Numerical Relativity Homework 2

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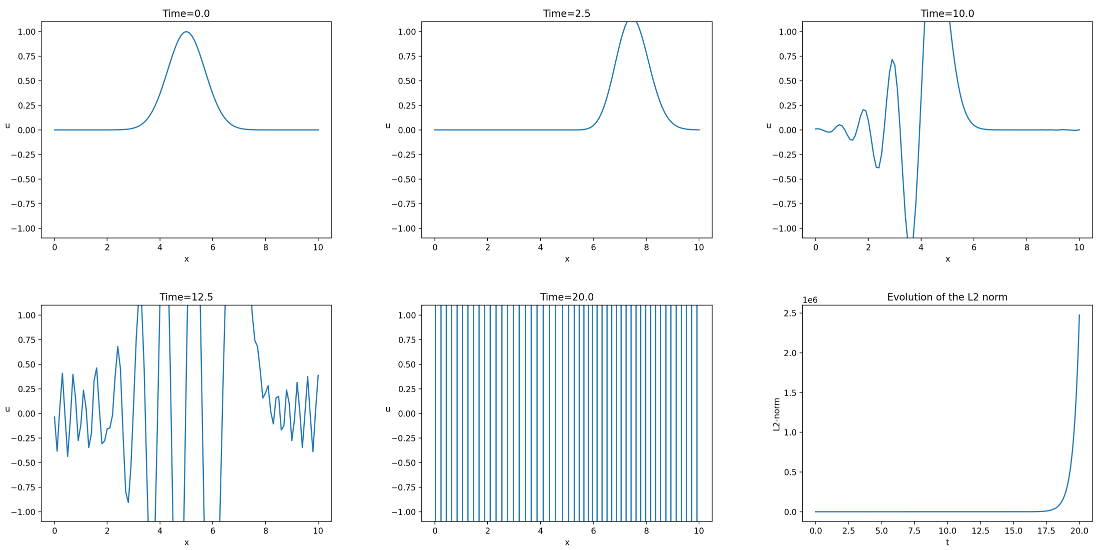
June 6, 2023

Question 1

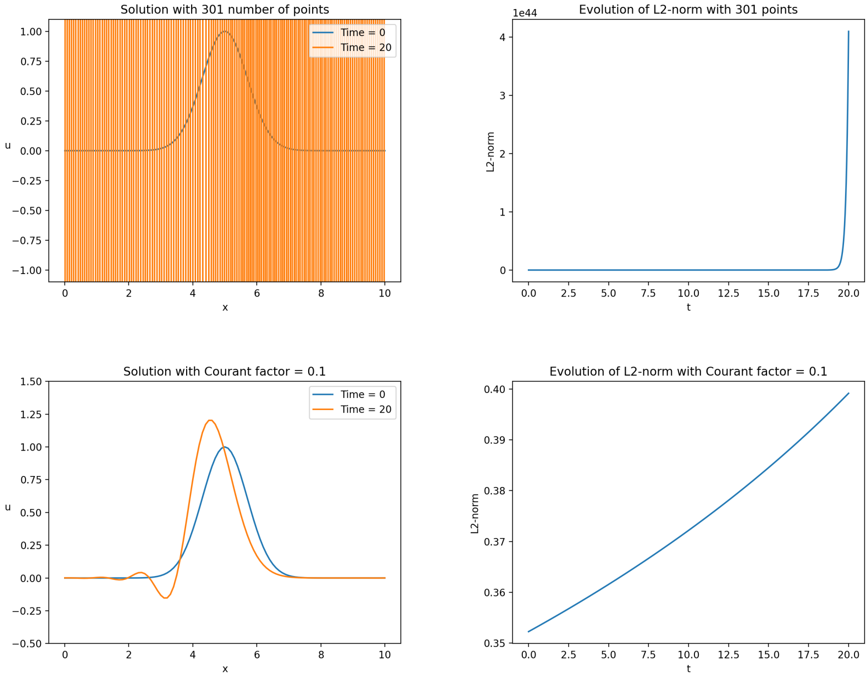
In all the following cases I solve the advection equation $u_t + u_x = 0$ using as initial condition a Gaussian $u(x, t = 0) = e^{-(x-x_0)^2}$ with $x_0 = 5$, a domain $x \in [0, 10]$ with periodic boundary conditions and a terminate time equal to 20. I will then vary the resolution scheme used, the number of points J and the Courant factor.

FTCS

I start wit $J = 101$ number of points and the Courant factor equal to 0.5, which lead to $dx = 0.1$ and $dt = 0.05$. The following figures show the solution of the advection equation $u(x, t)$ at different times. At first the initial condition travels towards the right, but soon it starts to heighten and in the end it explodes completely. This behaviour is due to the fact that FTCS is an unstable method, meaning that if you add a little perturbation the solution grows exponentially with time, as shown by the plot of the evolution of the L2 norm.

