Intro to

Computer Vision

with Prof. Kosta Derpanis

Image Formation

Part 2

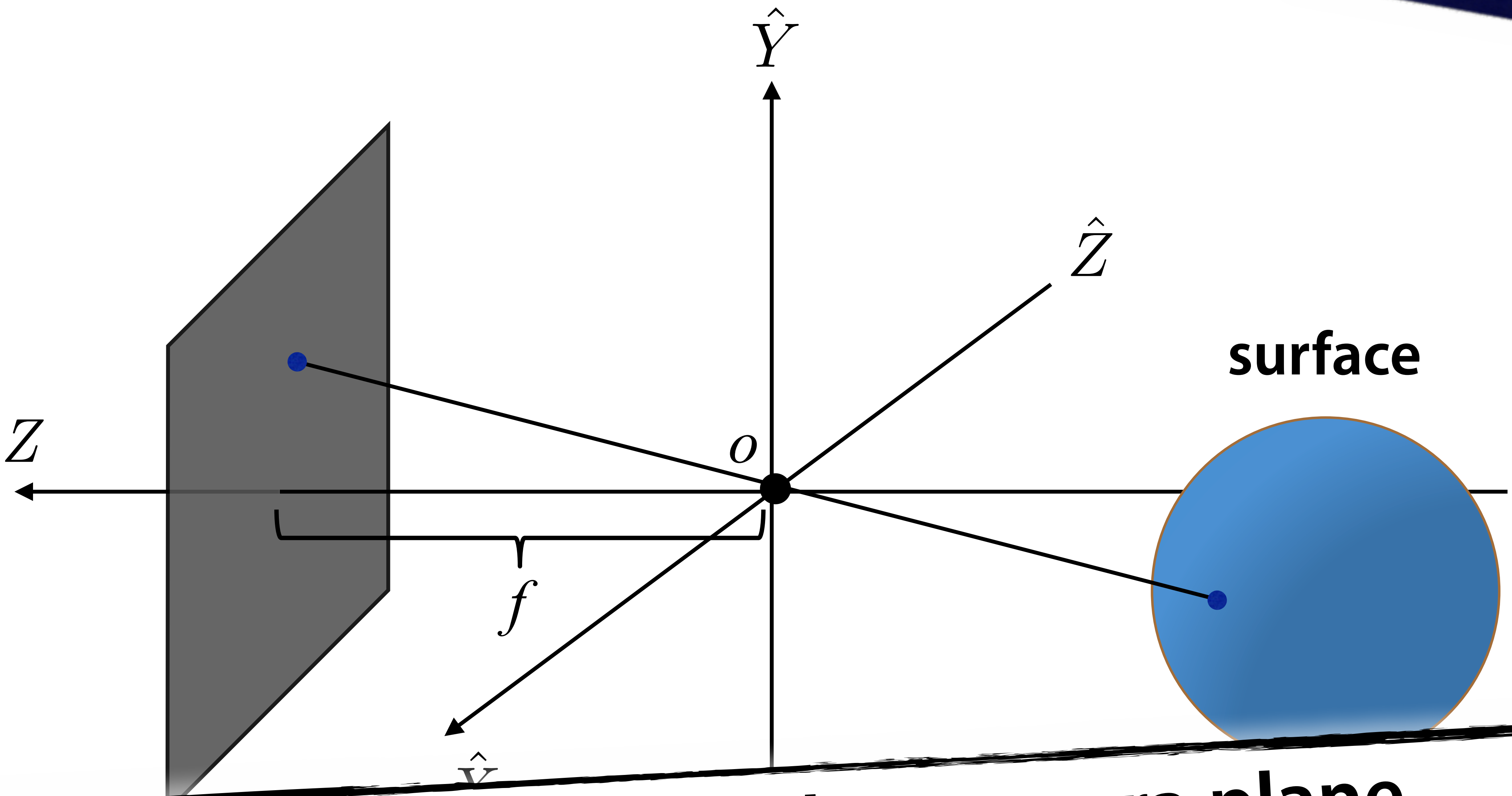
LECTURE TOPICS

Basic optics

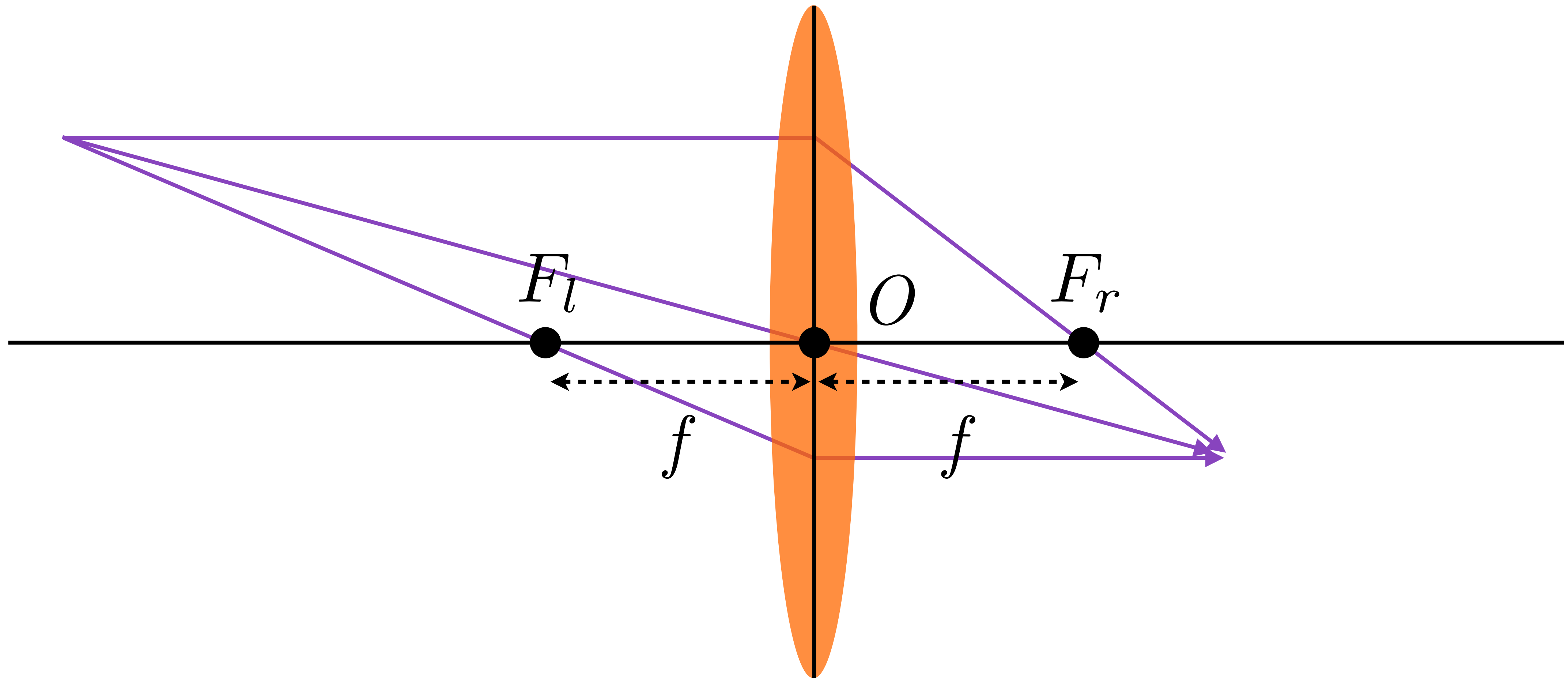
Image formation geometry

Image transformations

Pinhole Camera



mapping from world to camera plane

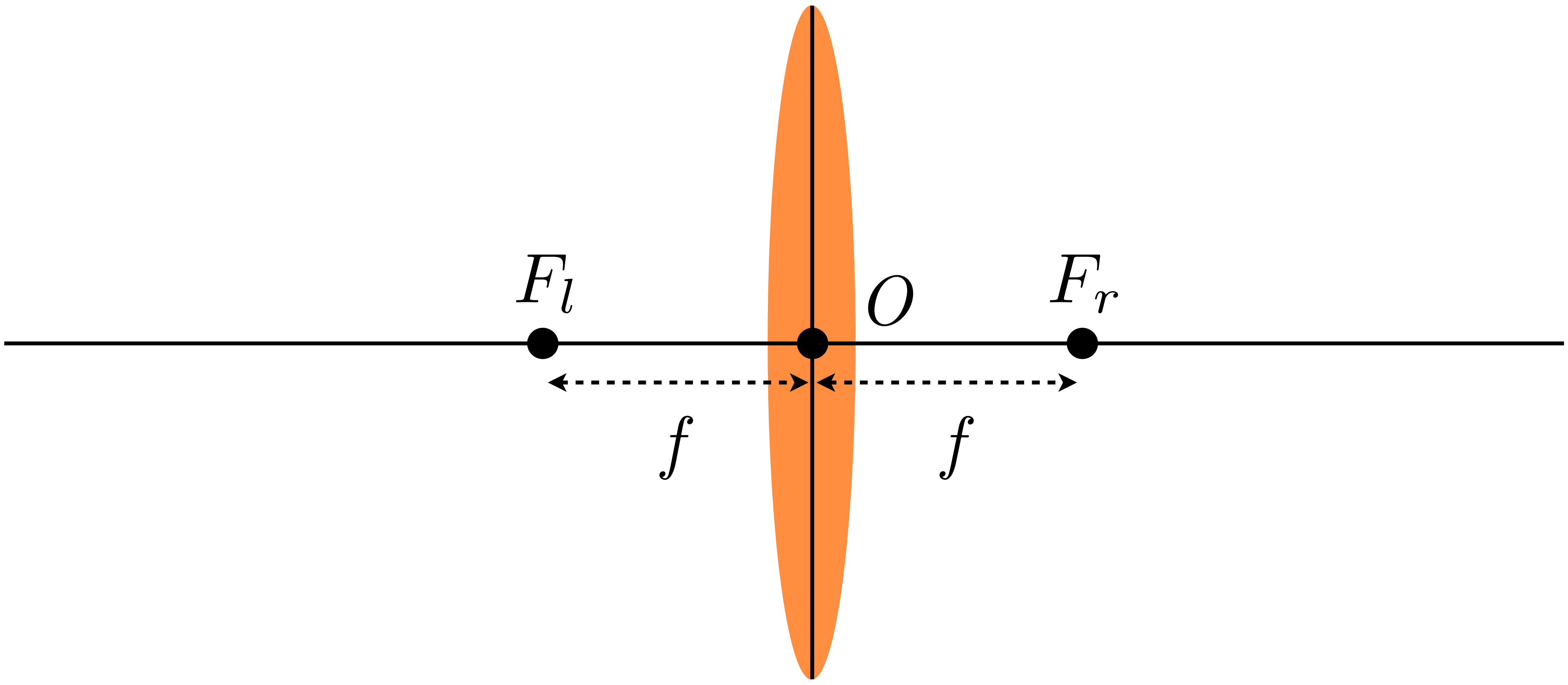


Basic thin lens properties:

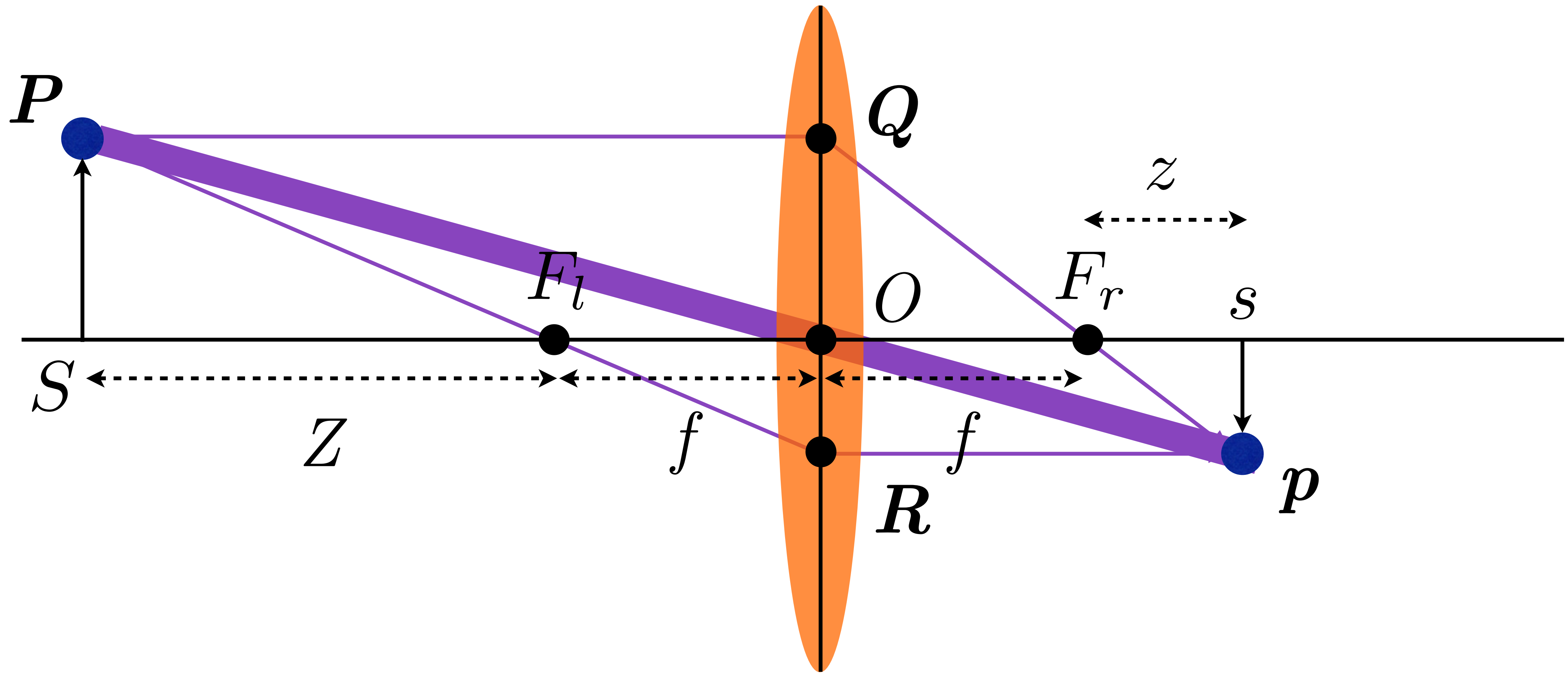
A ray entering parallel to the optical axis goes through the focus on other side

A ray entering through the focus on one side is parallel to the optical axis on the other side

A ray going through the lens centre goes undeflected



Derivation: Fundamental Equation of Thin Lenses

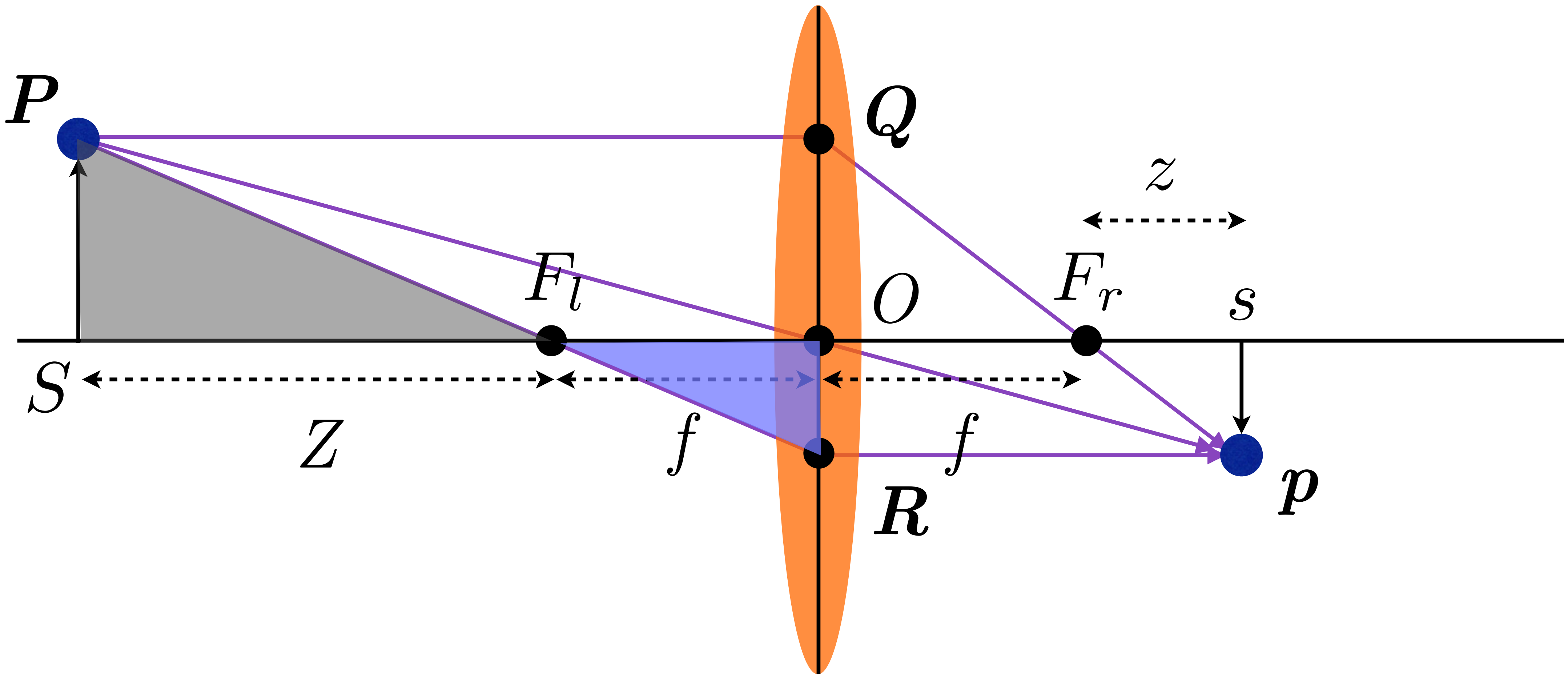


Derivation: Fundamental Equation of Thin Lenses

Consider a point P at a distance $Z+f$ from the lens

All rays from P are focused to the same point, p:

