Numerical Relativity Homeworks

Alice Perego MAT 858878

June 6, 2023

Question 1

The Euler equations describe the evolution of a perfect fluid in Newtonian physics. In one dimension, they are a system of three differential equations, written as conservation laws, involving the rest-mass density ρ, the 1D velocity v and the pressure P of the fluid. If at least one of these quantities has a piecewise constant initial data with a single discontinuity (while the others are constant) this becomes a Riemann problem. In a SOD Shock Tube problem, the initial condition is such that there is a discontinuity in the density and the pressure, while the velocity is set to zero. The left and right hand side initial data are:

$$\begin{cases} \rho_{\mathrm{L}} = 1 \\ v_{\mathrm{L}} = 0 \\ P_{\mathrm{L}} = 1 \end{cases} \qquad \begin{cases} \rho_{\mathrm{R}} = \frac{1}{8} \\ v_{\mathrm{R}} = 0 \\ P_{\mathrm{R}} = \frac{1}{10} \end{cases} \tag{1}$$

I solve this problem running a simulation with Einstein Toolkit. In order to solve the Euler equations I specify the equation of state as a polytrope with the exponent equal to 5/3. I also choose HLLE as the approximant Riemann solver. The duration of the simulation is set to 0.4 and the spatial domain is defined from $x = -0.5$ to $x = 0.5$, while it is narrower in the y and z direction since the problem is one dimensional. The spacing dx of the domain is the parameter that defines the number of points of the grid (which is an integer defined by the ratio between the length of the domain and dx) and so it sets also the resolution of the simulation. I've used three different values: $dx = 0.005$ (200 points), $dx = 0.0025$ (400 points), $dx = 0.00125$ (800 points), so I got three different solutions, depending on the resolution. I can compare them with the exact solution by making a plot of the final density curve (Figure 1). As expected, increasing the number of points makes the numerical solution approach the exact one, due to the better resolution.

The solution of a Riemann problem for the Euler equations gives three different waves: the one in the middle is always a contact discontinuity, in which the density has a jump while velocity and pressure remain constant; the other two can be rarefaction waves or shock waves. The Figure 1 shows that the SOD problem produces a rarefaction wave that propagates towards the left, a contact discontinuity in between and a shock wave propagating towards the right.

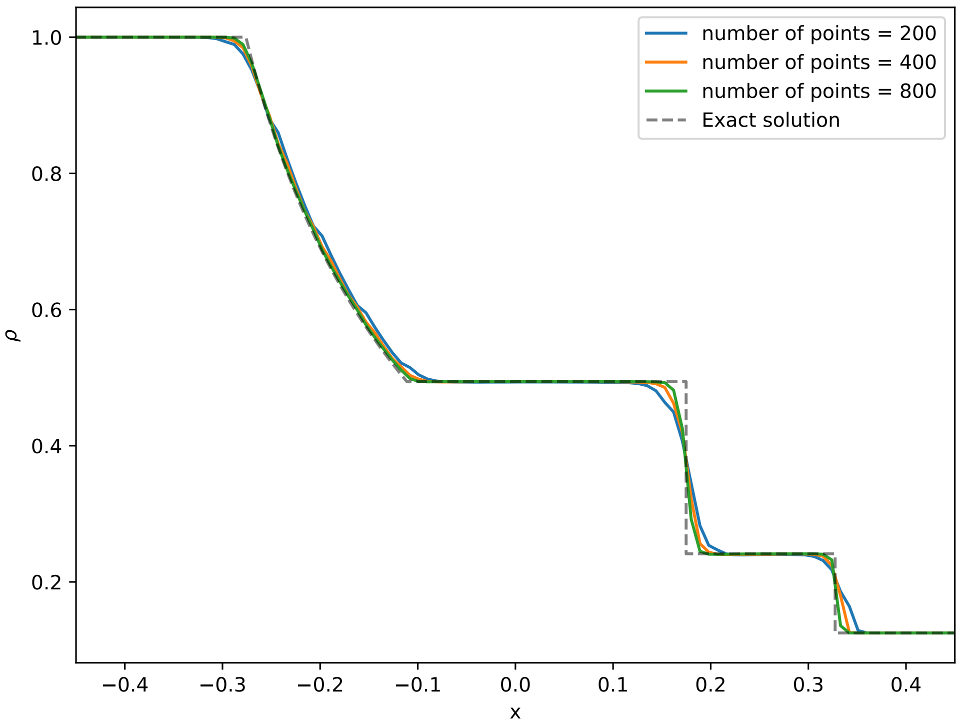

Figure 1: Rest-mass density at the last iteration.