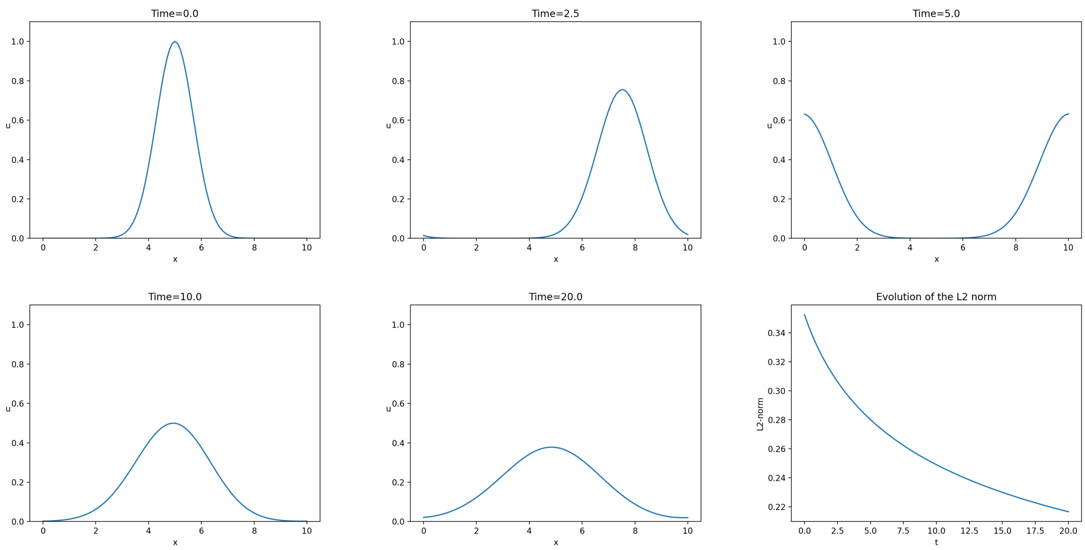


By changing the number of points to $J = 301$, ($dx = 0.033$ and $dt = 0.0166$), $u(x, t)$ still grows exponentially, because no matter the resolution you have, FTCS remains an unstable method. Note that in these cases the Courant factor is smaller than one, nevertheless the solution explodes due to the unconditionally instability of FTCS. However if I decrease it to $c_f = 0.1$, ($dx = 0.1$ and $dt = 0.01$), the solution still grows but less fast, since I reduced the amplification factor, although it will eventually blow up.

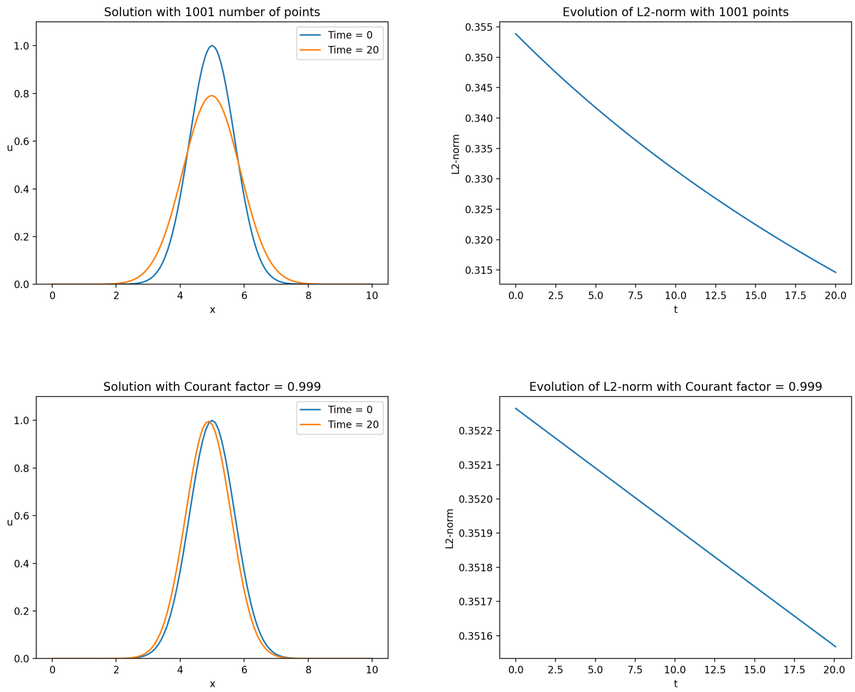


Lax-Friedrichs

I start with $J = 101$ number of points and the Courant factor equal to 0.5. The initial profile travels towards the right with speed 1, but the curve broadens and the height gets lower in time: this is because in the Lax-Friedrichs scheme is present a dissipative term. Since it is also conditionally stable and in this case $c_f < 1$, so the CFL condition is satisfied, the L2 norm doesn't grow like with FTCS case, but instead it decreases, as shown by the plot of its evolution.

