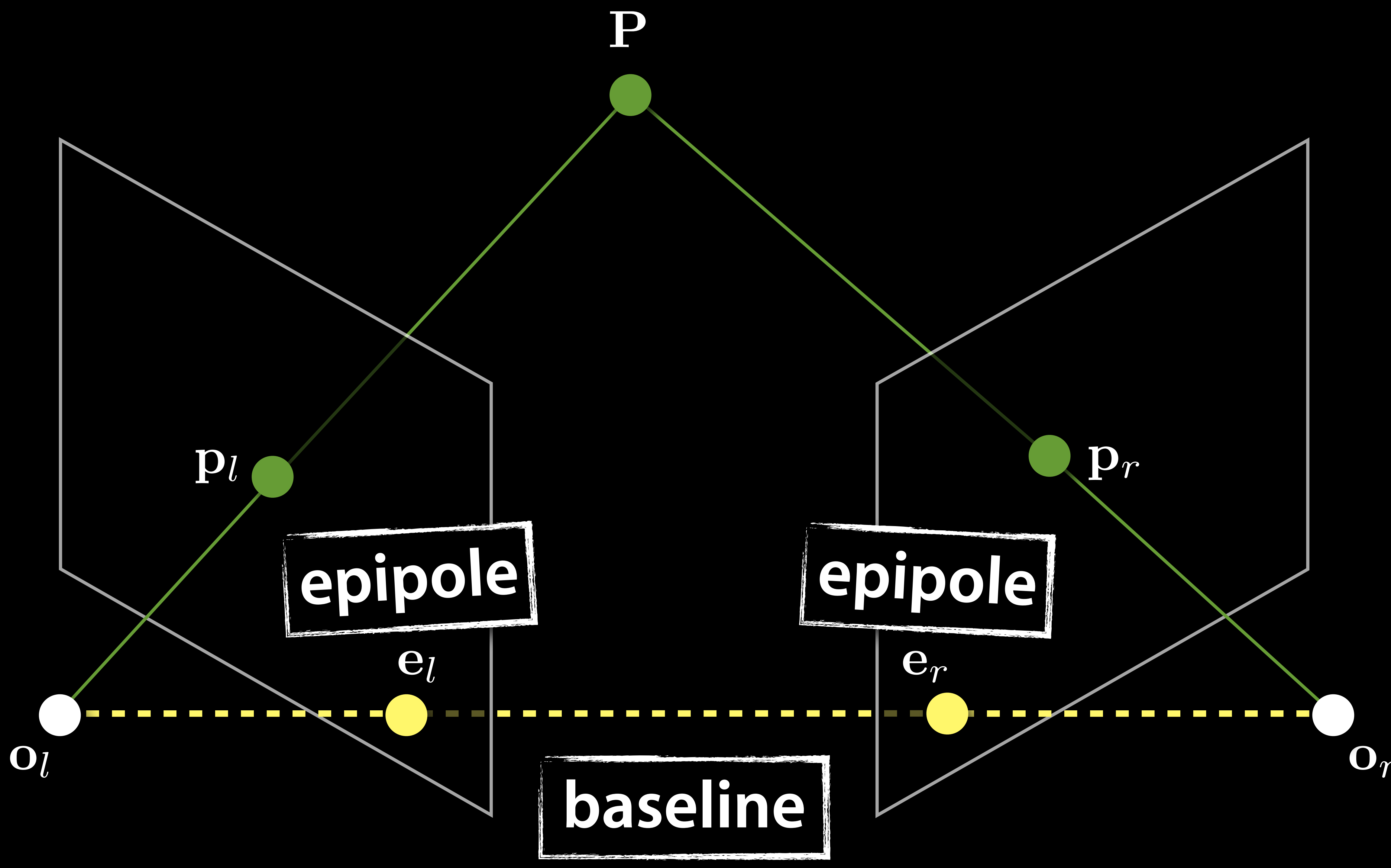
LECTURE TOPICS

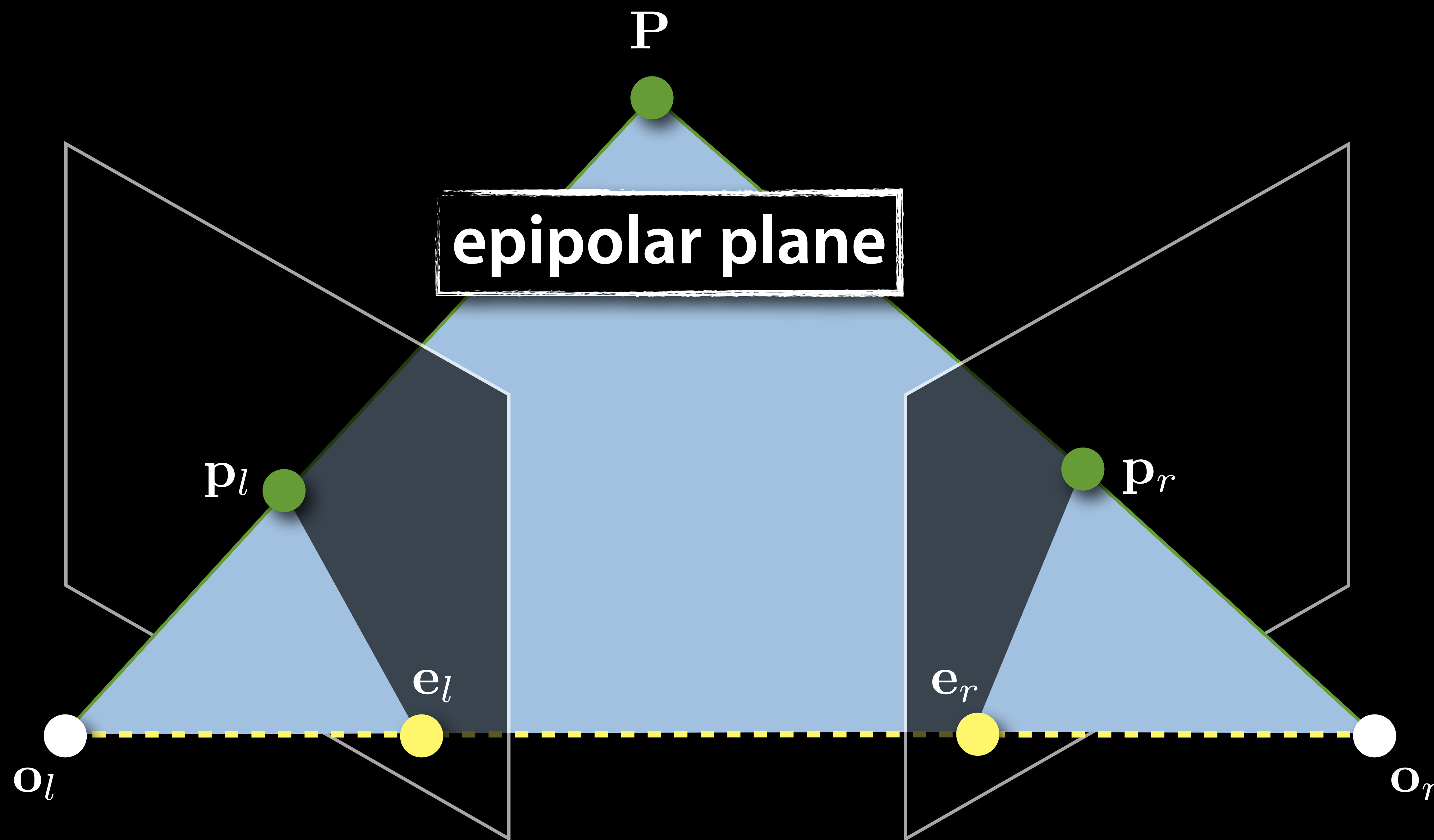
- Epipolar geometry
- Fundamental matrix
- Stereo matching

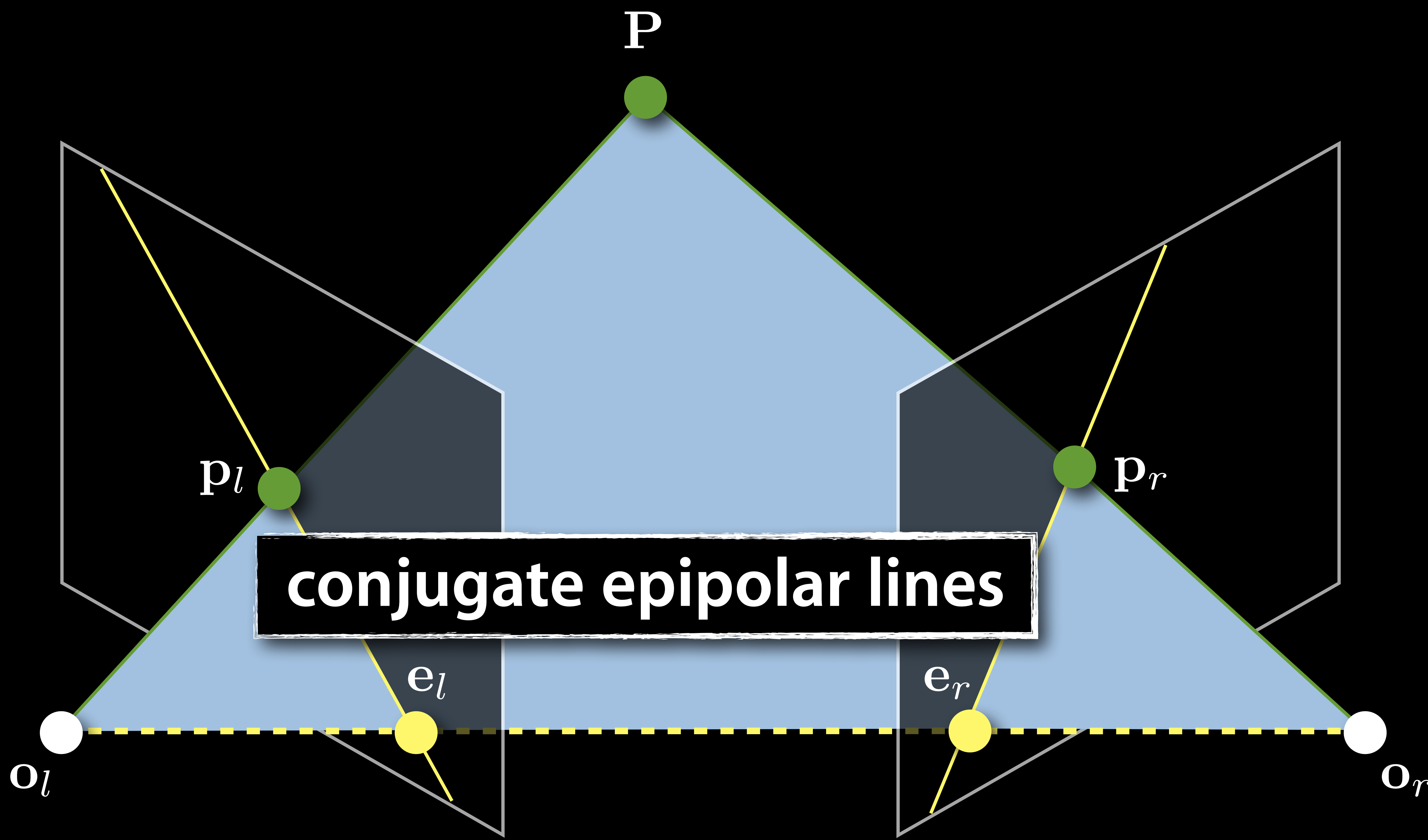
How can we recover 3D geometry from images?

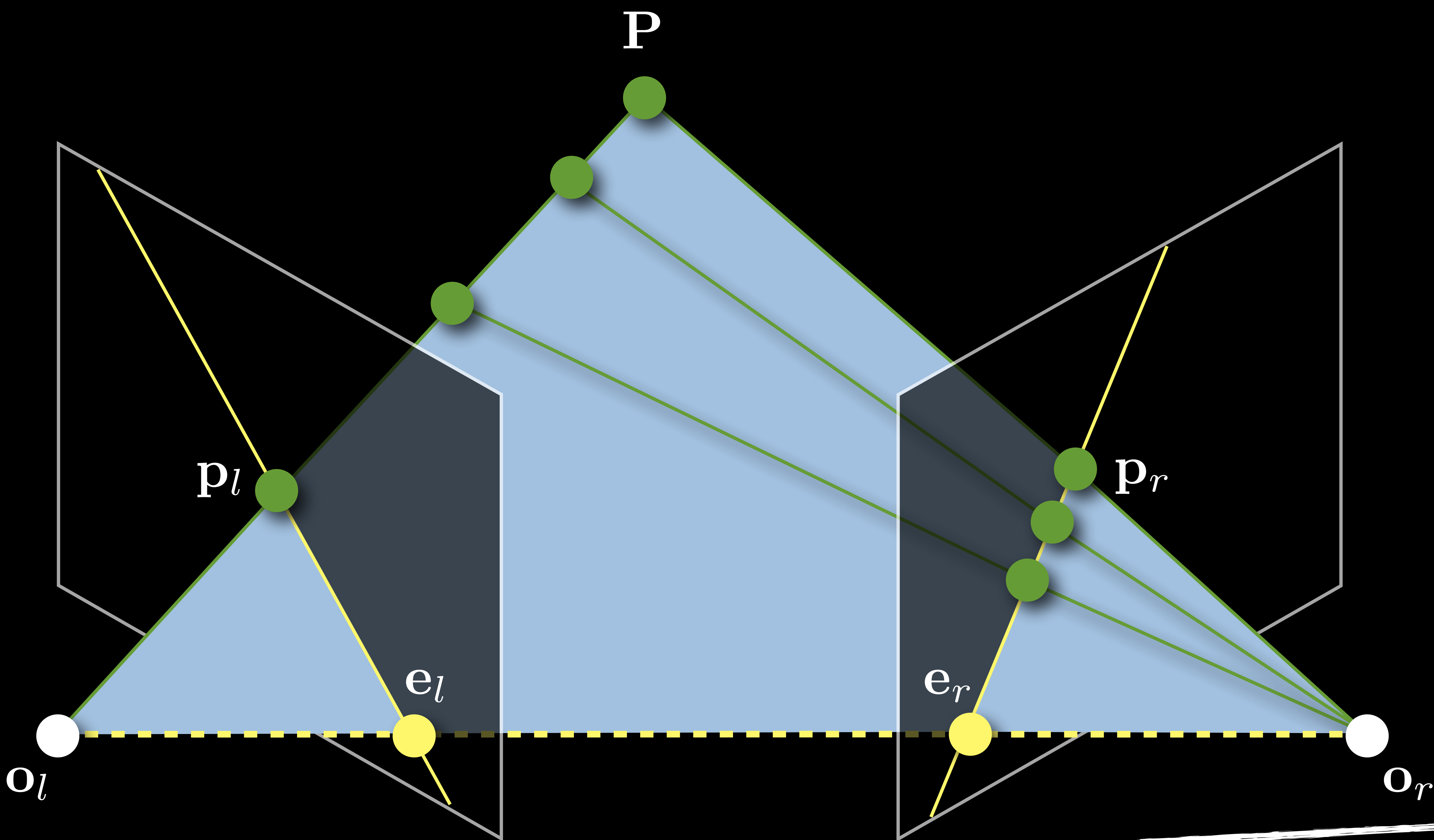
How can its match in the right image be found?



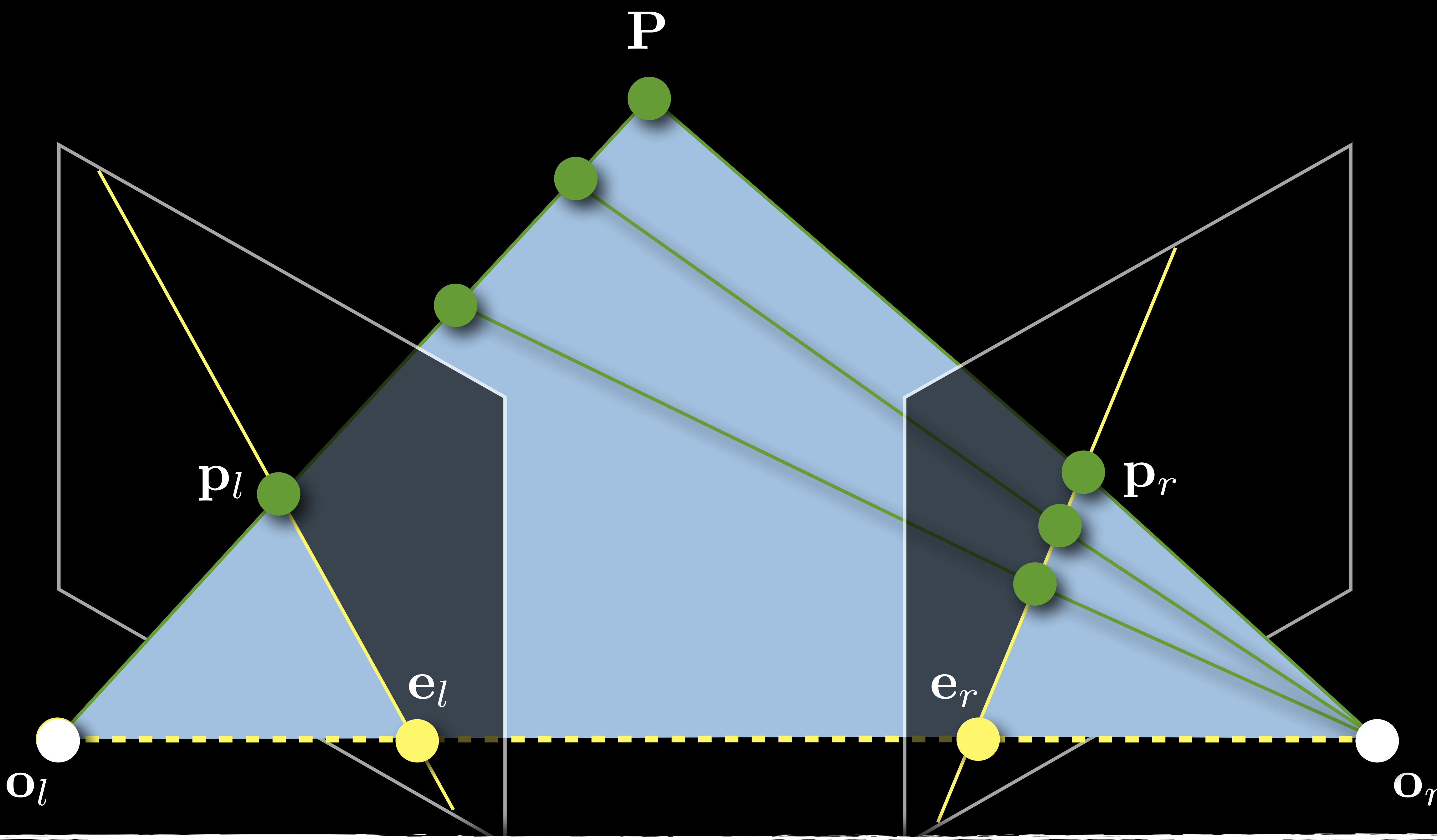
$\mathbf{P}$
$\mathbf{p}_l$
$\mathbf{p}_r$
epipole
epipole
$\mathbf{e}_l$
$\mathbf{e}_r$
$\mathbf{o}_l$
$\mathbf{o}_r$
baseline


$\mathbf{P}$
epipolar plane
$\mathbf{p}_l$
$\mathbf{p}_r$
$\mathbf{e}_l$
$\mathbf{e}_r$
$\mathbf{o}_l$
$\mathbf{o}_r$


P
$\mathbf{p}_l$
$\mathbf{p}_r$
conjugate epipolar lines
$\mathbf{e}_l$
$\mathbf{e}_r$
$\mathbf{o}_l$
$\mathbf{o}_r$

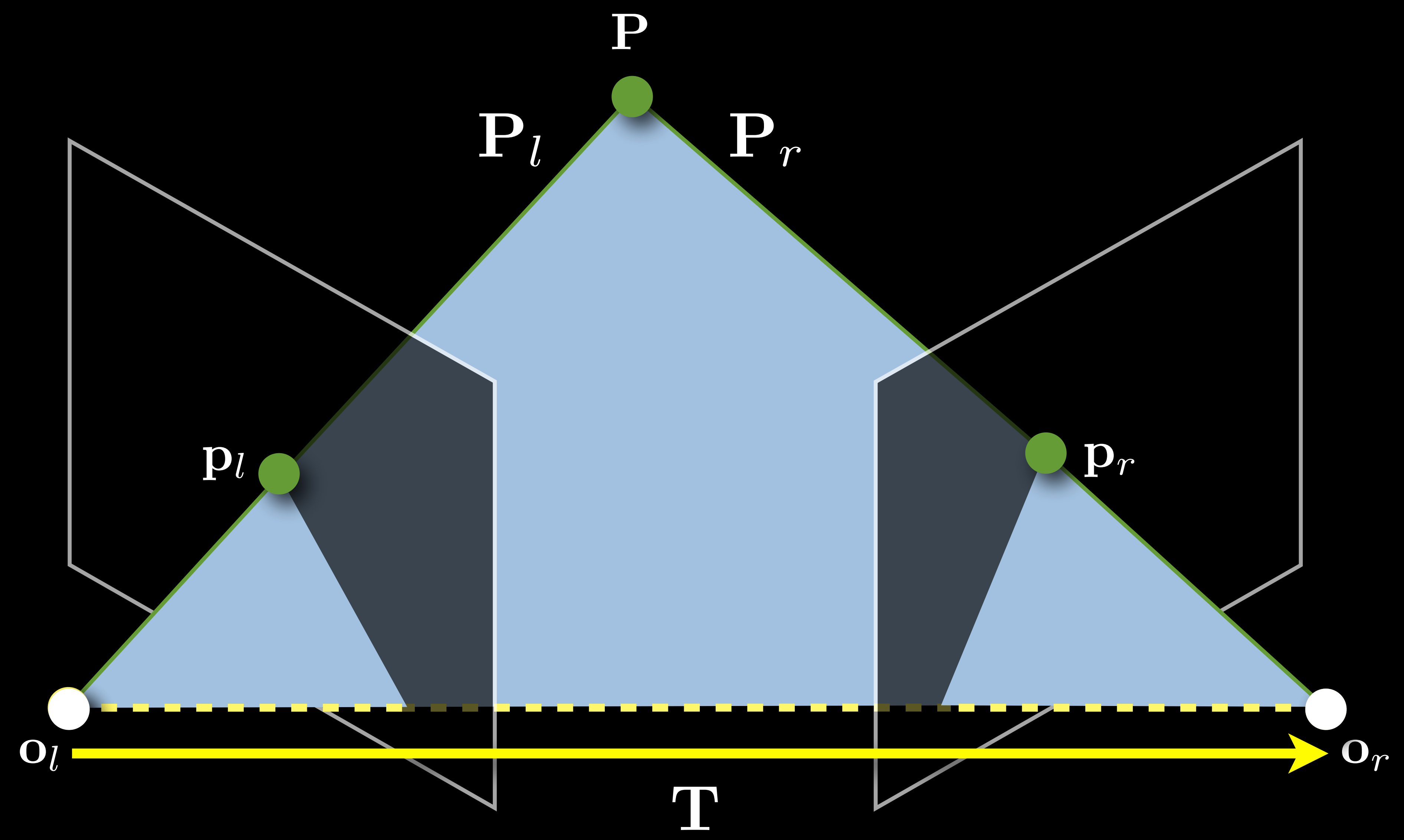


Given $\mathbf{p}_l$, where does it project in the right image?



Epipolar constraint

Matching points must lie on conjugated epipolar lines



$\mathrm{T} = \mathrm{O}_r - \mathrm{O}_l$ define the translation that shifts the left centre of projection to the right one

R define the rotation matrix that aligns the coordinate axes of the left and right views

$\mathrm{P}_r = \mathrm{R}(\mathrm{P}_l - \mathrm{T})$ define the transformation between the coordinates of P in the left and right views

Definition: The column rank of an $m \times n$ matrix $\mathbf{A}$ is the number of linearly independent columns of the matrix.

The row rank of matrix $\mathbf{A}$ is the number of linearly independent rows of $\mathbf{A}$.

row and column matrix ranks are always equal

Definition: The column rank of an $m \times n$ matrix $\mathbf{A}$ is the number of linearly independent columns of the matrix.

The row rank of matrix $\mathbf{A}$ is the number of linearly independent rows of $\mathbf{A}$.

row and column matrix ranks are always equal

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

What are the row and column ranks of this matrix?

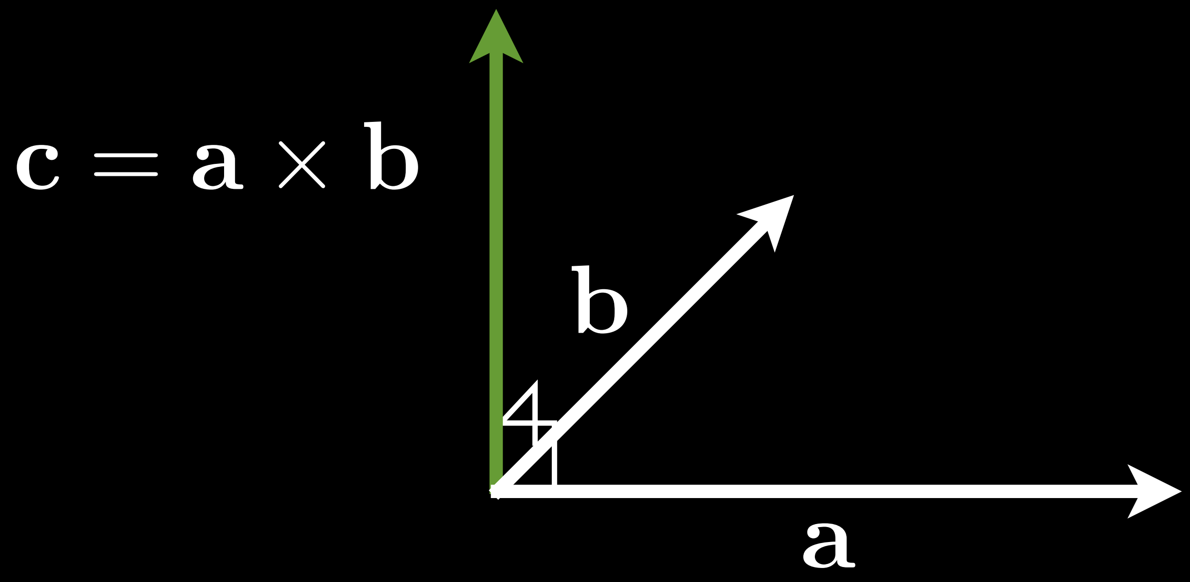
Definition: The null space of an $m \times n$ matrix $\mathbf{A}$ is the set of all solutions to the homogeneous equation

$$\mathbf{Ax} = \mathbf{0}.$$

$$\text{Null}(\mathbf{A}) = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{Ax} = \mathbf{0}\}$$

rank and nullity of a matrix add up to the number of columns of the matrix

Definition: The cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as a vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, i.e., $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$.