Question 2

The solution of the TOV equations describes a stable non-rotating neutron star, whose evolution can be computed with a numerical simulation using Einstein Toolkit. I set the initial central density to $\rho = 1.28\times10^{-3}\mathrm{M}_{\odot}{}^{-2}$ (equal to $7.99\times10^{14}\mathrm{g\,cm}^{-3}$) and the equation of state as a polytrope with $K = 100$ e $\gamma = 2$. The duration of the simulation is $400\mathrm{M}_{\odot}$ (1.97ms), while the spatial domain is defined as an octant of size $24.0\mathrm{M}_{\odot}$ (36km), which is then extended to a three-dimensional cube centered in the origin, thanks to the spherical symmetry of the system.

I set the same spacing of the spatial grid to $dx = 2.0$, so that the resolution remains constant. I firstly run three different simulations, varying the coefficient K of the polytropic equation for the evolution of the system:

1. I keep the same value $K = 100$
2. I set $K = 98$, which means that I reduce the pressure of the star of 2%
3. I set $K = 95$, which means that I reduce the pressure of the star of 5%

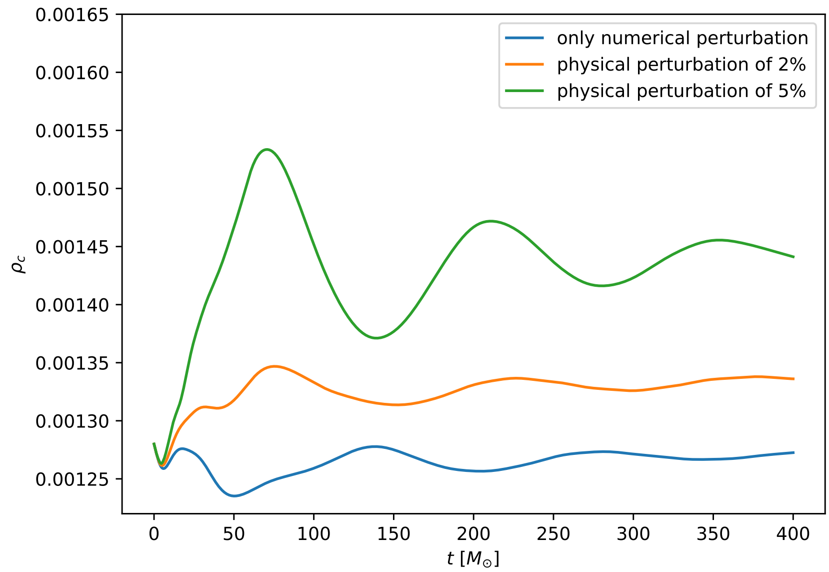

Figure 2: Evolution of the maximum density for a simulation with $dx = 2.0$.

Figure 2 shows the plot of the maximum rest-mass density in function of time, for the three different simulations.

In the first one, where K doesn't change, the oscillations are produced only by a numerical perturbation: since the coordinate system used is discrete, the surface of the star is adapted to the spatial grid and gets deformed; this results in an artificial perturbation that triggers a radial oscillatory motion in the system around the stable solution.

On the other hand in the second and third case, the change of K introduces a physical perturbation: the internal pressure is not sufficient to balance out the external one and so the radius decreases and the central density increases up to a new equilibrium state, around which the oscillatory motion of the star continues. The bigger the variation of K from the initial condition, the bigger the perturbation is and so the higher the amplitude of the oscillations is.

The difference between a physical perturbation and a numerical one can be shown by varying the resolution of the simulation. The first one is independent on it, which means that the oscillations will be present even in the limit of an infinite number of points; rather it will converge to the exact solution, where the star oscillate around the stable solution. Conversely since the numerical perturbation is produced by the discrete domain itself, a lower spacing of the grid will make the star less perturbed: the amplitude of the oscillations of the central density will tend to zero in the infinitesimal limit of dx, because the solution is stable.

This behaviour is shown by Figure 3, 4, 5, where I have reduced the size of the grid, from $dx = dy = dz = 2.0$ to $dx = dy = dz = 1.5$ and then to $dx = dy = dz = 1.0$, for all the three previous cases. For $K = 100$ the oscillations get smaller, while this doesn't happen for $K = 98$ and $K = 95$.