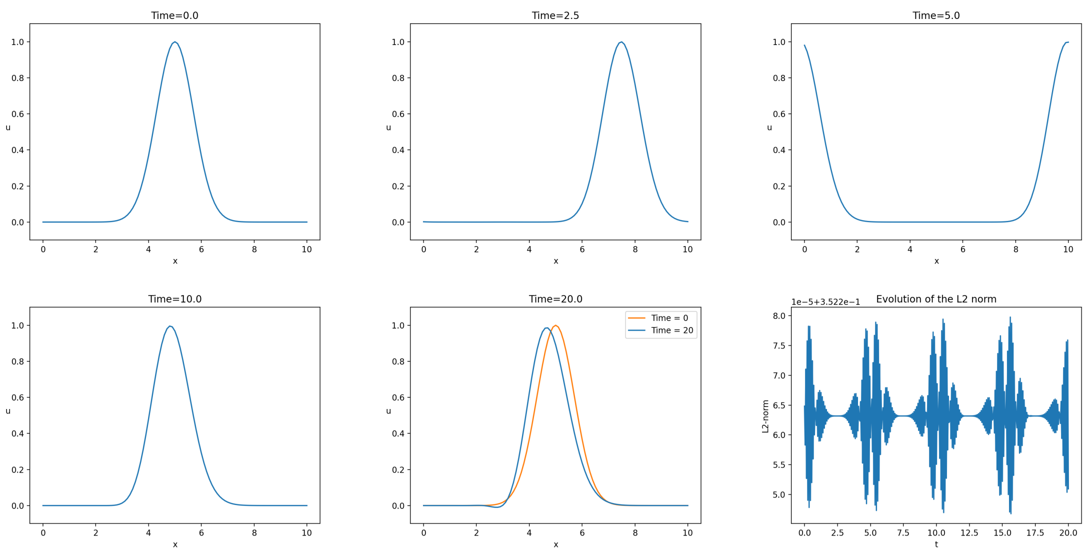


By increasing the number of points to $J = 1001$ ($dx = 0.01$ and $dt = 0.005$), the diffusive term reduces because it goes like dx and so the amplitude of the solution at the final time is higher than before and the curve is less dissipated. Moreover the L2 norm decreases less than the previous case. Then I vary the Courant factor to 0.999 ($dx = 0.1$ and $dt = 0.0999$), which produces an amplification factor almost equal to 1 so the amplitude of the Gaussian tends to be constant.