- PQ goes through F_r
- PR emerges parallel to the optical axis
- PO goes undeflected

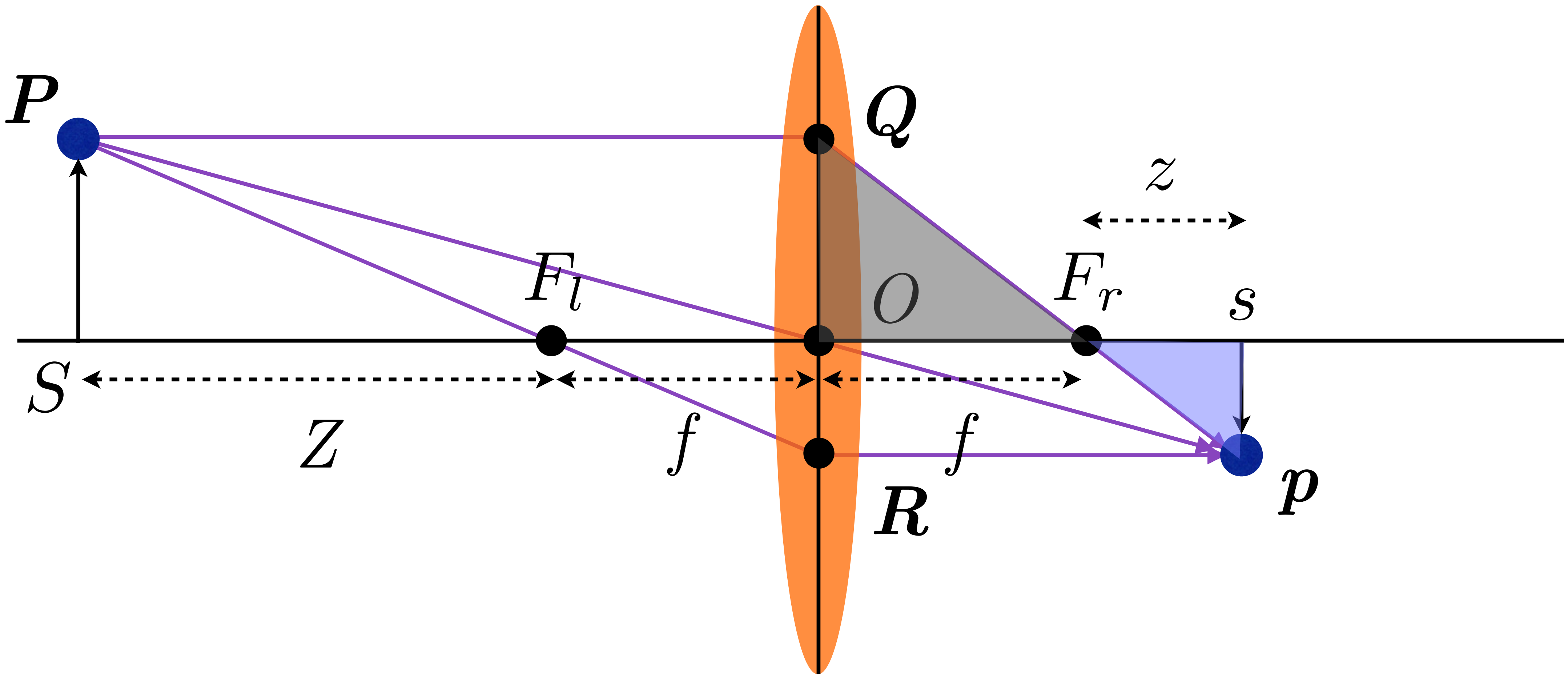


Derivation: Fundamental Equation of Thin Lenses

From similar triangles:

$$\triangle P F_l S \sim \triangle R F_l O$$

$$\frac{Z}{f} = \frac{PS}{OR}$$

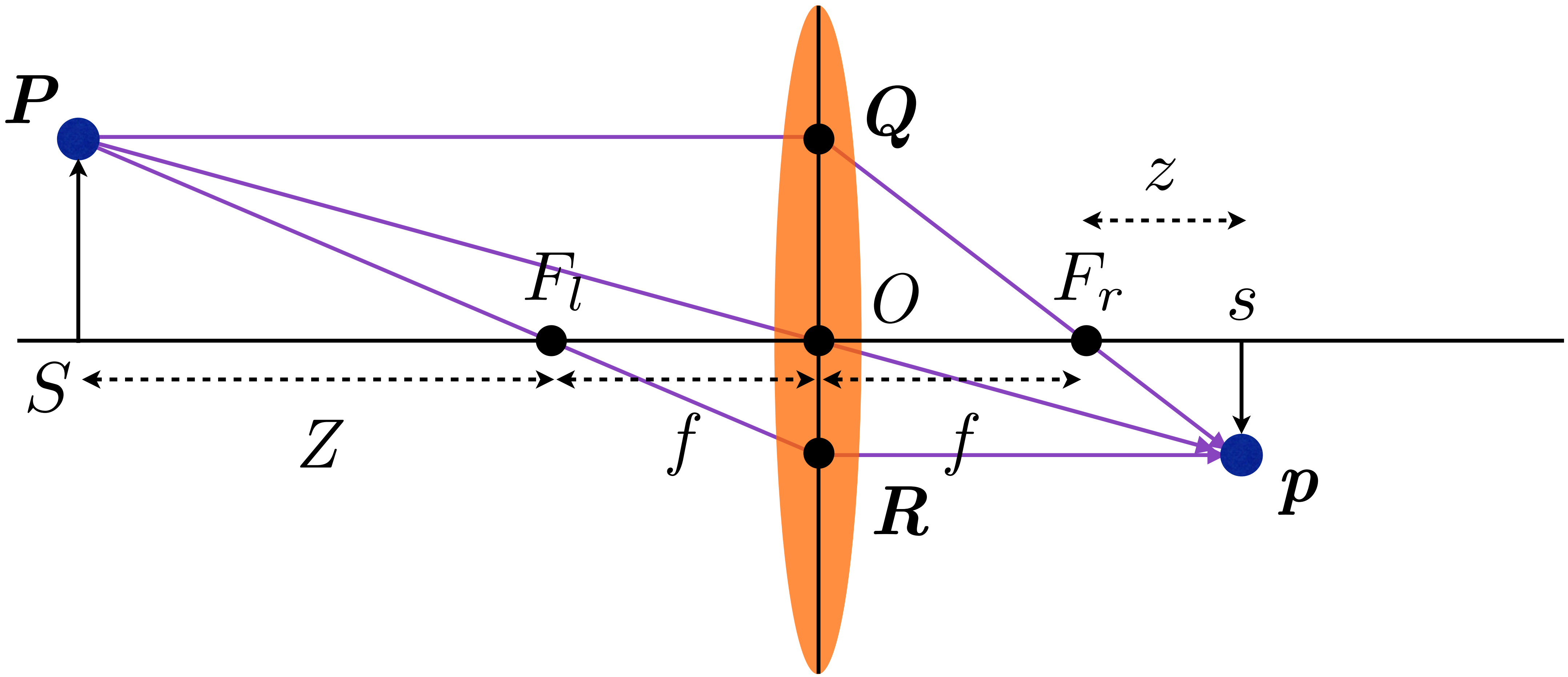


Derivation: Fundamental Equation of Thin Lenses

From similar triangles:

$$\triangle PF_lS \sim \triangle RF_lO \qquad\qquad \triangle psF_r \sim \triangle QOF_r$$

$$\frac{Z}{f} = \frac{PS}{OR} \qquad\qquad \frac{QO}{sp} = \frac{f}{z}$$

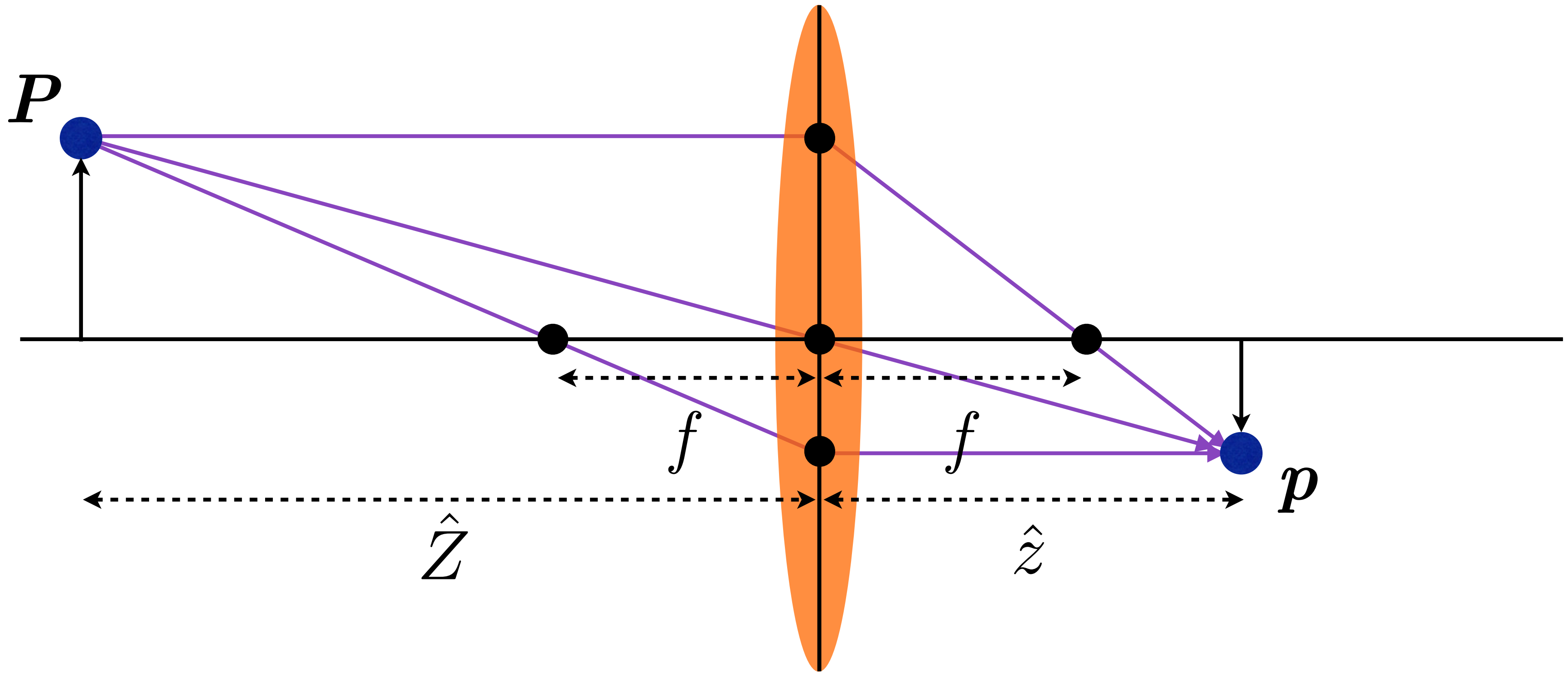


Derivation: Fundamental Equation of Thin Lenses

From similar triangles:

$$\triangle PF_lS \sim \triangle RF_lO \qquad\qquad \triangle psF_r \sim \triangle QOF_r$$

$$\frac{Z}{f} = \frac{PS}{OR} = \frac{QO}{sp} = \frac{f}{z}$$



Derivation: Fundamental Equation of Thin Lenses

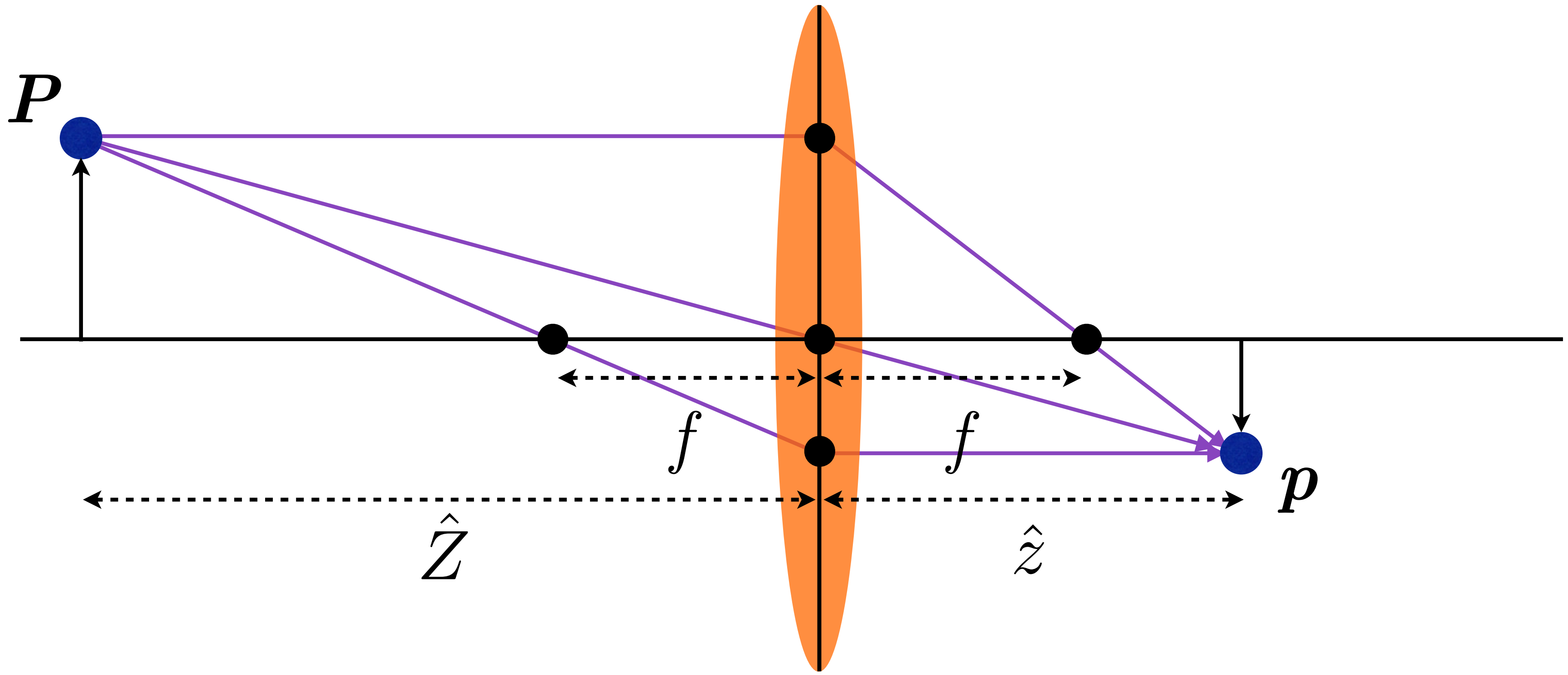
From similar triangles:

$$\triangle PF_lS \sim \triangle RF_lO \qquad \triangle psF_r \sim \triangle QOF_r$$

$$\frac{Z}{f} = \frac{f}{z}$$

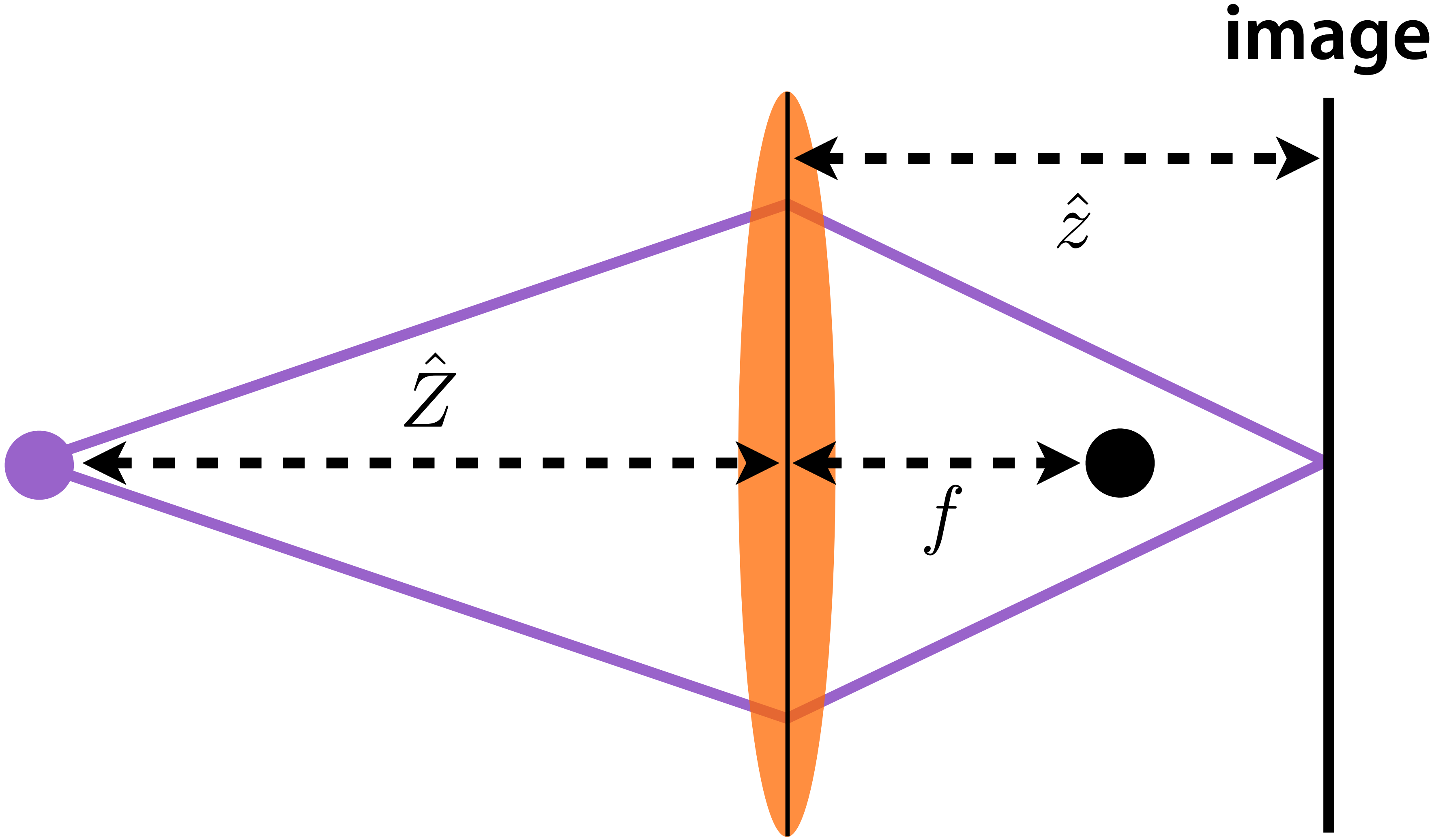
Let $\hat{Z} = Z + f$ **and** $\hat{z} = z + f$

$$\boxed{\text{Thin Lens Equation} \quad \frac{1}{\hat{Z}} + \frac{1}{\hat{z}} = \frac{1}{f}}$$

