand, if we consider the numerical point of view, rewriting the equations as a strongly hyperbolic system lead us to a well-posed problem whose solution doesn't explode in time and so can be solved numerically.

2 Question 2

The 3+1 formulation of Einstein Equations is based on the spacetime foliation, which consists in slicing the spacetime in a series of space-like hypersurfaces Σ_t in order to divide space and time components of the EE.
On each hypersurface you define:

- the *timelike normal*: $n_\mu = (-\alpha, 0, 0, 0)$, that is proportional to the gradient of the time vector $\nabla_\mu t = (1, 0, 0, 0)$. α is the *lapse function* and its value depends on the point of the hypersurface.

- the *3-metric* that lives on the hypersurface: $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$, where $g_{\mu\nu}$ is the spacetime metric tensor. Its components are $\gamma_{ij} = g_{ij}$ and $\gamma^{0\nu} = 0$, it is orthogonal to the normal n_μ and its inverse is γ^{ij} - which means that it behaves like a spatial metric and can be used to rise and lower spatial indexes.

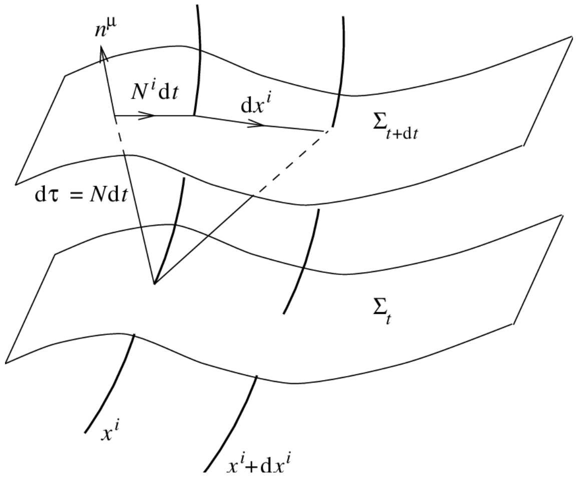

Figure 1: The 3 + 1 decomposition of the manifold, with lapse function N , and shift vector N^i [1].

To decompose a generic 4-vector u^μ in its spatial and time parts it is necessary to introduce two operators:

- the *time projector operator*: $N^\mu{}_\nu = -n^\mu n_\nu$, which gives the component of a vector parallel to the time coordinate vector.

- the *spatial projector operator*: $\gamma^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$, which gives the component of a vector that lies on the hypersurface.

Now u^μ can be written as $u^\mu = N^\mu{}_\nu u^\nu + \gamma^\mu{}_\nu u^\nu$.
In order to write the metric in the 3+1 formulation there is a last quantity that needs to be defined: the *shift vector* β^i. Starting from the time coordinate vector, this can be defined

[1] An introduction to quantum cosmology - Scientific Figure on ResearchGate. Available from: `https://www.researchgate.net/figure/The-3-1-decomposition-of-the-manifold-with-lapse-function-N-and-shift-vector-N-i_fig1_1963562`

as $t^\mu = \alpha\, n^\mu + \beta^\mu$ with $\beta^\mu = (0, \beta^i)$. In this way t^μ is the dual of $\nabla_\mu t$:

$$t^\mu \nabla_\mu t = \alpha n^\mu \left(\frac{n_\mu}{-\alpha}\right) + \beta^\mu \left(\frac{n_\mu}{-\alpha}\right) = 1$$

Since t^μ is the versor along t its components are $t^\mu = (1, 0, 0, 0)$ and so we can write the controvariant components of the normal: $n^\mu = \left(\frac{1}{\alpha}, \frac{-\beta^i}{\alpha}\right)$
To understand the meaning of the lapse function and the shift vector we consider two events along the normal $\bar{n}$, separated by a certain $d\tau$.

- we can compute the separation in coordinate time as

$$dt = (\nabla_\mu t)(n^\mu d\tau) = -\frac{n^\mu}{\alpha} n^\mu d\tau = \frac{d\tau}{\alpha} \;\rightarrow\; d\tau = \alpha dt$$

  This means that the lapse function α tells how proper time changes by going from one hypersurface to the next one along $\bar{n}$. Also α is choosen to be defined positive so that n^μ is pointing toward the positive direction of time.