Given 3×1 **vectors** $\mathbf{a} = (a_1, a_2, a_3)^\top$ **and** $\mathbf{b} = (b_1, b_2, b_3)^\top$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} -a_3 b_2 + a_2 b_3 \\ a_3 b_1 - a_1 b_3 \\ -a_2 b_1 + a_1 b_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$\boxed{\textbf{matrix multiplication}}$

Given 3×1 **vectors** $\mathbf{a} = (a_1, a_2, a_3)^\top$ **and** $\mathbf{b} = (b_1, b_2, b_3)^\top$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

skew symmetric matrix

$$\mathbf{S} = -\mathbf{S}^\top$$

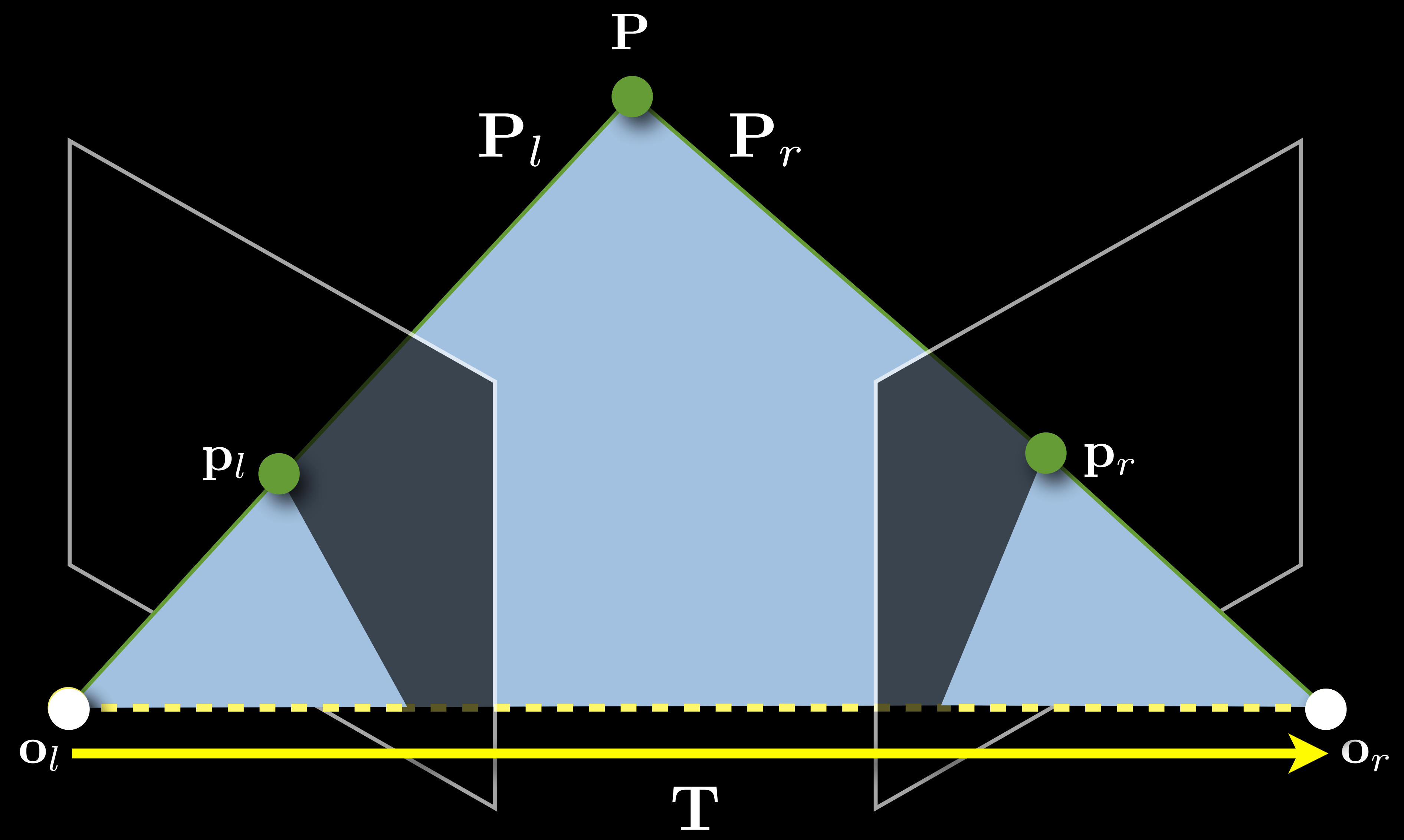
Given 3×1 **vectors** $\mathbf{a} = (a_1, a_2, a_3)^\top$ **and** $\mathbf{b} = (b_1, b_2, b_3)^\top$

$$\mathbf{a} \times \mathbf{b} = \underbrace{\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}}_{\text{rank 2 matrix}} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Given 3×1 **vectors** $\mathbf{a} = (a_1, a_2, a_3)^\top$ **and** $\mathbf{b} = (b_1, b_2, b_3)^\top$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= [\mathbf{a}_\times]\mathbf{b}$$

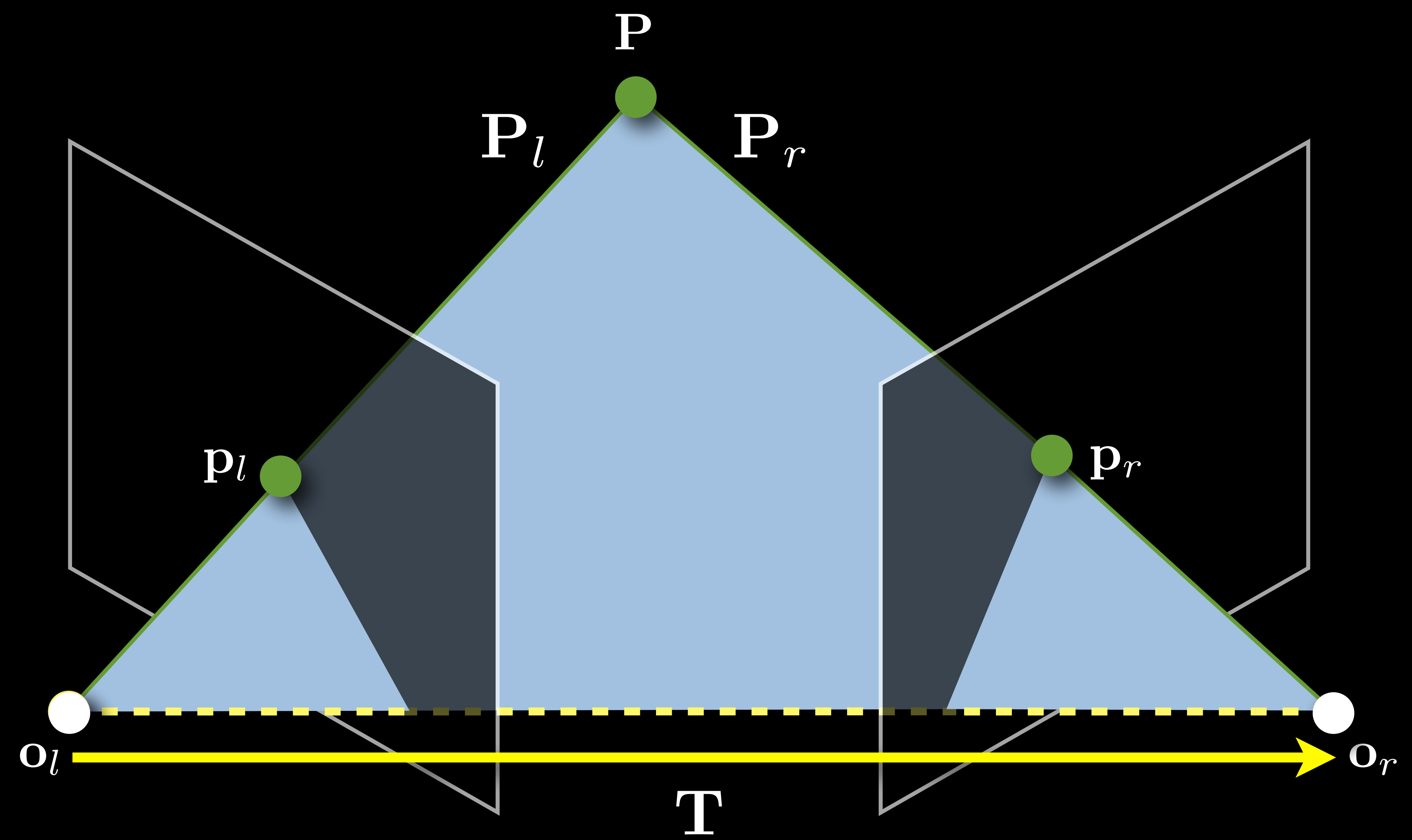


$\mathbf{P}_r = \mathbf{R}(\mathbf{P}_l - \mathbf{T})$ **define the transformation between the coordinates of $\mathbf{P}$ in the left and right views**

Coplanarity condition:

$$(\mathbf{P}_l - \mathbf{T}) \cdot (\mathbf{T} \times \mathbf{P}_l) = 0$$

normal to the plane



$\mathbf{P}_r = \mathbf{R}(\mathbf{P}_l - \mathbf{T})$ **define the transformation between the coordinates of $\mathbf{P}$ in the left and right views**

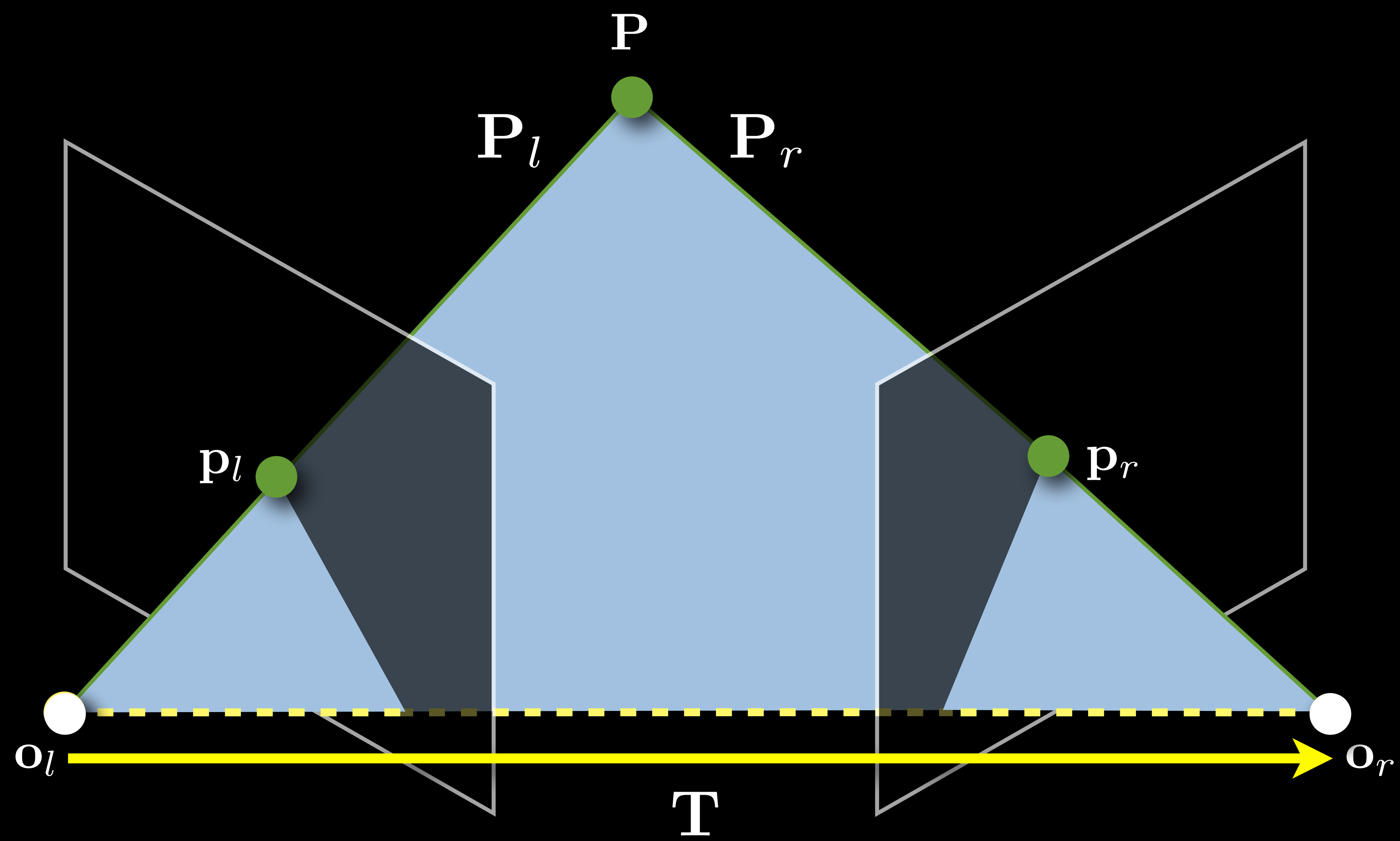
Coplanarity condition:

$$(\mathbf{P}_l - \mathbf{T}) \cdot (\mathbf{T} \times \mathbf{P}_l) = 0$$

substitute

$$\mathbf{P}_l = \mathbf{R}^\top \mathbf{P}_r + \mathbf{T}$$

$$(\mathbf{R}^\top \mathbf{P}_r) \cdot (\mathbf{T} \times \mathbf{P}_l) = 0$$



Coplanarity condition:

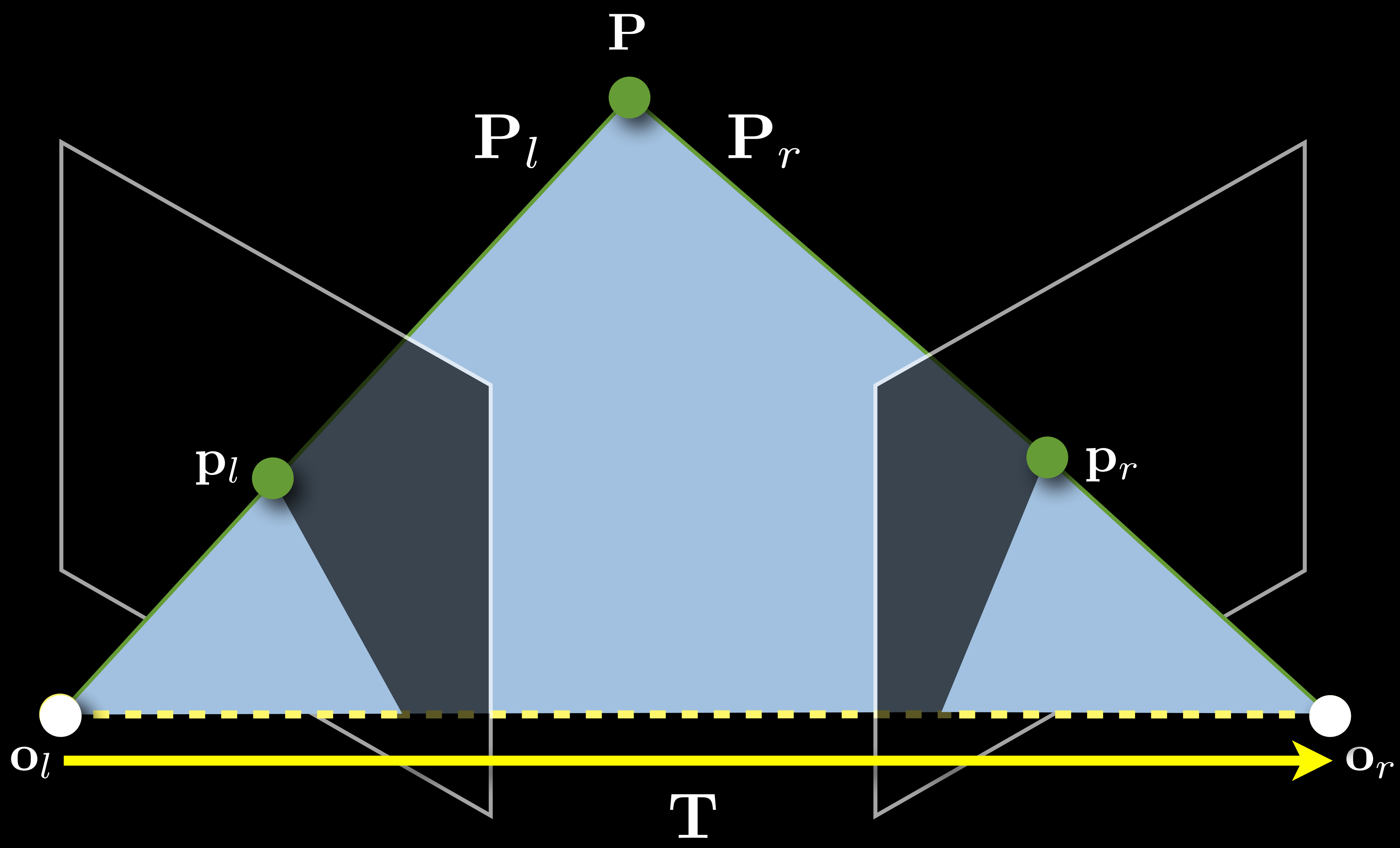
$$(\mathbf{R}^\top \mathbf{P}_r) \cdot (\mathbf{T} \times \mathbf{P}_l) = 0$$

cross product as matrix

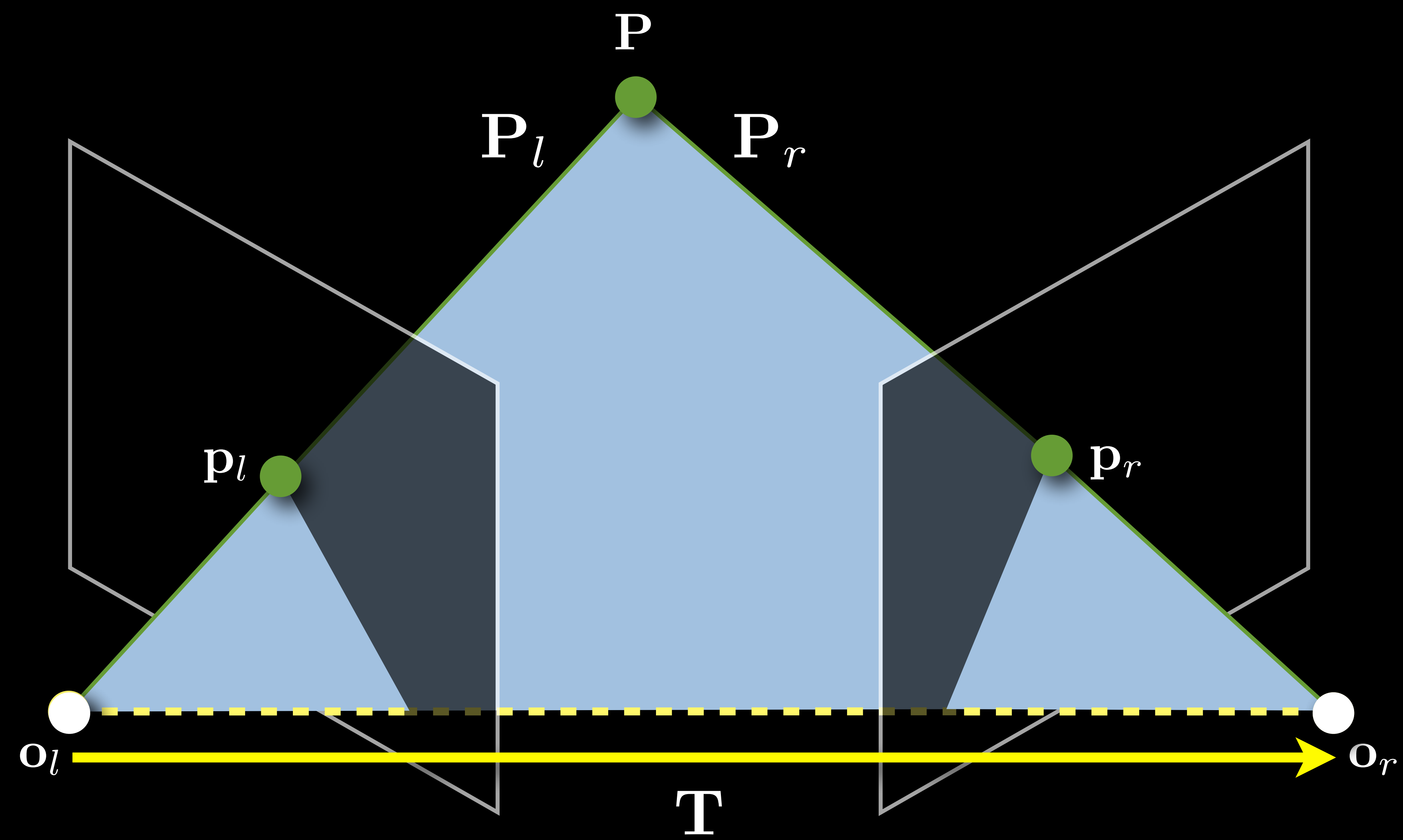
$$(\mathbf{R}^\top \mathbf{P}_r)^\top [\mathbf{T}_\times] \mathbf{P}_l = 0$$

rewrite

$$\mathbf{P}_r^\top \mathbf{R} [\mathbf{T}_\times] \mathbf{P}_l = 0$$



$$\mathbf{P}_r^\top \mathbf{R}[\mathbf{T}_\times]\mathbf{P}_l = 0$$

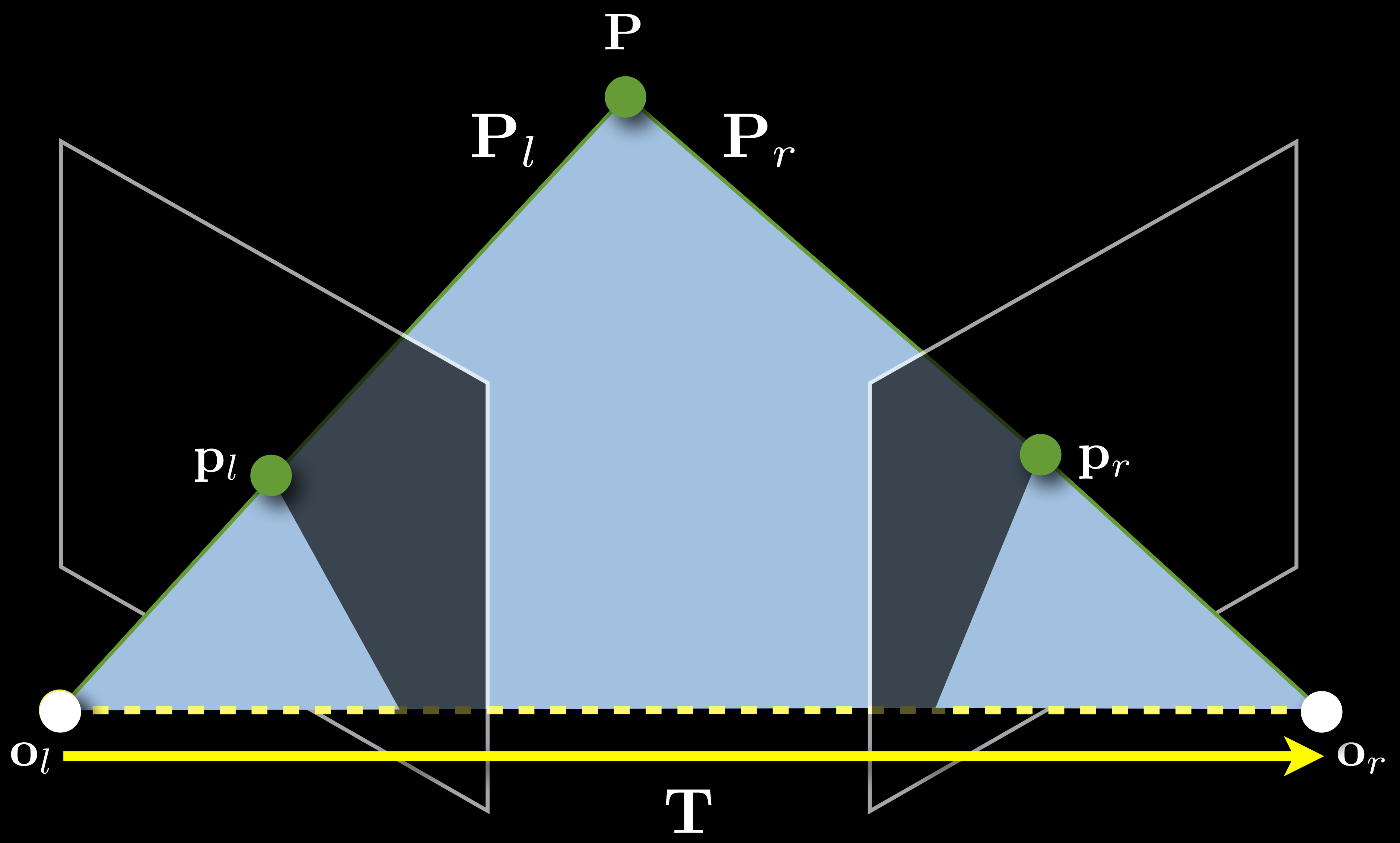


$$\mathbf{P}_r^\top \mathbf{R}[\mathbf{T}_\times]\mathbf{P}_l = 0$$

Let $\mathbf{E} = \mathbf{R}[\mathbf{T}_\times]$

$$\mathbf{P}_r^\top \mathbf{E}\mathbf{P}_l = 0$$

Essential Matrix

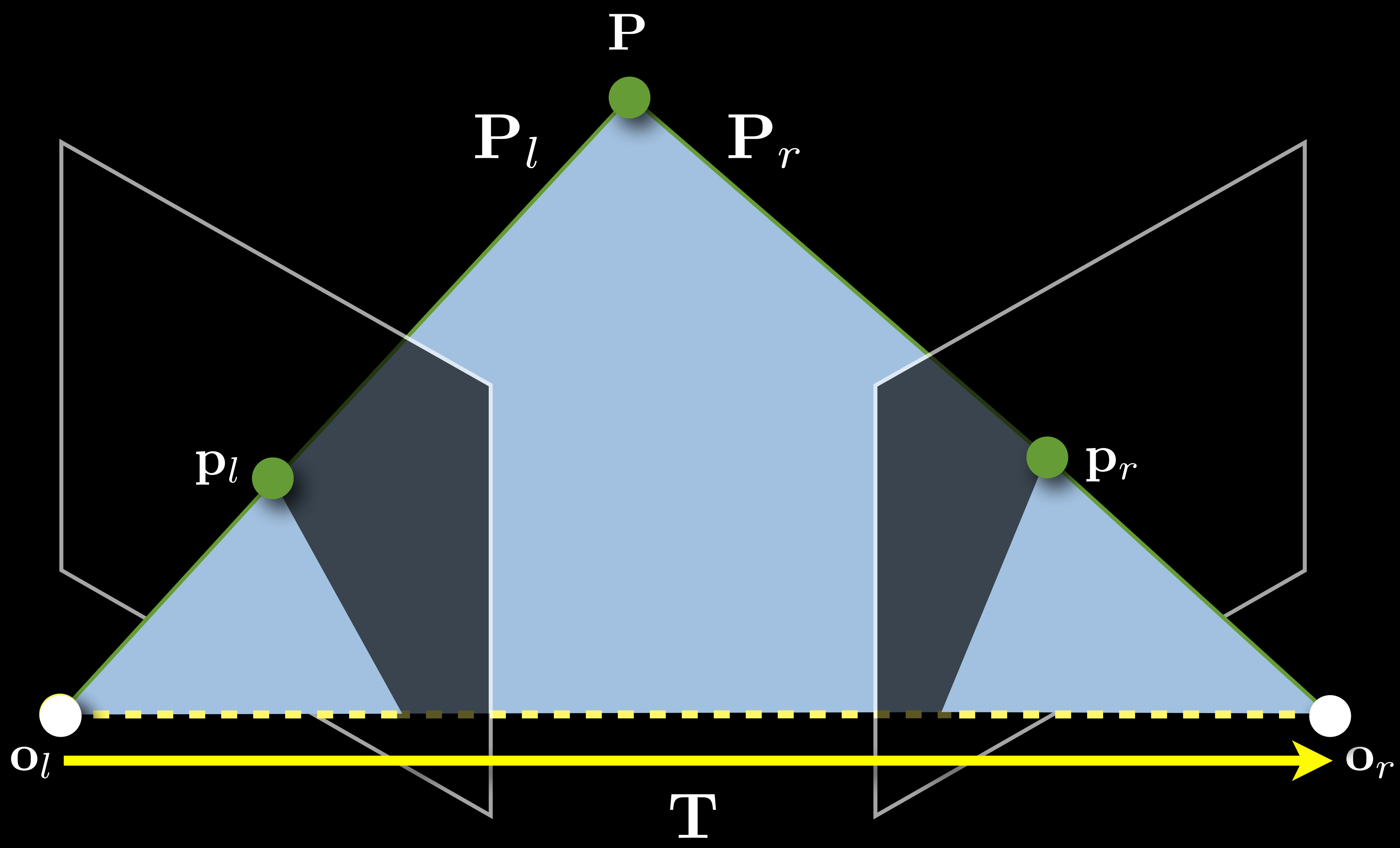


$$\mathbf{P}_r^\top \mathbf{R}[\mathbf{T}_\times]\mathbf{P}_l = 0$$

Let $\mathbf{E} = \mathbf{R}[\mathbf{T}_\times]$

$$\mathbf{P}_r^\top \mathbf{E}\mathbf{P}_l = 0$$

How do we revise constraint to relate image points (in camera coordinates)?