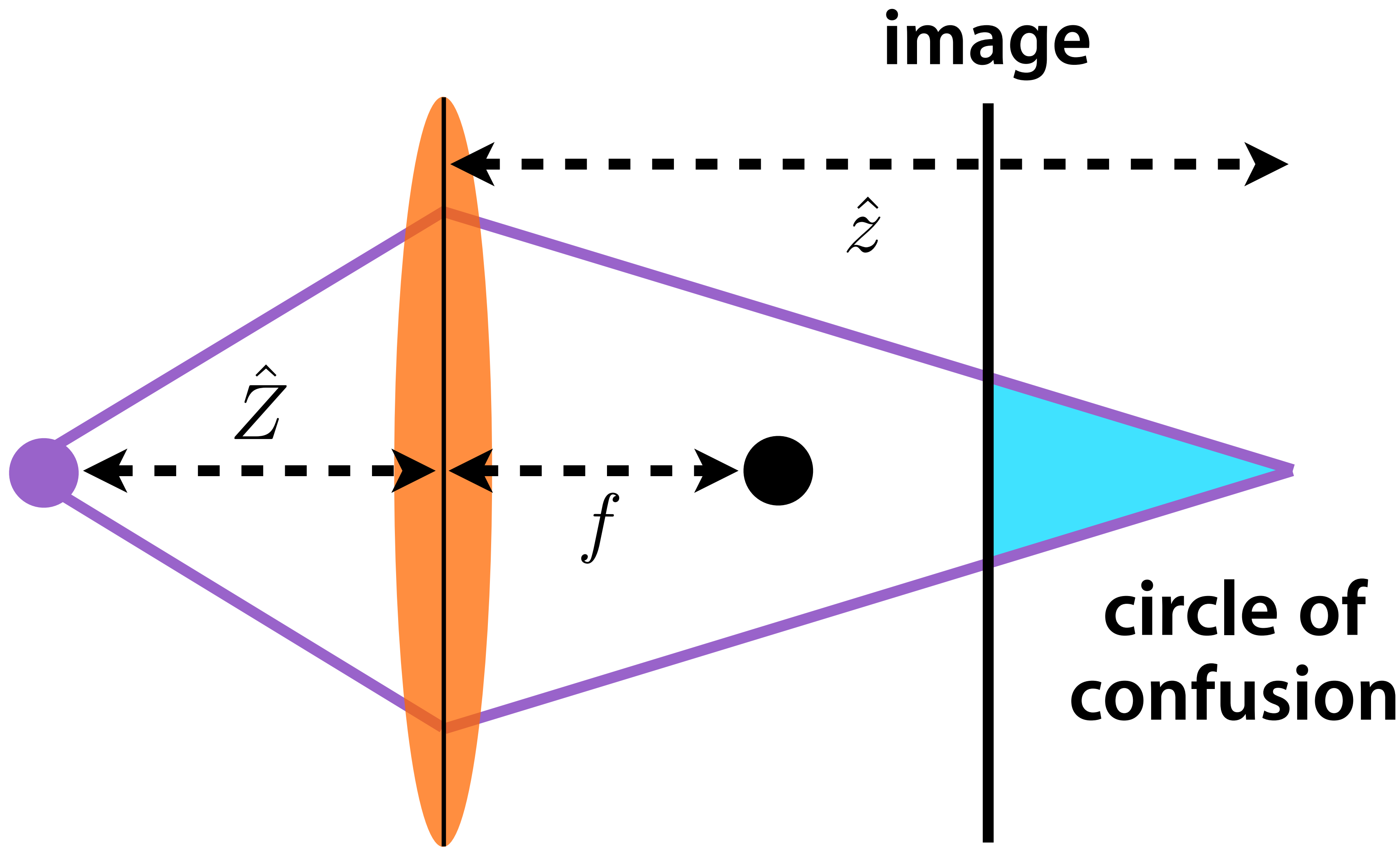


Thin Lens Equation $$\frac{1}{\hat{Z}} + \frac{1}{\hat{z}} = \frac{1}{f}$$

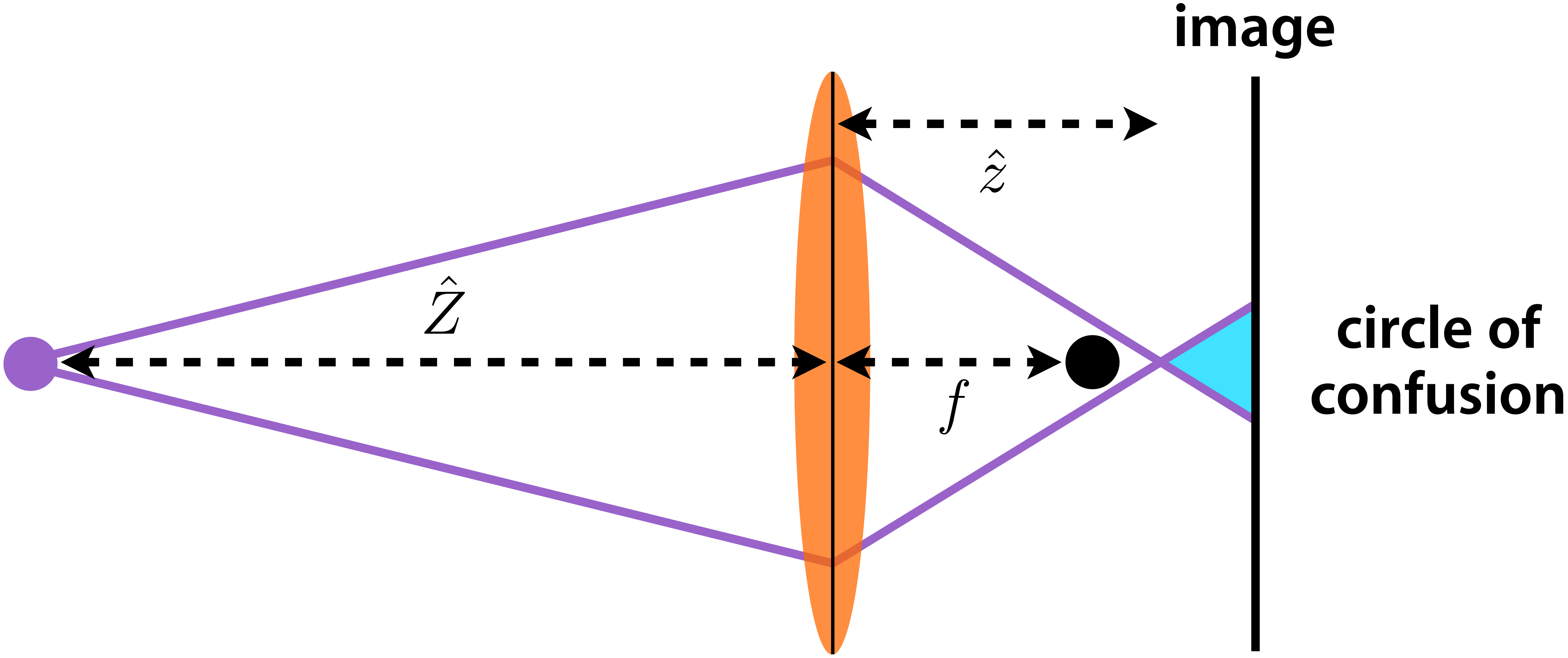
Any point in the world satisfying the thin lens equation is in focus



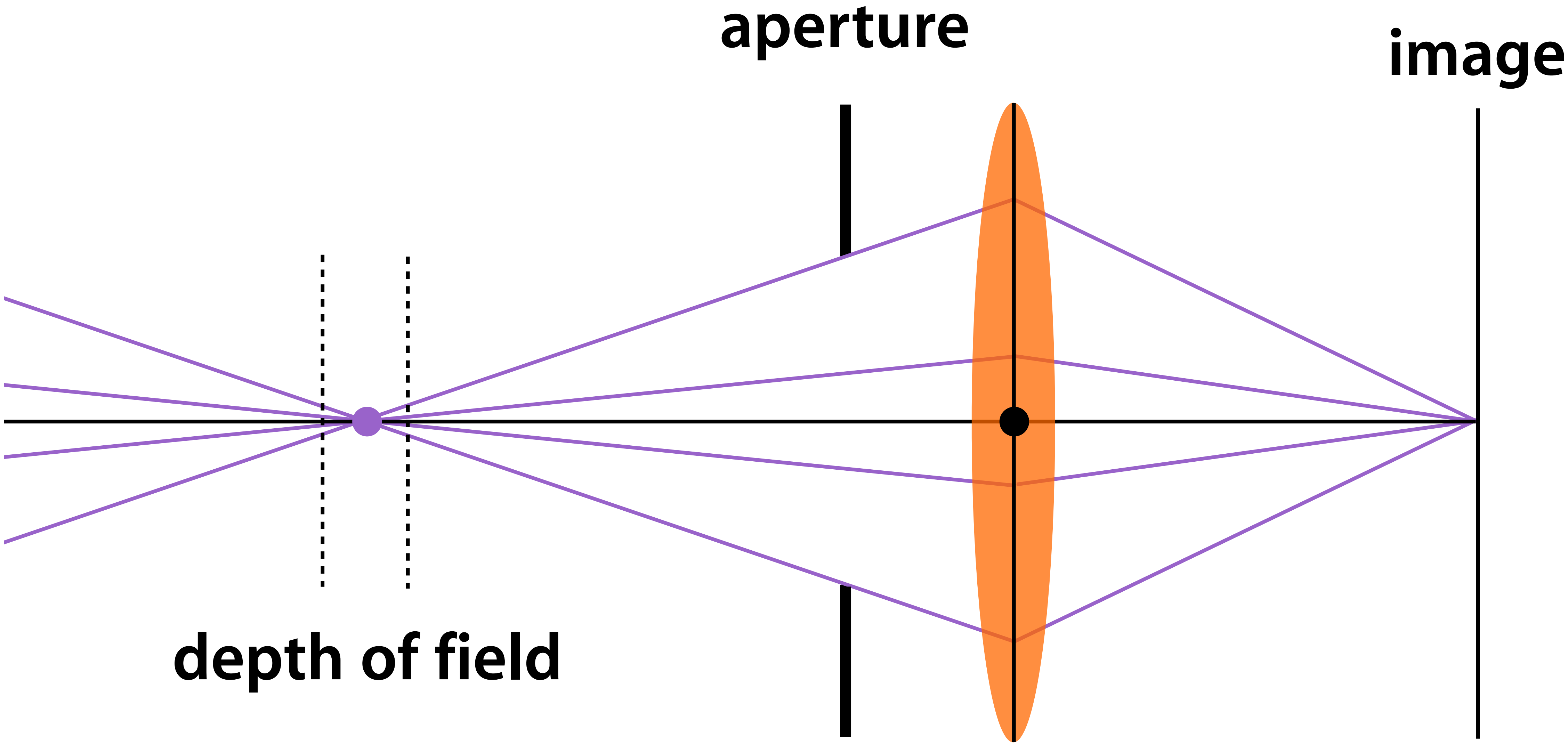
Thin Lens Equation $$\frac{1}{\hat{Z}} + \frac{1}{\hat{z}} = \frac{1}{f}$$