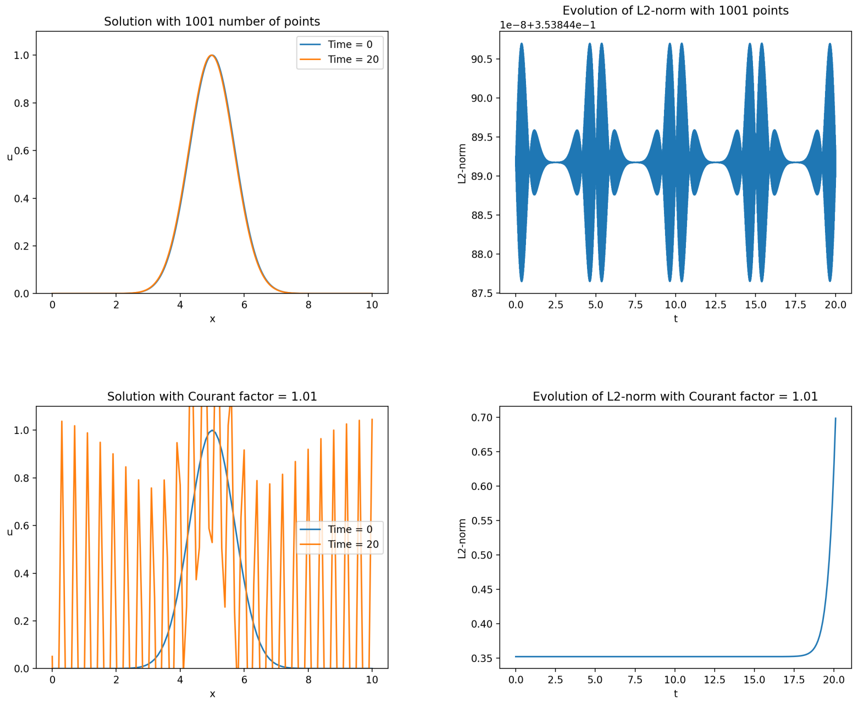


Leapfrog

I start wit $J = 101$ number of points and the Courant factor equal to 0.5. Since the Leapfrog scheme needs to know the solution at $t = -dt$, I initialized the exact solution: the gaussian profile shifted to the left by dt. The downsides of the Leapfrog scheme is that it has to keep in memory three time levels instead of two. The initial condition travels towards the right with speed 1, the height of the curve decreases a little and a very small oscillation in the tail is present. The L2 norm has variations of 10^{-5} in time, way smaller than the previous methods.

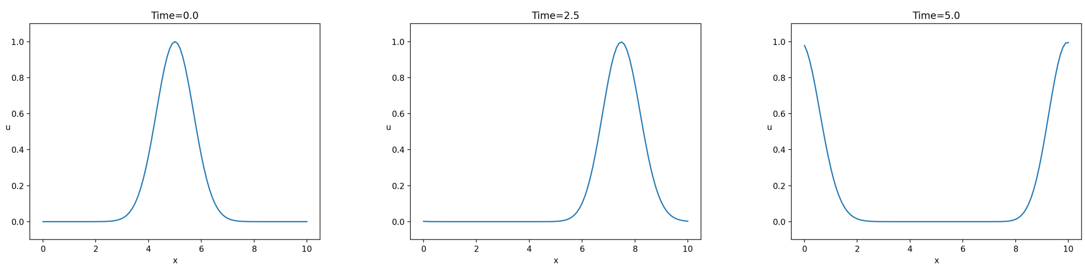


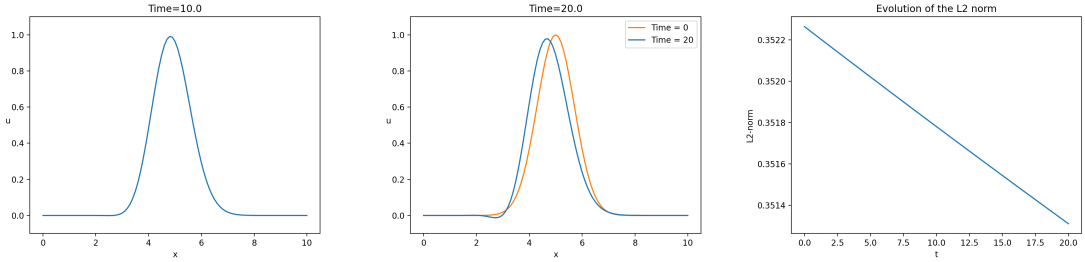
By increasing the number of points to $J = 1001$, the numerical solution at the final time almost coincide with the initial data, as it would be for the exact solution. The Leapfrog method is indeed a second order accurate scheme, which means that the error goes as $(dx)^2$, so the accuracy increases more with a lower grid spacing than first order methods like Lax-Friedrichs. The oscillations of the L2 norm are now of the order of 10^{-8}. Then I change the Courant factor to a value grater than 1: $c_f = 1.01$ ($dx = 0.1$ and $dt = 0.101$), making the scheme unstable: the solution explodes with time, as shown by the exponential growth of the L2 norm, like it was for the FTCS scheme. The Leapfrog method is indeed conditionally stable.