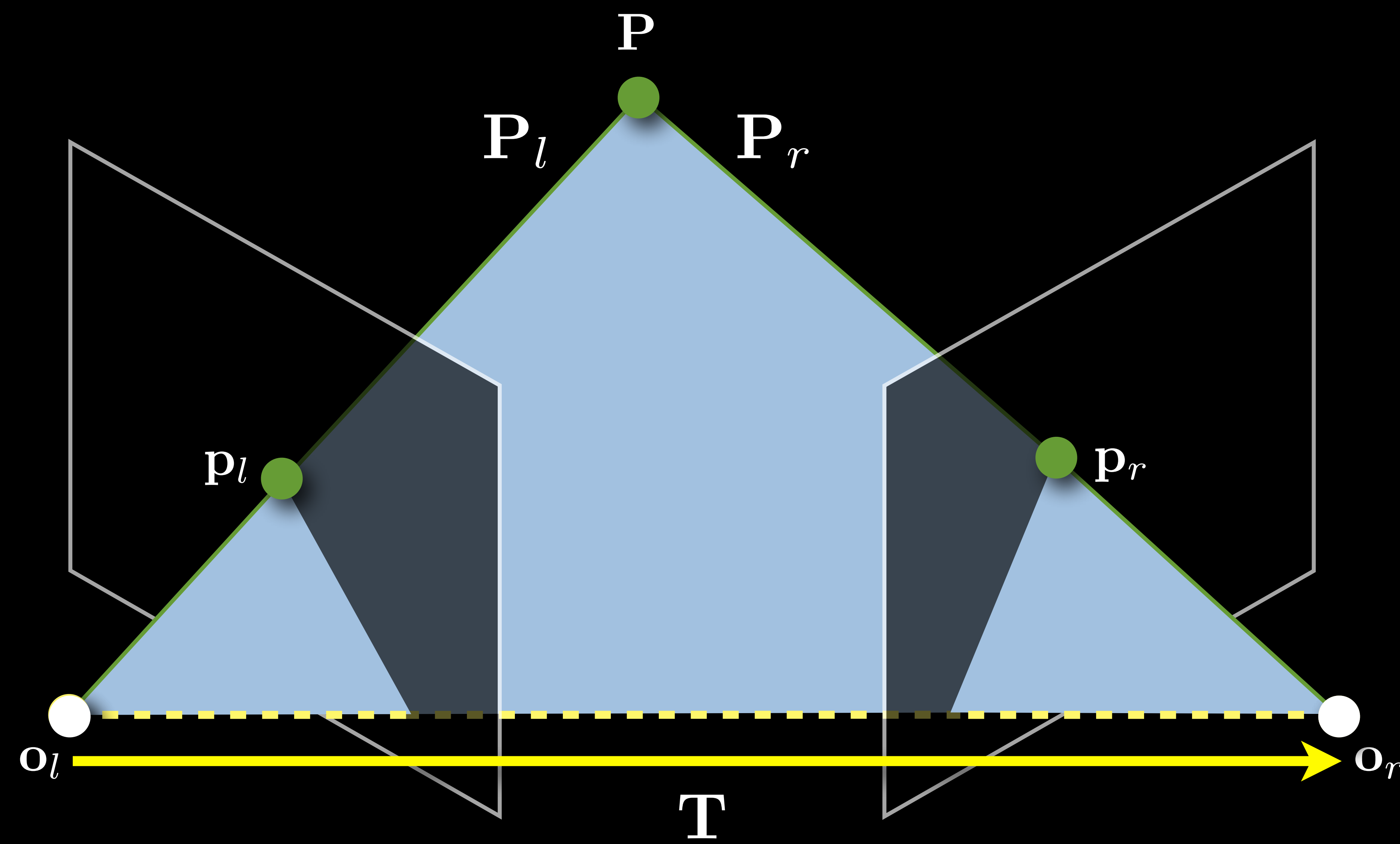


$$\mathbf{P}_r^\top \mathbf{E} \mathbf{P}_l = 0$$

$$\left(\frac{\mathbf{p}_r \cancel{Z_r}}{\cancel{f_r}}\right)^\top \mathbf{E} \left(\frac{\mathbf{p}_l \cancel{Z_l}}{\cancel{f_l}}\right) = 0$$

Recall: $\boxed{\mathbf{p} = f\frac{\mathbf{P}}{Z}}$

simplify



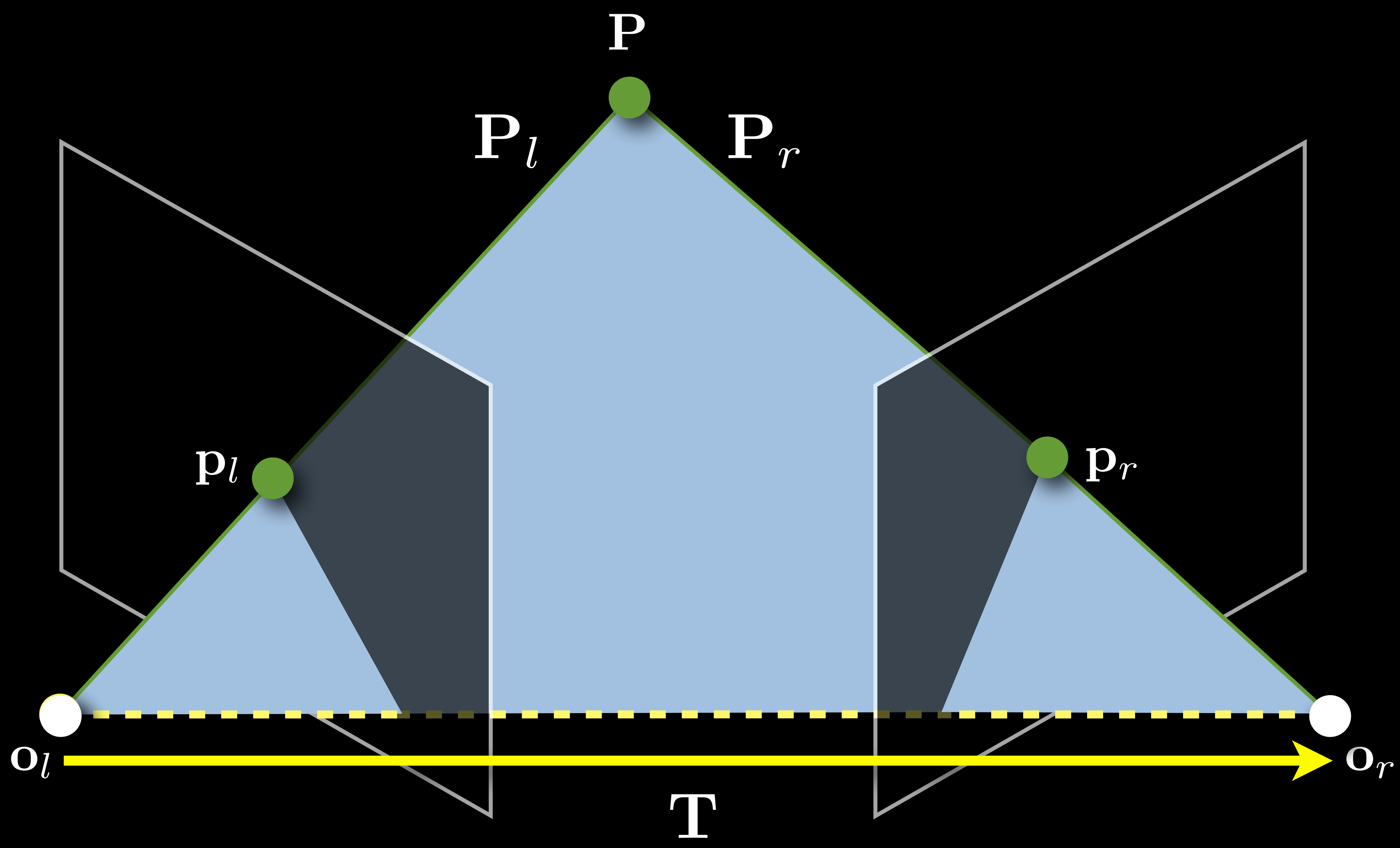
$$\mathbf{P}_r^\top \mathbf{E} \mathbf{P}_l = 0$$

$$\left(\frac{\mathbf{p}_r \cancel{Z_r}}{\cancel{f_r}}\right)^\top \mathbf{E} \left(\frac{\mathbf{p}_l \cancel{Z_l}}{\cancel{f_l}}\right) = 0$$

$$\mathbf{p}_r^\top \mathbf{E} \mathbf{p}_l = 0$$

Recall: $\mathbf{p} = f\dfrac{\mathbf{P}}{Z}$

simplify



$$\mathbf{p}_r^\top \mathbf{E} \mathbf{p}_l = 0$$

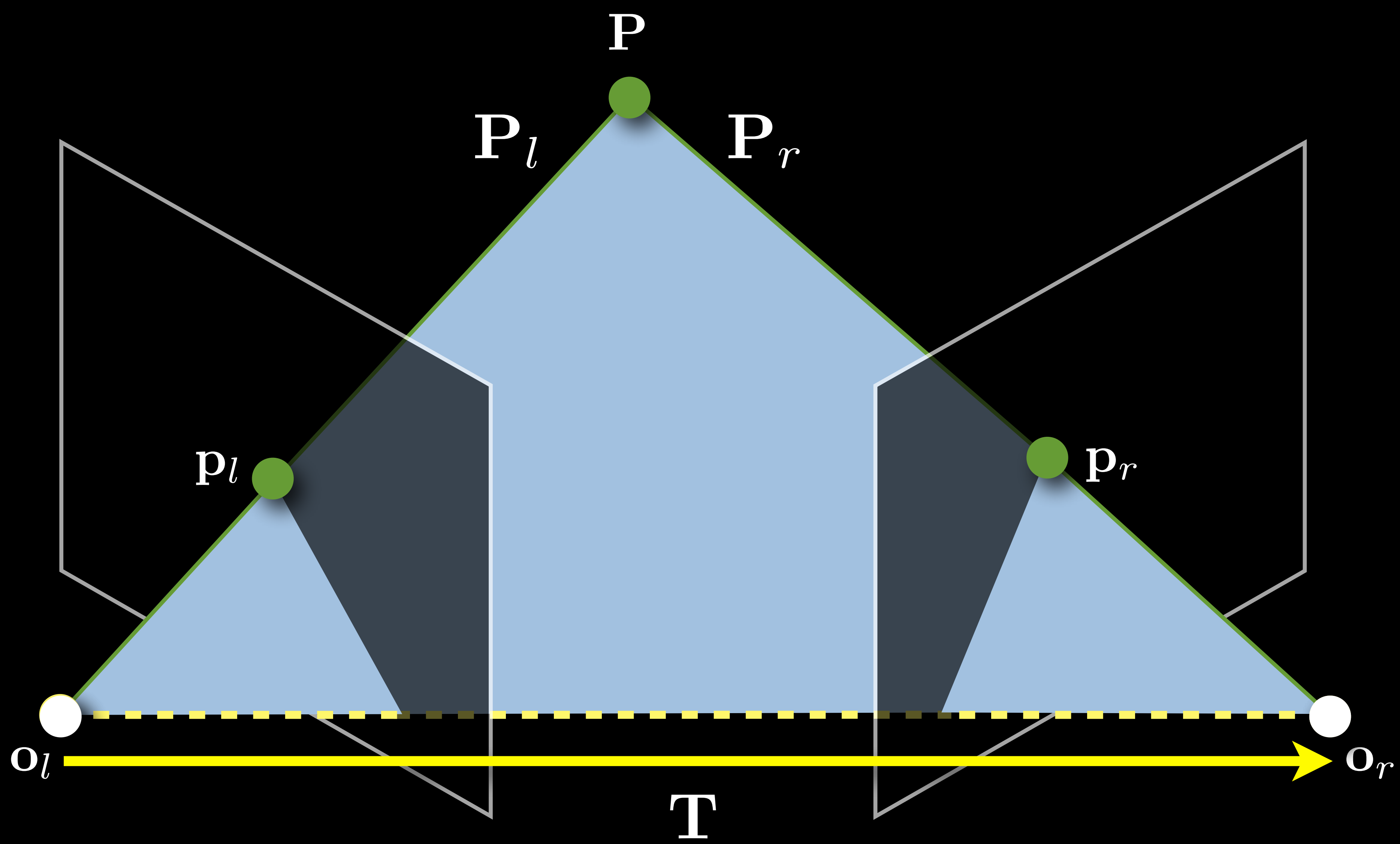
assumes that image points are measured in camera coordinates rather than pixel coordinates

$$\underbrace{\begin{pmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{pmatrix}}_{\substack{\textit{intrinsic matrix} \\ \mathbf{M}_{\text{int}}}} \underbrace{\begin{pmatrix} & -\mathbf{R}_1^\top \mathbf{T} \\ \mathbf{R} & -\mathbf{R}_2^\top \mathbf{T} \\ & -\mathbf{R}_3^\top \mathbf{T} \end{pmatrix}}_{\substack{\textit{extrinsic matrix} \\ \mathbf{M}_{\text{ext}}}} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

intrinsic matrix is invertible

$$\begin{pmatrix}\alpha x_{\mathrm{im}}\\ \alpha y_{\mathrm{im}}\\ \alpha\end{pmatrix} = \underbrace{\begin{pmatrix}-f/s_x & 0 & o_x\\ 0 & -f/s_y & o_y \\ 0 & 0 & 1\end{pmatrix}}_{\substack{\textit{intrinsic matrix}\\ \mathbf{M}_{\mathrm{int}}}} \underbrace{\begin{pmatrix} & -\mathbf{R}_1^\top\mathbf{T}\\ \mathbf{R} & -\mathbf{R}_2^\top \mathbf{T}\\ & -\mathbf{R}_3^\top\mathbf{T}\end{pmatrix}}_{\substack{\textit{extrinsic matrix}\\ \mathbf{M}_{\mathrm{ext}}}} \begin{pmatrix}X_w\\ Y_w\\ Z_w\\ 1\end{pmatrix}$$

divide by α

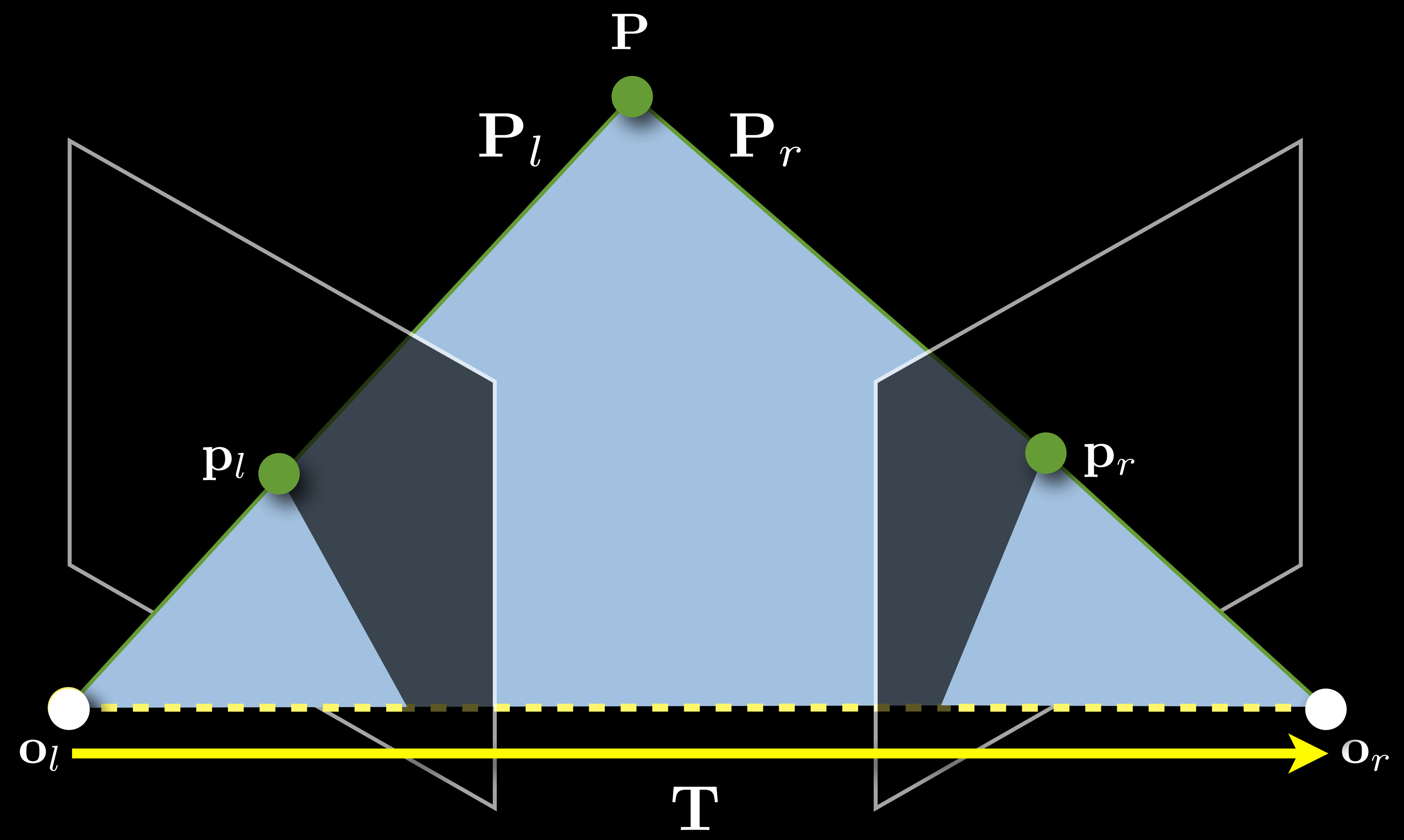


$$\mathbf{p}_r^\top \mathbf{E} \mathbf{p}_l = 0$$

We have:

$$\mathbf{p}_l = \mathrm{M}_{\text{int},l}^{-1} \tilde{\mathbf{p}}_l \qquad \mathbf{p}_r = \mathrm{M}_{\text{int},r}^{-1} \tilde{\mathbf{p}}_r$$

pixel coordinates



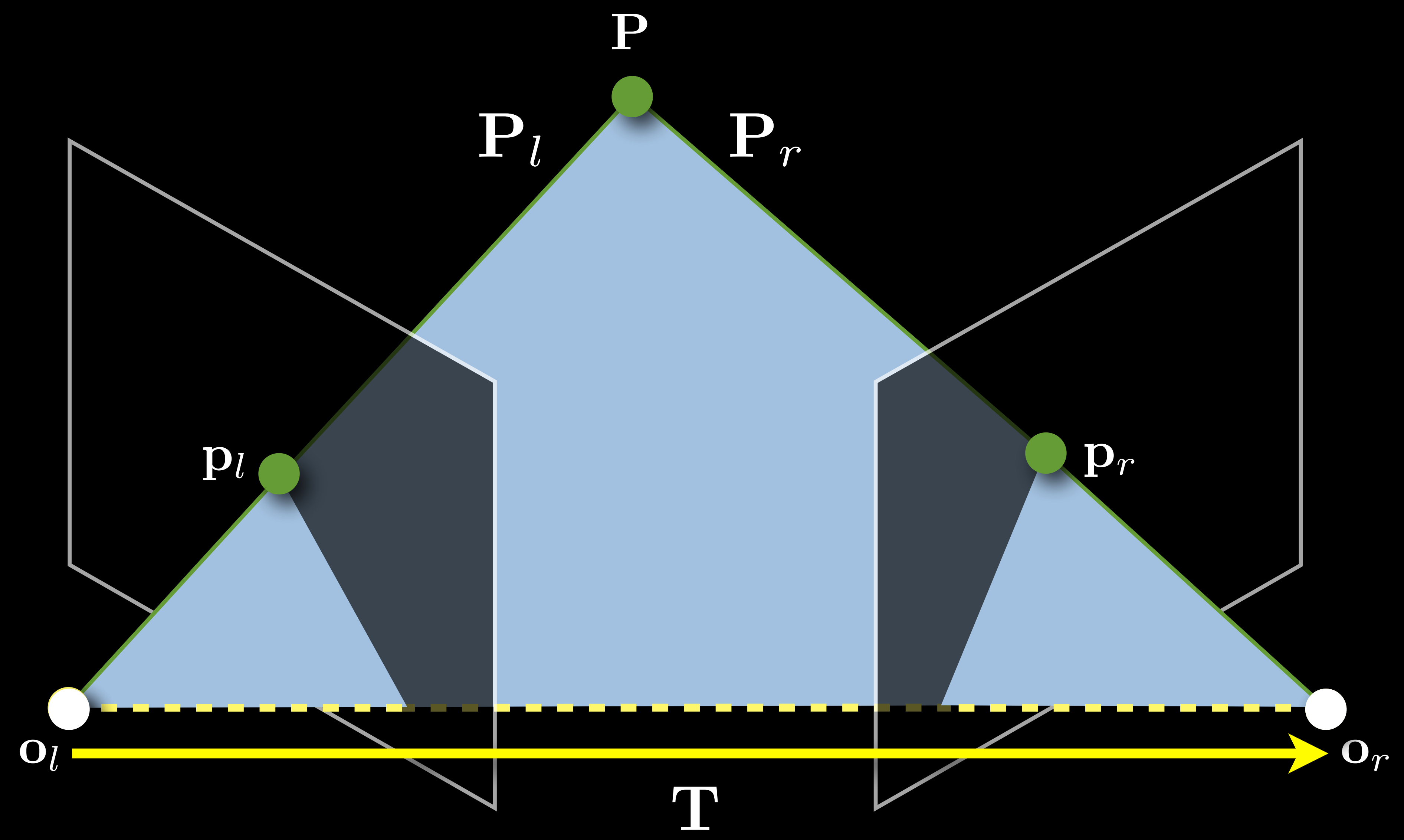
$$\mathbf{p}_r^\top \mathbf{E} \mathbf{p}_l = 0$$

We have:

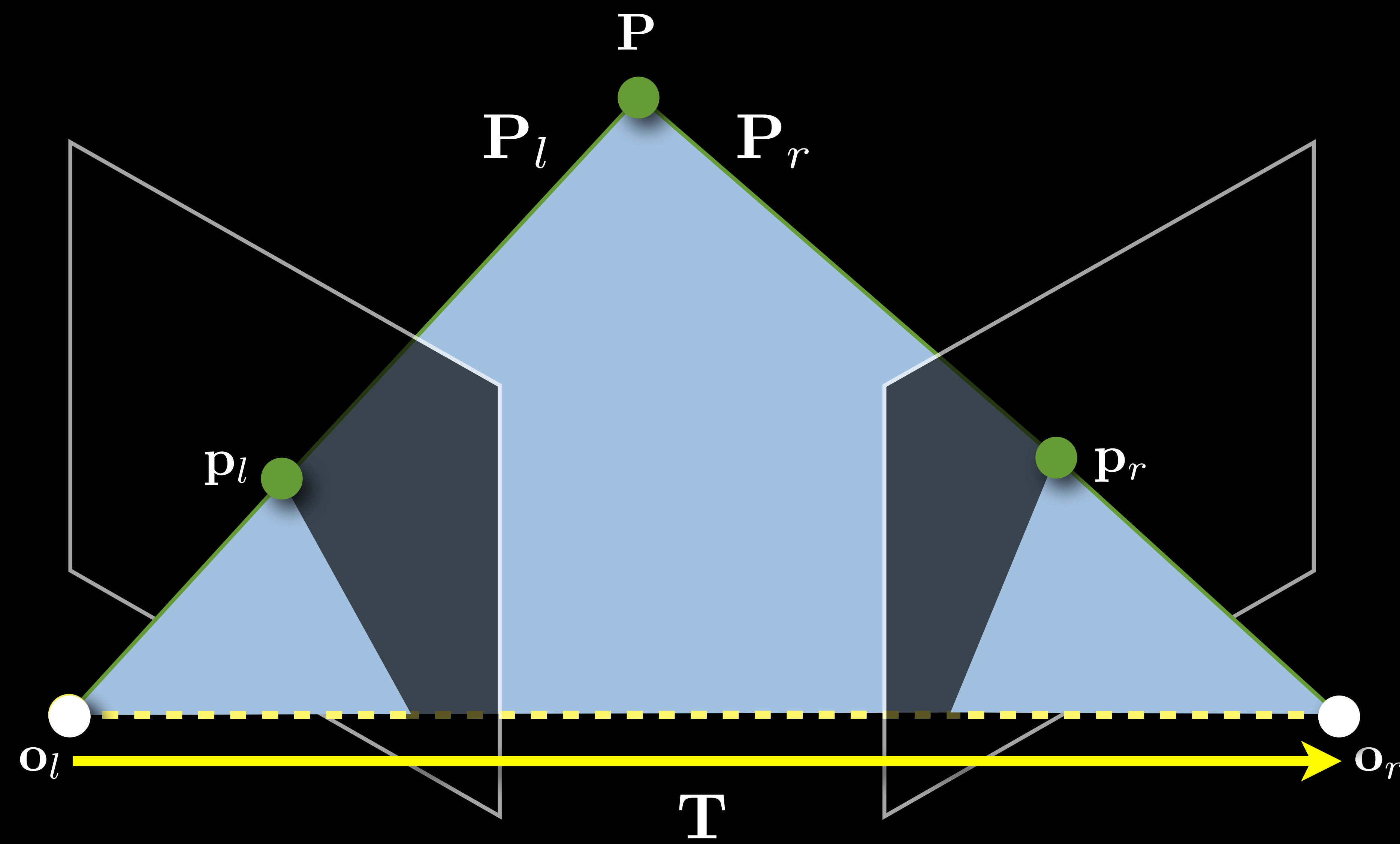
$$\mathbf{p}_l = \mathbf{M}_{\text{int},l}^{-1} \tilde{\mathbf{p}}_l \qquad \mathbf{p}_r = \mathbf{M}_{\text{int},r}^{-1} \tilde{\mathbf{p}}_r$$

substitute

$$\tilde{\mathbf{p}}_r^\top \mathbf{M}_{\text{int},r}^{-\top} \mathbf{E} \mathbf{M}_{\text{int},l}^{-1} \tilde{\mathbf{p}}_l = 0$$



$$\tilde{\mathbf{p}}_r^\top \mathbf{M}_{\text{int},r}^{-\top} \mathbf{E} \mathbf{M}_{\text{int},l}^{-1} \tilde{\mathbf{p}}_l = 0$$



$$\tilde{\mathbf{p}}_r^\top \mathbf{M}_{\text{int},r}^{-\top} \mathbf{E} \mathbf{M}_{\text{int},l}^{-1} \tilde{\mathbf{p}}_l = 0$$

Let $\mathbf{F} = \mathbf{M}_{\text{int},r}^{-\top} \mathbf{E} \mathbf{M}_{\text{int},l}^{-1}$

Fundamental Matrix Constraint $\tilde{\mathbf{p}}_r^\top \mathbf{F} \tilde{\mathbf{p}}_l = 0$

Fundamental Matrix Constraint

$$\tilde{\mathbf{p}}_r^\top \mathbf{F} \tilde{\mathbf{p}}_l = 0$$

Captures the two view projective geometry

Independent of scene structure

Can be computed from correspondences without knowledge of the camera intrinsics and extrinsics