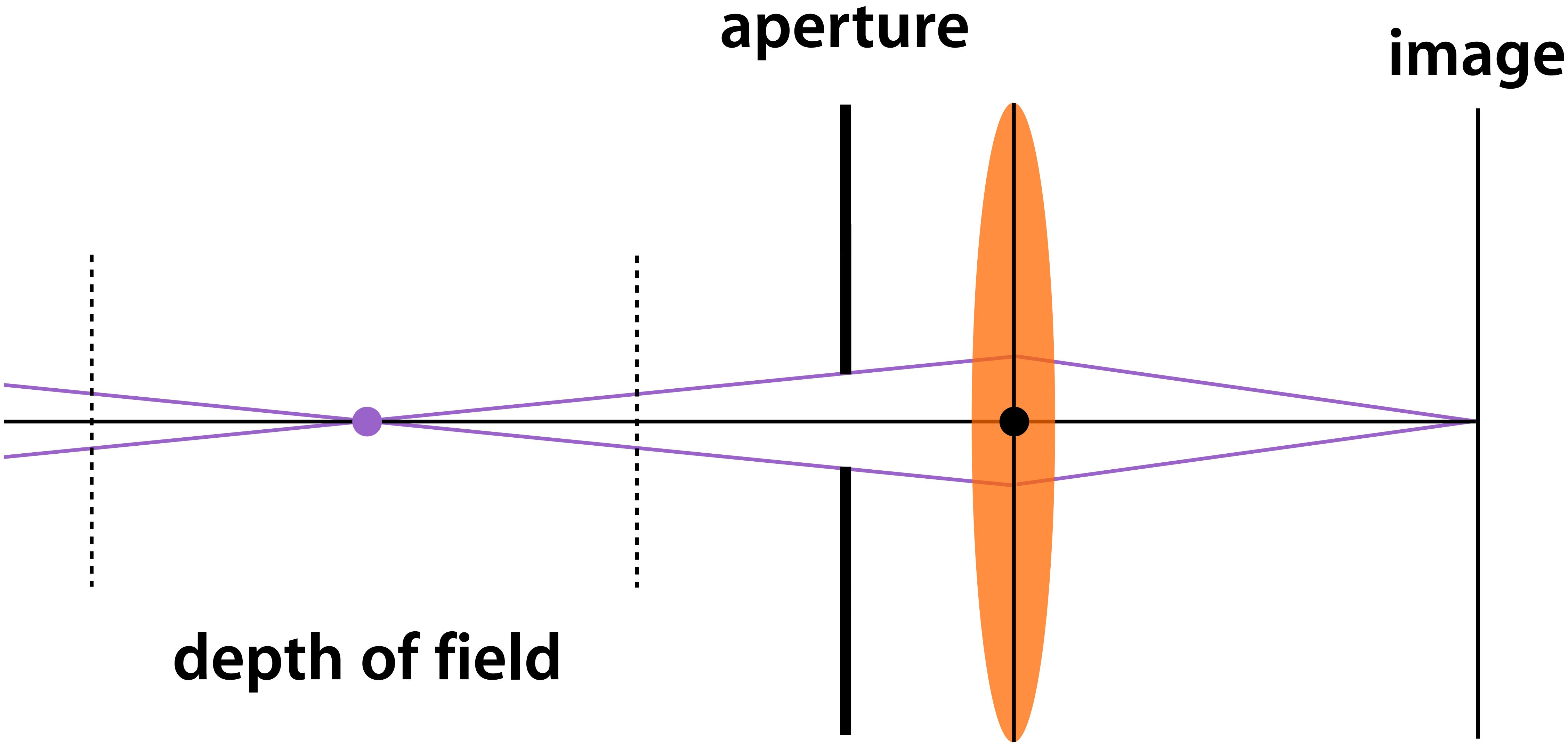


Thin Lens Equation $$\frac{1}{\hat{Z}} + \frac{1}{\hat{z}} = \frac{1}{f}$$



Thin Lens Equation $$\frac{1}{\hat{Z}} + \frac{1}{\hat{z}} = \frac{1}{f}$$


aperture
image
depth of field

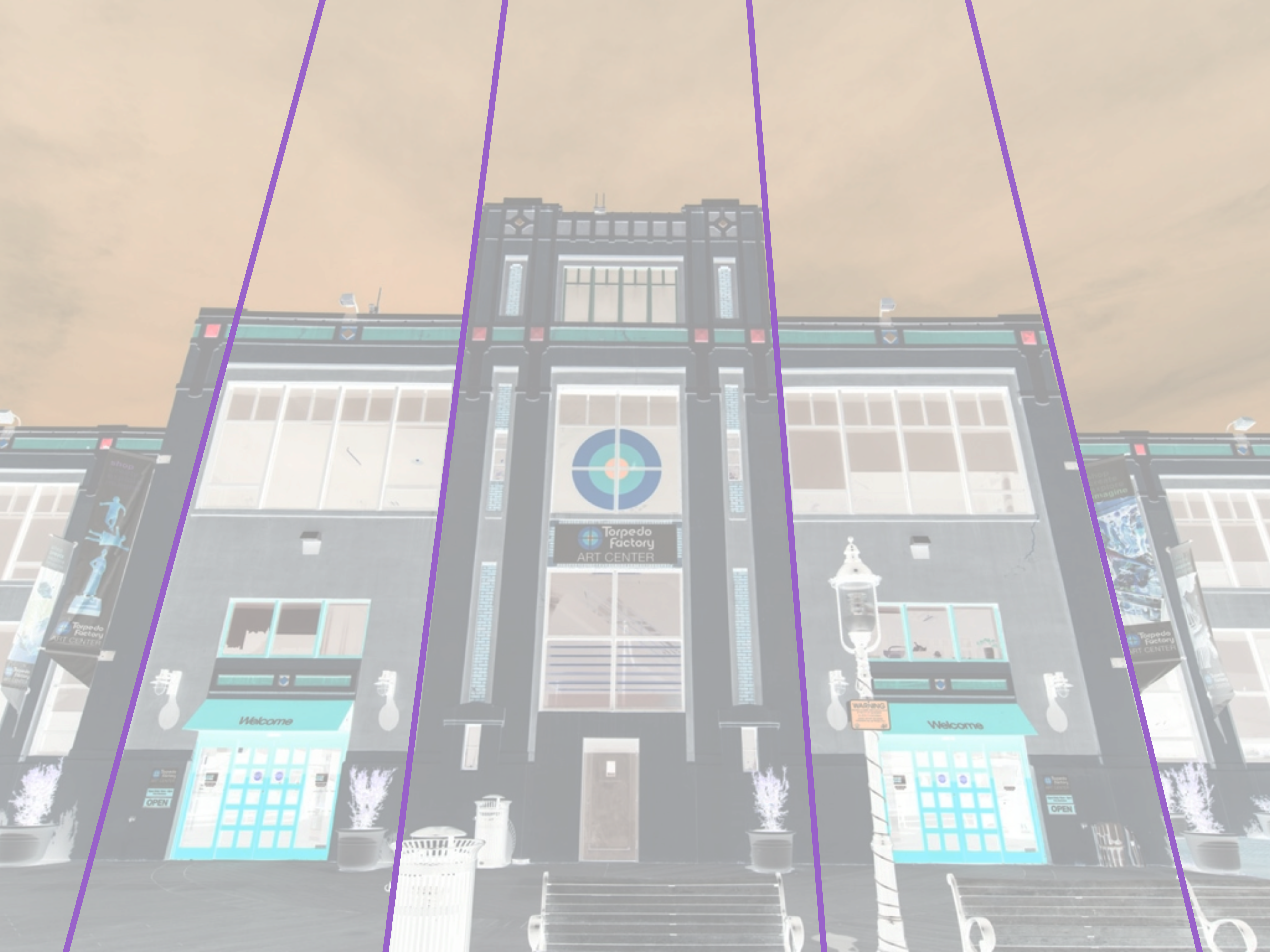

aperture
image
depth of field

$$D = f/N$$

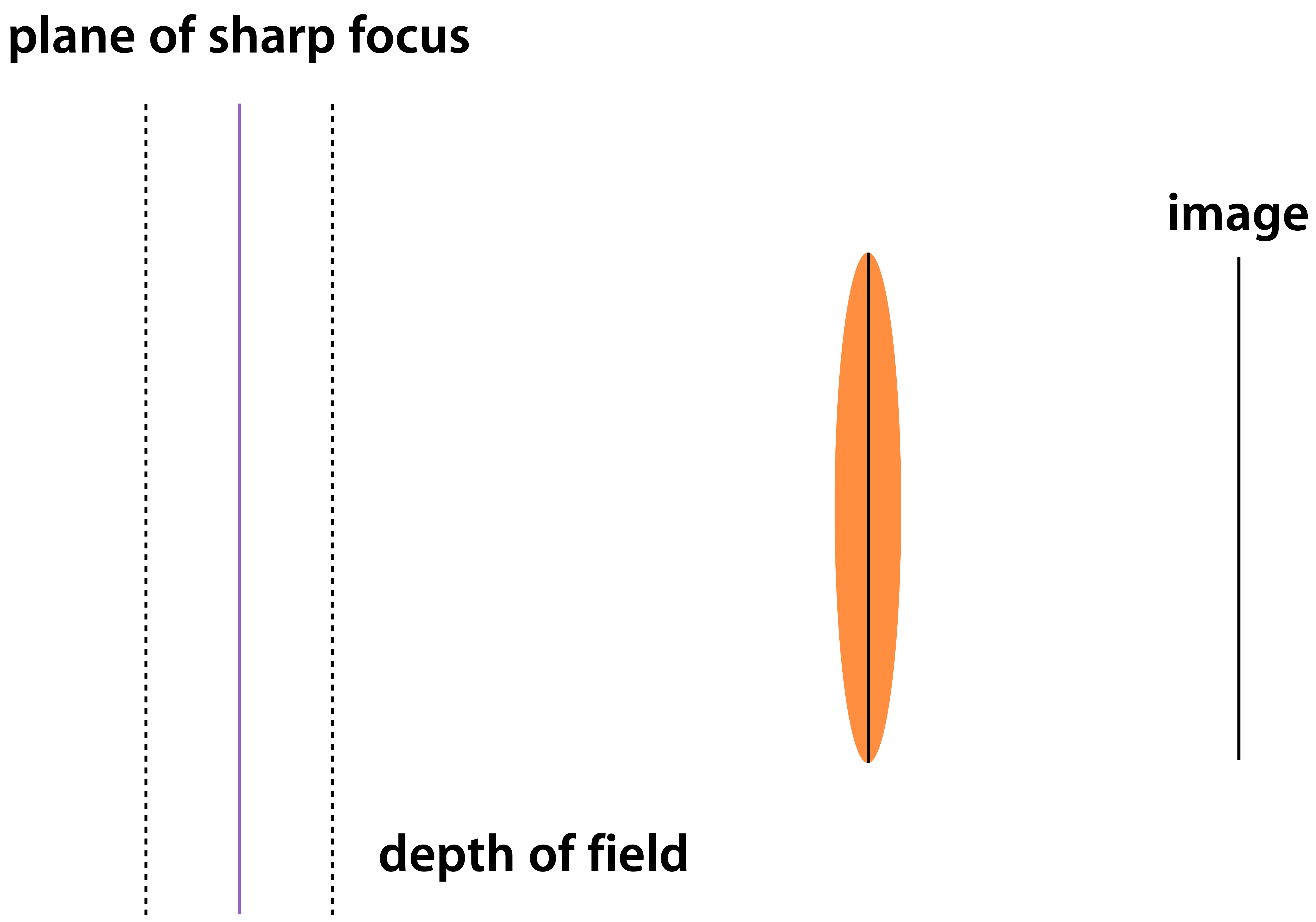
aperture diameter

$$D = f/N$$

f-number


Torpedo Factory
ART CENTER
Welcome
Welcome
OPEN
OPEN
WARNING


Torpedo Factory
ART CENTER
Welcome
Welcome
shop
create
explore
imagine
OPEN
OPEN
WARNING



plane of sharp focus
image
depth of field

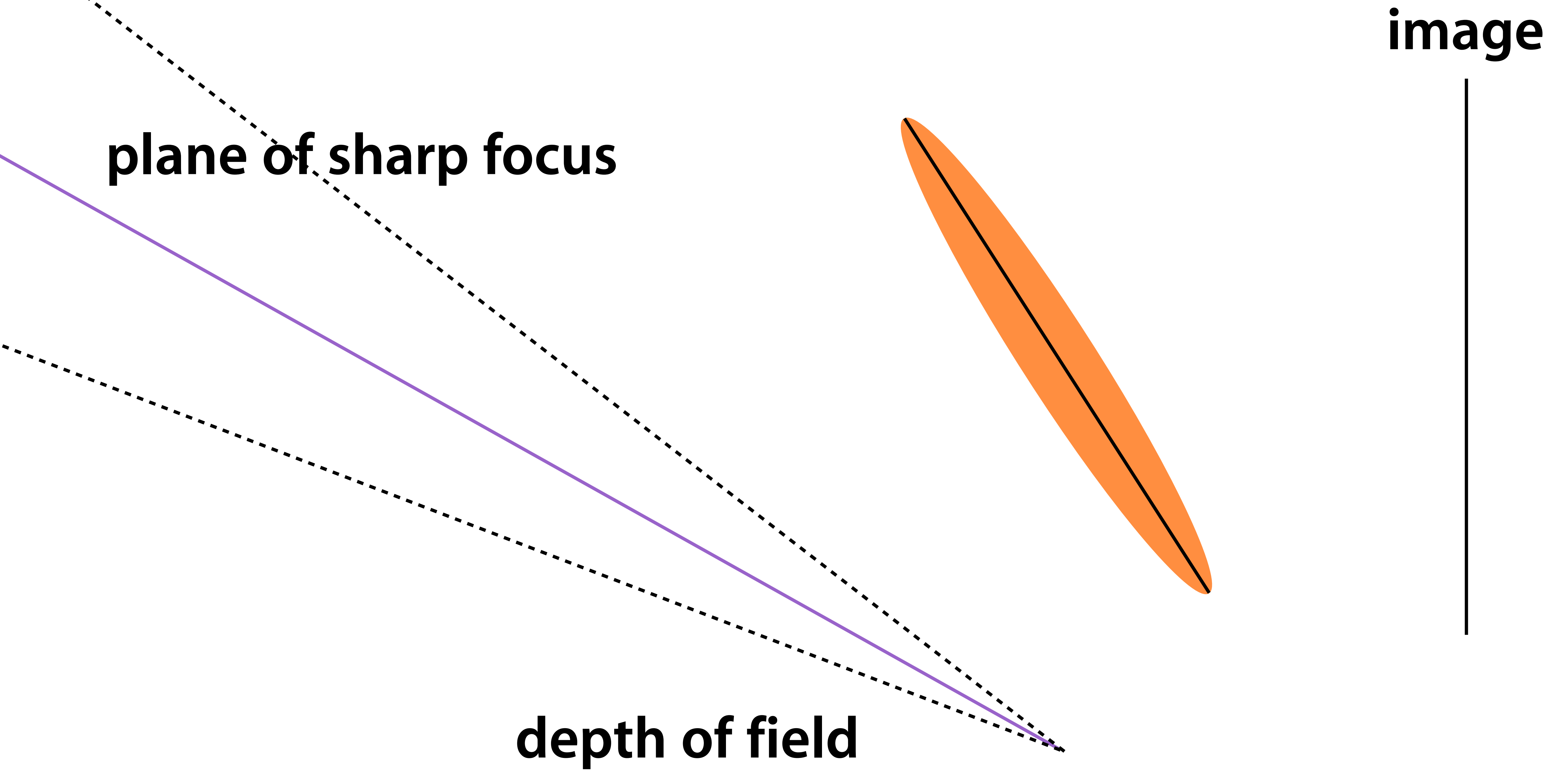
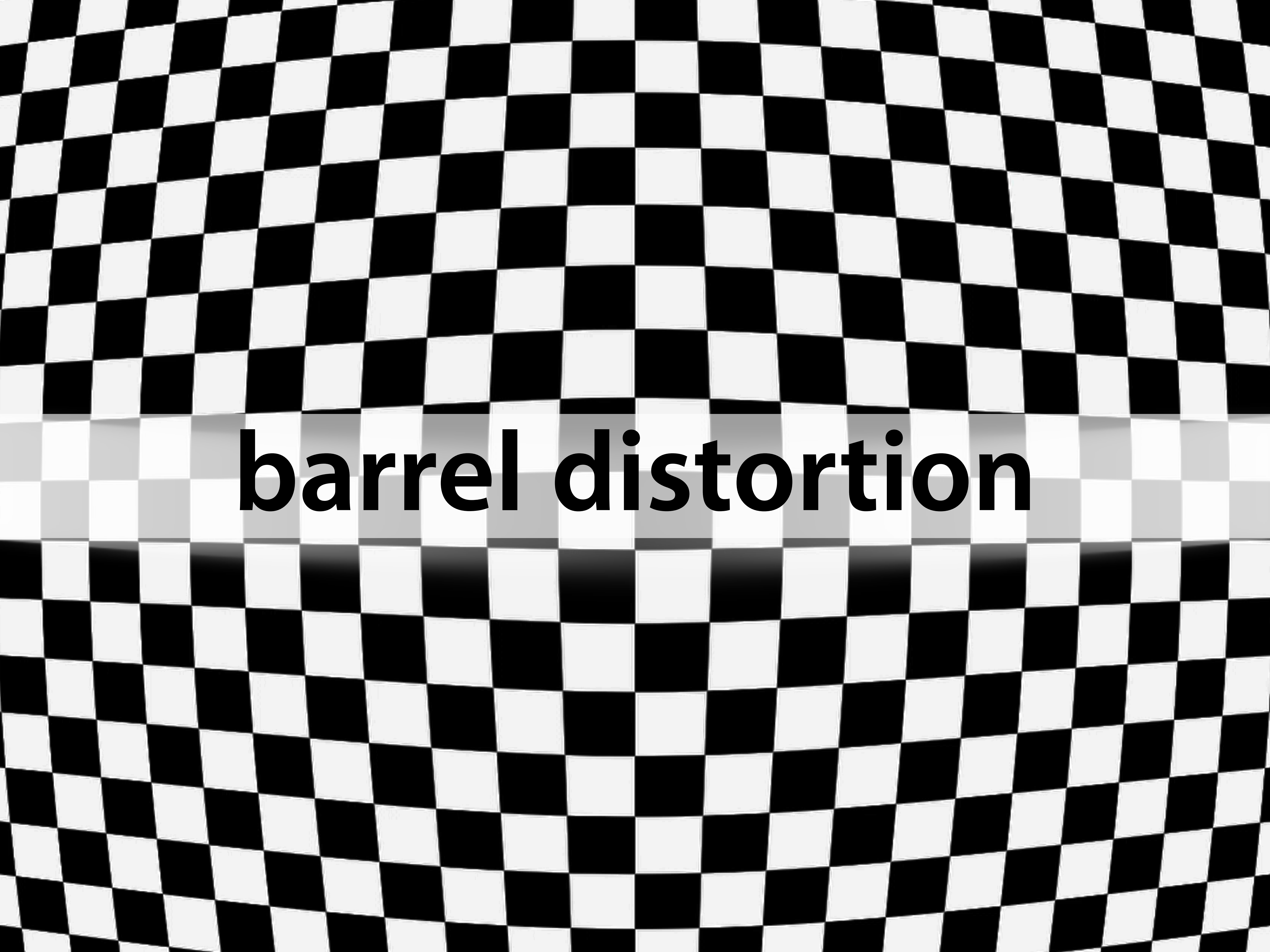
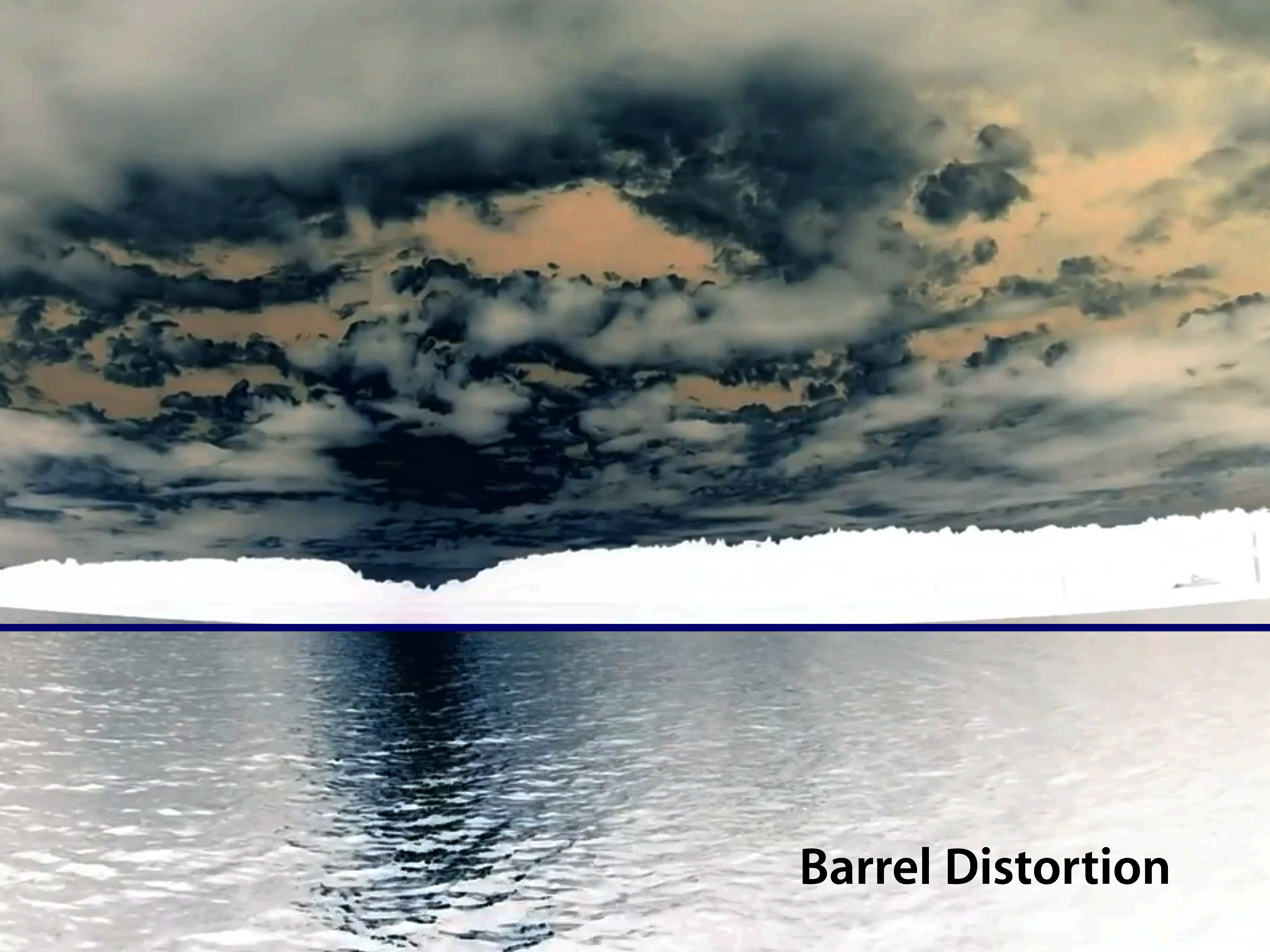