- we can compute the spatial separation as:

$$dx^i = (\nabla_\mu x^i)(n^\mu d\tau) = \delta^i{}_\mu n^\mu d\tau = n^i d\tau = -\frac{\beta^i}{\alpha} \alpha dt = -\beta^i dt \;\rightarrow\; dx^i = -\beta^i dt$$

  From this we can see that the shift vector measures the shift in spatial coordinates by going from one hypersurface to the next one along the normal vector.

The metric in the 3+1 formulation is:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

The choice of both the lapse function and the shift vector is fundamental for the numerical solution of the problem. For example, we can consider the simulation of a neutron star collapsing into a black hole. If the *Geodesic Slicing* is applied, which means that the lapse and the shift are constant ($\alpha = 1$ and $\beta^i = 0$) then the numerical computation will develop some problems. The metric of the cell at the center of the star will diverge when the singularity starts, causing the code to crash. Moreover, the proper volume of the cells near the compact object will diverge too as space curves, giving rise to a loss of accuracy in computing numerical derivatives.

By contrast, the most used choice in numerical relativity is the *Singularity Avoiding Slicing Conditions*, for which α and β^i are functions of space and time coordinates. At the beginning of the simulation, when the metric is still smooth, the lapse and the shift resemble the Geodesic Slicing. As time evolves and the metric becomes more curve, the lapse function decreases to avoid as possible the formation of the singularity while the shift vector is introduced to counteract the change of the proper volume of the cells, making it constant.

3 Question 3

3.1 ADM Formulation

The ADM Formulation is a way to rewrite the Einstein Equation in the 3+1 formulation. First we need to introduce some objects that live on the hypersurfaces:

- the *3D Covariant Derivative* is the spatial component of the standard 4D covariant derivative:
$$D_\alpha f \equiv \gamma^\mu_\alpha \; f_{;\mu}$$
The 3D metric has the property that $D_\alpha \gamma_{\mu\nu} = 0$
- the *3D Connection Coefficients*, defined as in 4D but using the 3D metric: ${}^{(3)}\Gamma^\alpha{}_{\beta\gamma}$
- the *3D Riemann Tensor*, defined as in 4D but using the 3D connection: ${}^{(3)}R^\sigma{}_{\alpha\beta\gamma}$
- the *3D Ricci Tensor*: ${}^{(3)}R_{\alpha\beta} = {}^{(3)}R^\sigma{}_{\alpha\sigma\beta}$
- the *3D Ricci Scalar*: ${}^{(3)}R = {}^{(3)}R^\alpha{}_\alpha$
- the *Extrinsic Curvature*:
$$K_{\alpha\beta} \equiv -\frac{1}{2}\mathcal{L}_{\bar{n}}\gamma_{\alpha\beta}$$
where $\mathcal{L}_{\bar{n}}$ is the Lie derivative of $\gamma_{\alpha\beta}$ along the normal and it tells how the 3-metric changes by moving from one hypersurface to the next. Differently from ${}^{(3)}R^\tau{}_{\alpha\beta\sigma}$, which gives the intrinsic curvature of the hypersurface, $K_{\alpha\beta}$ tells how the hypersurface is embedded in the 4D spacetime. Therefore if at a particular time the spacetime is sliced so that the hypersurface is flat, then the Riemann tensor will be zero, while $K_{\alpha\beta}$ will not, unless the whole spacetime is flat.

Knowing that $\bar{t} = \alpha\bar{n} + \bar{\beta}$, the Lie derivative becomes: $\mathcal{L}_{\bar{n}} = \frac{1}{\alpha}(\mathcal{L}_{\bar{t}} - \mathcal{L}_{\bar{\beta}})$
Now, using $\mathcal{L}_{\bar{n}}\gamma_{\alpha\beta} = -2K_{\alpha\beta}$, we can write $\partial_t\gamma_{\alpha\beta} = -2\alpha K_{\alpha\beta} + D_\alpha\beta_\beta + D_\beta\beta_\alpha$. Since these vectors have only spatial components, we take:

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

This is the evolution equation for γ_{ij} and it's first order in time and first order in space.

By combining the EE equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ and the Ricci equation - which contains $\mathcal{L}_{\bar{n}}K_{\alpha\beta}$ that gives the time derivative of K - we can write the evolution equation for K_{ij}:

$$\begin{aligned}\partial_t K_{ij} = -D_iD_j\alpha + (\beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k) \\ + \alpha({}^{(3)}R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j) + 4\pi\alpha\left[\gamma_{ij}(S-E) - 2S_{ij}\right]\end{aligned}$$

$$E \equiv n^\alpha n^\beta T_{\alpha\beta} \quad S_{\mu\nu} \equiv \gamma^\alpha{}_\mu\gamma^\beta{}_\nu T_{\alpha\beta} \quad S_\mu \equiv -\gamma^\alpha{}_\mu n^\beta T_{\alpha\beta} \quad S = S^\mu{}_\mu \quad K = K^\mu{}_\mu$$

which is first order in time and second order in space (because of D_iD_j, ${}^{(3)}R_{ij}$).
These equations are weakly hyperbolic, meaning that solving them numerically can cause some problems.