Defined up to a scalar factor

Rank is 2

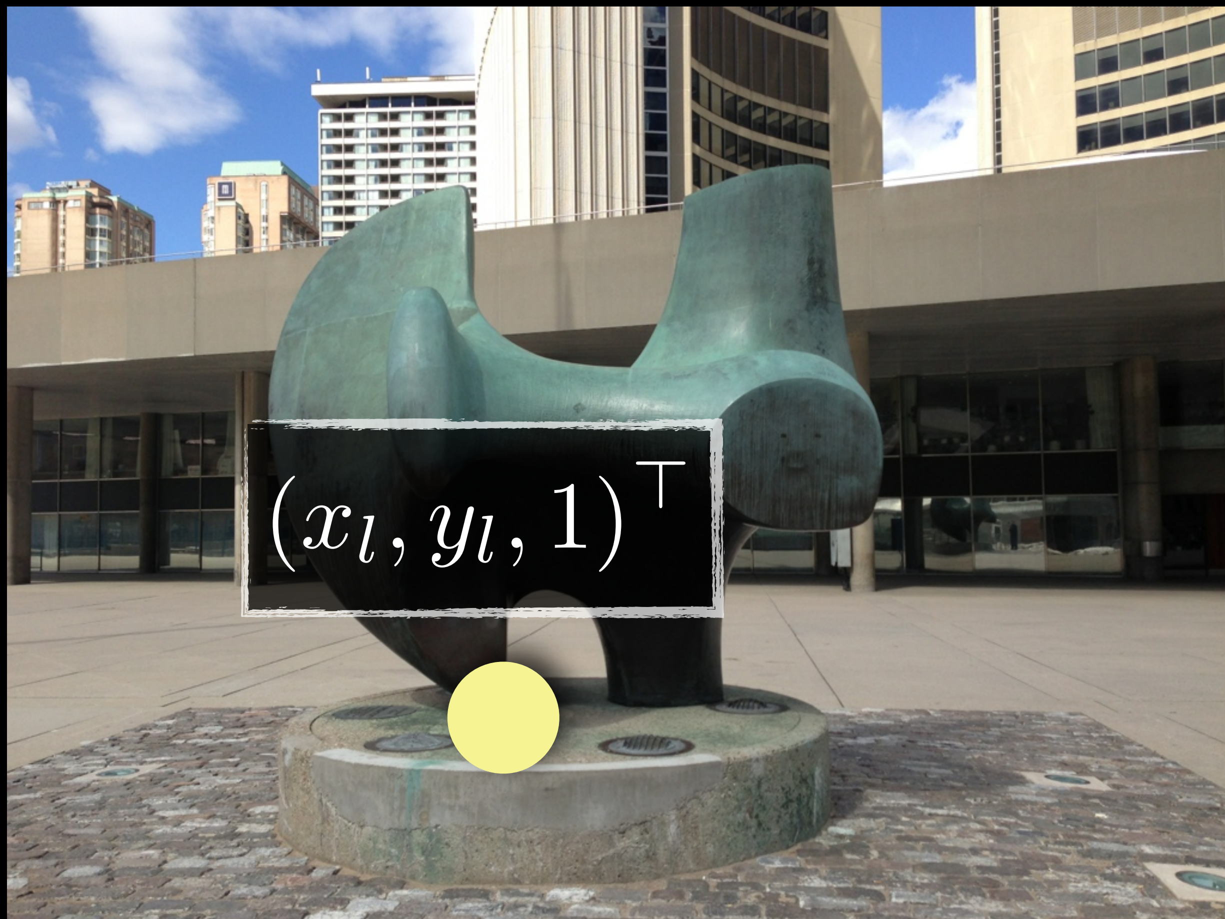
Fundamental Matrix Constraint

$$\tilde{\mathbf{p}}_r^\top \mathbf{F} \tilde{\mathbf{p}}_l = 0$$

Rank is 2

Epipoles are the left and right null vectors

$$\mathbf{F}\mathbf{e}_l = \mathbf{0} \text{ and } \mathbf{e}_r^\top \mathbf{F} = \mathbf{0}$$



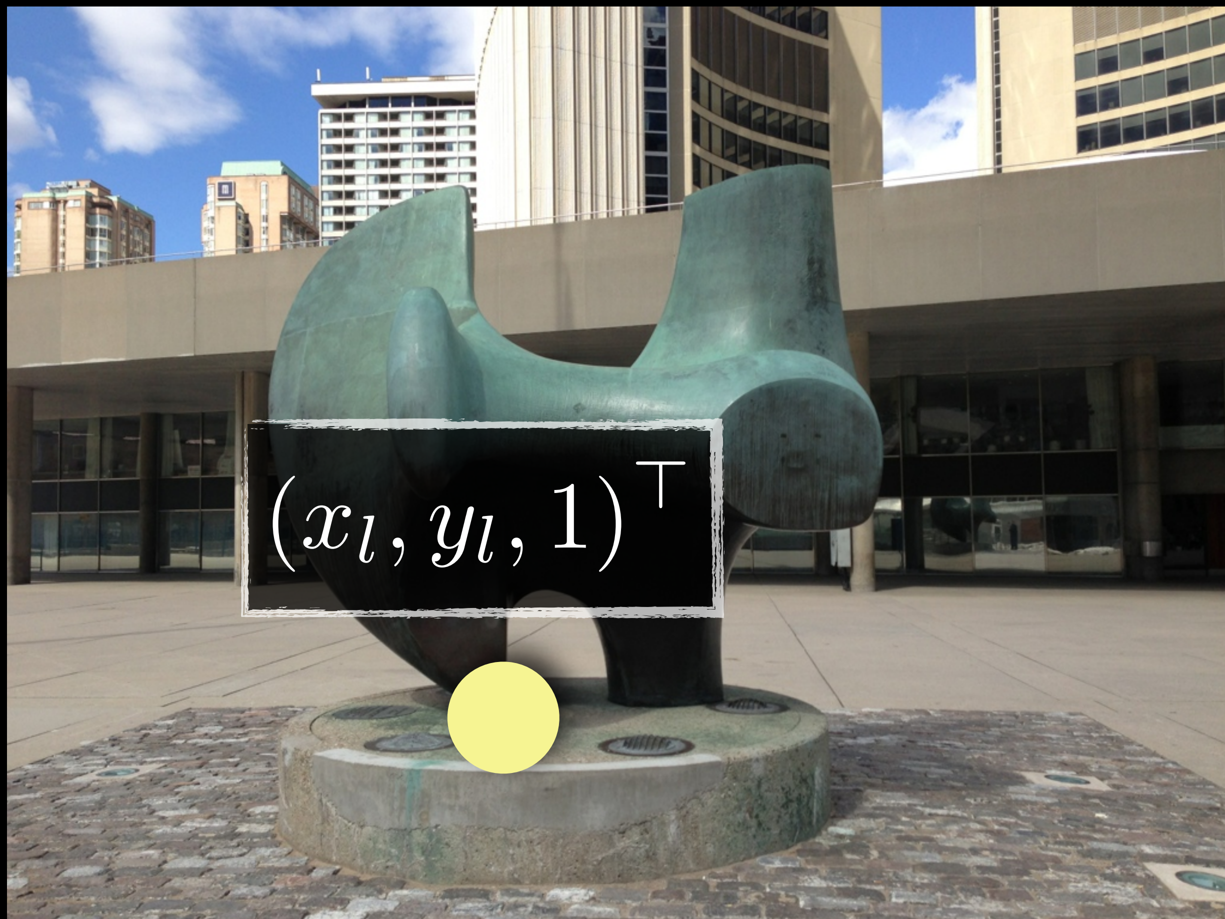
$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$\begin{pmatrix} x_r & y_r & 1 \end{pmatrix} \mathbf{F} \begin{pmatrix} x_l & y_l & 1 \end{pmatrix}^\top = 0$$

Let $\mathbf{k} = \mathbf{F} \begin{pmatrix} x_l & y_l & 1 \end{pmatrix}^\top$

$$\begin{pmatrix} x_r & y_r & 1 \end{pmatrix} \mathbf{k} = 0$$

What does this constraint mean geometrically?



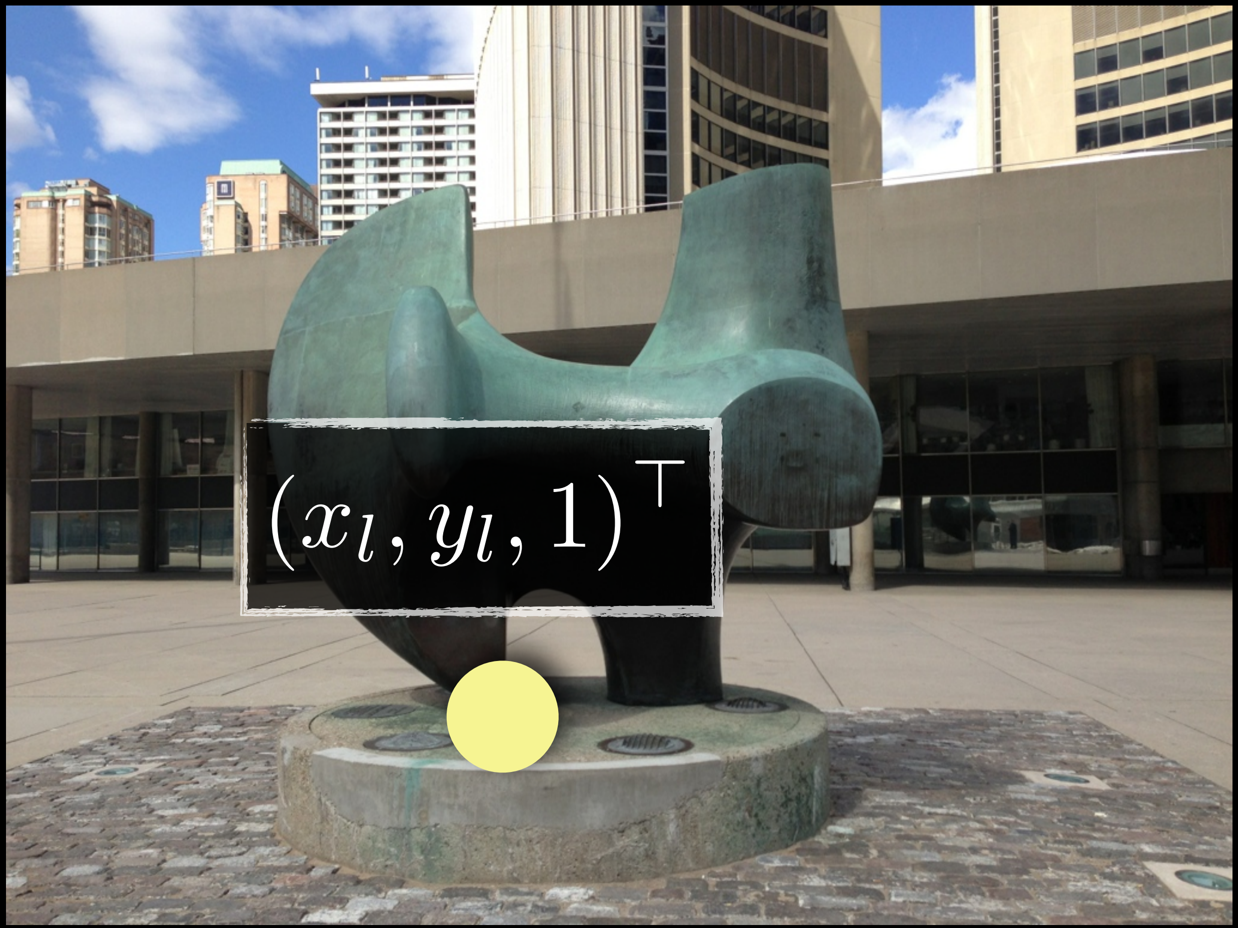
$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$\begin{pmatrix} x_r & y_r & 1 \end{pmatrix} \mathbf{F} \begin{pmatrix} x_l & y_l & 1 \end{pmatrix}^\top = 0$$

Let $\mathbf{k} = \mathbf{F} \begin{pmatrix} x_l & y_l & 1 \end{pmatrix}^\top$

$$\begin{pmatrix} x_r & y_r & 1 \end{pmatrix} \mathbf{k} = 0$$

search space reduced to a line



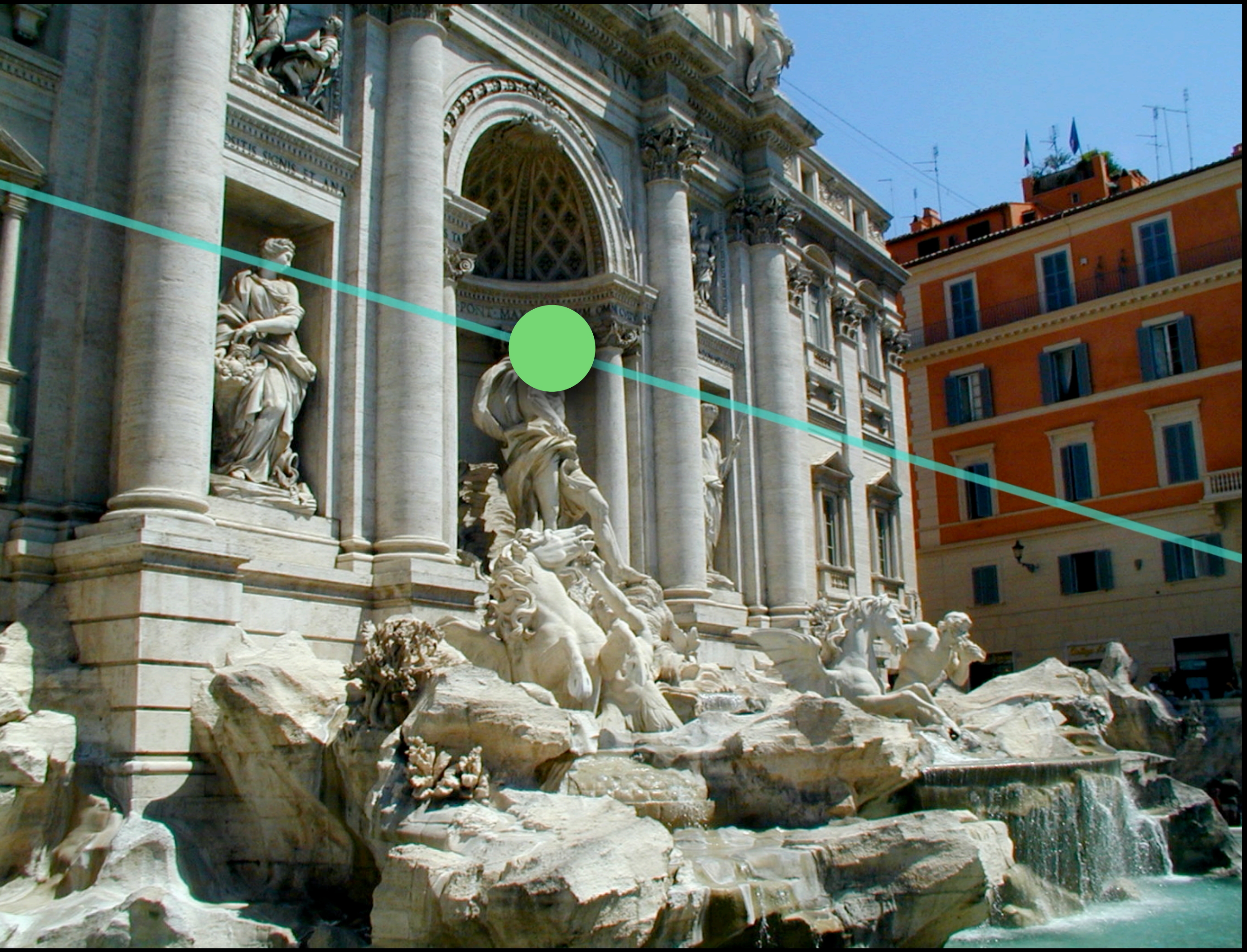
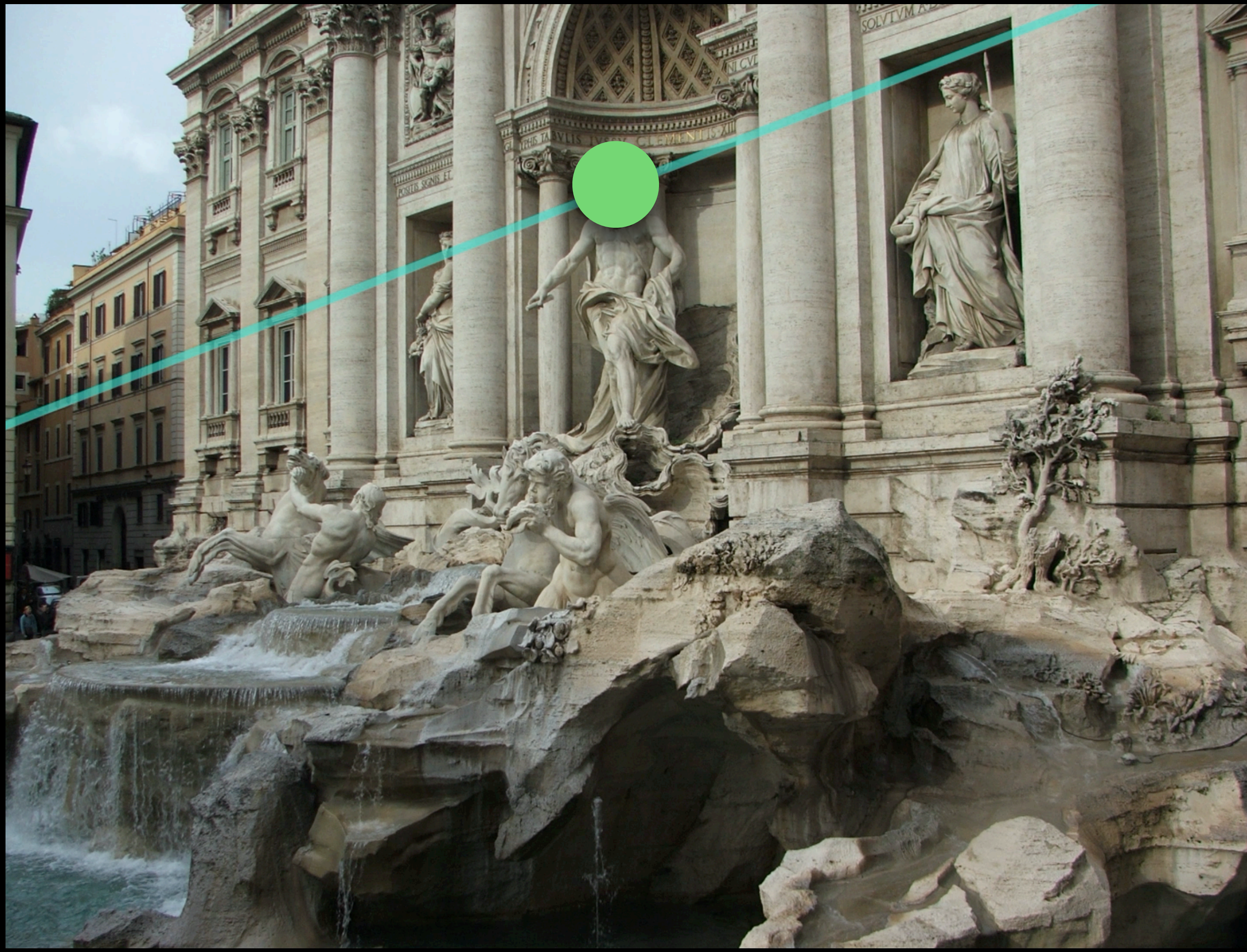
$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$\begin{pmatrix} x_r & y_r & 1 \end{pmatrix} \mathbf{F} \begin{pmatrix} x_l & y_l & 1 \end{pmatrix}^\top = 0$$

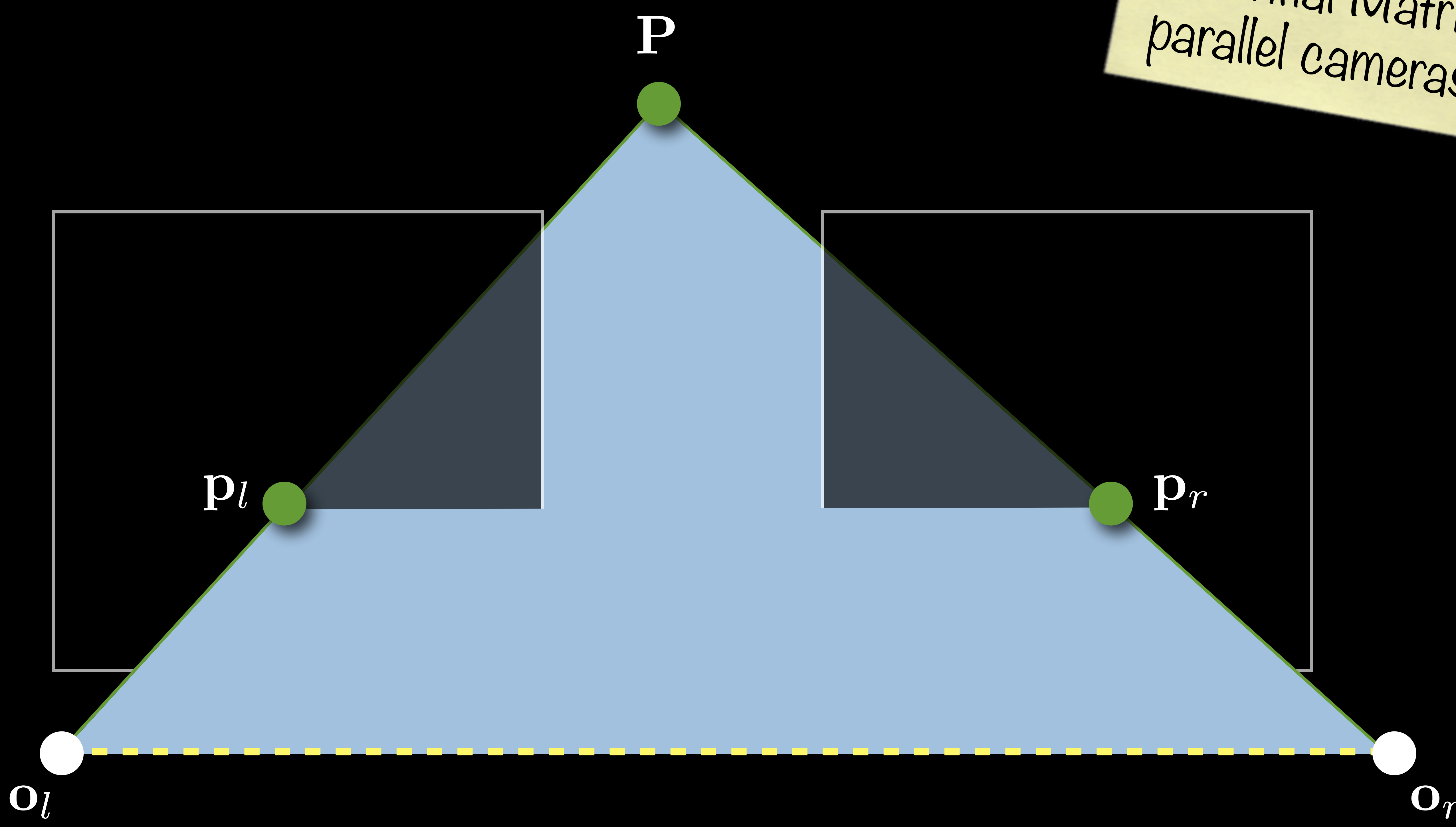
Let $\mathbf{k} = \mathbf{F} \begin{pmatrix} x_l & y_l & 1 \end{pmatrix}^\top$

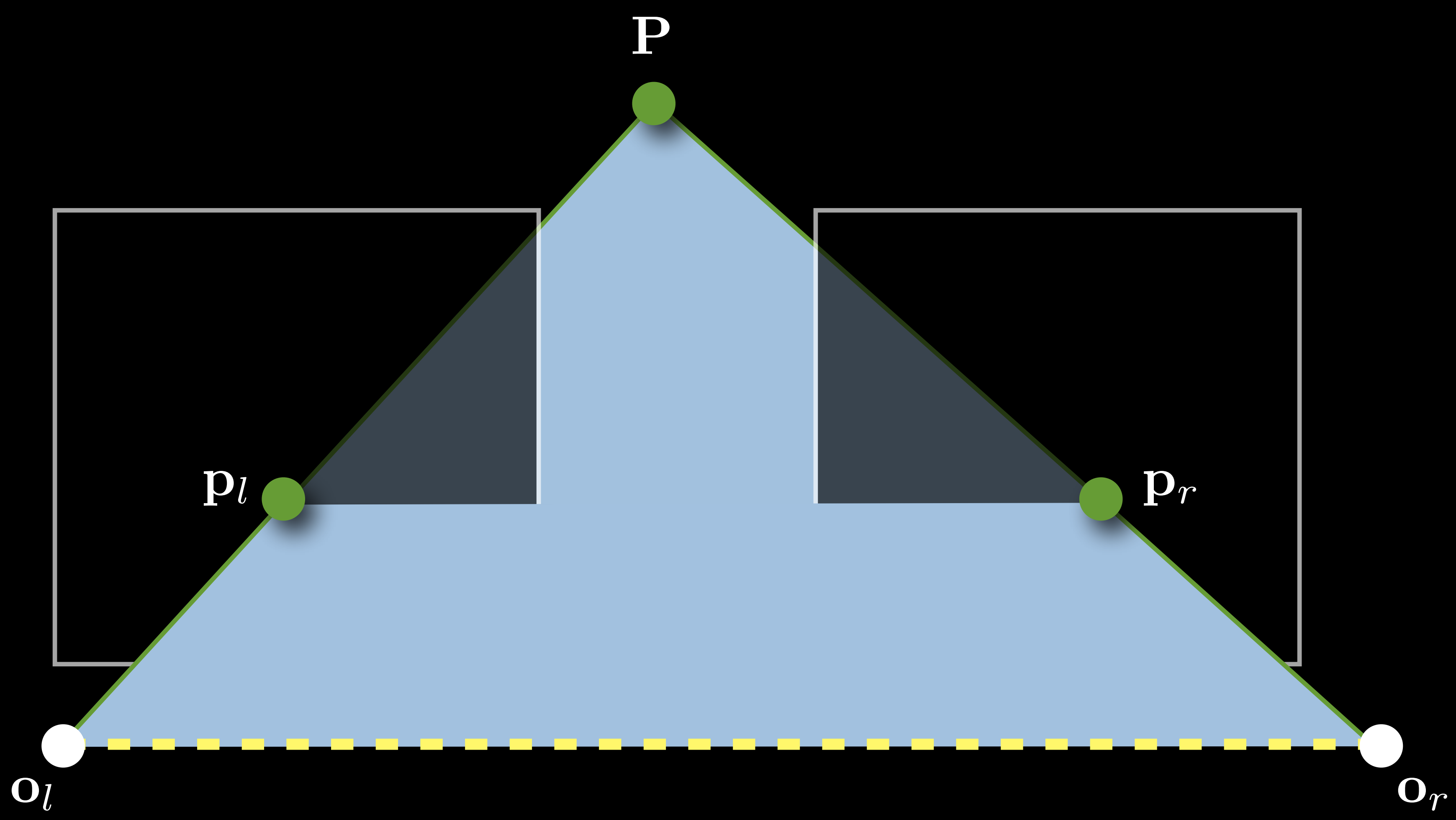
$$\begin{pmatrix} x_r & y_r & 1 \end{pmatrix} \mathbf{k} = 0$$