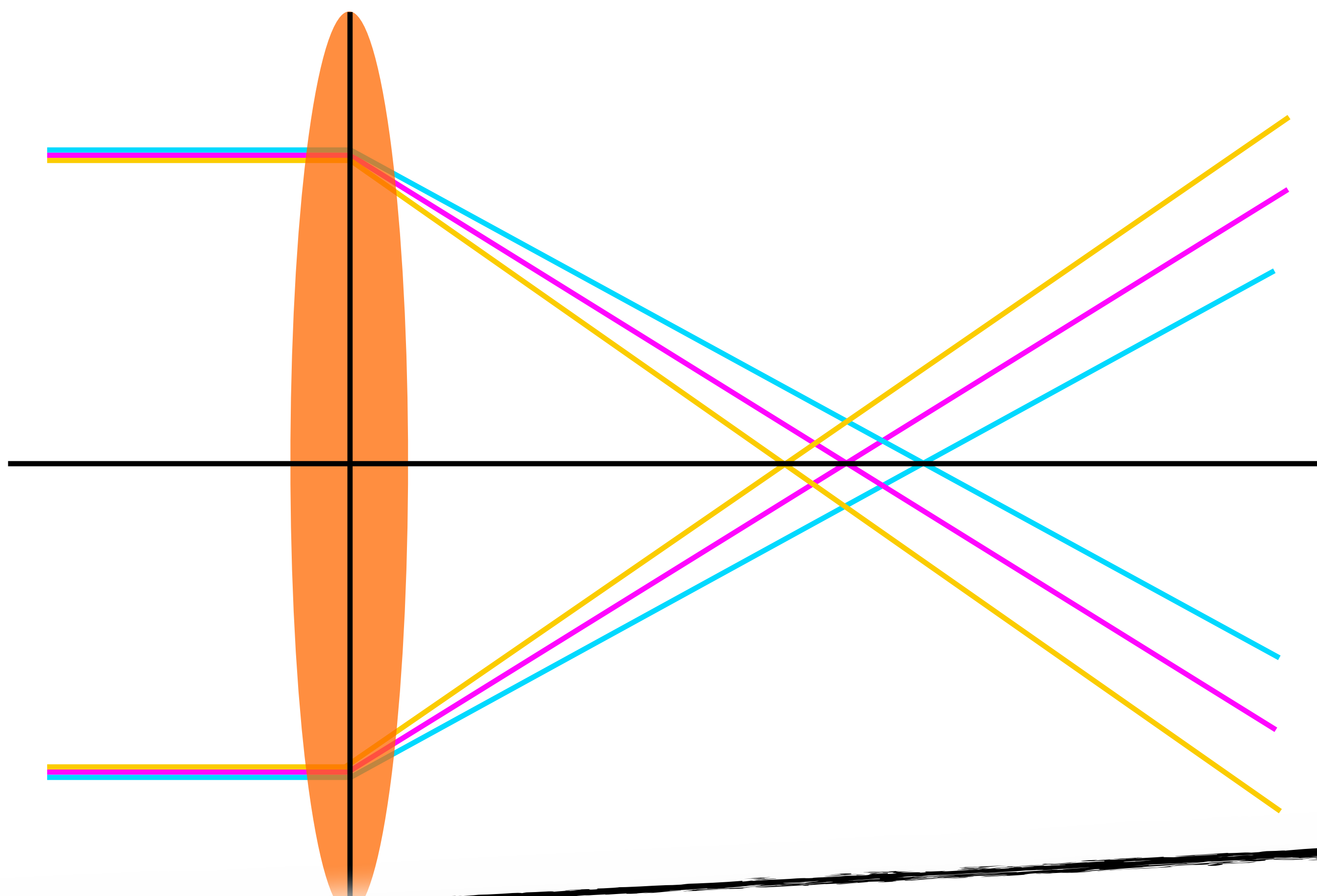
image
plane of sharp focus
depth of field


barrel distortion


pincushion distortion

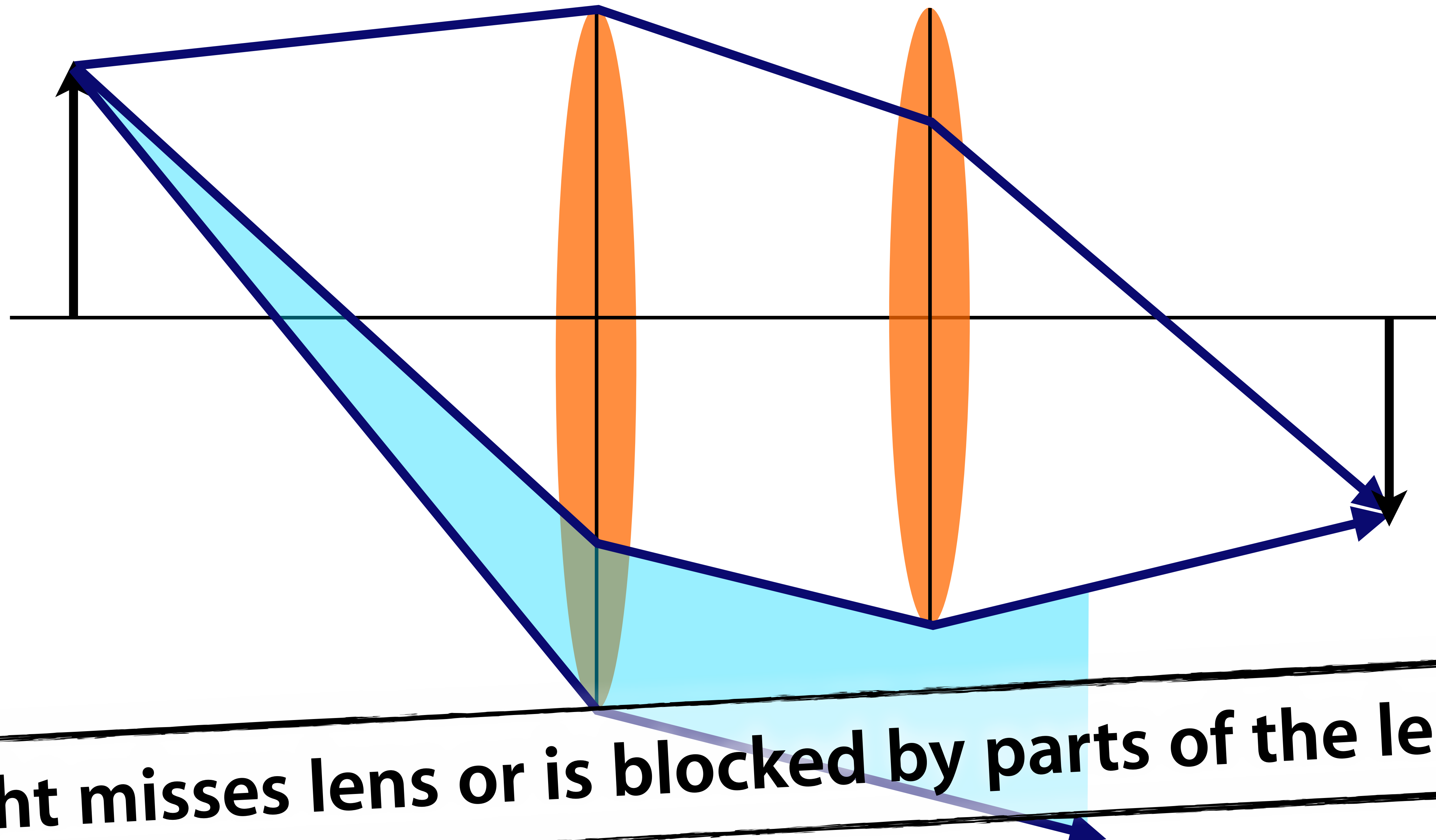

Barrel Distortion

Chromatic Aberration



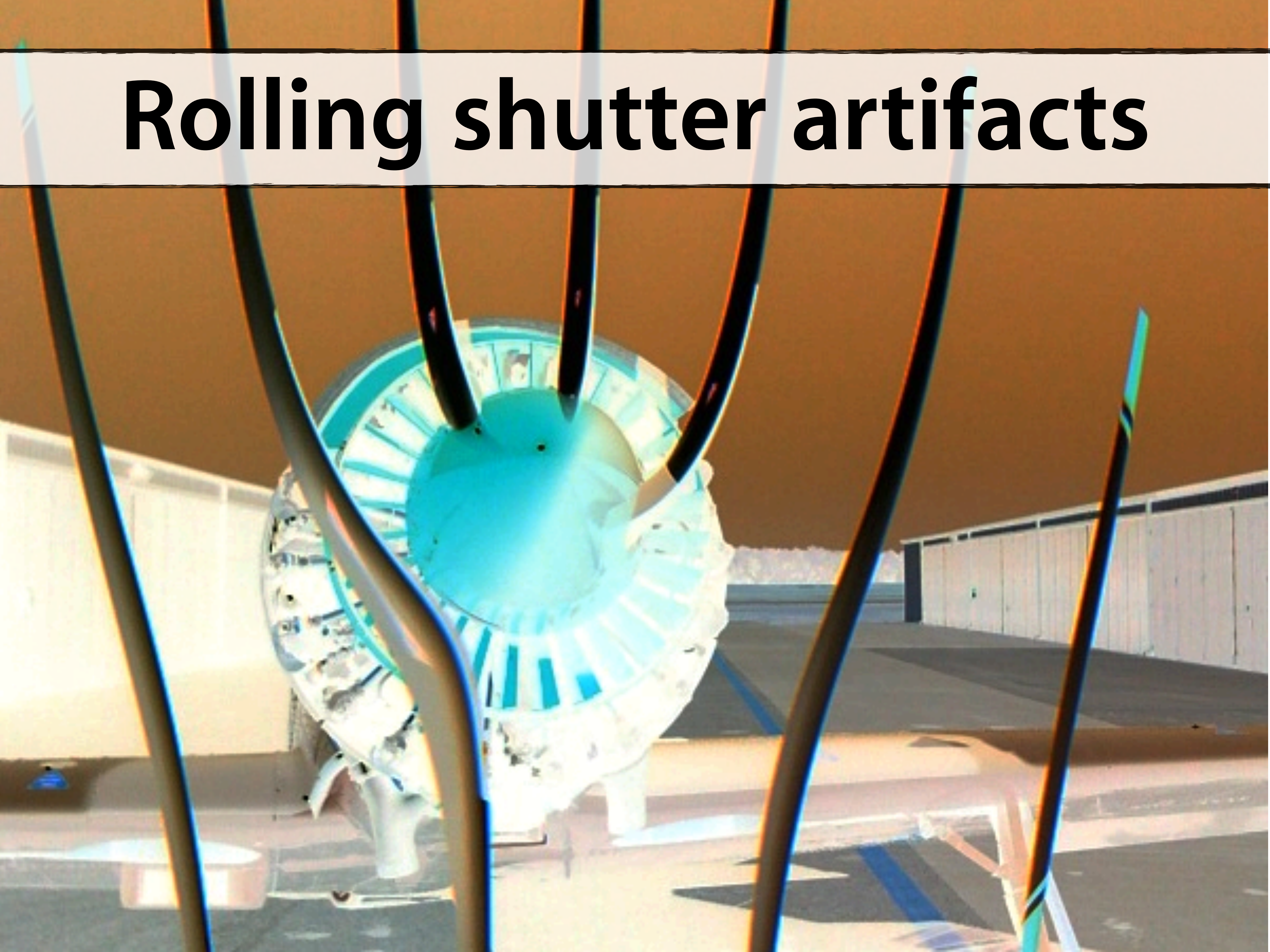
rays of different wavelength focus in different planes