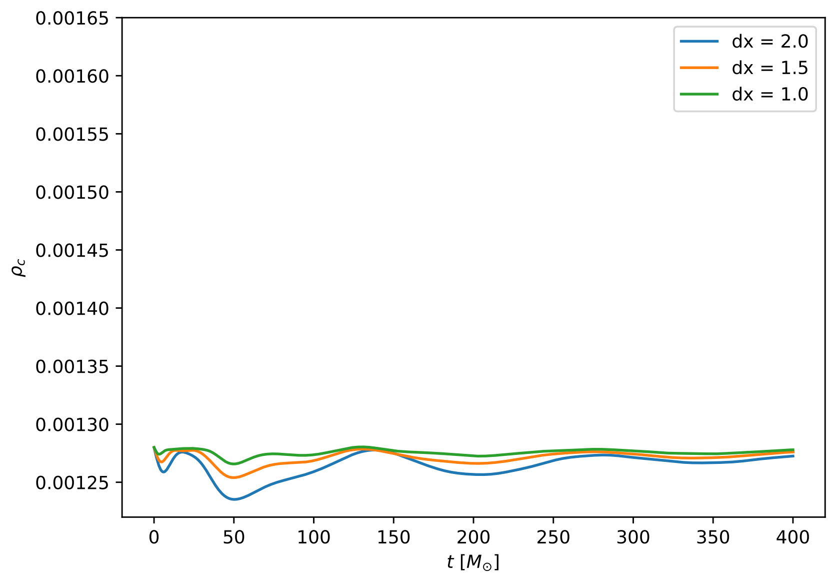

Figure 3: Evolution of the maximum density for a simulation with $K = 100$.

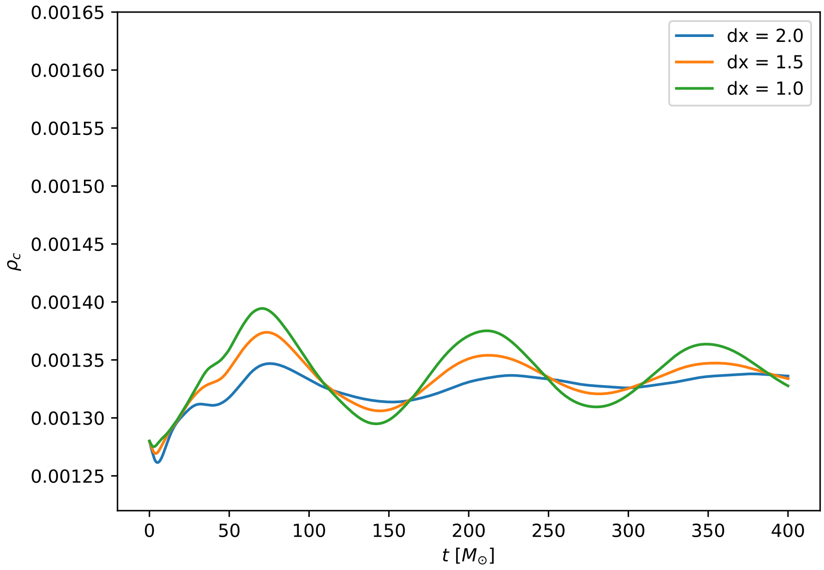

Figure 4: Evolution of the maximum density for a simulation with $K = 98$.

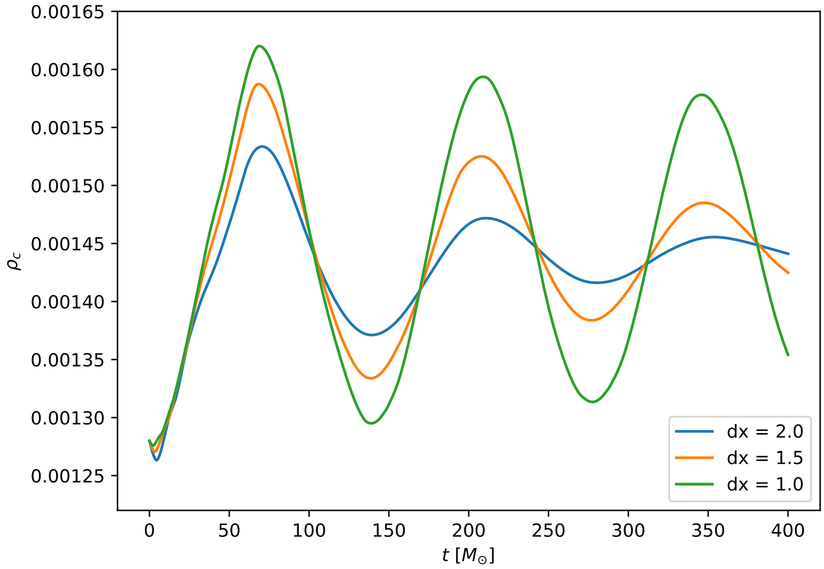

Figure 5: Evolution of the maximum density for a simulation with $K = 95$.