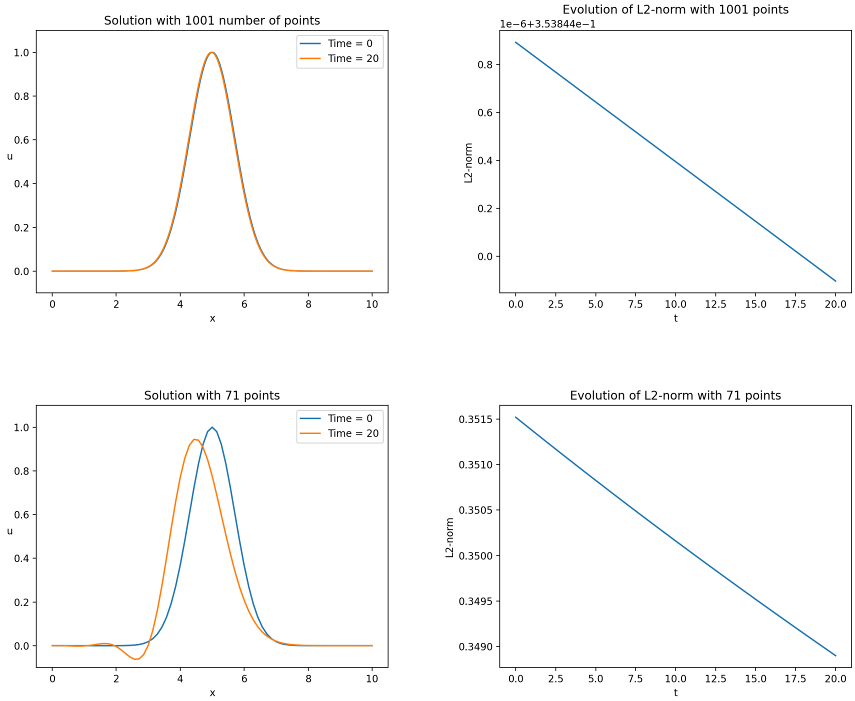
Lax-Wendroff

I start wit $J = 101$ number of points and the Courant factor equal to 0.5. Similarly to the Leapfrog scheme, as the initial profile travels towards the right the height of the curve gets a little lower and a little osccilation appears. Lax-Wendroff is a second order scheme that presents both a dispersive term (that goes like $(dx)^2$) and a diffusive term (that goes like $(dx)^3$, so it affects less the solution). The first one is responsible for the oscillations. The evolution of the L2 norm still shows that the method is stable since the CFL condition is satisfied.





I firstly raised the number of points to $J = 1001$, making both the dissipative term and the dispersive term decrease: the solution is almost the exact one. Then I instead reduced the number of points to $J = 71$ ($dx = 0.14$ and $dt = 0.07$): the two terms got bigger and so the dissipation and the oscillations are more relevant, even if they increase with a different power of the grid spacing.

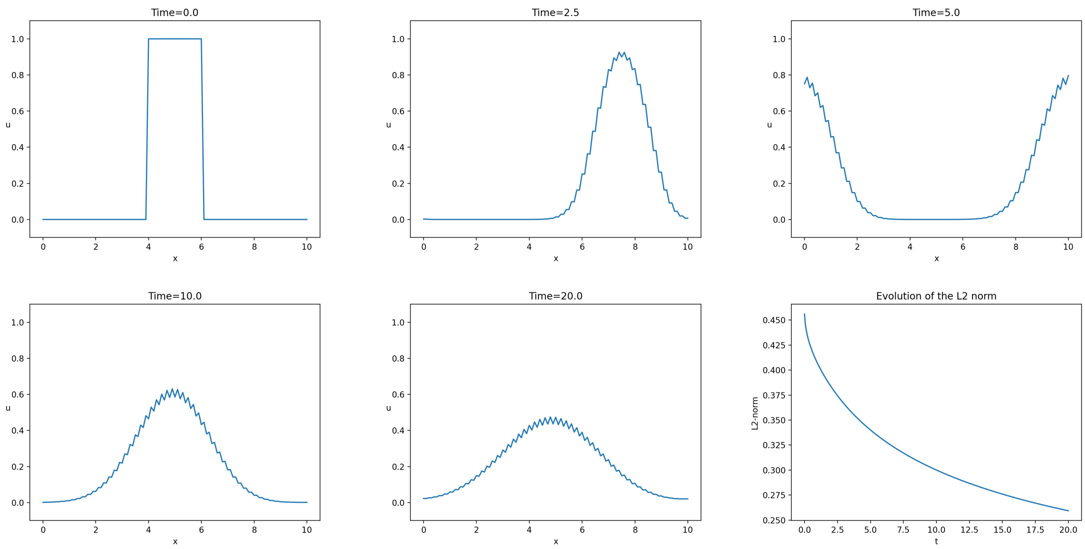


Question 2

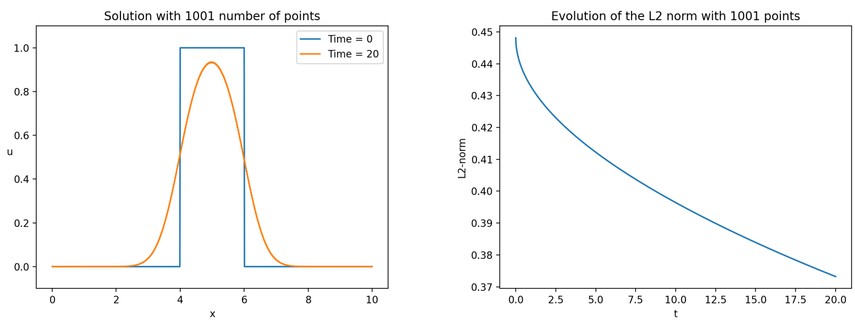
In all the following cases I solve the advection equation $u_t + u_x = 0$ using as initial condition a step function: $u(x, t = 0) = 1$ for $x \in [4, 6]$ and $u(x, t = 0) = 0$ in the rest of the domain, with periodic boundary conditions and a terminate time equal to 20.