search space reduced to a line

Special case:
Essential Matrix
parallel cameras





$$\mathbf{R} = \mathbf{I}_{3\times 3}$$

$$\mathbf{T} = \begin{pmatrix} B & 0 & 0 \end{pmatrix}^\top$$

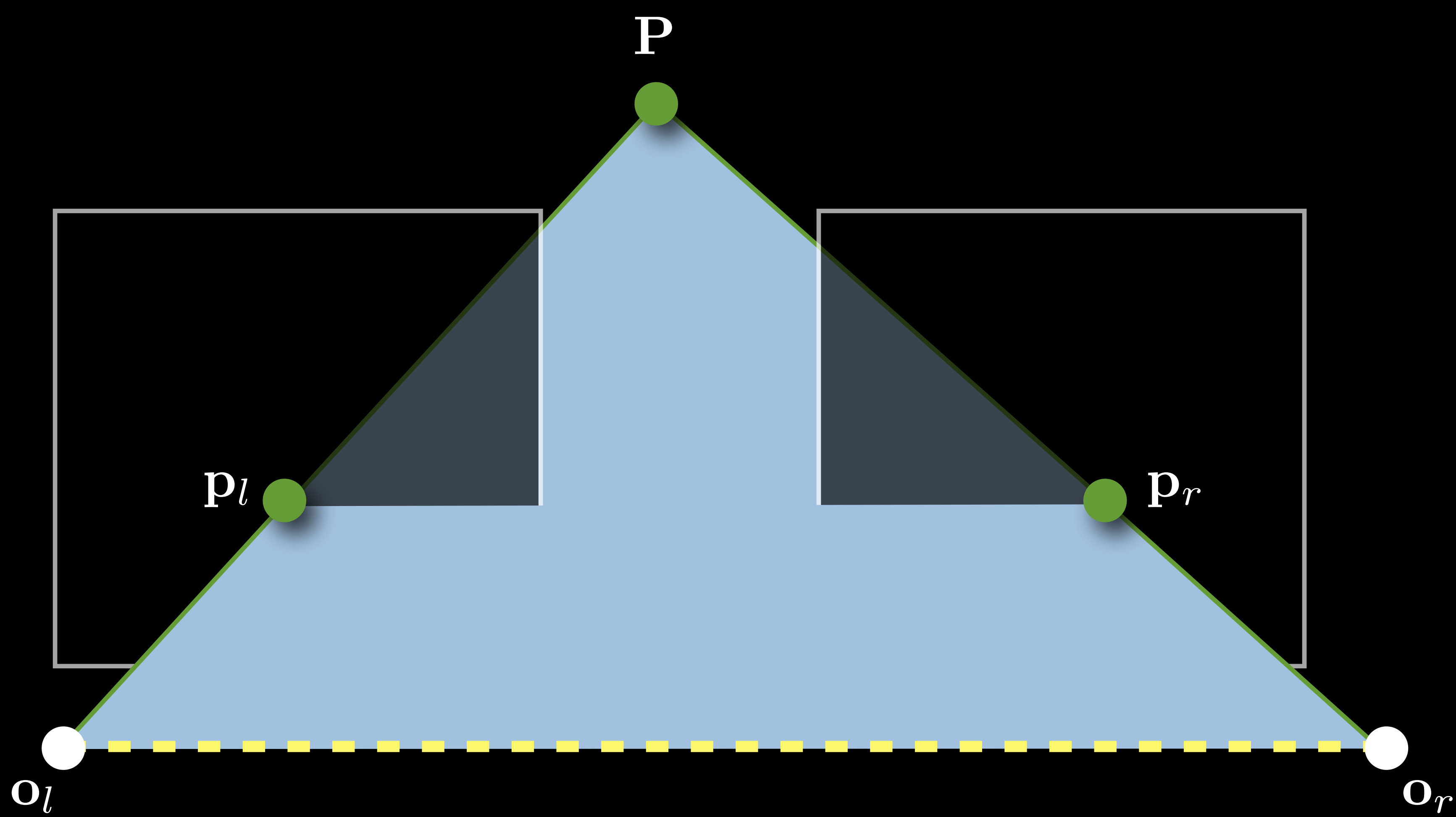
$$\mathbf{p}_r^\top \mathbf{E} \mathbf{p}_l = 0$$

$$\mathbf{p}_r^\top \mathbf{R} [\mathbf{T}_\times] \mathbf{p}_l = 0$$

$$\begin{pmatrix} x_r & y_r & 1 \end{pmatrix}^\top \cancel{\mathbf{R}} [\mathbf{T}_\times] \begin{pmatrix} x_l & y_l & 1 \end{pmatrix} = 0$$

simplify

$$\begin{pmatrix} x_r & y_r & 1 \end{pmatrix}^\top [\mathbf{T}_\times] \begin{pmatrix} x_l & y_l & 1 \end{pmatrix} = 0$$



$$\mathbf{R} = \mathbf{I}_{3\times 3}$$

$$\mathbf{T} = \begin{pmatrix} B & 0 & 0 \end{pmatrix}^\top$$

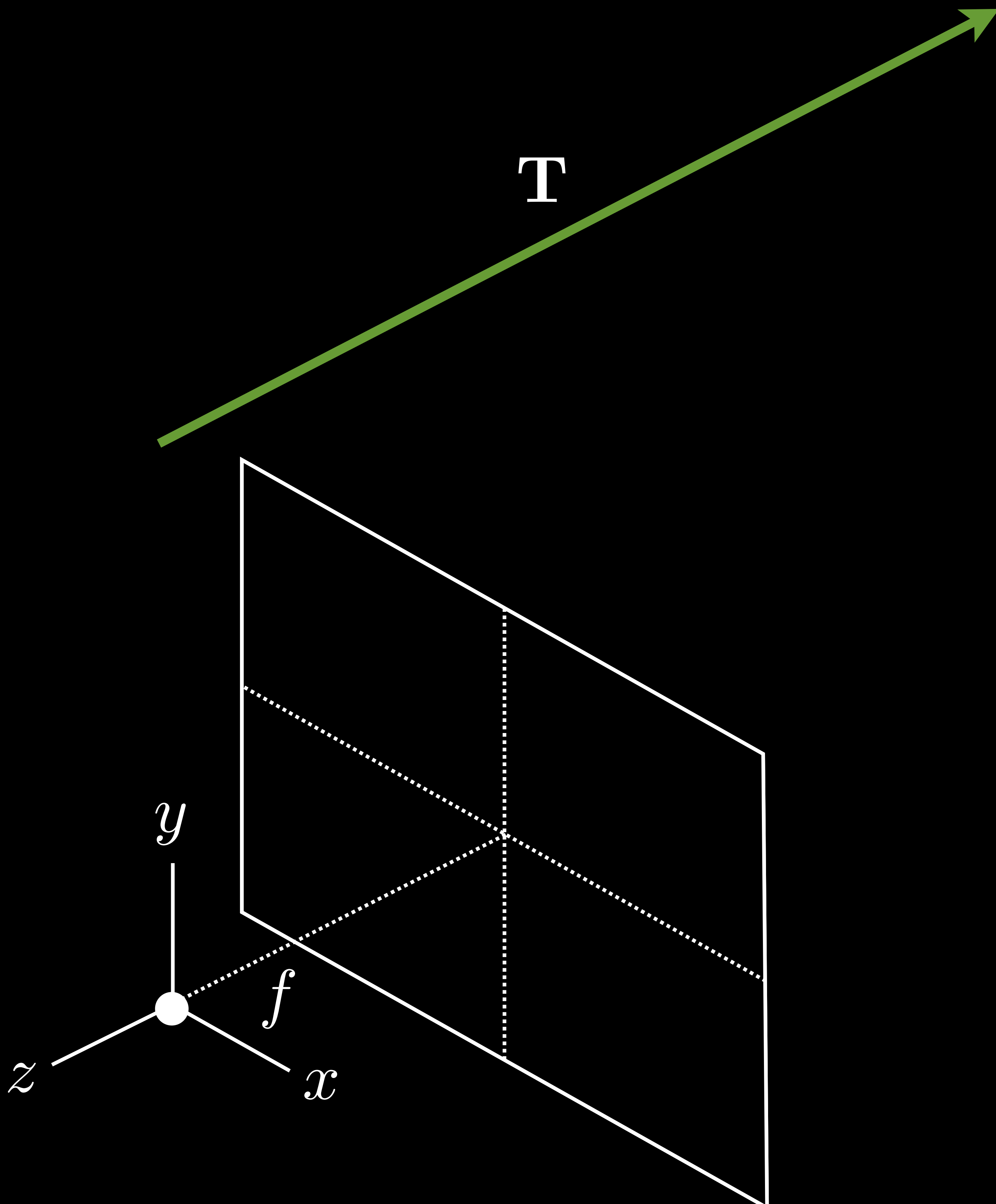
$$\mathbf{p}_r^\top \mathbf{E} \mathbf{p}_l = 0$$

$$\begin{pmatrix} x_r & y_r & 1 \end{pmatrix}^\top [\mathbf{T}_\times] \begin{pmatrix} x_l & y_l & 1 \end{pmatrix} = 0$$

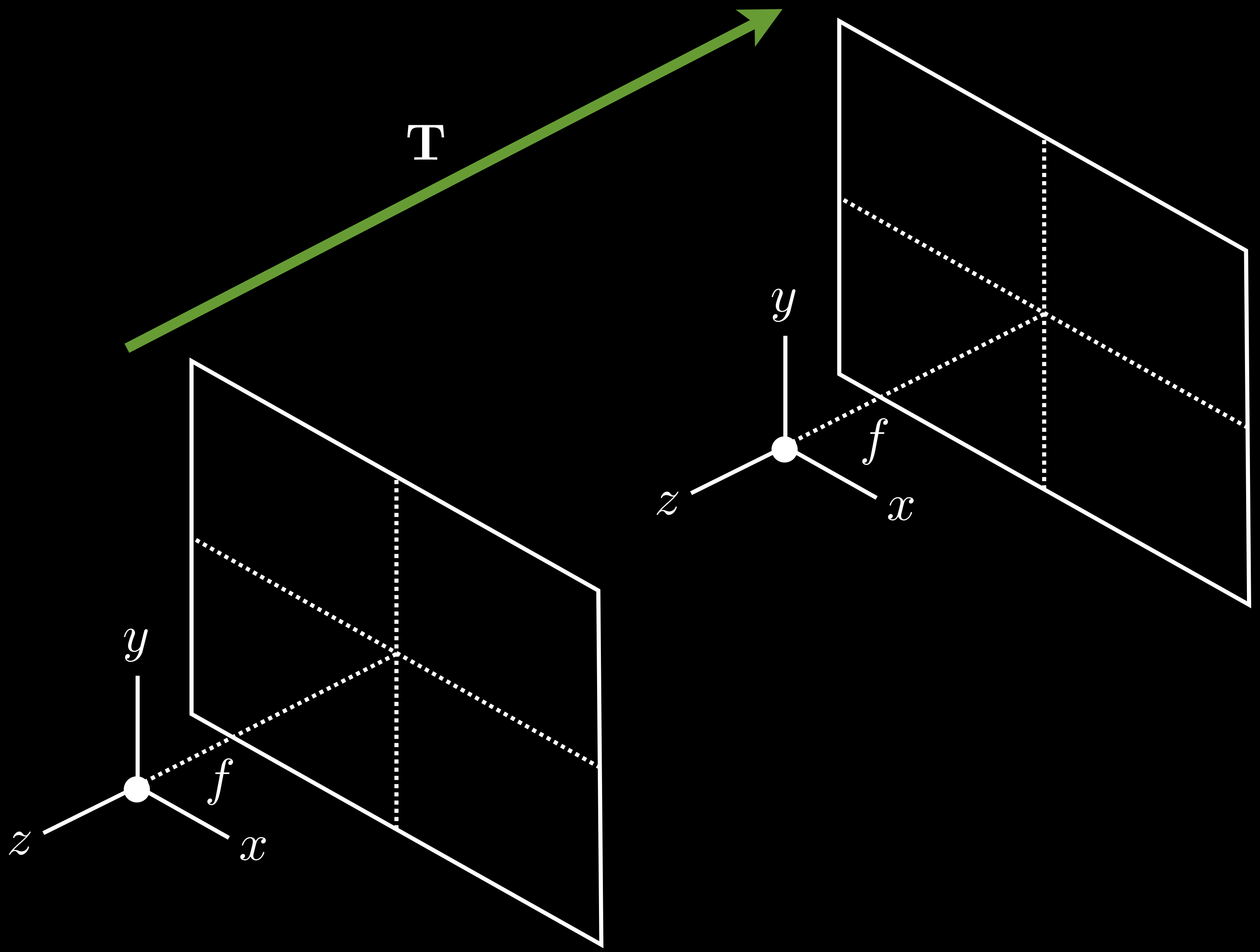
expand and simplify

$$y_r = y_l$$

corresponding points lie on same horizontal line



Special case:
Essential Matrix
forward motion

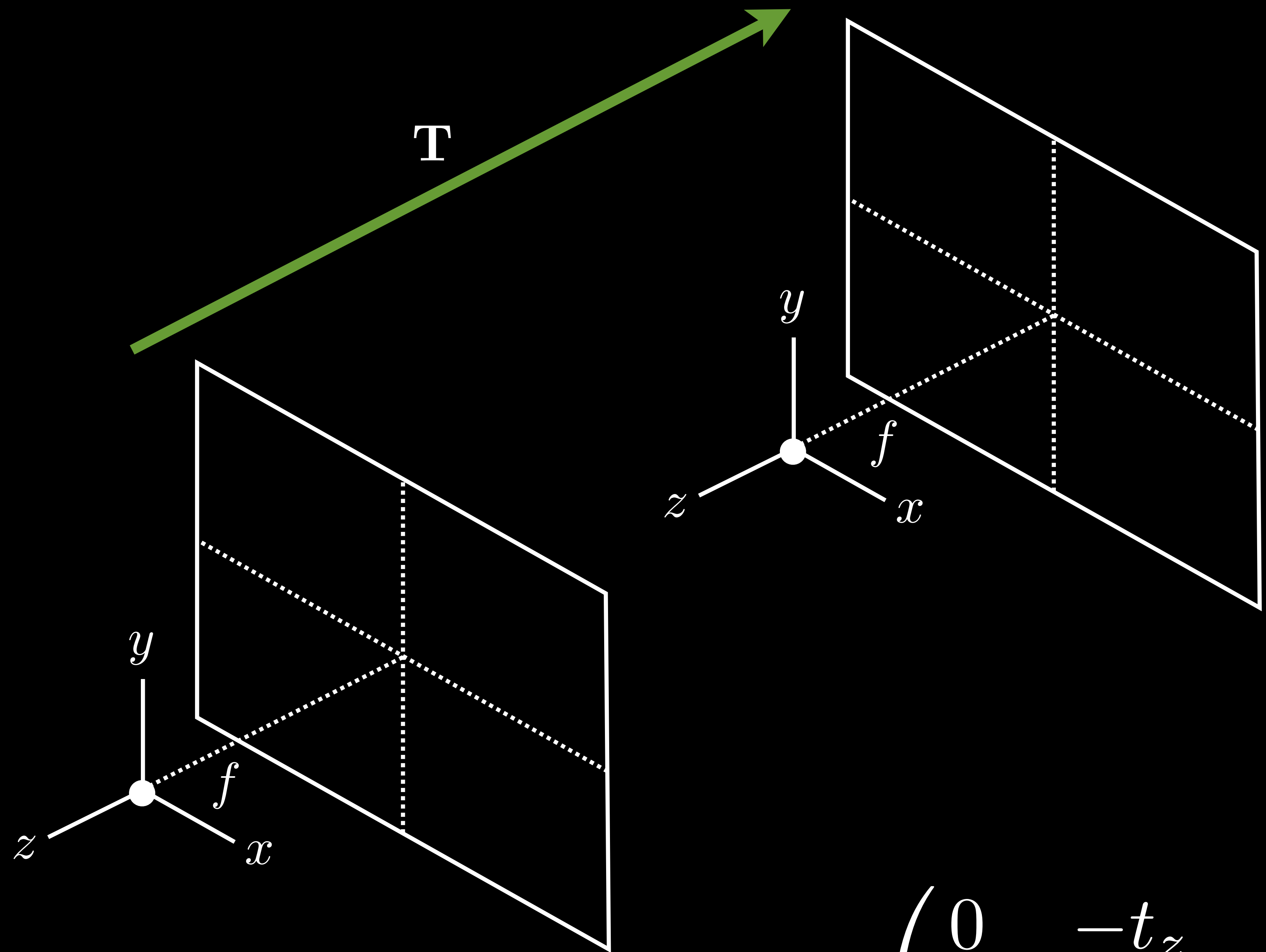


$$\mathbf{R} = \mathbf{I}_{3\times 3}$$

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & t_z \end{pmatrix}^\top$$

$$\mathbf{E} = [\mathbf{T}_\times]\mathbf{R}$$

$$= \begin{pmatrix} 0 & -t_z & 0 \\ t_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{I}_{3\times 3}$$

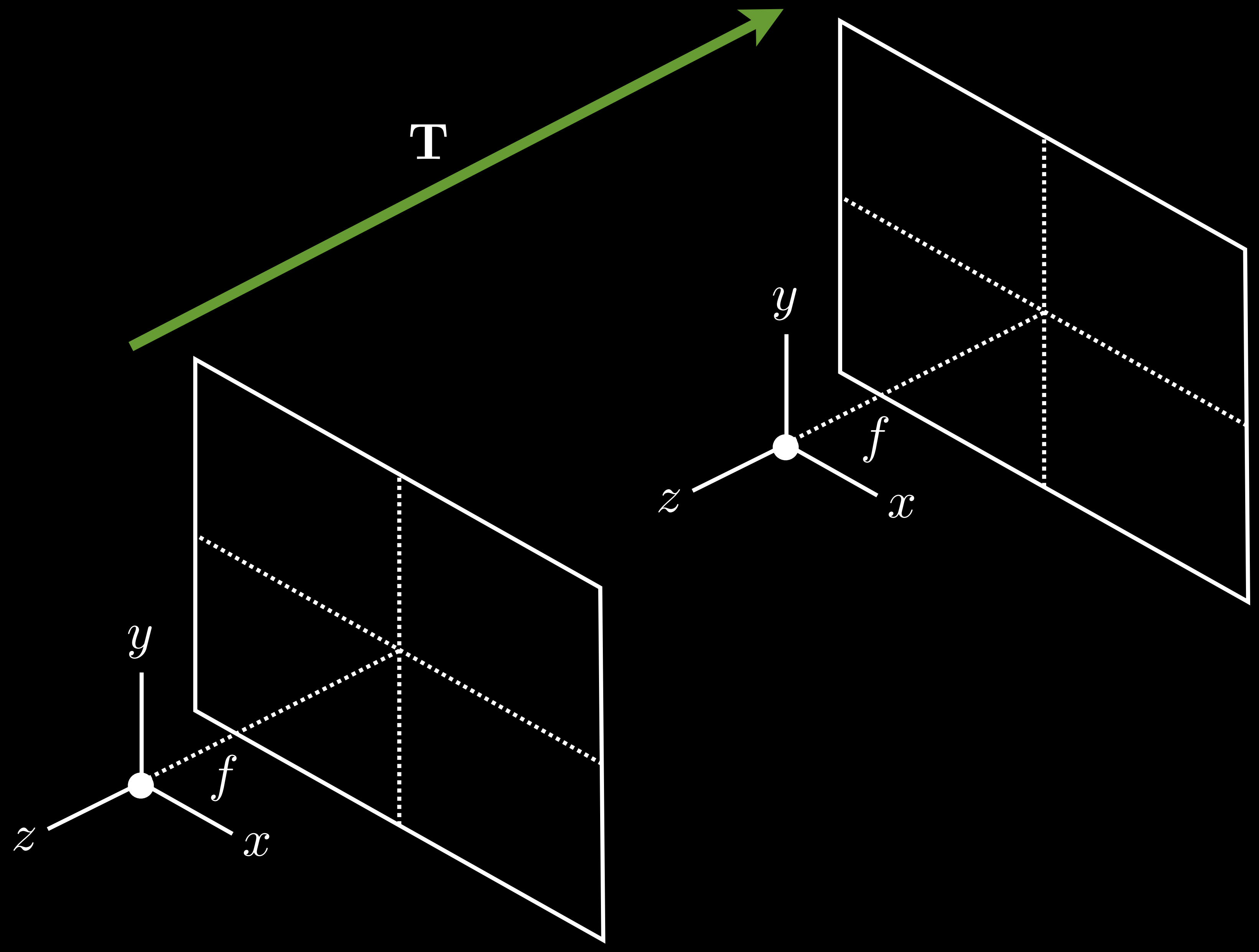


$$\mathbf{R} = \mathbf{I}_{3\times 3}$$

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & t_z \end{pmatrix}^\top$$

$$\mathbf{E} = \begin{pmatrix} 0 & -t_z & 0 \\ t_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Essential Matrix is defined up to a scalar multiple

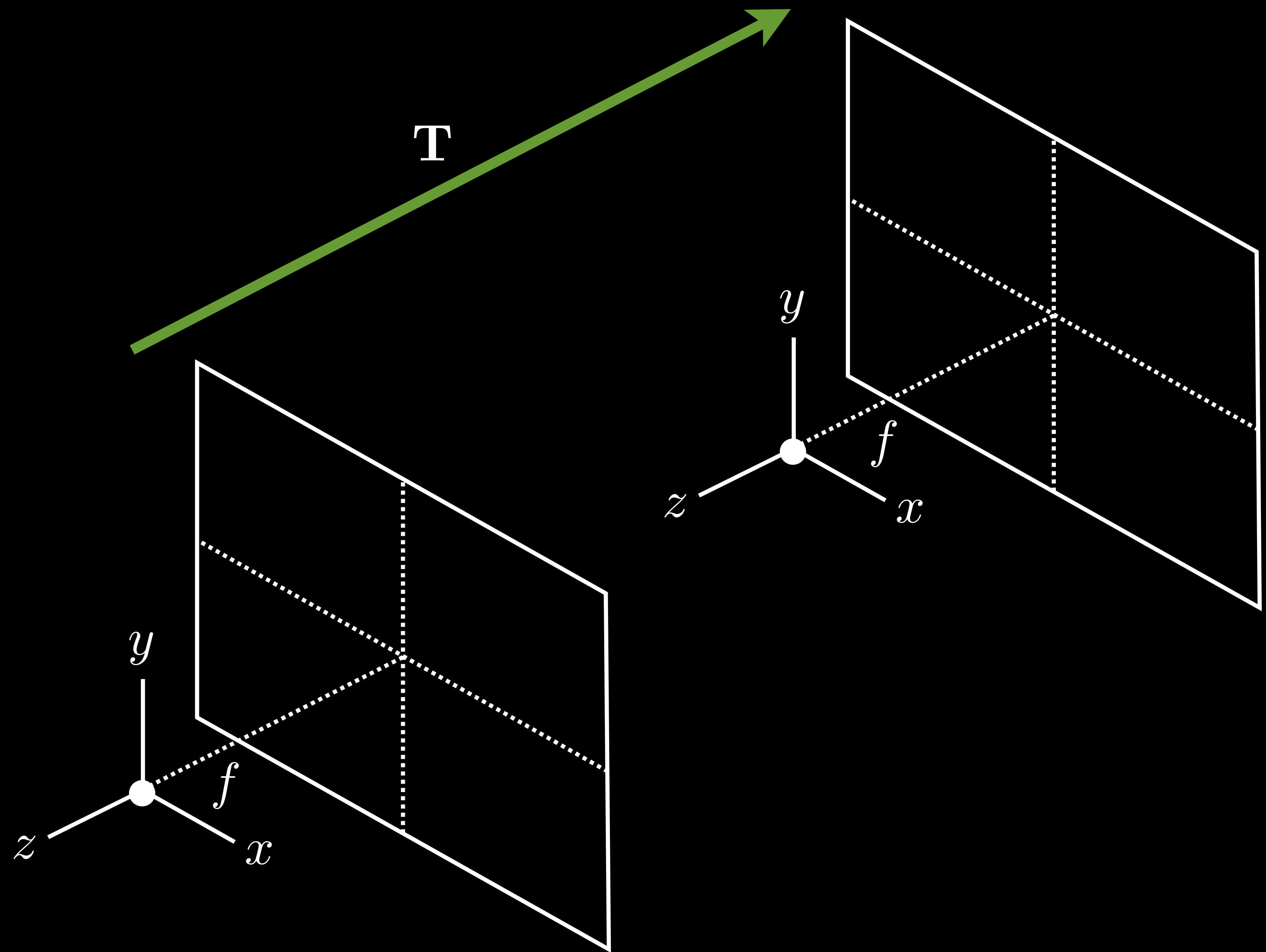


$$\mathbf{R} = \mathbf{I}_{3\times 3}$$

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & t_z \end{pmatrix}^\top$$

$$\mathbf{E} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{l}' = \mathbf{E}\mathbf{x}$$



$$\mathbf{R} = \mathbf{I}_{3\times 3}$$

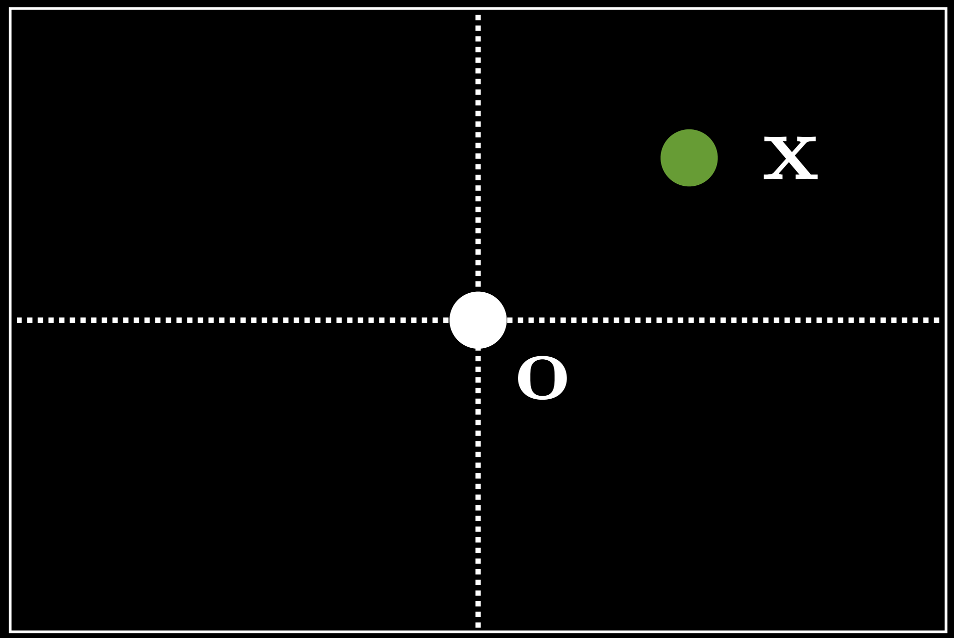
$$\mathbf{T} = \begin{pmatrix} 0 & 0 & t_z \end{pmatrix}^\top$$

$$\mathbf{E} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

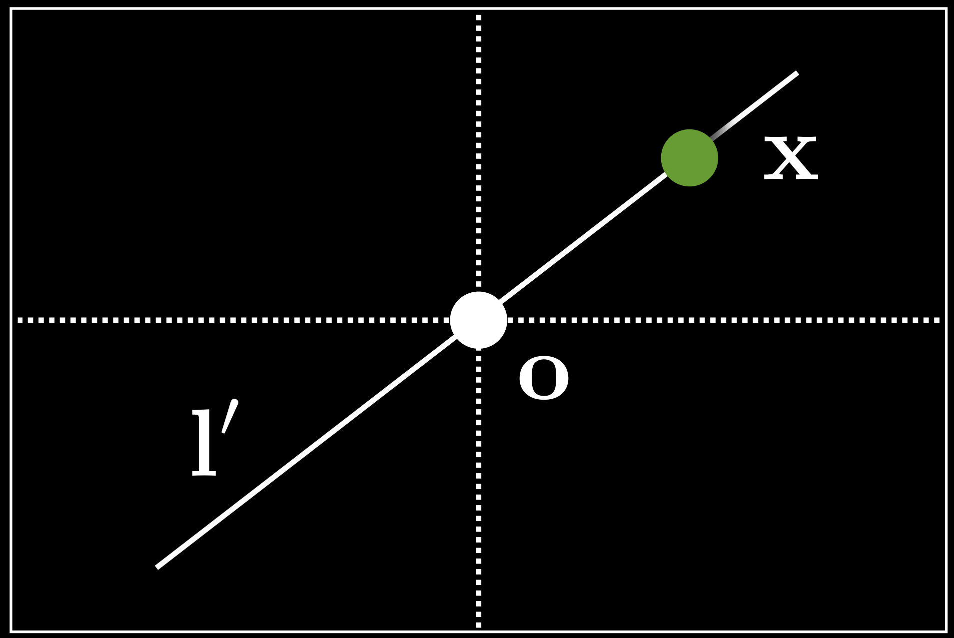
$$\mathbf{l}' = \mathbf{E}\mathbf{x} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

image origin is the epipole in both images

$$\mathbf{l}' = \mathbf{E}\mathbf{x} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$



initial image



second camera

points move along lines radiating from epipole

H vs. F

Homography

Fundamental

$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

Homography matrix maps a point to a point

$$\mathbf{l}' = \mathbf{F}\mathbf{x}$$

Fundamental Matrix maps a point to a line

Estimation

Fundamental Matrix

8-point algorithm

$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$\begin{pmatrix} x_r & y_r & 1 \end{pmatrix} \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} x_l \\ y_l \\ 1 \end{pmatrix} = 0$$