Vignetting

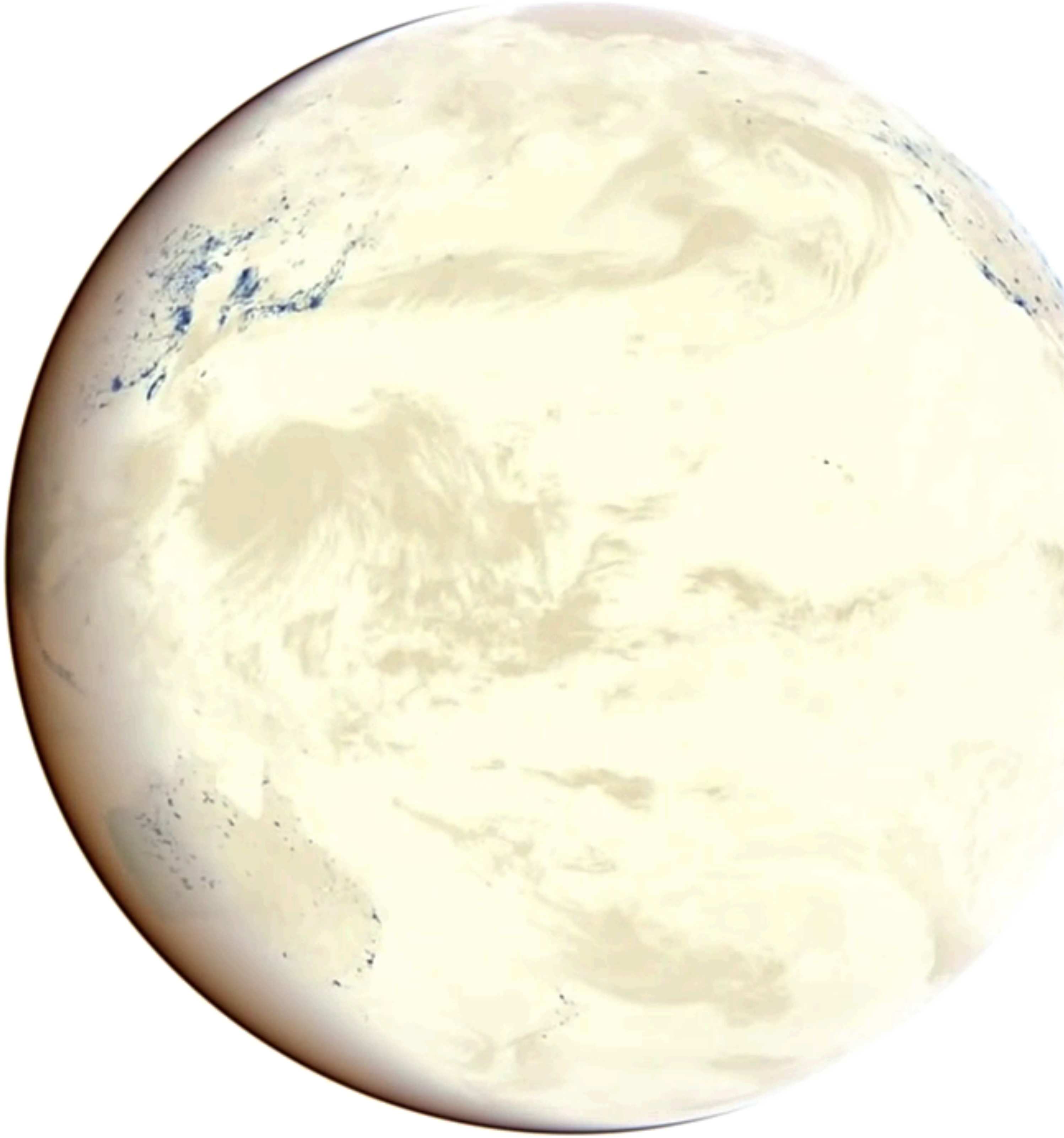


light misses lens or is blocked by parts of the lens

Rolling shutter artifacts



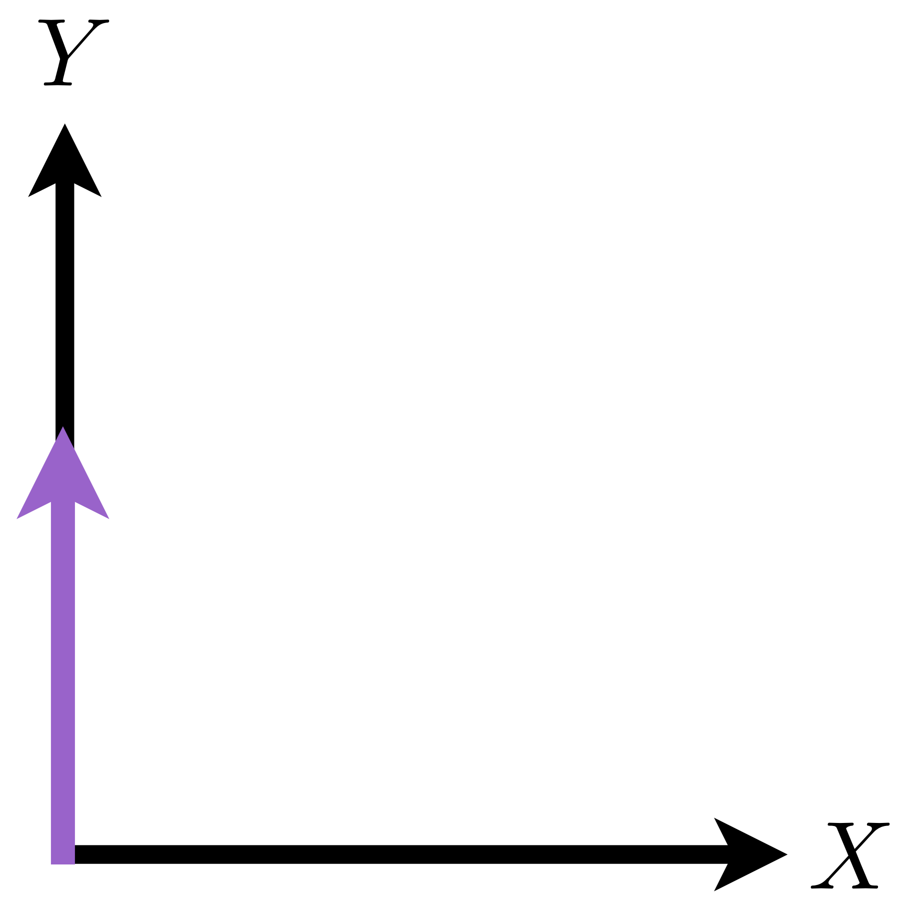
rotation



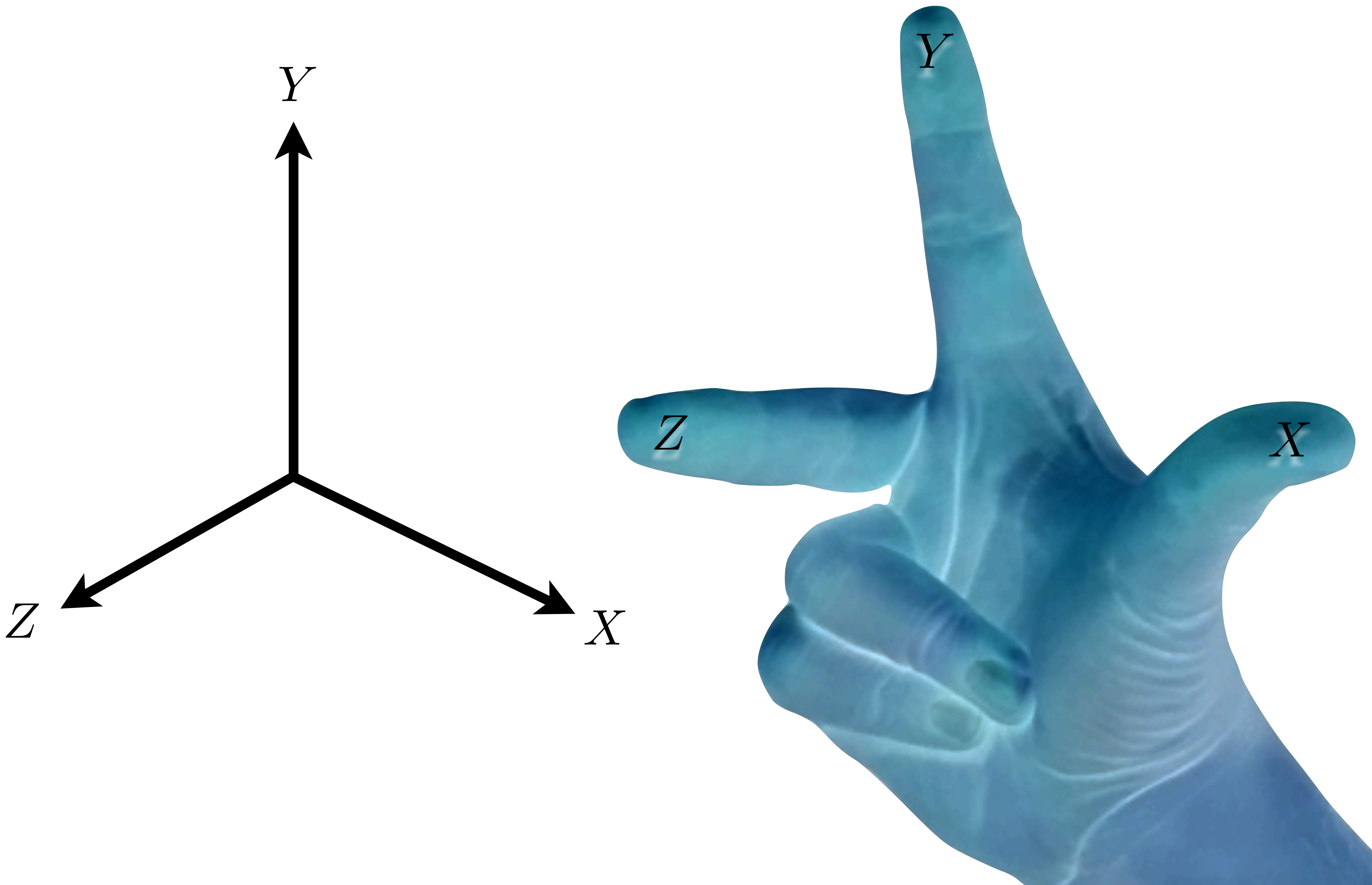
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

positive rotation is counterclockwise

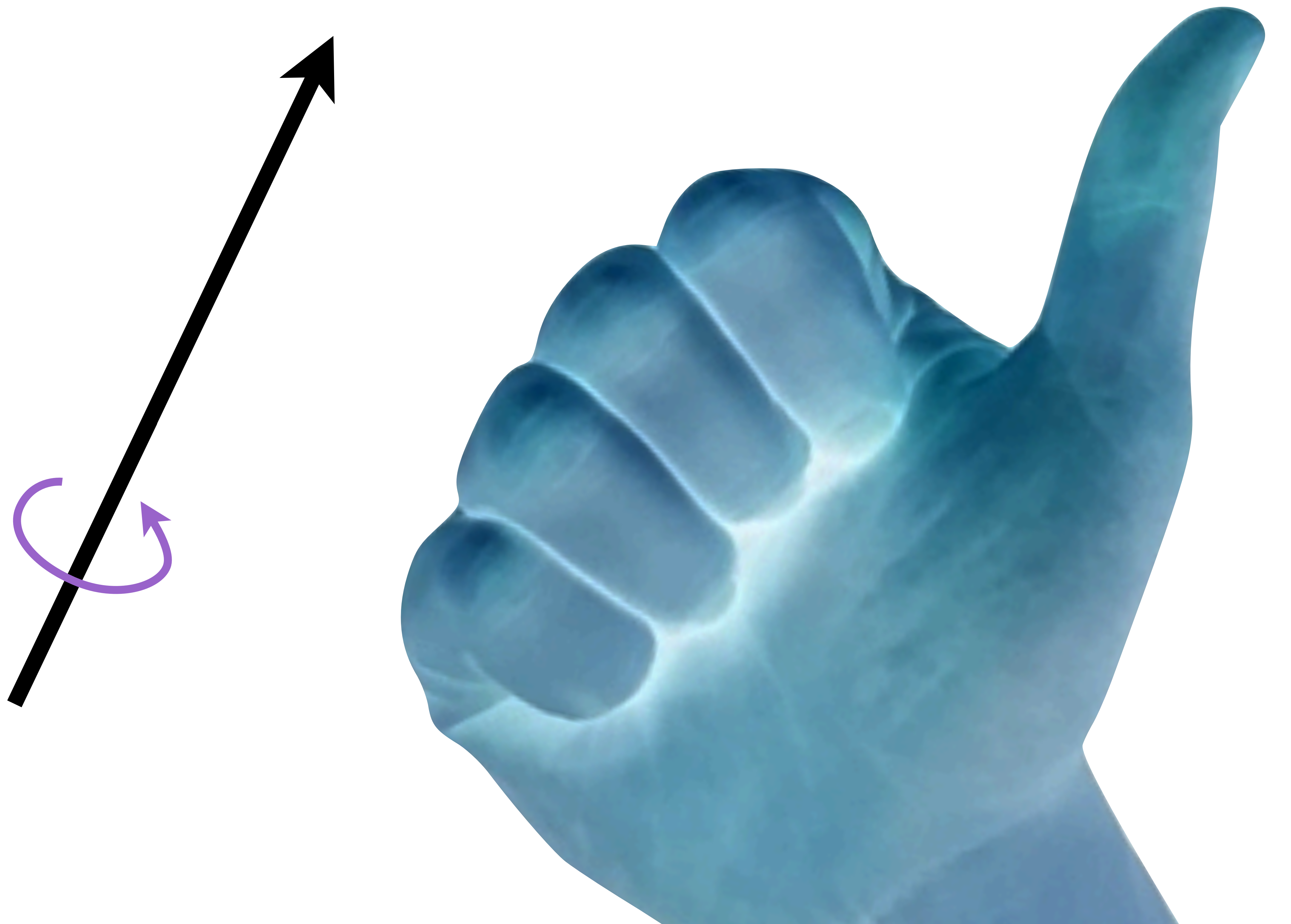
Example: Rotation

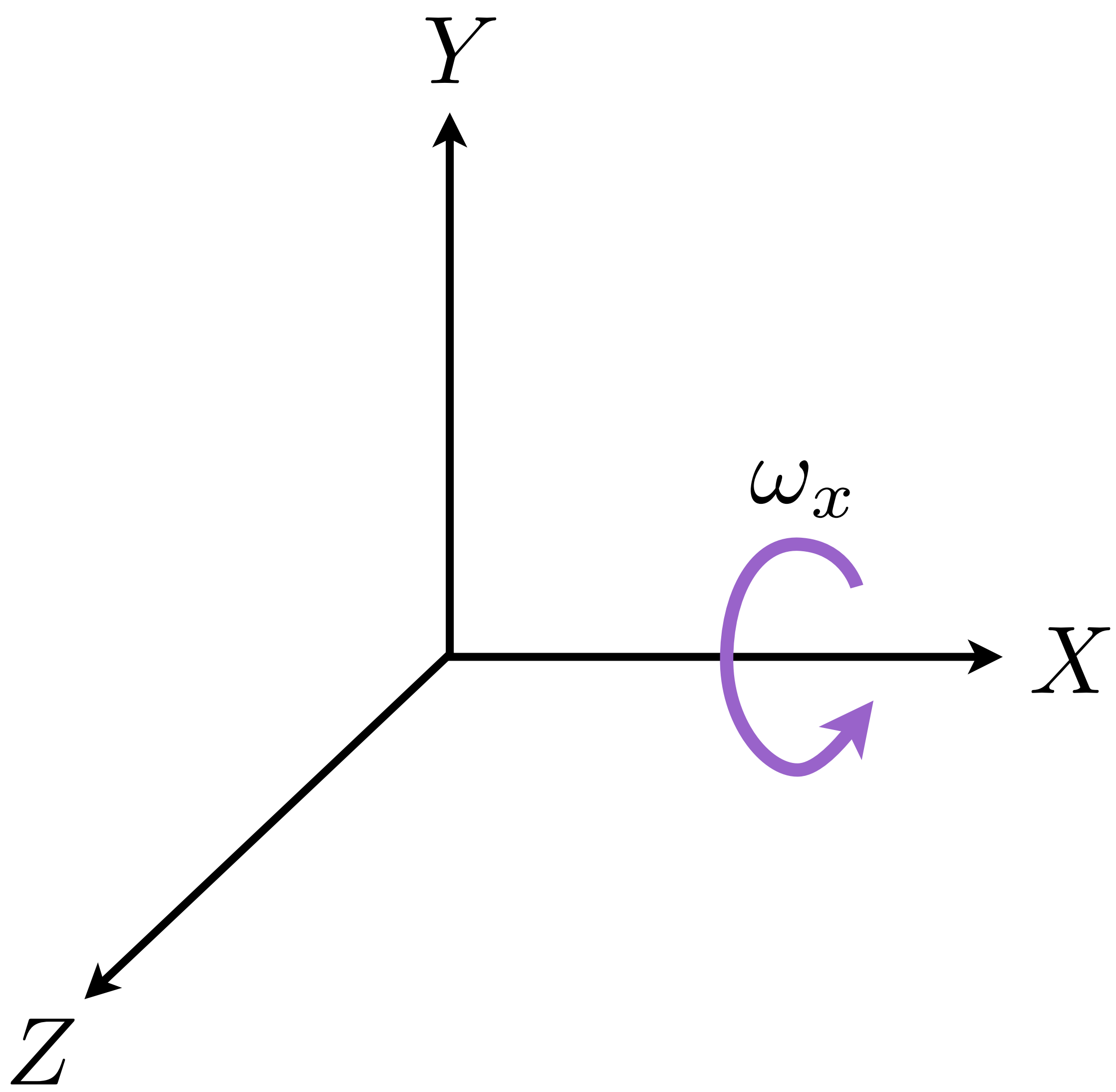


$$\begin{pmatrix} \cos \pi/2 & -\sin \pi/2 \\ \sin \pi/2 & \cos \pi/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$


Y
Z
X
Y
Z
X

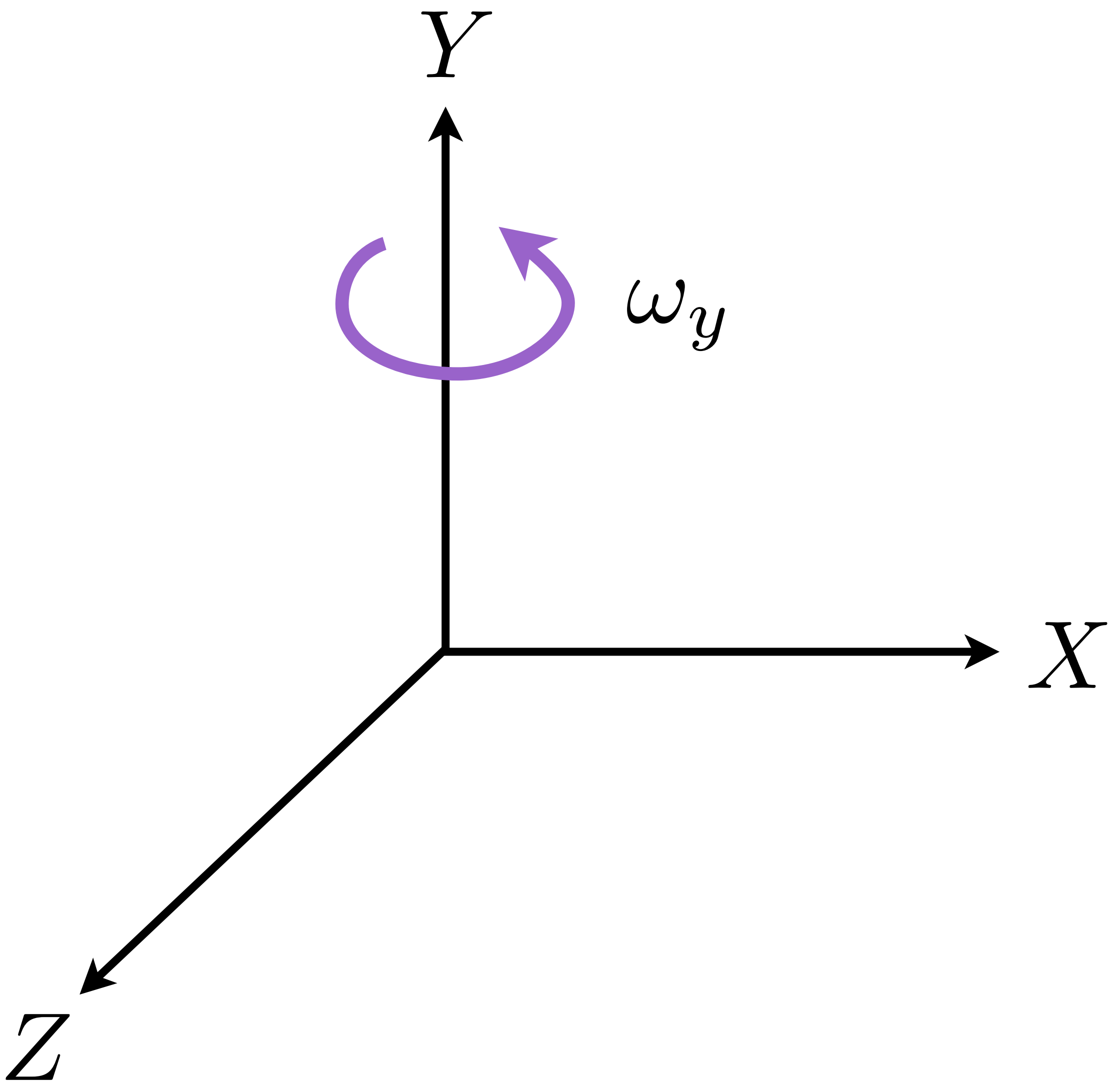
Which way is the positive rotation direction?