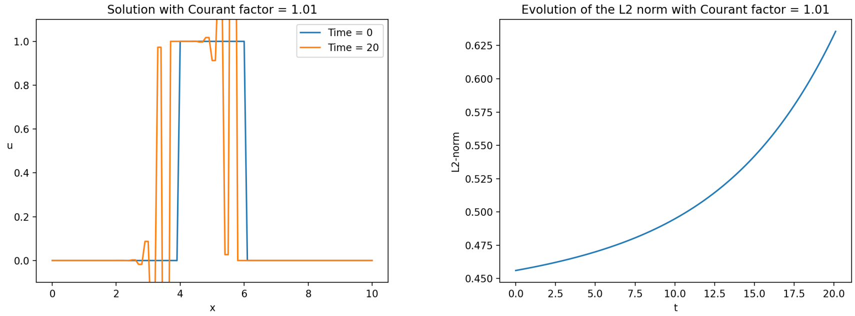
Lax-Friedrichs

I start with $J = 101$ number of points and the Courant factor equal to 0.5 ($dx = 0.1$ and $dt = 0.05$). The initial data travels towards the right with speed 1, but the shape of the step function is deformed due to the dissipative term of the Lax-Friedrichs method: the step function gets lower and wider in time and the L2 norm decreases with time. Lax-Friedrichs is a first order, linear, monotone scheme and indeed it doesn't introduce oscillations.



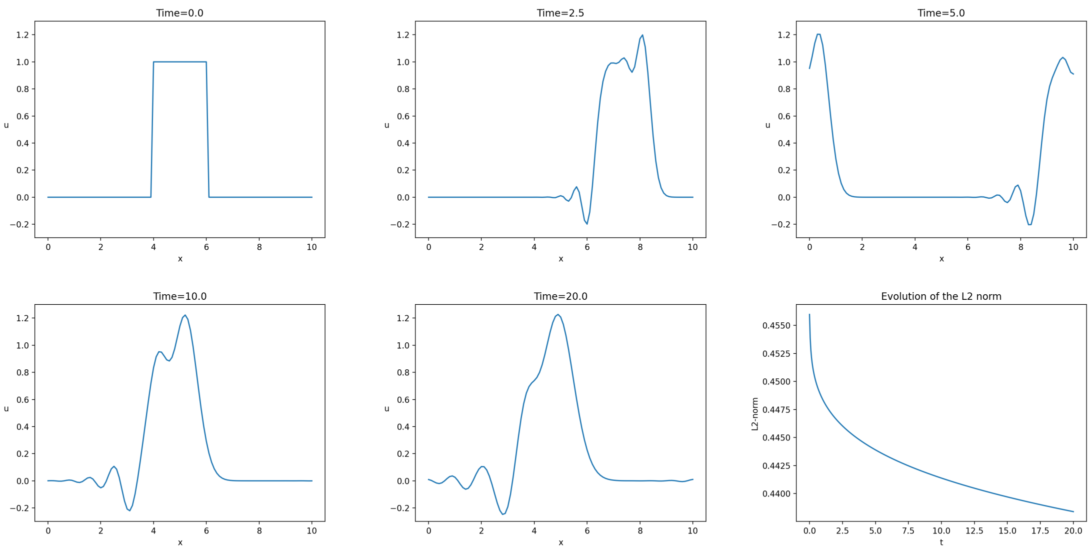
I increase the resolution by using $J = 1001$, so that the dissipative term impacts less the solution: the height of the step function decreases less and in general the shape is more preserved at the final time. In addition the L2 norm decreases less from the initial value. To check that the Lax-Friedrichs is conditionally stable I change the Courant factor to 1.01 ($dx = 0.1$ and $dt = 0.101$) so that the CFL condition isn't satisfied anymore: the scheme gets unstable, the solution explodes and the L2 norm grows exponentially with time.