8-point algorithm

$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$\begin{pmatrix} x_r x_l & x_r y_l & x_r & y_r x_l & y_r y_l & y_r & x_l & y_l & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

one point correspondence across two views

8-point algorithm

$$\mathbf{p}_r^\top \mathbf{F}\mathbf{p}_l = 0$$

$$\begin{pmatrix} x_{r1}x_{l1} & x_{r1}y_{l1} & x_{r1} & y_{r1}x_{l1} & y_{r1}y_{l1} & y_{r1} & x_{l1} & y_{l1} & 1 \\ & & & & \vdots & & & & \\ x_{rn}x_{ln} & x_{rn}y_{ln} & x_{rn} & y_{rn}x_{ln} & y_{rn}y_{ln} & y_{rn} & x_{ln} & y_{ln} & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = \mathbf{0}$$

n **point correspondences**

$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$\begin{pmatrix} x_{r1}x_{l1} & x_{r1}y_{l1} & x_{r1} & y_{r1}x_{l1} & y_{r1}y_{l1} & y_{r1} & x_{l1} & y_{l1} & 1 \\ & & & & \vdots & & & & \\ x_{rn}x_{ln} & x_{rn}y_{ln} & x_{rn} & y_{rn}x_{ln} & y_{rn}y_{ln} & y_{rn} & x_{ln} & y_{ln} & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = \mathbf{0}$$

matrix rank is 8

eight point correspondences yield a non-zero solution

solution is null vector

$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$\begin{pmatrix} x_{r1}x_{l1} & x_{r1}y_{l1} & x_{r1} & y_{r1}x_{l1} & y_{r1}y_{l1} & y_{r1} & x_{l1} & y_{l1} & 1 \\ & & & & \vdots & & & & \\ x_{rn}x_{ln} & x_{rn}y_{ln} & x_{rn} & y_{rn}x_{ln} & y_{rn}y_{ln} & y_{rn} & x_{ln} & y_{ln} & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = \mathbf{0}$$

matrix rank is 8

use more than eight points and solve with homogeneous least-squares

2

common solutions

1

Lagrange Multipliers

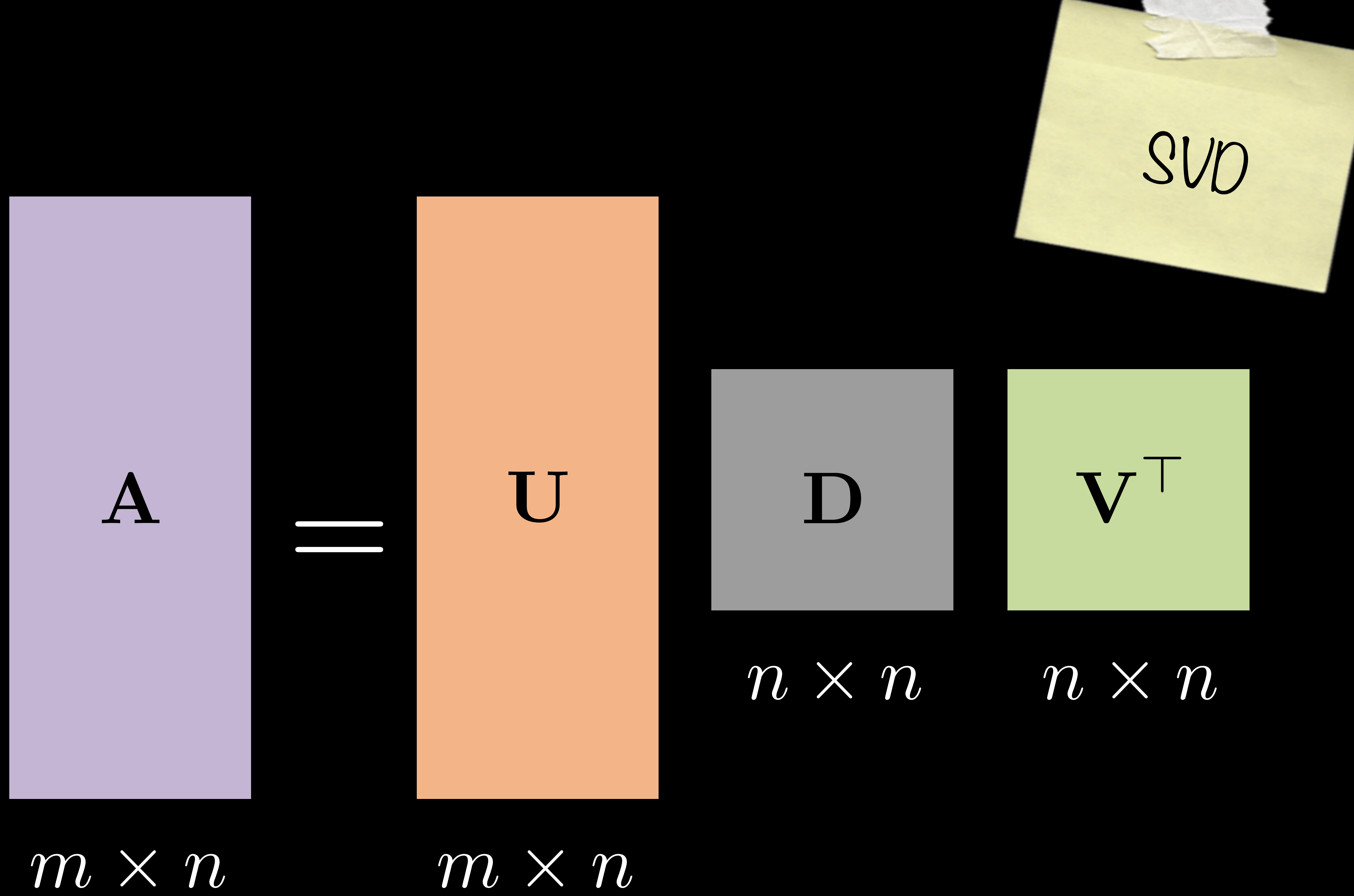
2

Singular Value Decomposition

SVD

Singular Value Decomposition

Review: SVD


SVD
$\mathbf{A} = \mathbf{U}\,\mathbf{D}\,\mathbf{V}^\top$
$m \times n$
$m \times n$
$n \times n$
$n \times n$

Definition: For any given matrix $\mathrm{A} \in \mathbb{R}^{m \times n}$ its Singular Value Decomposition (SVD) is defined as

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$$

such that

$\mathbf{U}$ is an $m \times n$ matrix with orthogonal columns

$\mathbf{V}^\top$ is an $n \times n$ orthogonal matrix

$\mathbf{D}$ is an $n \times n$ matrix with non-negative entries, termed the singular values

assume diagonal values of D are sorted in descending order,

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$$

SVD for Optimization

Find a vector $\mathbf{x}$ such that it minimizes

$$\|\mathbf{A}\mathbf{x}\|^2$$

SVD for Optimization

Find a vector $\mathbf{x}$ such that it minimizes

$$\|\mathbf{A}\mathbf{x}\|^2$$

subject to the constraint

$$\|\mathbf{x}\| = 1$$

Homogeneous least squares

SVD for Optimization

Find a vector $\mathbf{x}$ such that it minimizes

$$\|\mathbf{A}\mathbf{x}\|^2$$

subject to the constraint

$$\|\mathbf{x}\| = 1$$

Why do we need the unit vector constraint?

avoid trivial solution

Find a vector $\mathbf{x}$ such that it minimizes

$$\|\mathbf{A}\mathbf{x}\|^2$$

subject to the constraint

$$\|\mathbf{x}\| = 1$$

$$\begin{aligned}\|\mathbf{A}\mathbf{x}\|^2 &= \|\mathbf{U}\mathbf{D}\mathbf{V}^\top\mathbf{x}\|^2 \\ &= (\mathbf{U}\mathbf{D}\mathbf{V}^\top\mathbf{x})^\top(\mathbf{U}\mathbf{D}\mathbf{V}^\top\mathbf{x}) \\ &= (\mathbf{x}^\top\mathbf{V}\mathbf{D}\mathbf{U}^\top)(\mathbf{U}\mathbf{D}\mathbf{V}^\top\mathbf{x})\end{aligned}$$

What does this product evaluate to?

identity matrix

Find a vector x such that it minimizes

$$\|\mathbf{A}\mathbf{x}\|^2$$

subject to the constraint

$$\|\mathbf{x}\| = 1$$

$$\|\mathbf{A}\mathbf{x}\|^2 = \mathbf{x}^\top \mathbf{V}\mathbf{D}\mathbf{D}\mathbf{V}^\top \mathbf{x}$$

Find a vector $\mathbf{x}$ such that it minimizes

$$\|\mathbf{A}\mathbf{x}\|^2$$

subject to the constraint

$$\|\mathbf{x}\| = 1$$

$$\|\mathbf{A}\mathbf{x}\|^2 = \mathbf{x}^\top \mathbf{V}\mathbf{D}\mathbf{D}\mathbf{V}^\top \mathbf{x}$$

Let $\mathbf{y} = \mathbf{V}^\top \mathbf{x}$

Now we minimize

$$\mathbf{y}^\top \mathbf{D}\mathbf{D}\mathbf{y}$$

subject to the constraint

$$\|\mathbf{y}\| = 1$$

$$\boxed{\|\mathbf{y}\| = \|\mathbf{V}^\top \mathbf{x}\| = \|\mathbf{x}\| = 1}$$

Find a vector $\mathbf{x}$ such that it minimizes

$$\|\mathbf{A}\mathbf{x}\|^2$$

subject to the constraint

$$\|\mathbf{x}\| = 1$$

Now we minimize

$$\mathbf{y}^\top \mathbf{D}\mathbf{D}\mathbf{y}$$

subject to the constraint

$$\|\mathbf{y}\| = 1$$

Find a vector $\mathbf{x}$ such that it minimizes

$$\|\mathbf{A}\mathbf{x}\|^2$$

subject to the constraint

$$\|\mathbf{x}\| = 1$$

Now we minimize

$$\mathbf{y}^\top \mathbf{D}\mathbf{D}\mathbf{y}$$

subject to the constraint

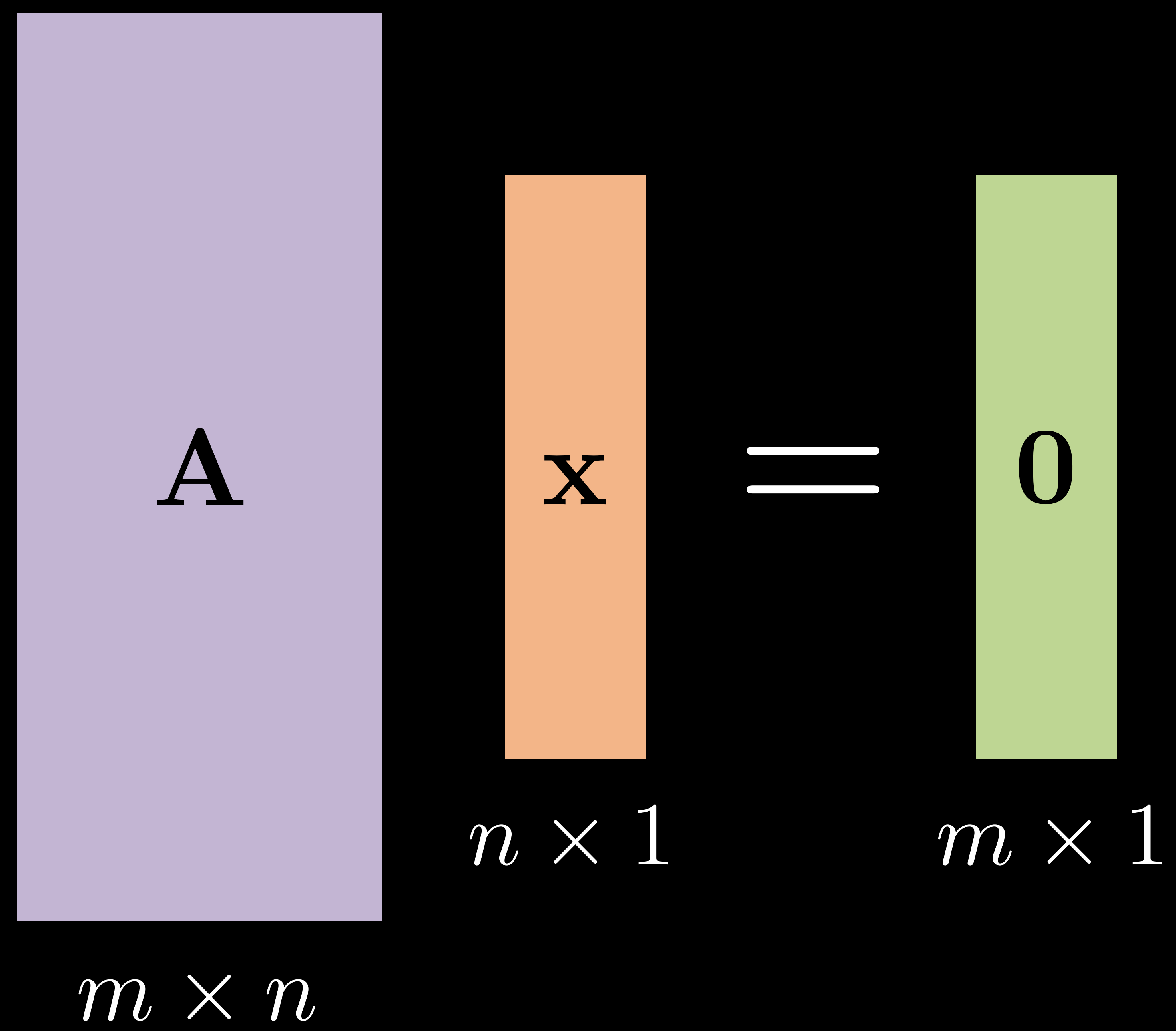
$$\|\mathbf{y}\| = 1$$

$$d_1^2 y_1^2 + d_2^2 y_2^2 + \cdots + d_n^2 y_n^2$$

What is the value of $\mathbf{y}$ that yields the minimum?

$$\boxed{\mathbf{y} = (0, 0, \ldots, 1)^\top}$$

tl;dl



$$\mathbf{x}^* = \arg\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}\|^2 \;\; \text{subject to} \;\; \|\mathbf{x}\| = 1$$

take SVD of A, solution is last column of V

Fundamental Matrix via SVD

$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$
\underbrace{\begin{pmatrix} x_{r1}x_{l1} & x_{r1}y_{l1} & x_{r1} & y_{r1}x_{l1} & y_{r1}y_{l1} & y_{r1} & x_{l1} & y_{l1} & 1 \\ & & & & \vdots & & & & \\ x_{rn}x_{ln} & x_{rn}y_{ln} & x_{rn} & y_{rn}x_{ln} & y_{rn}y_{ln} & y_{rn} & x_{ln} & y_{ln} & 1 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix}}_{\mathbf{f}} = \mathbf{0}
$$

Fundamental Matrix must be rank 2

estimated F matrix may be FULL RANK

$$\tilde{\mathbf{F}}^* = \underset{\tilde{\mathbf{F}}}{\operatorname{argmin}} \|\tilde{\mathbf{F}} - \hat{\mathbf{F}}\|^2 \ \text{ subject to } \ \det(\tilde{\mathbf{F}}) = 0$$

Enforce
rank 2

3

major steps

Step 1

take SVD

$$\mathbf{F} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top$$

Step 2

adjust singular value

$$\boldsymbol{\Sigma}' = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Step 3

reassemble Fundamental Matrix

$$\mathbf{F}' = \mathbf{U}\mathbf{\Sigma}'\mathbf{V}^\top$$

Enforce rank 2 summary

1. Take SVD of $\mathbf{F}$

2. Set smallest singular value of $\mathbf{F}$ to $\mathbf{0}$

3. Recompute $\mathbf{F}$

$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$\underbrace{\begin{pmatrix} x_{r1}x_{l1} & x_{r1}y_{l1} & x_{r1} & y_{r1}x_{l1} & y_{r1}y_{l1} & y_{r1} & x_{l1} & y_{l1} & 1 \\ & & & & \vdots & & & & \\ x_{rn}x_{ln} & x_{rn}y_{ln} & x_{rn} & y_{rn}x_{ln} & y_{rn}y_{ln} & y_{rn} & x_{ln} & y_{ln} & 1 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \end{pmatrix}}_{\mathbf{f}} = \mathbf{0}$$

A is ill-conditioned

$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$\underbrace{\begin{pmatrix} x_{r1}x_{l1} & x_{r1}y_{l1} & x_{r1} & y_{r1}x_{l1} & y_{r1}y_{l1} & y_{r1} & x_{l1} & y_{l1} & 1 \\ & & & & \vdots & & & & \\ x_{rn}x_{ln} & x_{rn}y_{ln} & x_{rn} & y_{rn}x_{ln} & y_{rn}y_{ln} & y_{rn} & x_{ln} & y_{ln} & 1 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \end{pmatrix} = \mathbf{0}$$