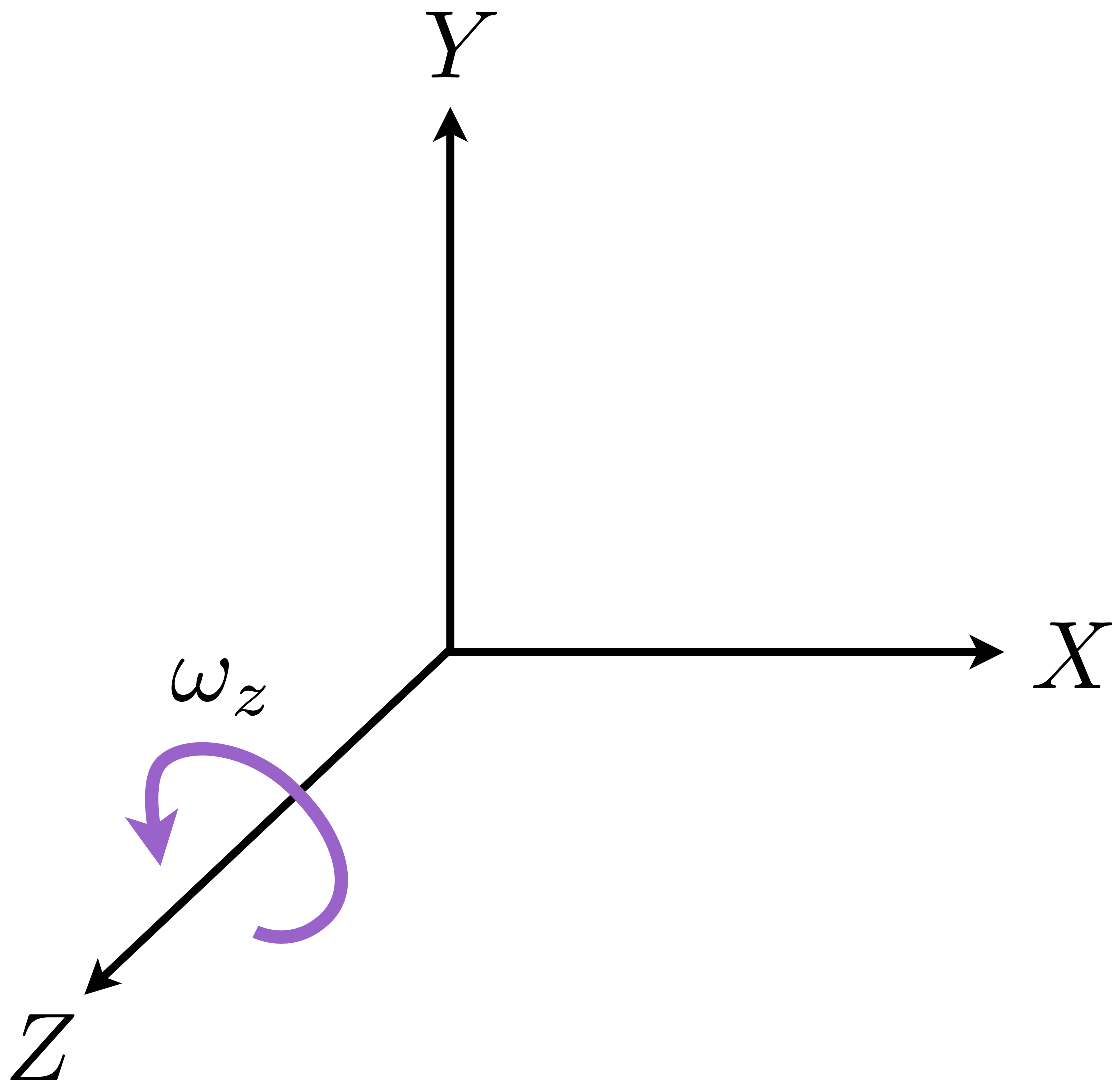
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_x & -\sin\omega_x \\ 0 & \sin\omega_x & \cos\omega_x \end{pmatrix}$$

rotation about the X-axis



$$\begin{pmatrix} \cos\omega_y & 0 & \sin\omega_y \\ 0 & 1 & 0 \\ -\sin\omega_y & 0 & \cos\omega_y \end{pmatrix}$$

rotation about the Y-axis



$$\begin{pmatrix} \cos\omega_z & -\sin\omega_z & 0 \\ \sin\omega_z & \cos\omega_z & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

rotation about the z-axis

An arbitrary rotation, R, can be given as:

$$\mathbf{R}(\omega_x)\mathbf{R}(\omega_y)\mathbf{R}(\omega_z)$$

$$=$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_x & -\sin\omega_x \\ 0 & \sin\omega_x & \cos\omega_x \end{pmatrix} \begin{pmatrix} \cos\omega_y & 0 & \sin\omega_y \\ 0 & 1 & 0 \\ -\sin\omega_y & 0 & \cos\omega_y \end{pmatrix} \begin{pmatrix} \cos\omega_z & -\sin\omega_z & 0 \\ \sin\omega_z & \cos\omega_z & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$=$$

$$\begin{pmatrix} \cos\omega_y\cos\omega_z & -\cos\omega_y\sin\omega_z & \sin\omega_y \\ \sin\omega_x\sin\omega_y\cos\omega_z + \cos\omega_x\sin\omega_z & -\sin\omega_x\sin\omega_y\sin\omega_z + \cos\omega_x\cos\omega_z & -\sin\omega_x\cos\omega_y \\ -\cos\omega_x\sin\omega_y\cos\omega_z + \sin\omega_x\sin\omega_z & \cos\omega_x\sin\omega_y\sin\omega_z + \sin\omega_x\cos\omega_z & \cos\omega_x\cos\omega_z \end{pmatrix}$$

3D Rotation

An arbitrary rotation, R, can be given as:

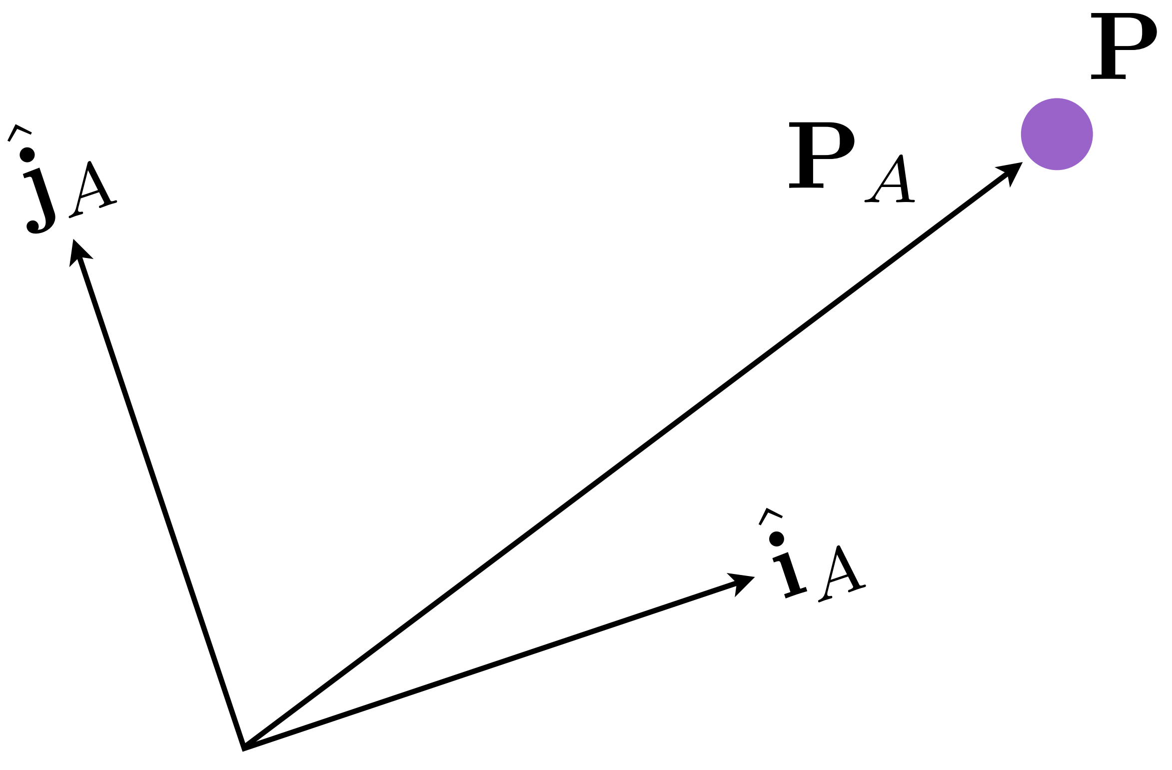
$$\mathbf{R}(\omega_x)\mathbf{R}(\omega_y)\mathbf{R}(\omega_z)$$

order of rotations matters

$$\mathbf{R}^\top\mathbf{R} = \mathbf{R}\mathbf{R}^\top = \mathbf{I}$$

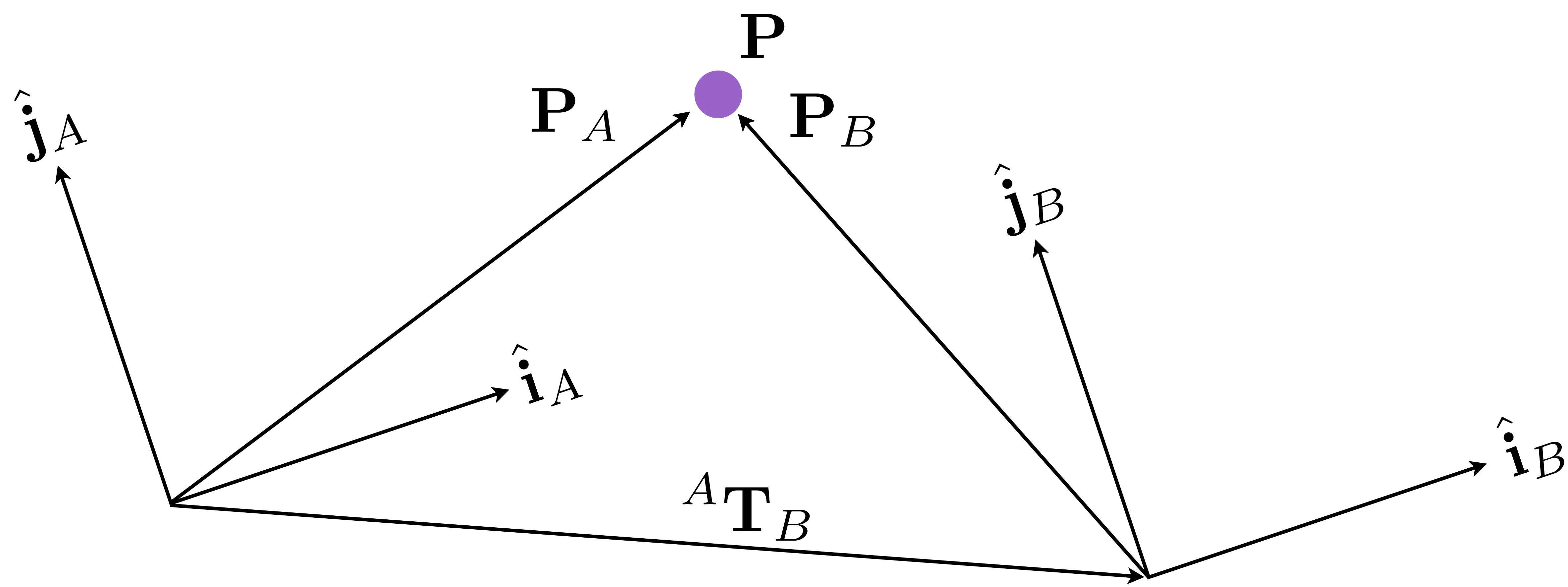
$$\det \mathbf{R} = 1$$

several alternative ways to represent 3D rotations



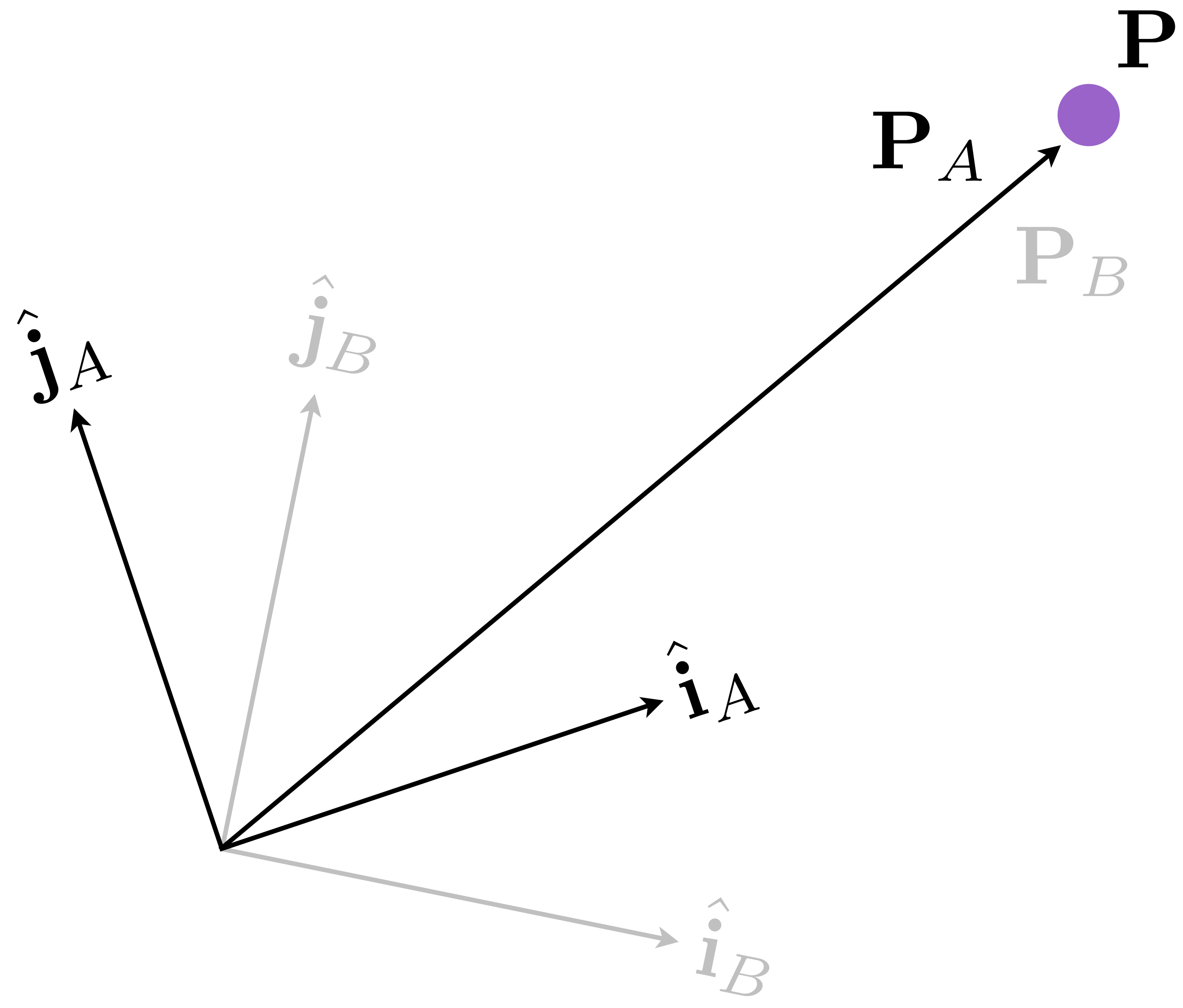
$$\mathbf{P} = x_A \hat{\mathbf{i}}_A + y_A \hat{\mathbf{j}}_A$$

translation



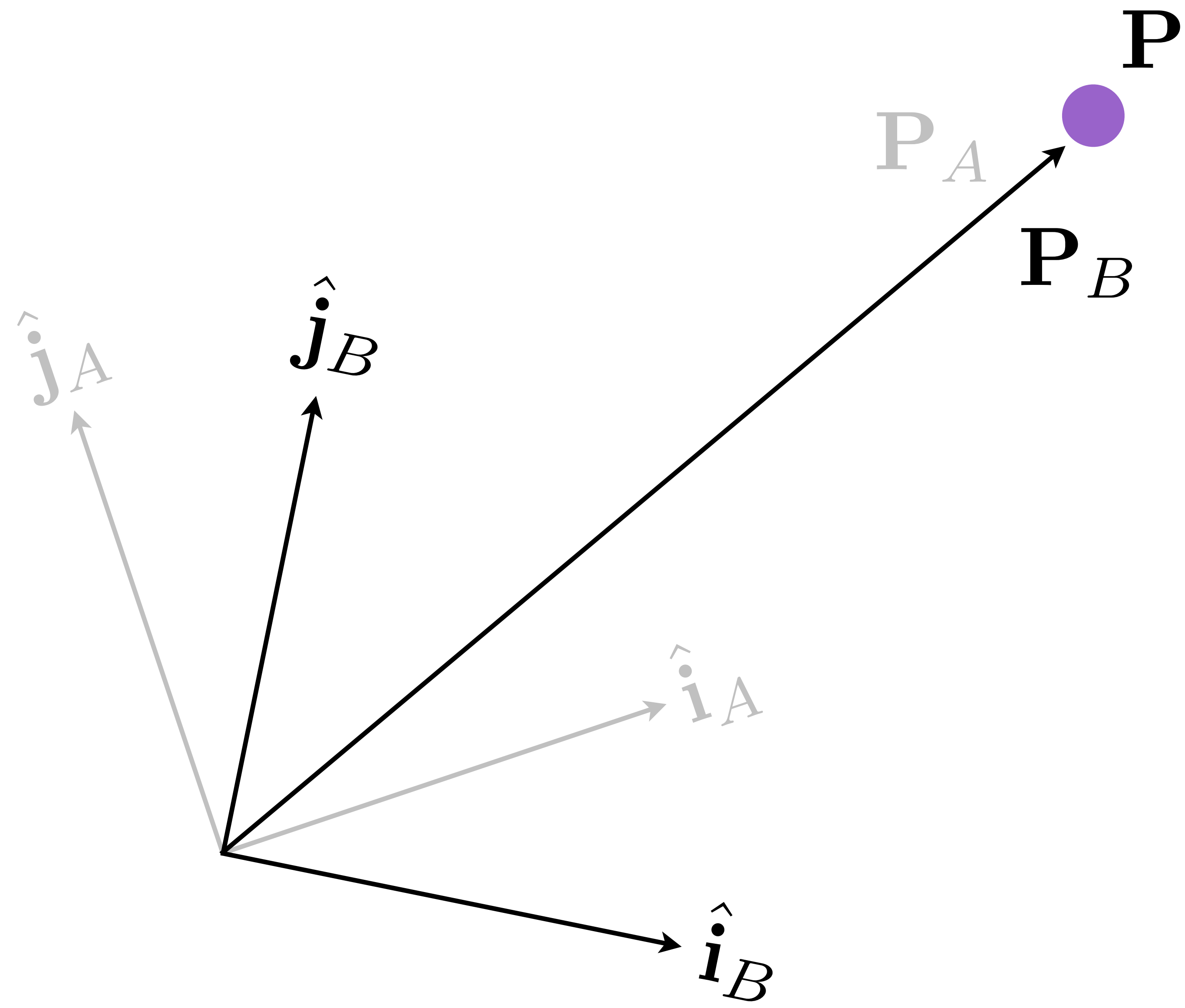
$$\mathbf{P}_A = \mathbf{P}_B + {}^A\mathbf{T}_B$$

rotation



$$\mathbf{P} = x_A \hat{\mathbf{i}}_A + y_A \hat{\mathbf{j}}_A$$

rotation



$$\mathbf{P} = x_A \hat{\mathbf{i}}_A + y_A \hat{\mathbf{j}}_A = x_B \hat{\mathbf{i}}_B + y_B \hat{\mathbf{j}}_B$$