The Ricci equation belongs to a more wide set of equations called the Gauss-Codazzi-Ricci equations, obtained by combining a certain times ${}^{(3)}R_{\alpha\beta\mu\nu}$ with $\gamma^{\mu}{}_{\nu}$ and n^{μ}. The other two equations produce the Hamiltonian constraint and the Momentum constraint, that don't contain time derivatives and that have to be satisfied at each time (analytically).

$$ {}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi E \qquad D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i $$

3.2 BSSN Formulation

This formulation introduces new variables to reduce the order of derivatives in space in the right hand side of the equations.

- the *conformal transformation* is $\tilde{\gamma}^{ij} = e^{-4\phi}\gamma^{ij}$, where ϕ is a function of space and time chosen so that $e^{4\phi} = \gamma^{1/3}$;
- the *trace-free extrinsic curvature* is $A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij}K$, so that $\tilde{A}_{ij} = e^{-4\phi}A_{ij}$.

By using a mathematical identity that results in $Tr(ln(\gamma ij)) = ln(\gamma)$ and the ADM equation for γij, we obtain the evolution equation for the determinant γ:

$$ \partial_t[ln(\sqrt{\gamma})] = -\alpha K + D_i\beta^i $$

We can do the same for $\partial_t K_{ij}$ and obtain:

$$ \partial_t K = -D^i D_i \alpha + \alpha[K_{ij}K^{ij} + 4\pi(S+E)] + \beta^i D_i K $$

From these we can find the evolution equation for ϕ and K, by knowing for example that $ln(\sqrt{\gamma}) = 6\phi$ and $D_i\beta^i = \partial_i\beta^i + 6\beta i\partial_i\phi$.

$$ \partial_t\phi = -\frac{1}{6}\alpha K + \frac{1}{6}\partial_i\beta^i + \beta^i\partial_i\phi $$

$$ \partial_t K = -D^i D_i\alpha + \alpha[\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2 + 4\pi(S+E)] + \beta^i D_i K $$

These equations can be subtracted from the ADM equations $\partial_t\gamma_{ij}$ and $\partial_t K_{ij}$ in order to get the evolution equations for the new conformal transformations introduced:

$$ \partial_t\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \beta^k\partial_k\tilde{\gamma}_{ij} + \tilde{\gamma}_{ik}\partial_j\beta^k + \tilde{\gamma}_{kj}\partial_i\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k $$

$$ \begin{aligned} \partial_t\tilde{A}_{ij} = e^{-4\phi}[-(D_iD_j\alpha)^{TF} + \alpha({}^{(3)}R_{ij}^{TF} - 8\pi S_{ij}^{TF}] + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j) \\ + \beta^k\partial_k\tilde{A}_{ij} + \tilde{A}_{ik}\partial_j\beta^k + \tilde{A}_{kj}\partial_i\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k \end{aligned} $$

where ${}^{(3)}R_{ij}^{TF} = {}^{(3)}R_{ij} - \frac{1}{3}\gamma_{ij}{}^{(3)}R$ and in general TF means "trace-free".
To summarize, the BSSN formalism evolves the variables ϕ, $\tilde{\gamma}_{ij}$, K and $\tilde{A}_{ij}$ and from them it reconstructs γ_{ij} and K_{ij}.

Then we split the Ricci tensor in two terms, ${}^{(3)}\bar{R}_{ij}$ containing second derivatives of the metric and ${}^{(3)}R^{\phi}_{ij}$ containing only derivatives in respect to ϕ. In order to remove second derivatives of γ_{ij} from ${}^{(3)}\bar{R}_{ij}$ we introduce a new variable $\tilde{\Gamma}^i = \partial_i \tilde{\gamma}^{ij}$ (similarly as it's done for the wave equation) and we find its evolution equation.

The BSSN equations found are strongly hyperbolic so they are suitable for numerical codes. The downward is that there are new variables that have to be evolved, so it's more expensive and complicated to implement.