transform points such that A is better conditioned

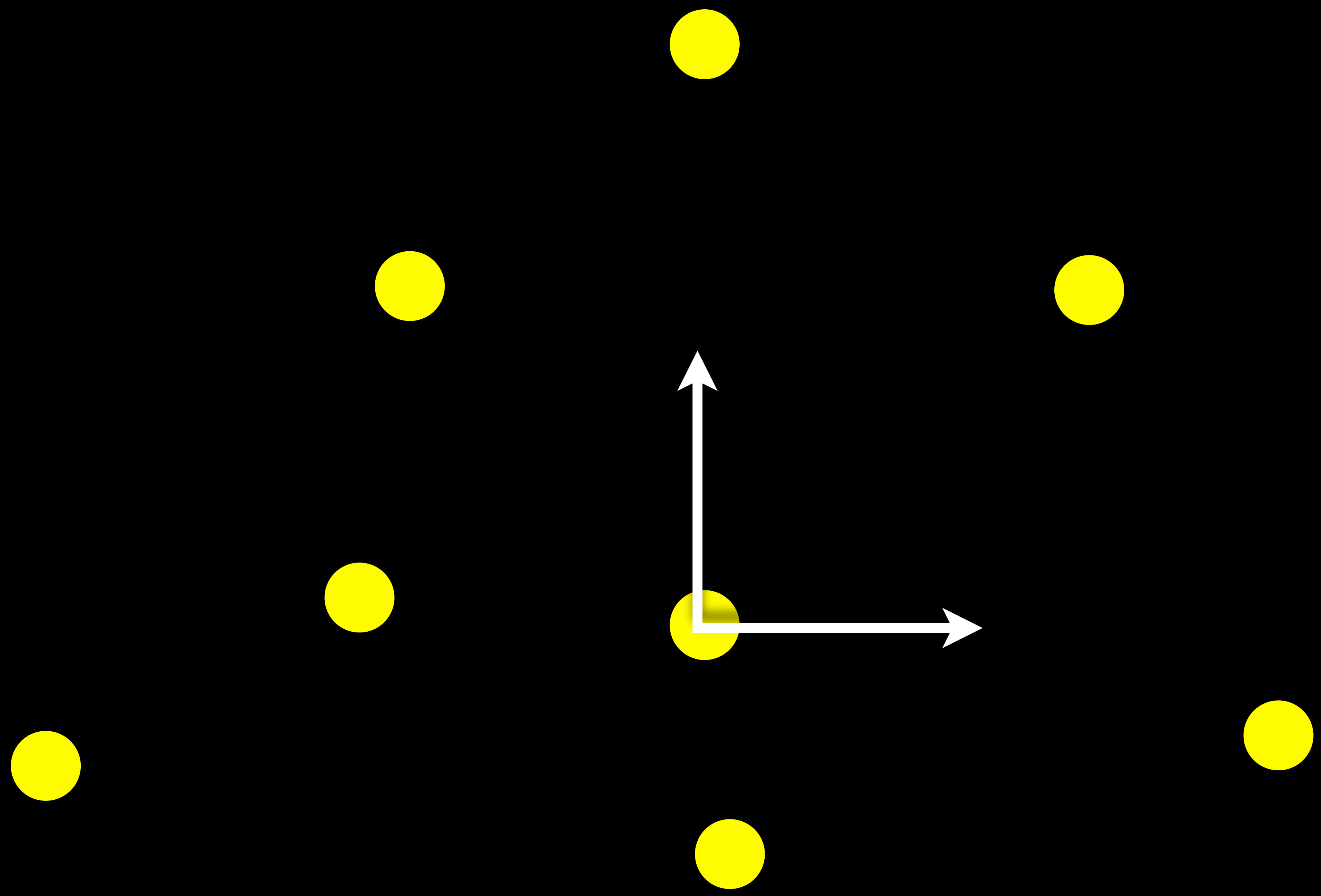
Normalized 8-point algorithm

4

major steps

Step 1

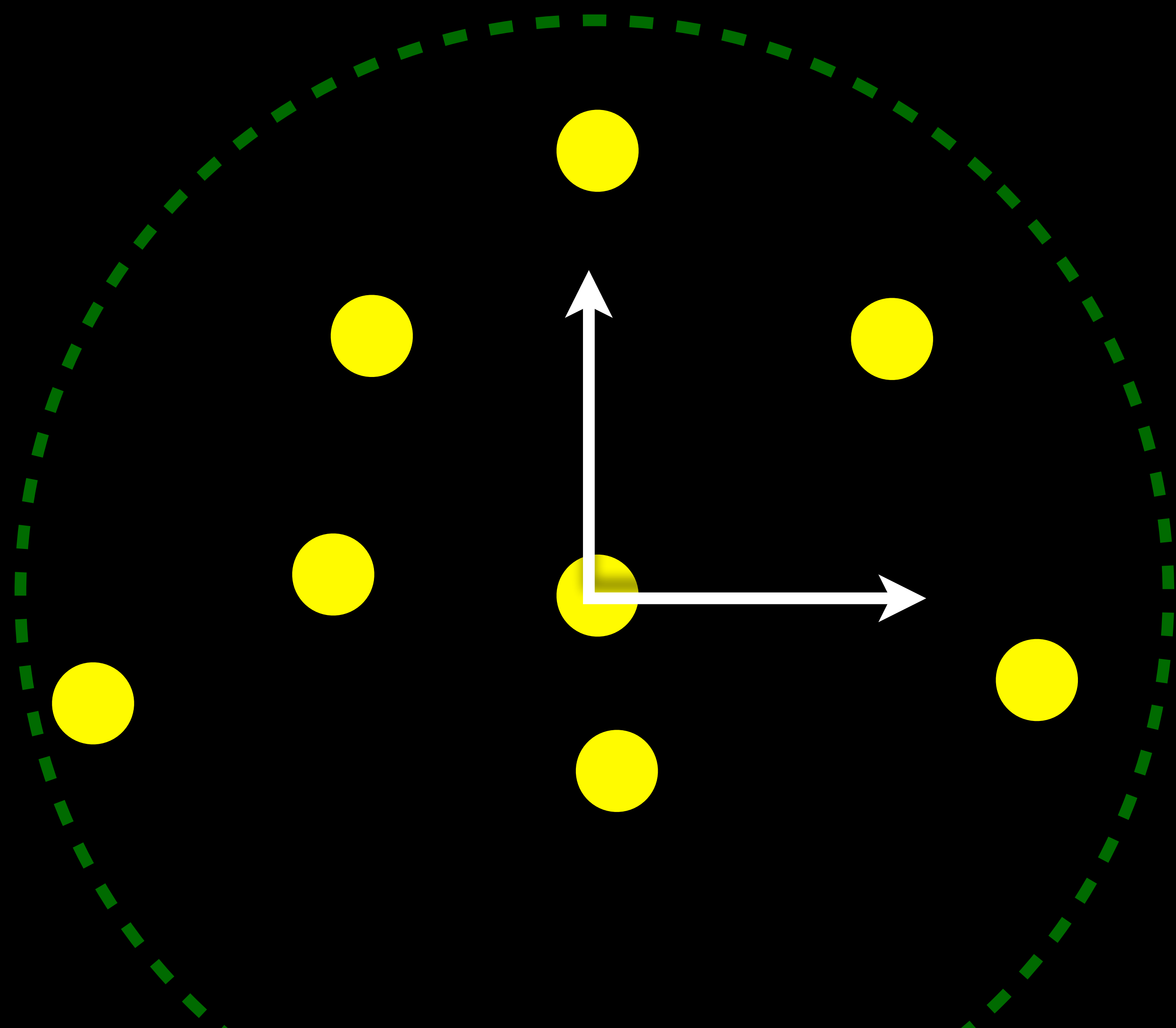
centre data



translate centroid of points to origin

Step 2

scale data



scale points to make mean square distance ~2 pixels

Step 3

compute Fundamental Matrix

$$(\mathbf{T}_r \mathbf{p}_r)^\top \mathbf{F} (\mathbf{T}_l \mathbf{p}_l) = 0$$

enforce rank 2 constraint on estimated $\mathbf{F}$

Step 4

de-normalize Fundamental Matrix

$$\mathbf{F} = \mathbf{T}_r^\top \hat{\mathbf{F}} \mathbf{T}_l$$

Normalized 8-point algorithm summary

1. Translate points with centroid centred at origin

2. Apply scaling to image coordinates

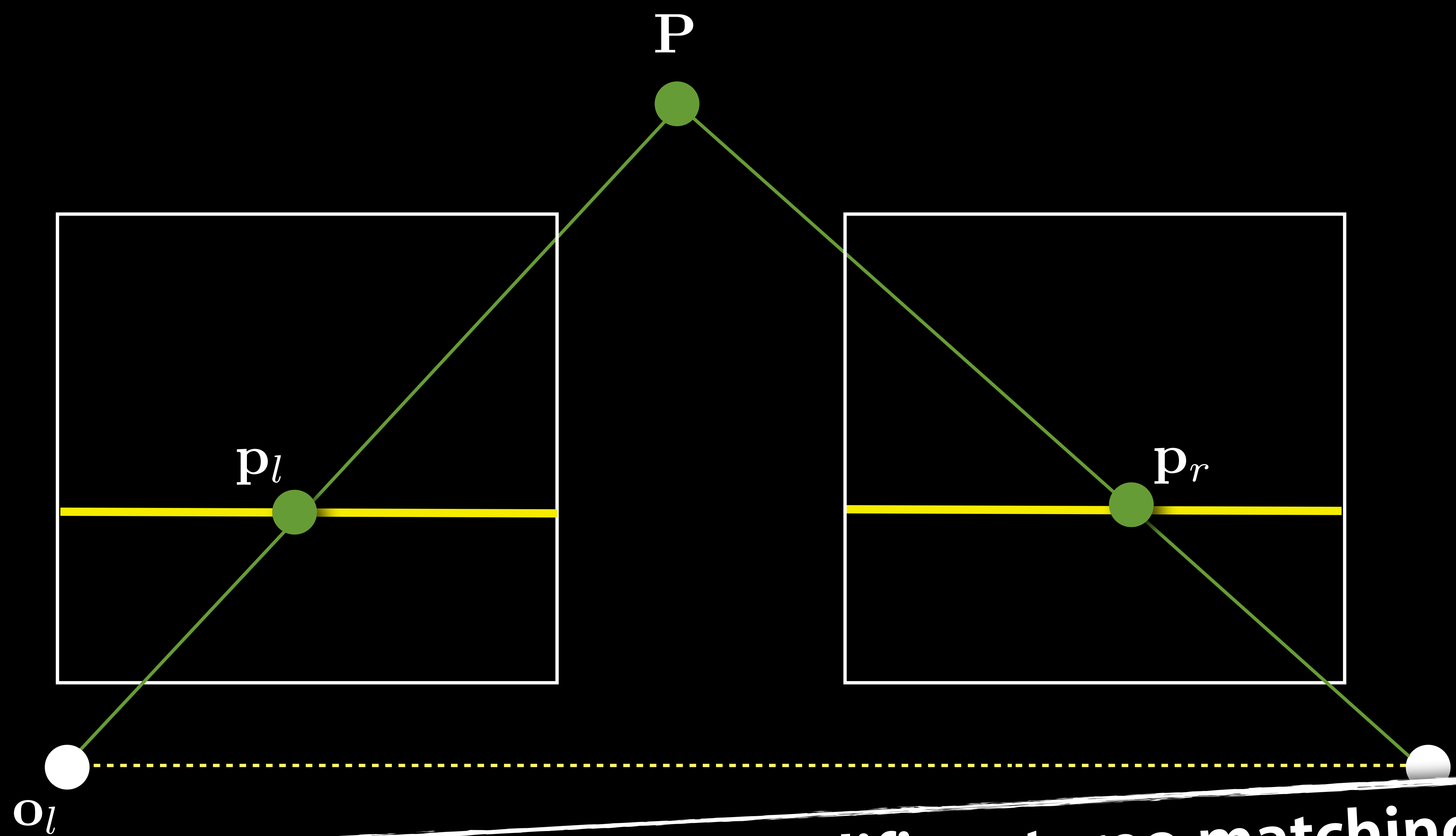
3. Estimate Fundamental Matrix

4. De-normalize estimated Fundamental Matrix

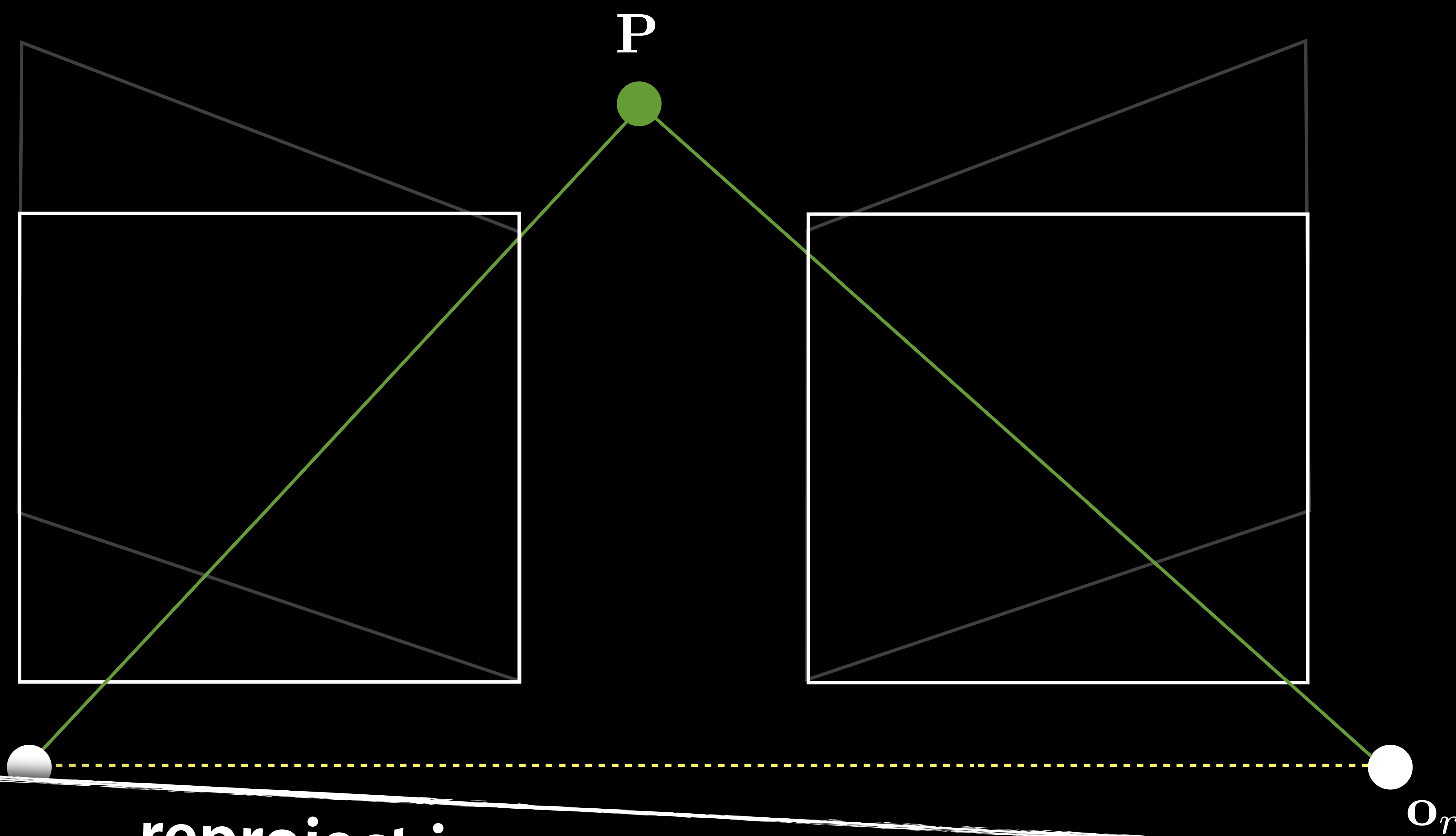
8-point algorithm

```
% build constraint matrix
A = [x2(1,:)'.*x1(1,:)' x2(1,:)'.*x1(2,:)' x2(1,:)' ...
    x2(2,:)'.*x1(1,:)' x2(2,:)'.*x1(2,:)' x2(2,:)' ...
    x1(1,:)' x1(2,:)' ones(npts,1) ];

% compute SVD matrix factorization
[U,D,V] = svd(A);

% extract Fundamental Matrix from the column of V
% corresponding to the smallest singular value
F = V(:,9);
F = reshape(F,3,3)';

% Enforce rank 2 constraint
[U,D,V] = svd(F);
F = U*diag([D(1,1) D(2,2) 0])*V';
```

Image
Rectification



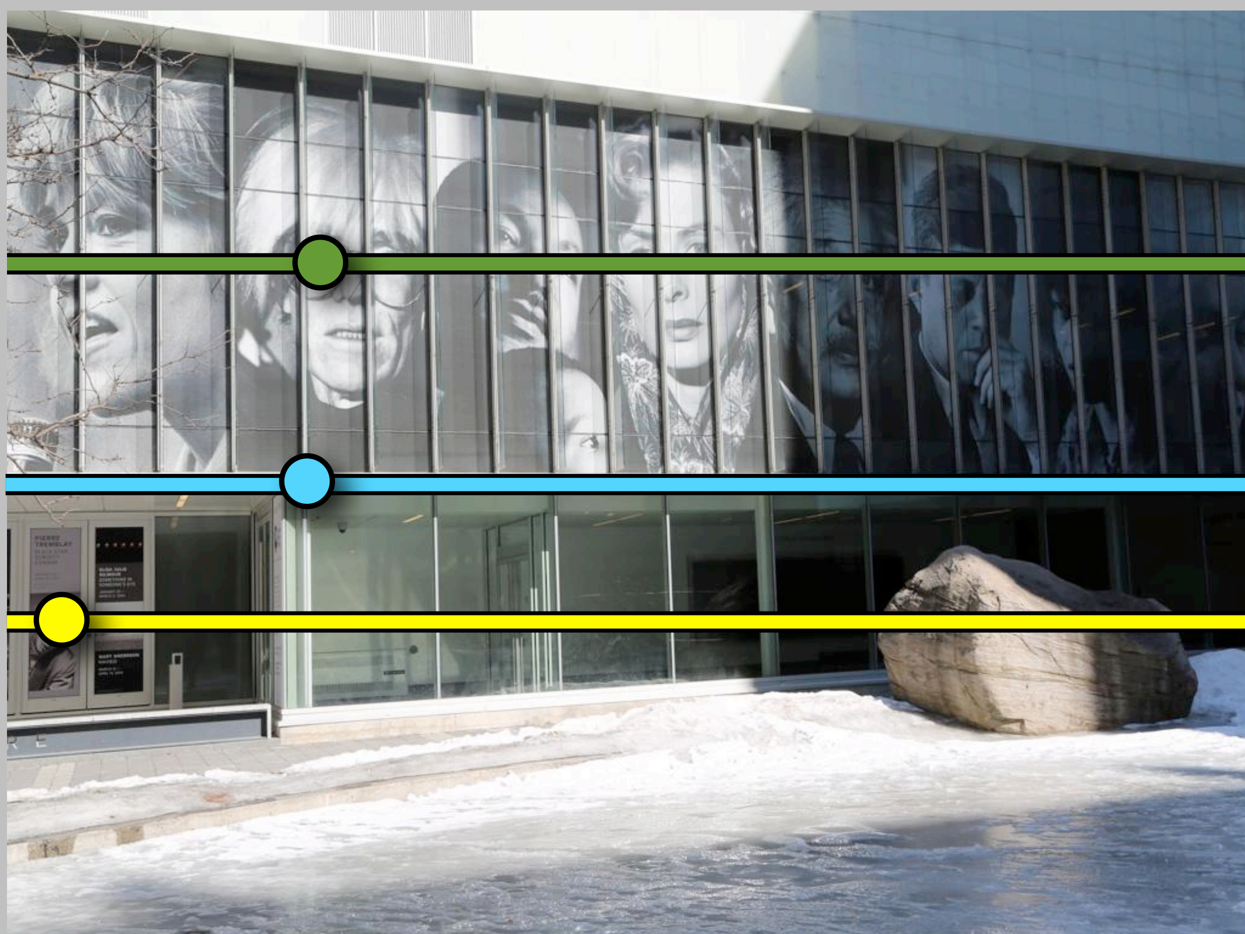
parallel image planes simplifies stereo matching