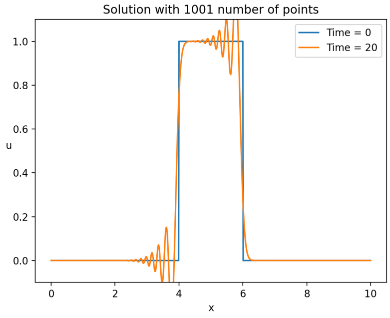


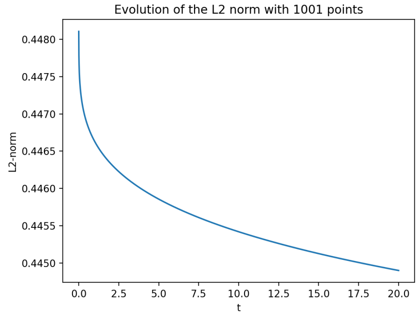
Lax-Wendroff

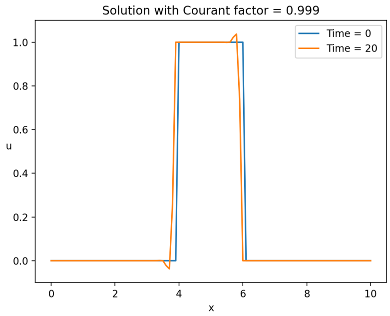
I start wit $J = 101$ number of points and the Courant factor equal to 0.5. As before the initial data travels towards the right with speed 1, but soon some oscillations start showing off due to the dispersive term of the Lax-Wendroff scheme. This one is a linear method but since it is second order accurate it can't be monotonic too, due to the Godunov theorem. It should be noted that the presence of the oscillations doesn't mean that the scheme is unstable, indeed the L2 norm doesn't explode. In the Lax-Wendroff method there is also term that dissipates the solution, but it's way less relevant because it goes as $((dx)^3)$.

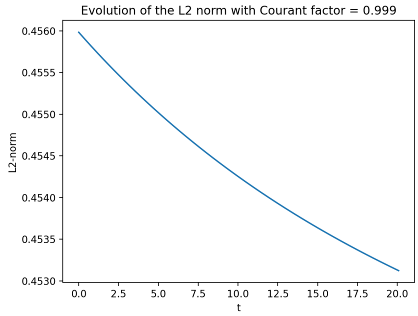


By increasing the number of points to $J = 1001$ the oscillations are still present but their amplitude diminished, since the dispersive term has reduced with a lower dx. The dissipative term now is way less evident too. It has to be noted that by improving the resolution, the L2 norm decreases less from the initial value at $t = 0$. Since Lax-Wendroff is a second order scheme, this change of the slope of the L2 norm, due to the lower dx, is bigger than the one that happens with Lax-Friedrichs, that is first order accurate. This proves that by adding more points the accuracy of Lax-Wendroff increases more than Lax-Friedrichs. Then I try a Courant factor near 1, $c_f = 0.999$ ($dx = 0.01$ and $dt = 0.0999$): this makes both the dispersive and dissipative term tend to zero, as shown by the solution at the final time that almost resemble the exact one.


Solution with 1001 number of points
Time = 0
Time = 20
u
x
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10


Evolution of the L2 norm with 1001 points
L2-norm
t
0.4450
0.4455
0.4460
0.4465
0.4470
0.4475
0.4480
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0

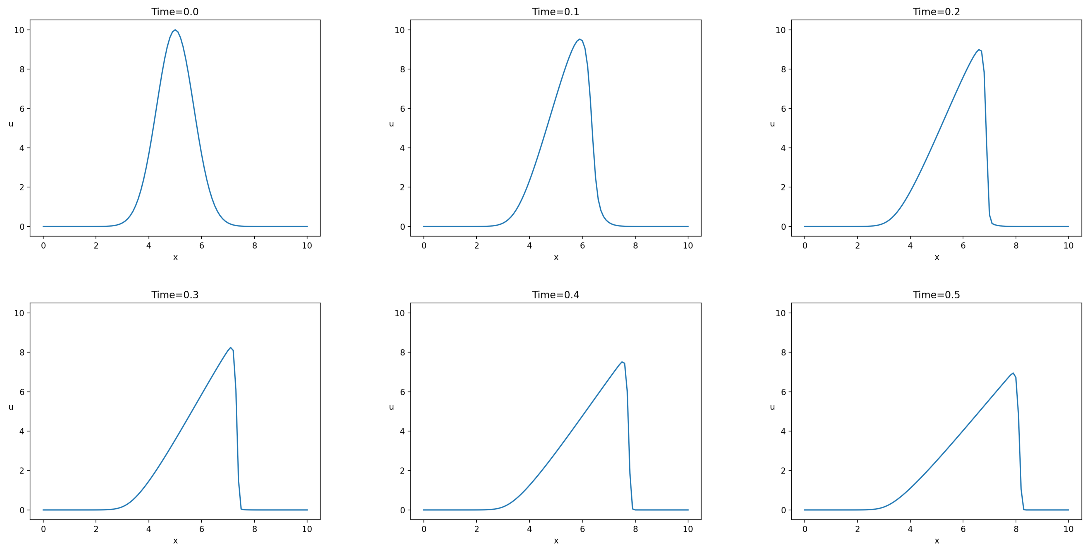

Solution with Courant factor = 0.999
Time = 0
Time = 20
u
x
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10