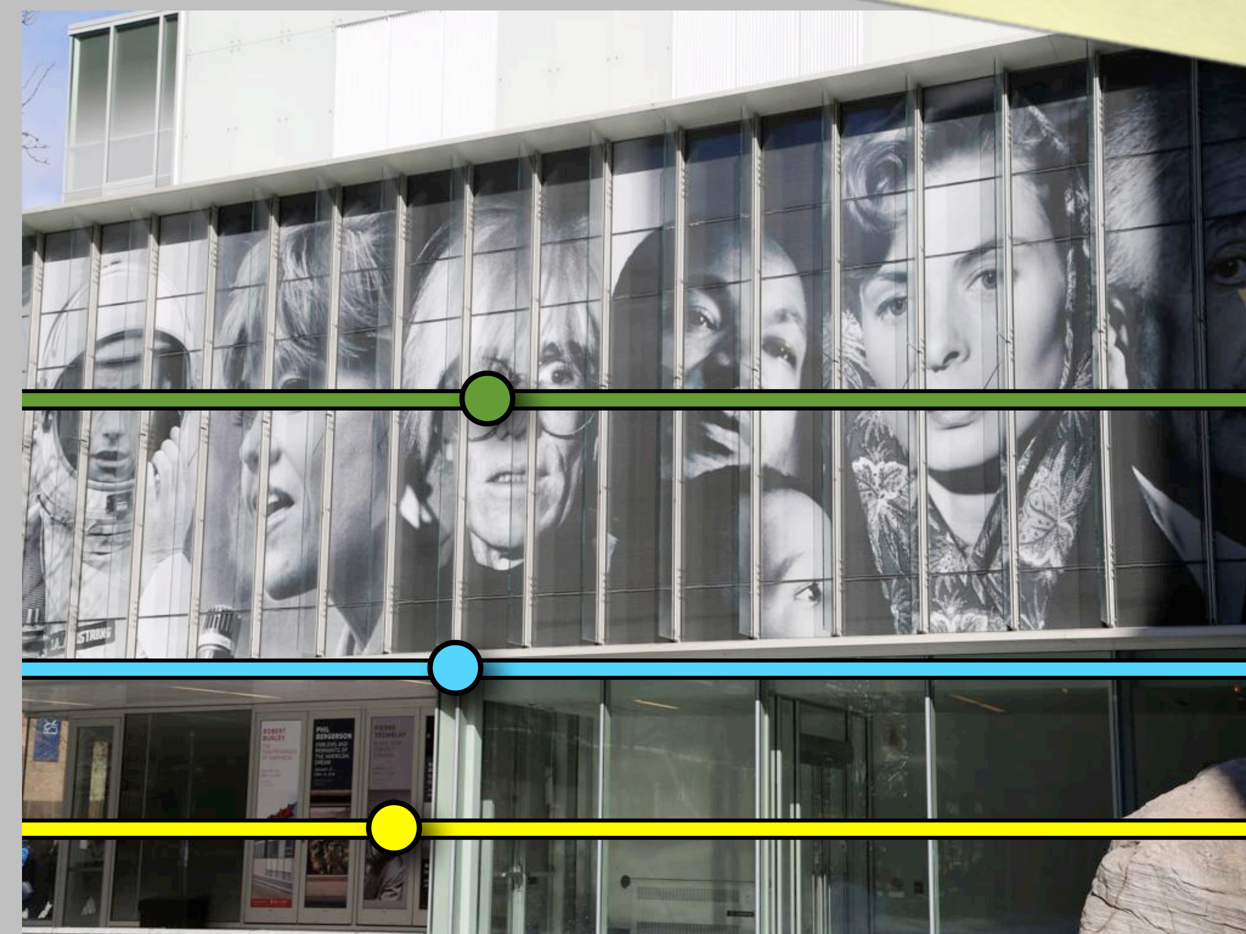


reproject images onto a common plane parallel to baseline

apply a **homography** to each image

Original Image Pair

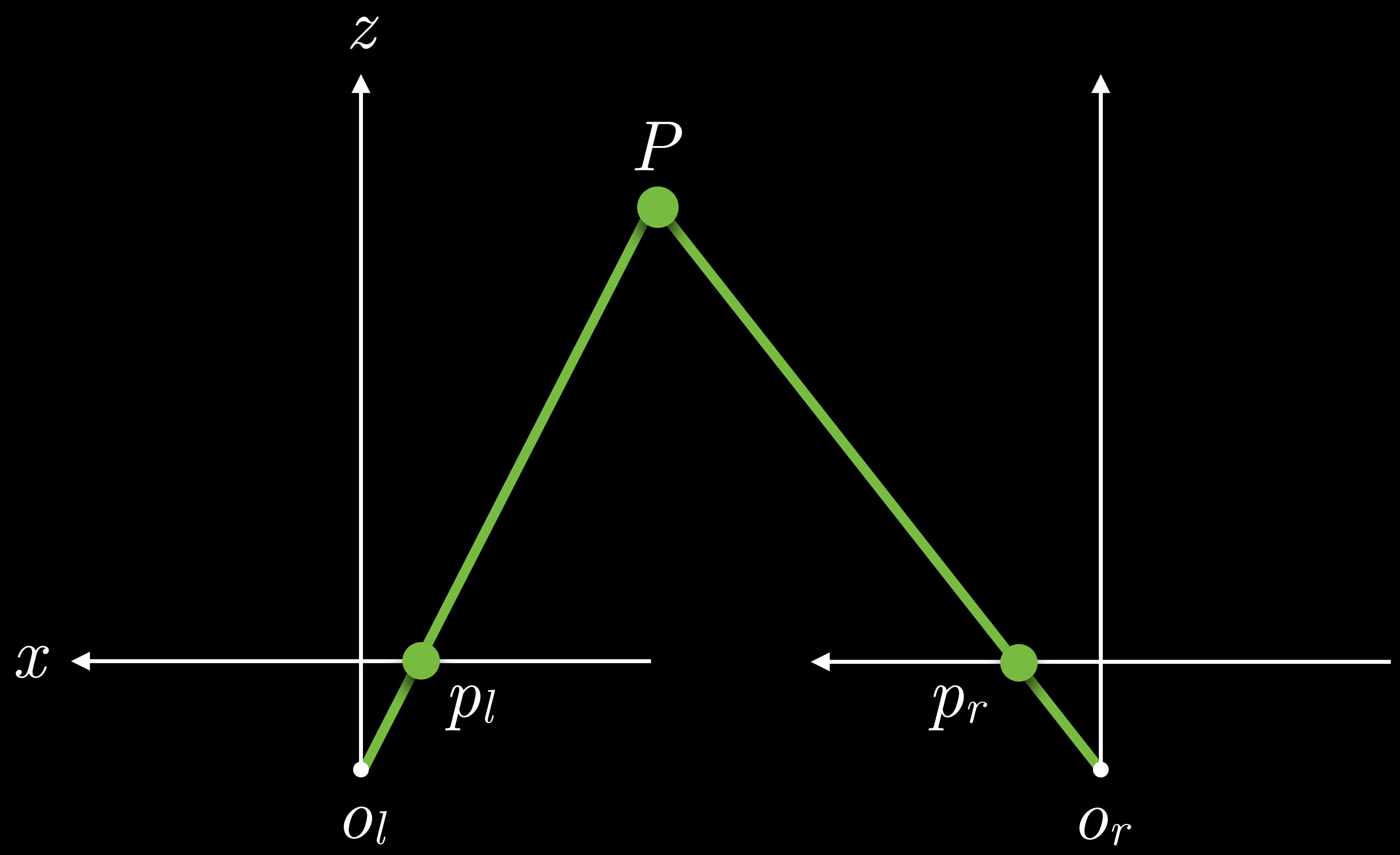




Rectified Image Pair

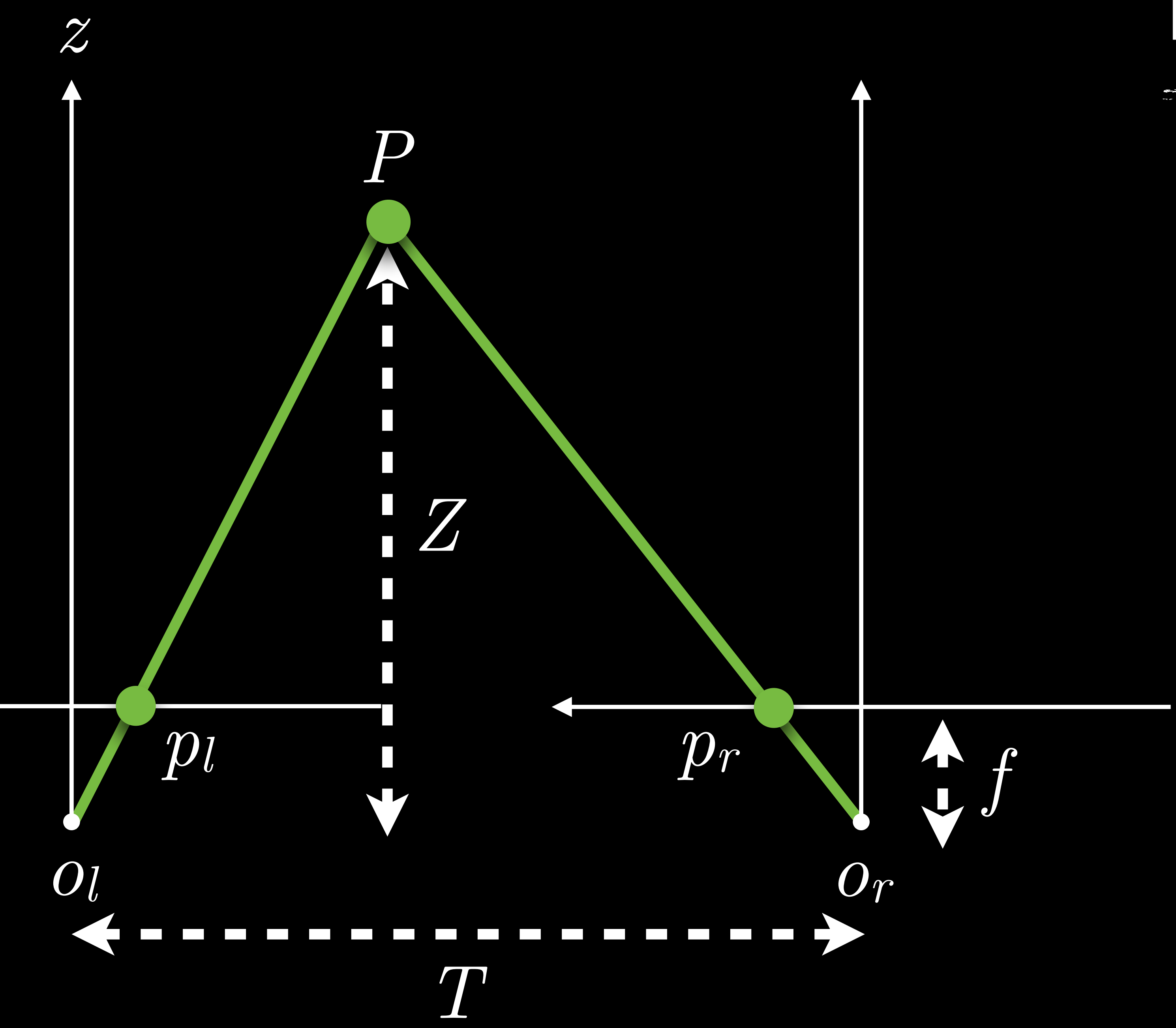
Rectified Image Pair

corresponding points are on the **same scan line**



3D position can be determined by the intersection of rays

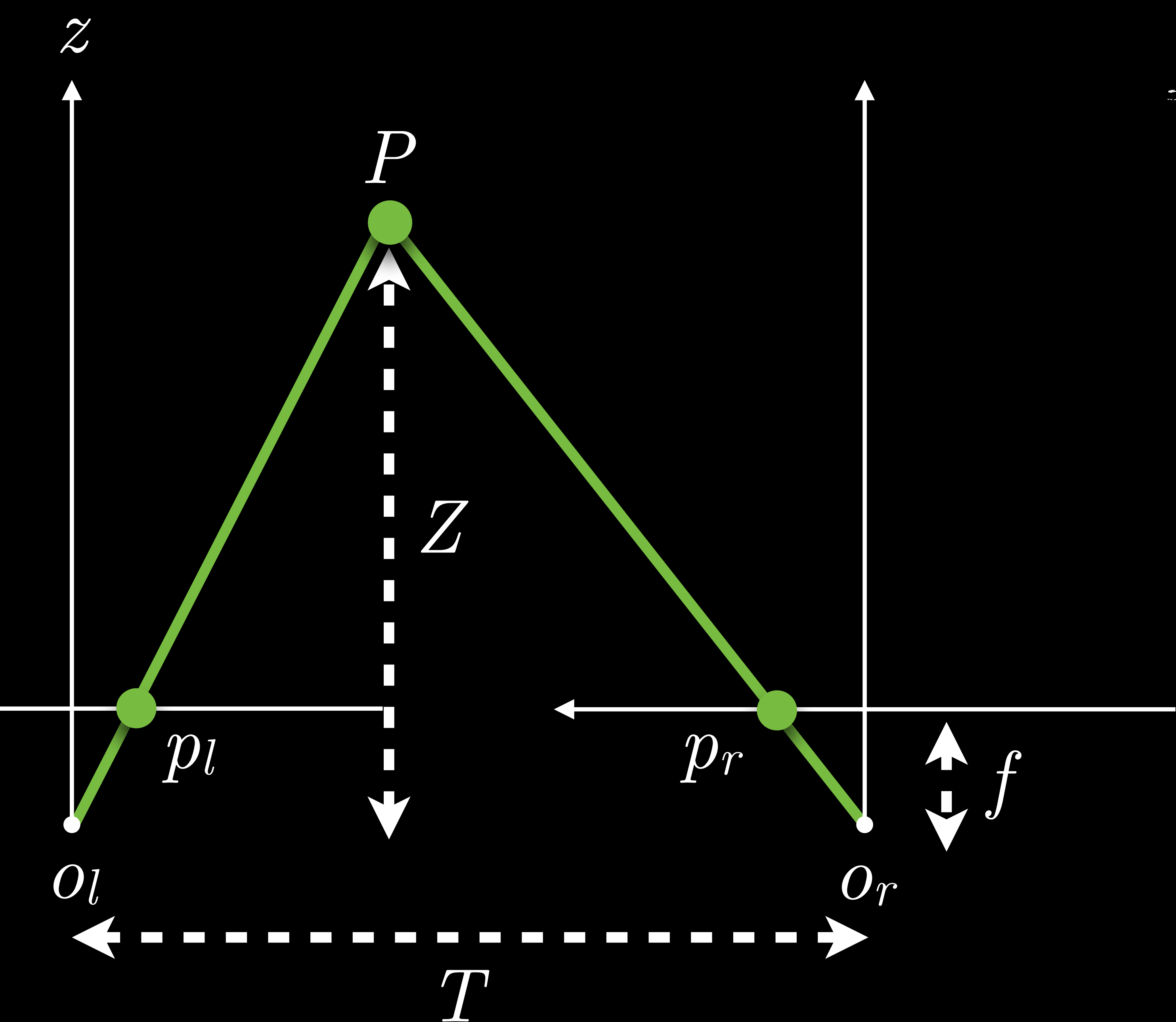
process known as triangulation



Triangulation derivation

Definitions

- T camera centre distance
- Z distance to baseline
- f **common focal length**



Triangulation derivation

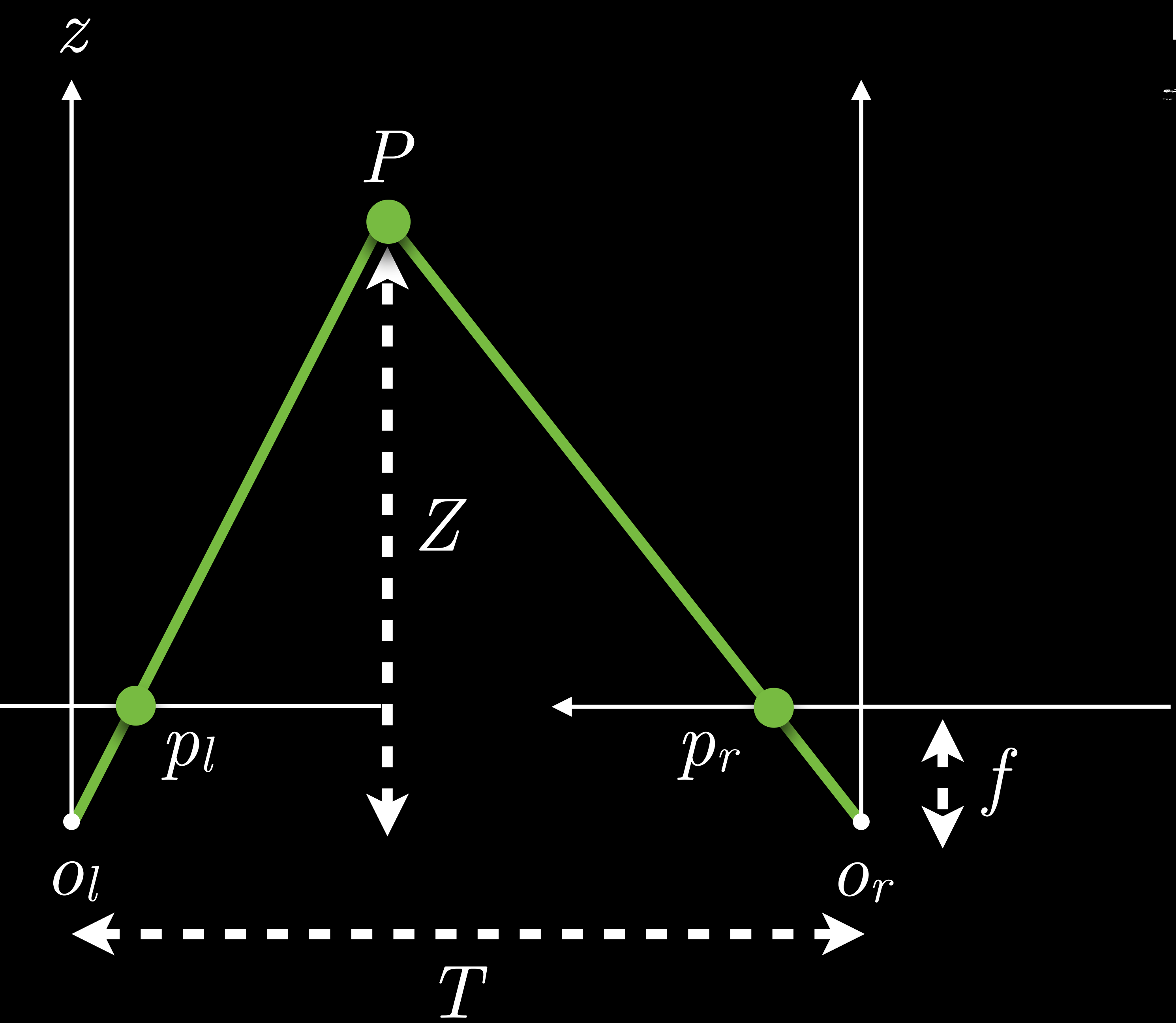
$\triangle p_l P p_r$

and

$\triangle o_l P o_r$

via similar triangles

$$\frac{T + x_l - x_r}{Z - f} = \frac{T}{Z}$$



Triangulation derivation

$$\frac{T + x_l - x_r}{Z - f} = \frac{T}{Z}$$

Let $x_r - x_l$ **be the disparity**

$$\frac{T - d}{Z - f} = \frac{T}{Z}$$

rearrange

Disparity Equation $\boxed{Z = f\frac{T}{d}}$

Disparity Equation $$Z = f\frac{T}{d}$$

Assumes stereo cameras are exactly parallel

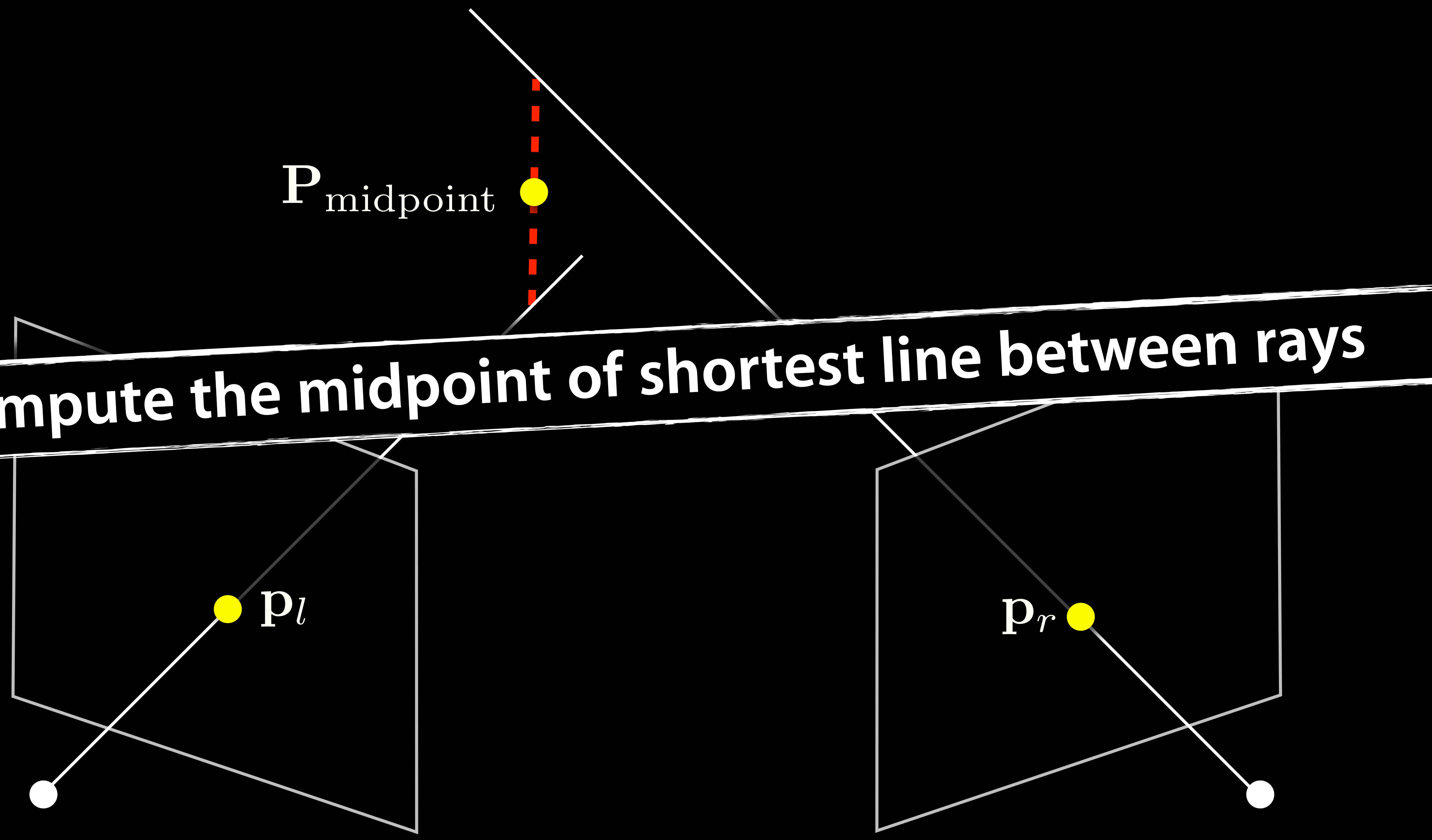
Assumes perfect correspondence

Depth is inversely proportional to disparity

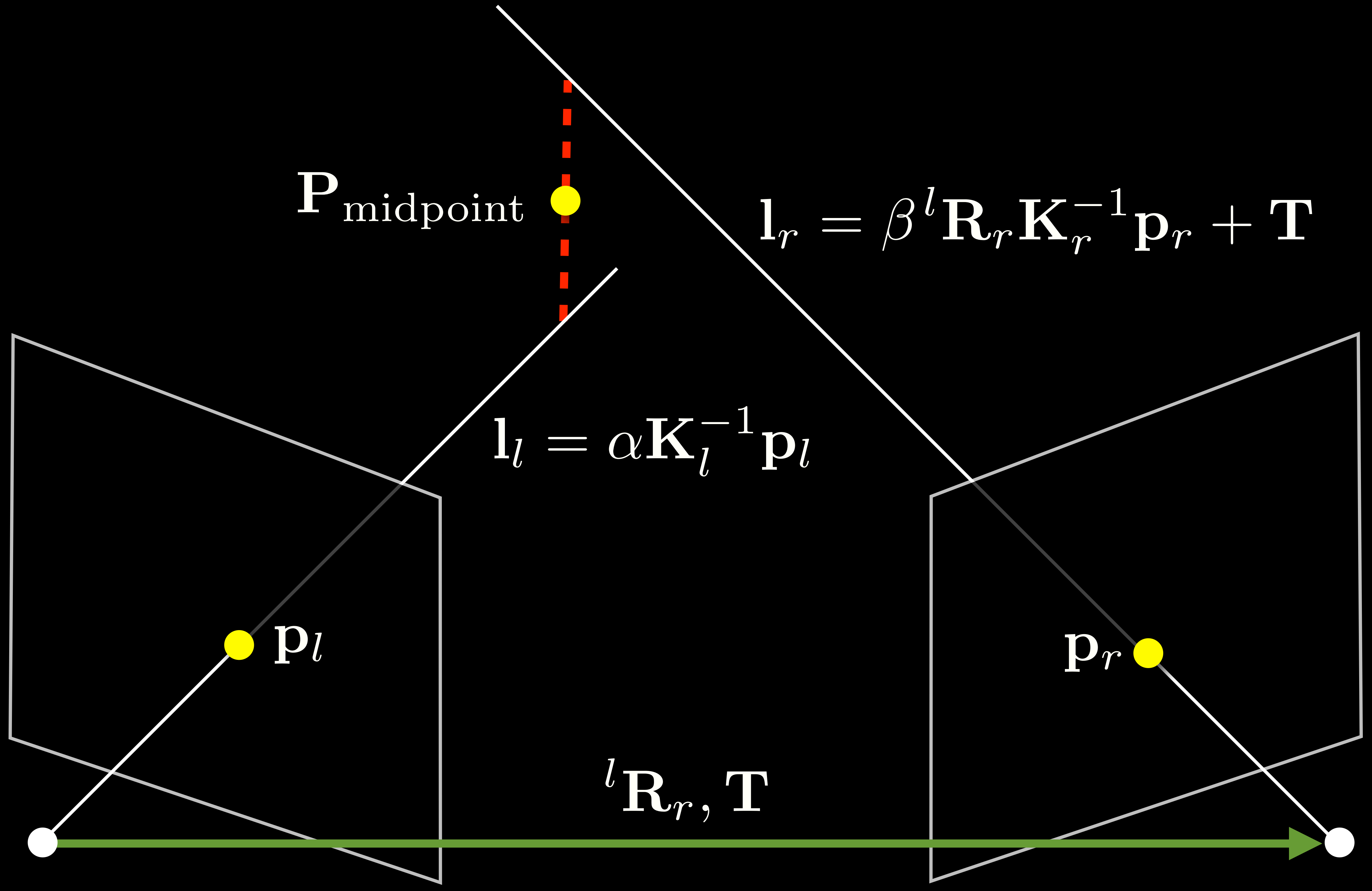
Assume we have a fully calibrated stereo rig

Assume we have a fully calibrated stereo rig

intrinsic and extrinsic parameters of cameras known



$\mathbf{P}_{\mathrm{midpoint}}$
compute the midpoint of shortest line between rays
$\mathbf{p}_l$
$\mathbf{p}_r$

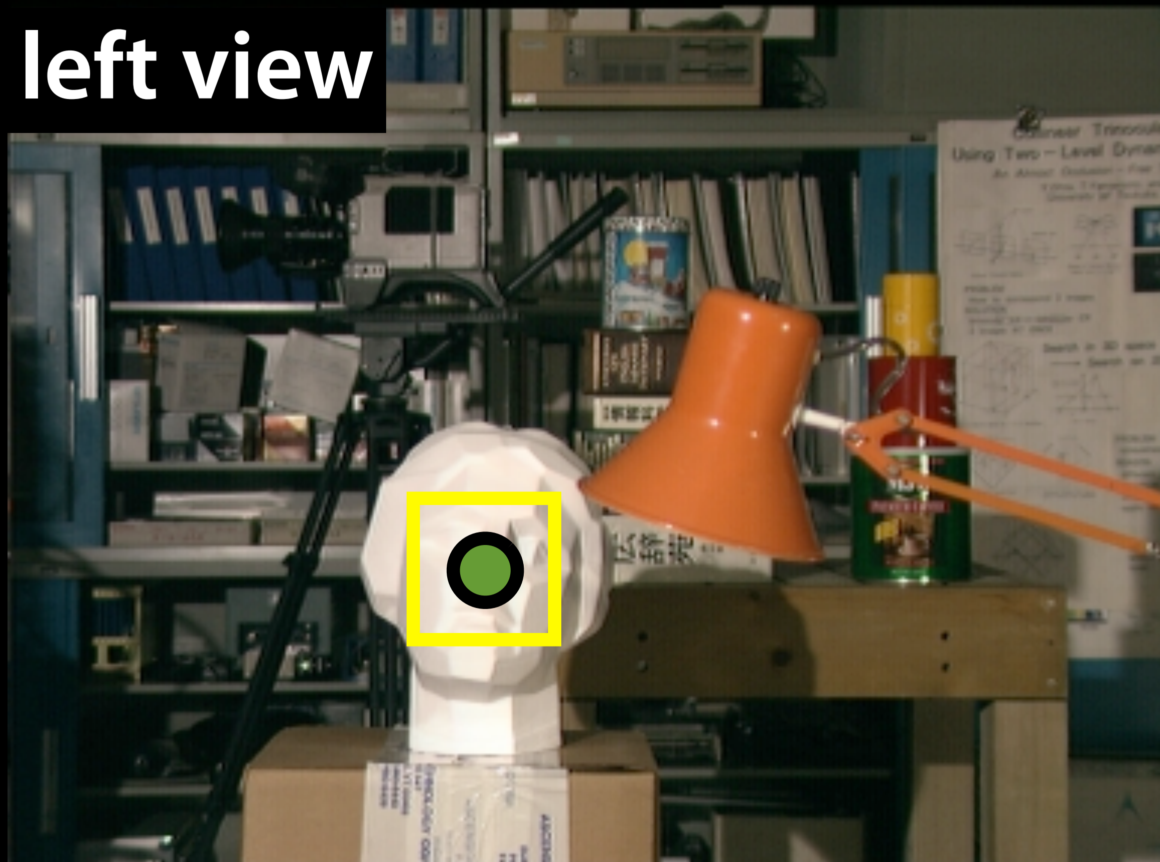


$$\arg\min_{\alpha,\beta} \left\| \alpha \mathbf{K}_l^{-1} \mathbf{p}_l - \left(\beta\, {}^{l}\mathbf{R}_r \mathbf{K}_r^{-1} \mathbf{p}_r + \mathbf{T}\right) \right\|$$

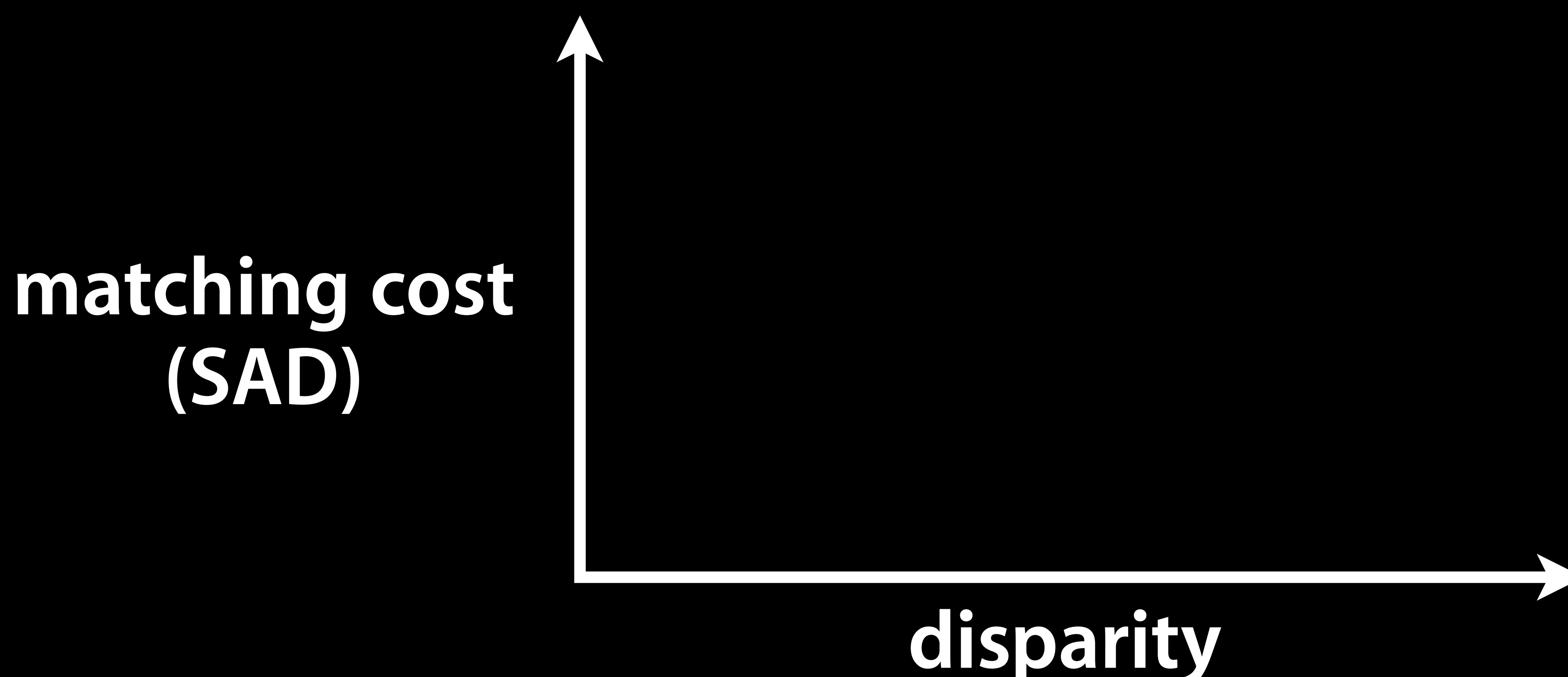
Stereo Matching

region-based methods

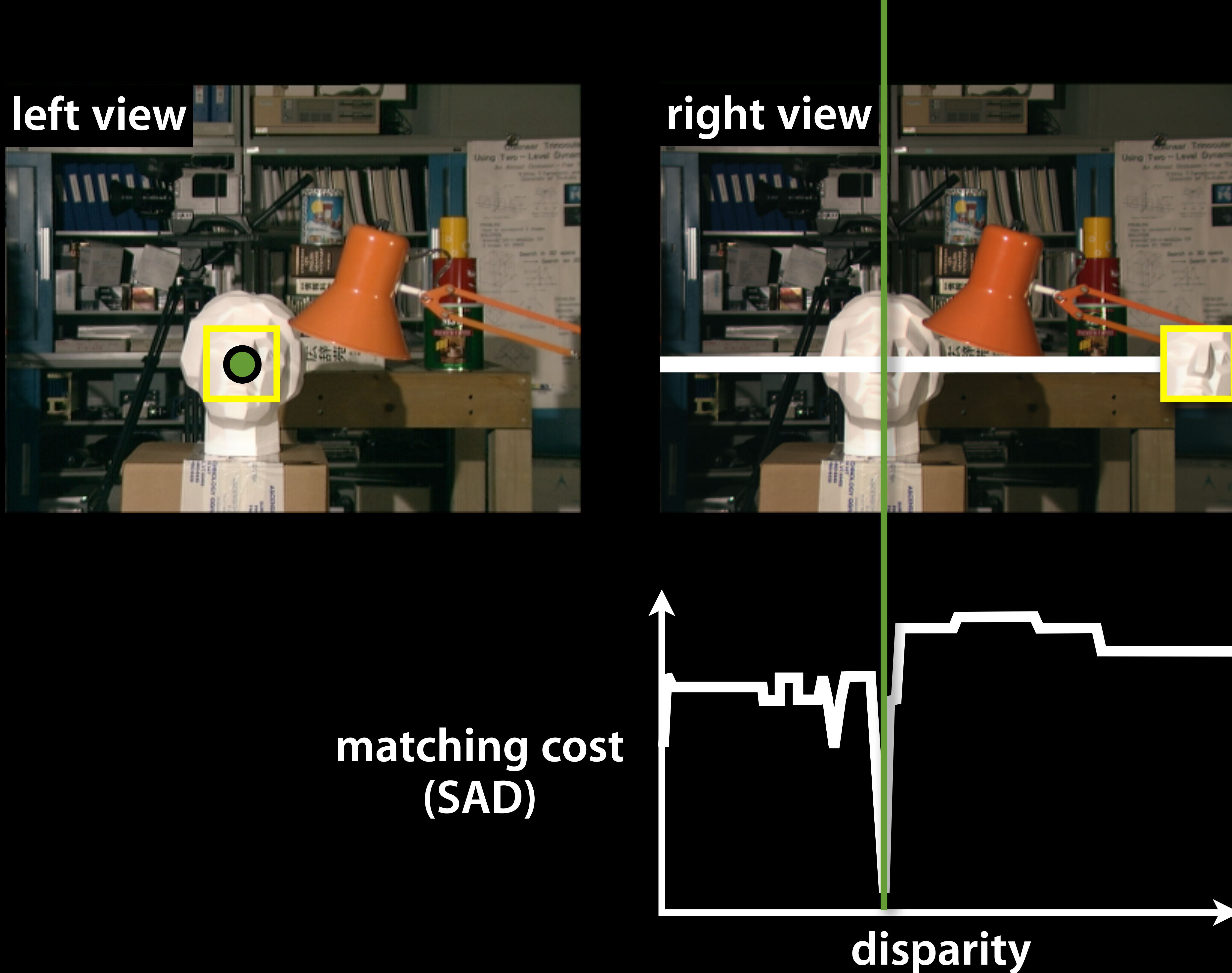
for each point in left view, find “best” matching region in right view


left view

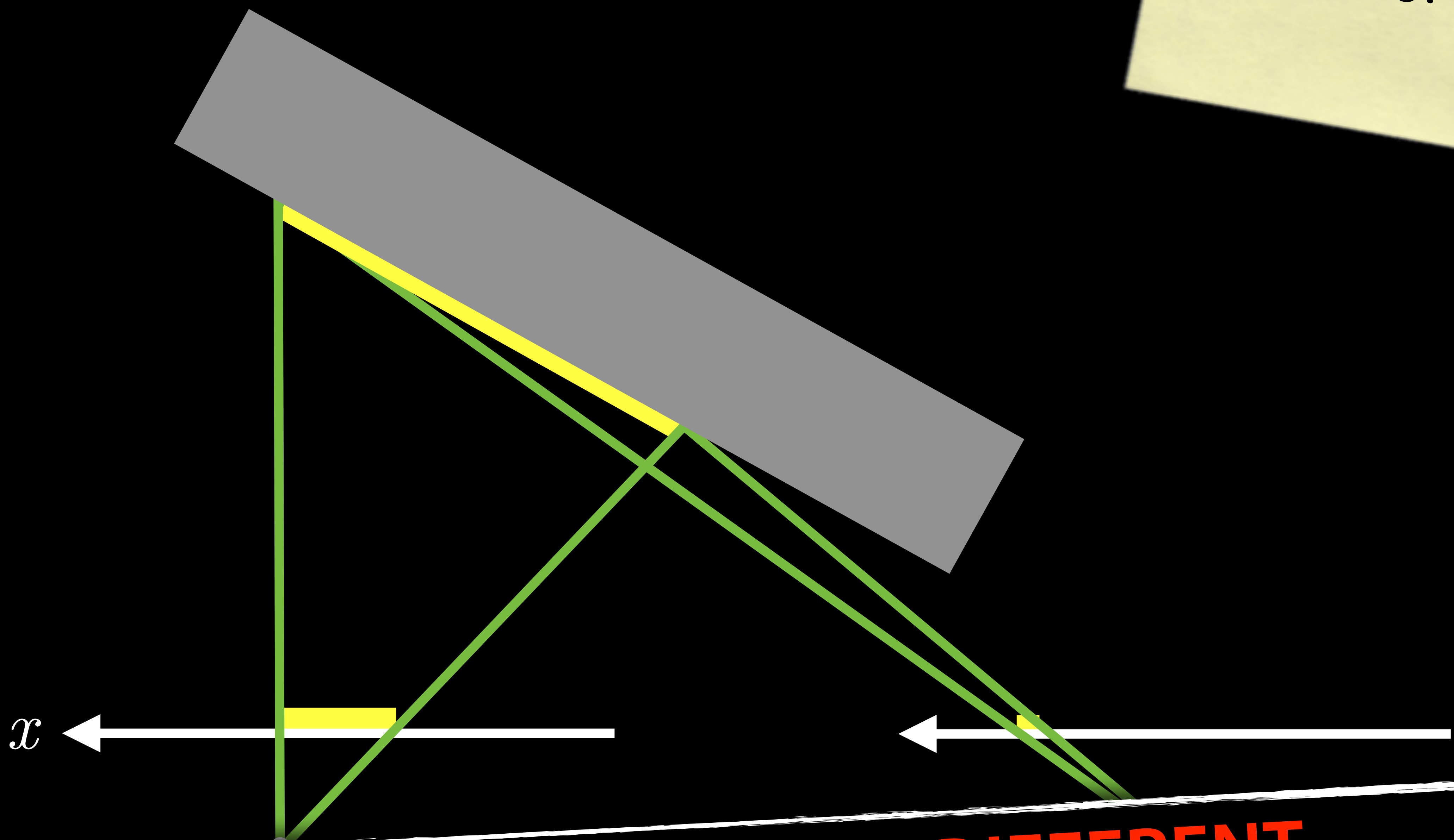

right view


matching cost
(SAD)
disparity

$$\mathrm{SAD}(I, J) = \sum_{i,j} |I(i,j) - J(i,j)|$$

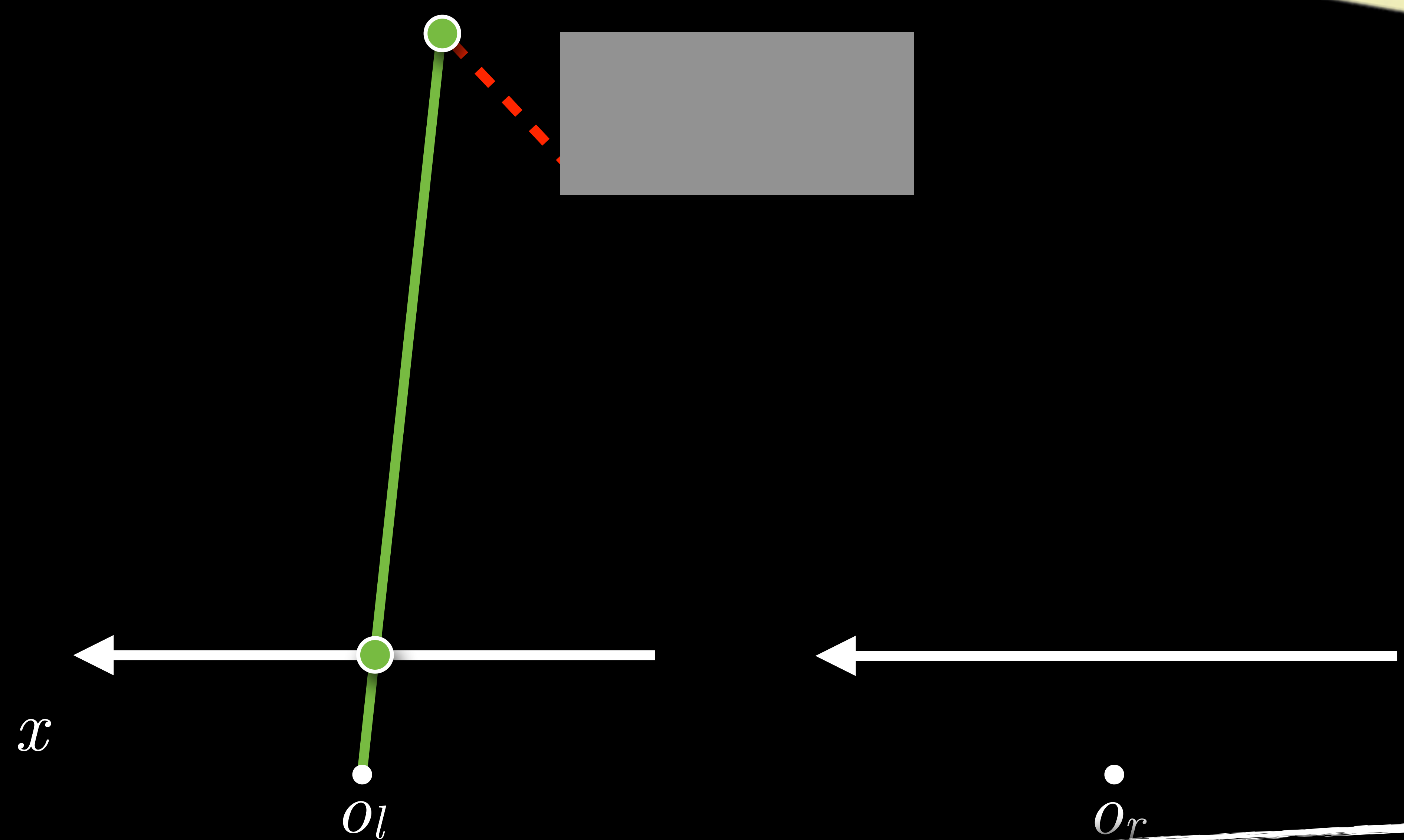

left view
right view
matching cost
(SAD)
disparity

Foreshortening



projection lengths are **DIFFERENT**

Binocular half-occlusion



all points are not visible from both views

Repetitive structures

region matches are **ambiguous**