Evolution of the L2 norm with Courant factor = 0.999
L2-norm
t
0.4530
0.4535
0.4540
0.4545
0.4550
0.4555
0.4560
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0

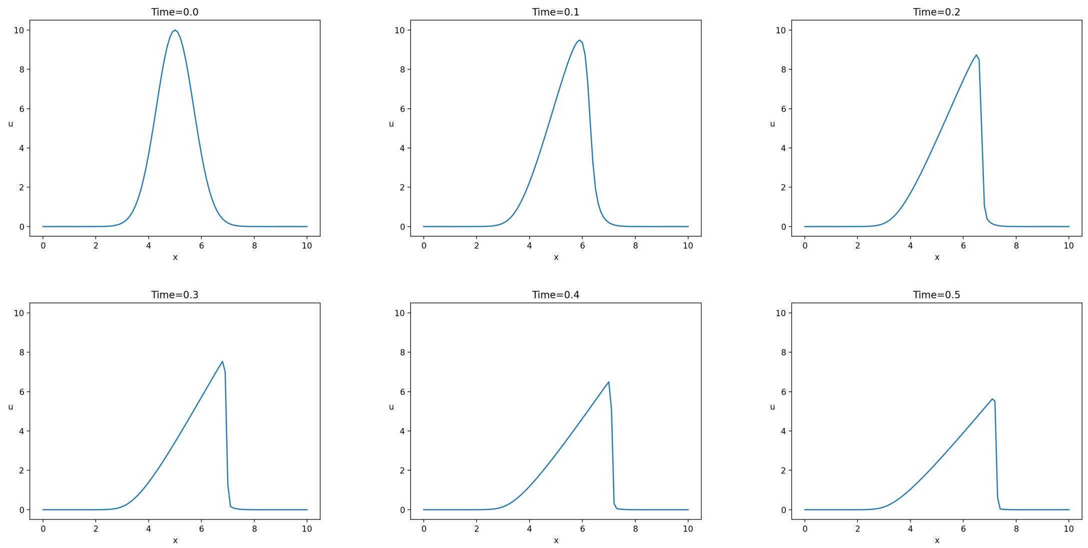
Question 3

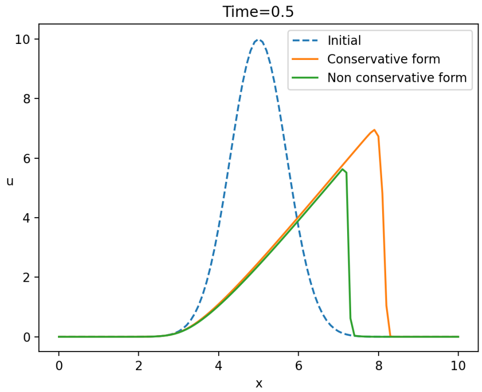
I solve the Burgers' equation $u_t + uu_x = 0$ using as initial condition a gaussian profile $u(x, t = 0) = 10e^{-(x-x_0)^2}$ with $x_0 = 5$. The domain is $x \in [0, 10]$, the final time is $t = 0.5$ and the Courant factor is $c_f = 0.5$.

I start with $J = 101$ points for the grid ($dx = 0.1$ and $dt = 0.005$). I first use the flux-conservative form of the upwind scheme. The following figures show the evolution in time of the solution $u(x, t)$: each point of the curve moves at different speed, depending on the value of u itself. In particular the right part of the Gaussian produces a shock wave and so a discontinuity, while the left part produces a rarefaction wave.



Then I change to the upwind scheme written in non flux-conservative form. The solution evolves as before, producing a shock wave and a rarefaction wave, but for the fact that the position of the discontinuity is different. It has to be noted also that in both the previous cases the solution doesn't explode and it doesn't display any oscillation, due to the fact that the upwind method is conditionally stable and monotone.





Afterwards I raised the resolution by using a higher number of points: $J = 201$ ($dx = 0.05$, $dt = 0.025$) and $J = 501$ ($dx = 0.02$, $dt = 0.001$). With a lower value of the grid spacing, the solution computed with the flux-conservative version of the upwind scheme converges to the exact solution of the Burgers' equation, because it satisfies the Lax-Wendroff theorem. On the contrary, by increasing the resolution the non flux-conservative form produces a solution with a more precise discontinuity, but still located in the wrong position. This means that the non-conservative form converges to the wrong solution for dx that tends to zero, as stated by the Hou-LeFlock theorem.