$$x_A\hat{\mathbf{i}}_A + y_A\hat{\mathbf{j}}_A = x_B\hat{\mathbf{i}}_B + y_B\hat{\mathbf{j}}_B$$

rewrite as matrix

$$\begin{pmatrix}\hat{\mathbf{i}}_A & \hat{\mathbf{j}}_A\end{pmatrix}\begin{pmatrix}x_A \\ y_A\end{pmatrix} = \begin{pmatrix}\hat{\mathbf{i}}_B & \hat{\mathbf{j}}_B\end{pmatrix}\begin{pmatrix}x_B \\ y_B\end{pmatrix}$$

matrix inversion

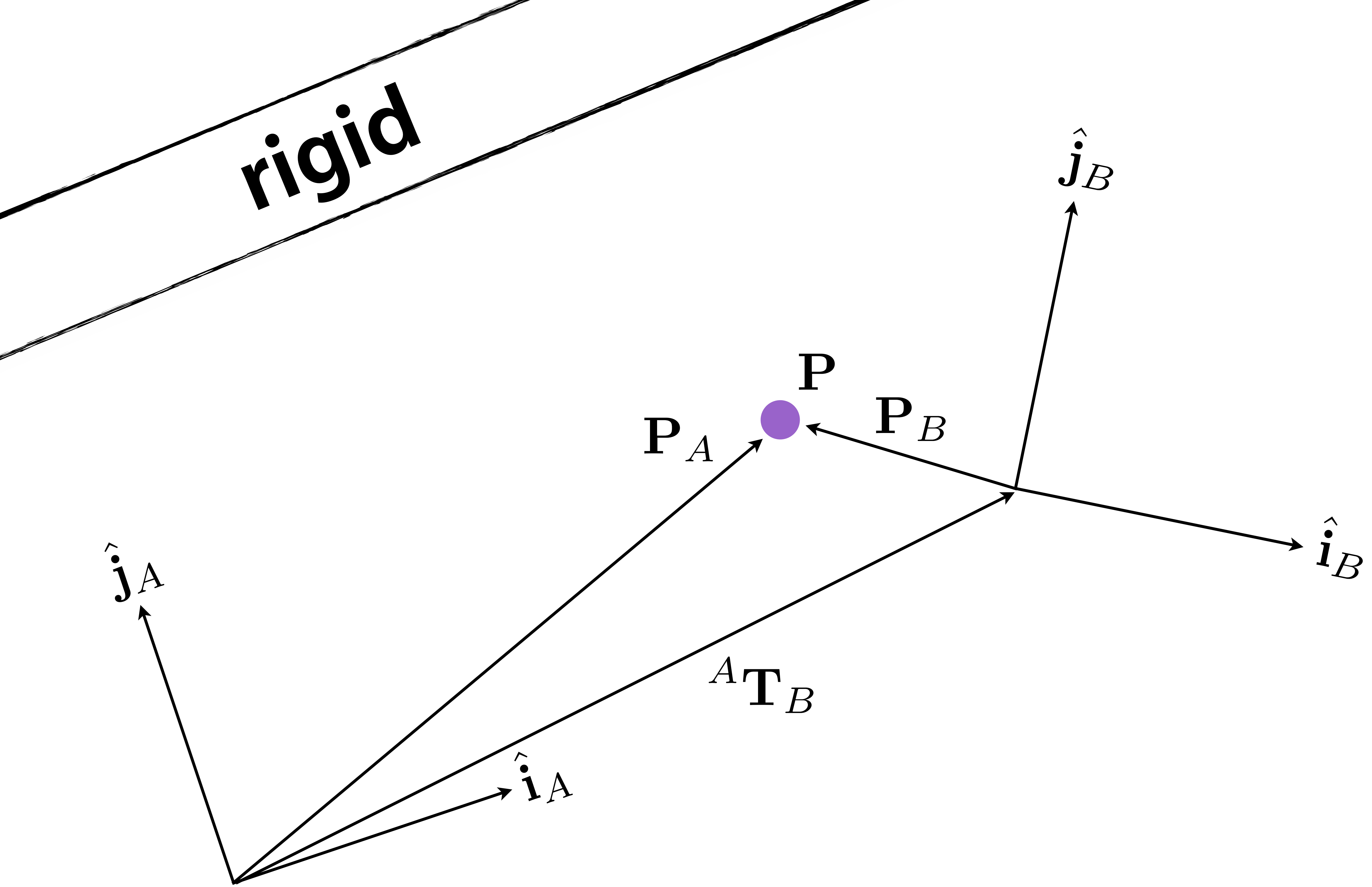
$$\begin{pmatrix}x_A \\ y_A\end{pmatrix} = \begin{pmatrix}\hat{\mathbf{i}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{i}}_A \cdot \hat{\mathbf{j}}_B \\ \hat{\mathbf{j}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{j}}_A \cdot \hat{\mathbf{j}}_B\end{pmatrix}\begin{pmatrix}x_B \\ y_B\end{pmatrix}$$

$$\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{i}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{i}}_A \cdot \hat{\mathbf{j}}_B \\ \hat{\mathbf{j}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{j}}_A \cdot \hat{\mathbf{j}}_B \end{pmatrix} \begin{pmatrix} x_B \\ y_B \end{pmatrix}$$

matrix columns are the axes of frame B expressed in frame A

$$\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \underbrace{\begin{pmatrix} \hat{\mathbf{i}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{i}}_A \cdot \hat{\mathbf{j}}_B \\ \hat{\mathbf{j}}_A \cdot \hat{\mathbf{i}}_B & \hat{\mathbf{j}}_A \cdot \hat{\mathbf{j}}_B \end{pmatrix}}_{{}^{A}\mathbf{R}_B} \begin{pmatrix} x_B \\ y_B \end{pmatrix}$$

$$\mathbf{P}_A = {}^{A}\mathbf{R}_B \mathbf{P}_B$$



$$\mathbf{P}_A = {}^A\mathbf{R}_B \mathbf{P}_B + {}^A\mathbf{T}_B$$

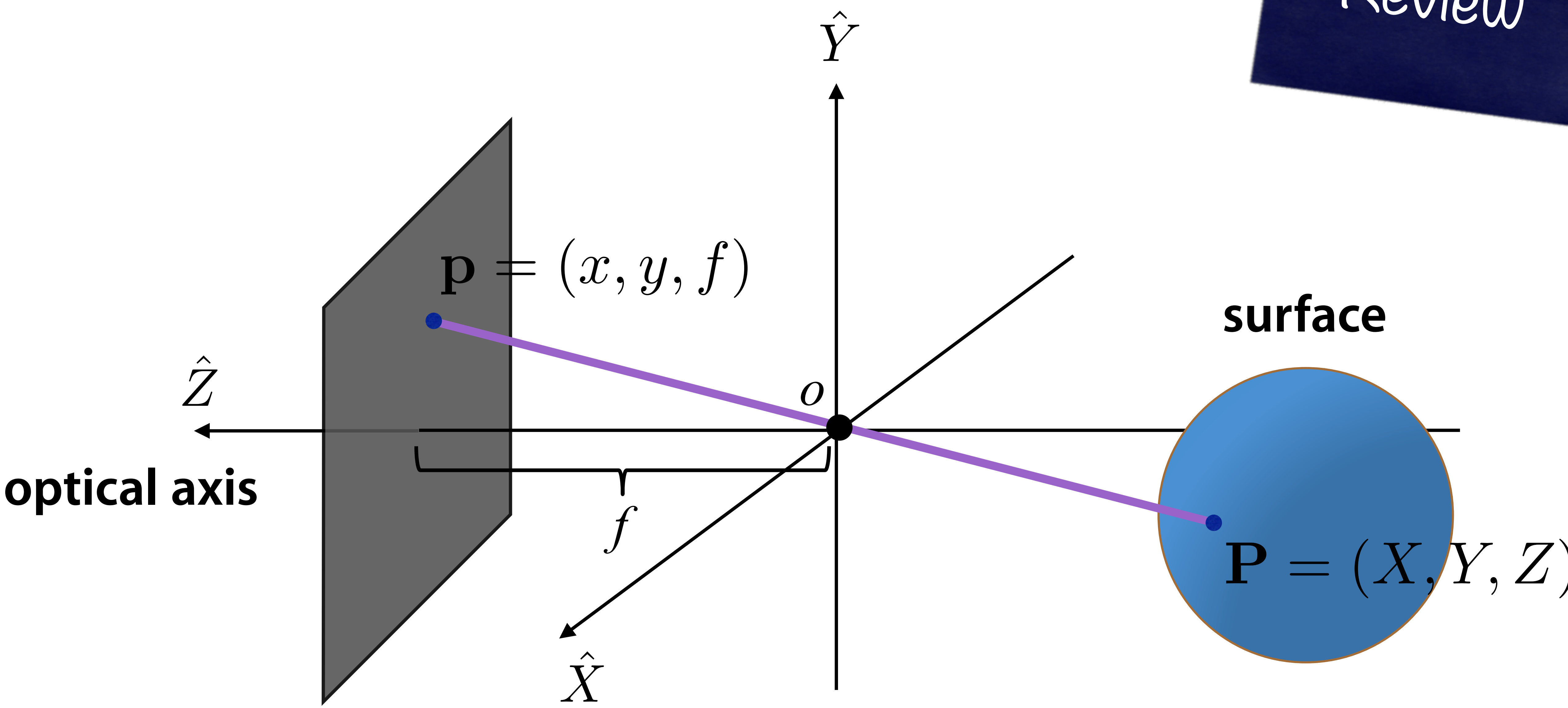
$$\mathbf{P}_A = {}^A\mathbf{R}_B \mathbf{P}_B + {}^A\mathbf{T}_B$$

rewrite as matrix multiplication

$$\begin{pmatrix} \mathbf{P}_A \\ 1 \end{pmatrix} = \begin{pmatrix} {}^A\mathbf{R}_B & {}^A\mathbf{T}_B \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{P}_B \\ 1 \end{pmatrix}$$

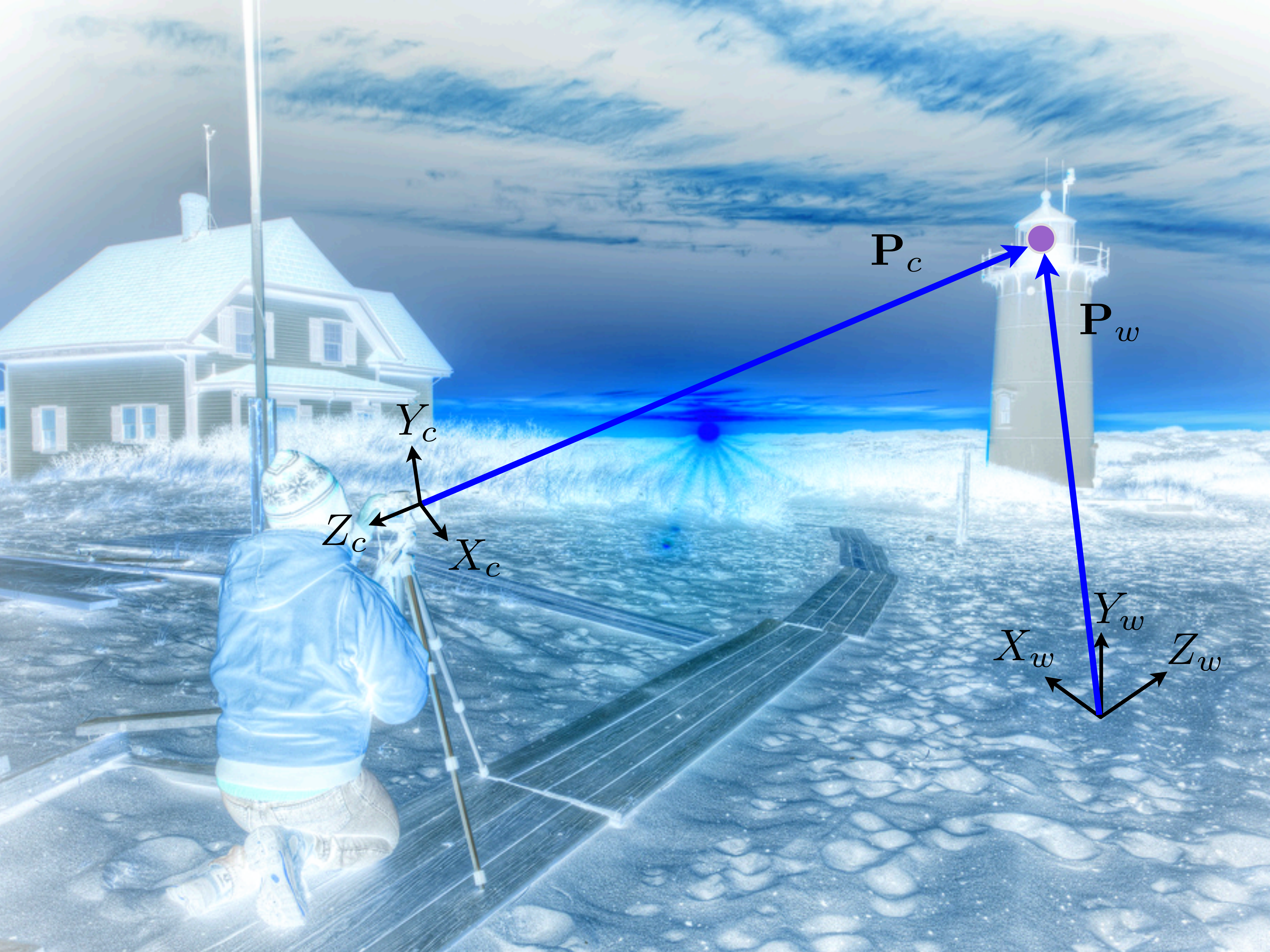
Homogeneous coordinates

Review

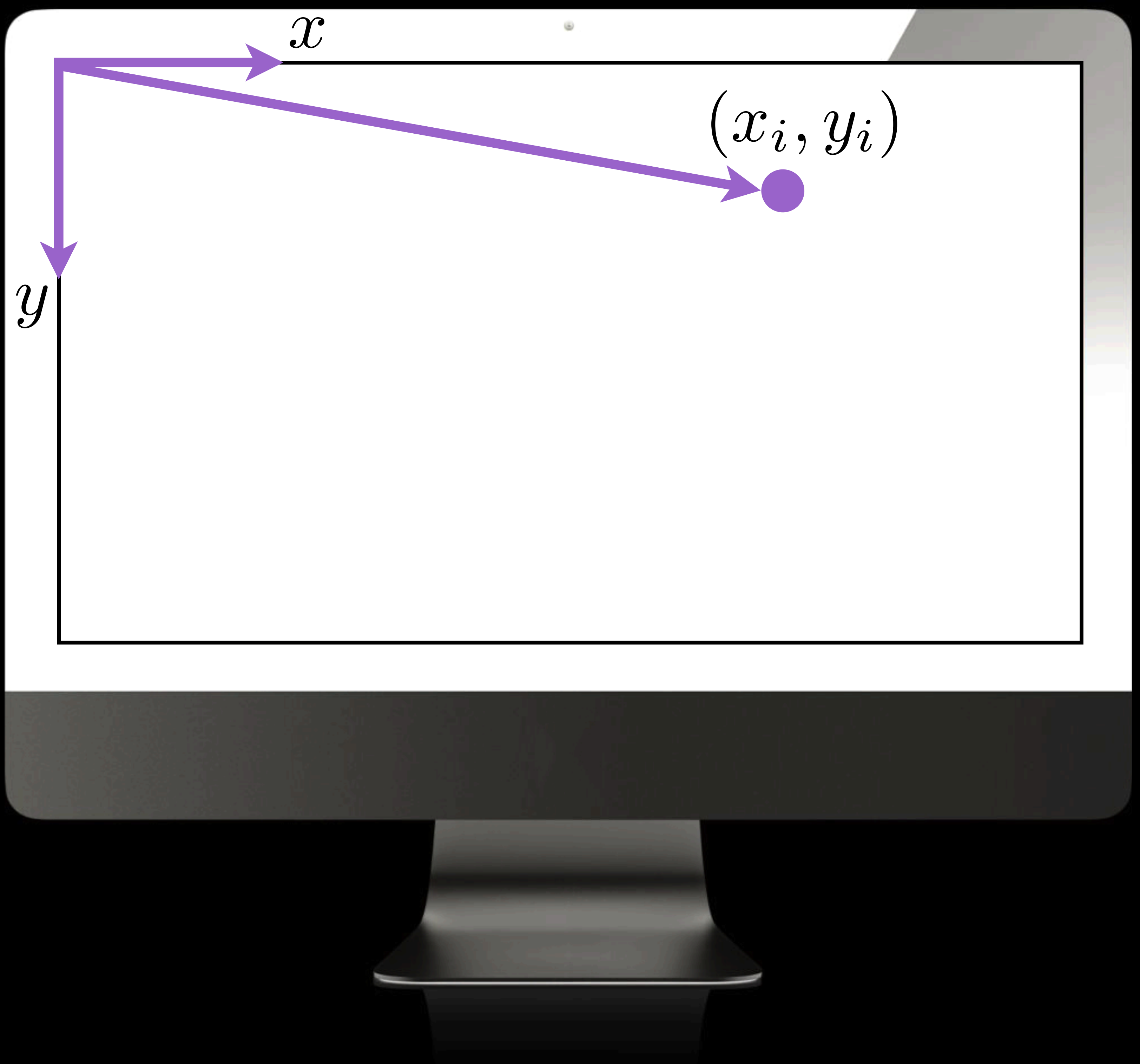


Perspective Projection

$$x = f\frac{X}{Z}, \qquad y = f\frac{Y}{Z}$$



$\mathbf{P}_c$
$\mathbf{P}_w$
Y_c
Z_c
X_c
Y_w
X_w
Z_w



x
y
(x_i, y_i)

Definition

extrinsic camera parameters

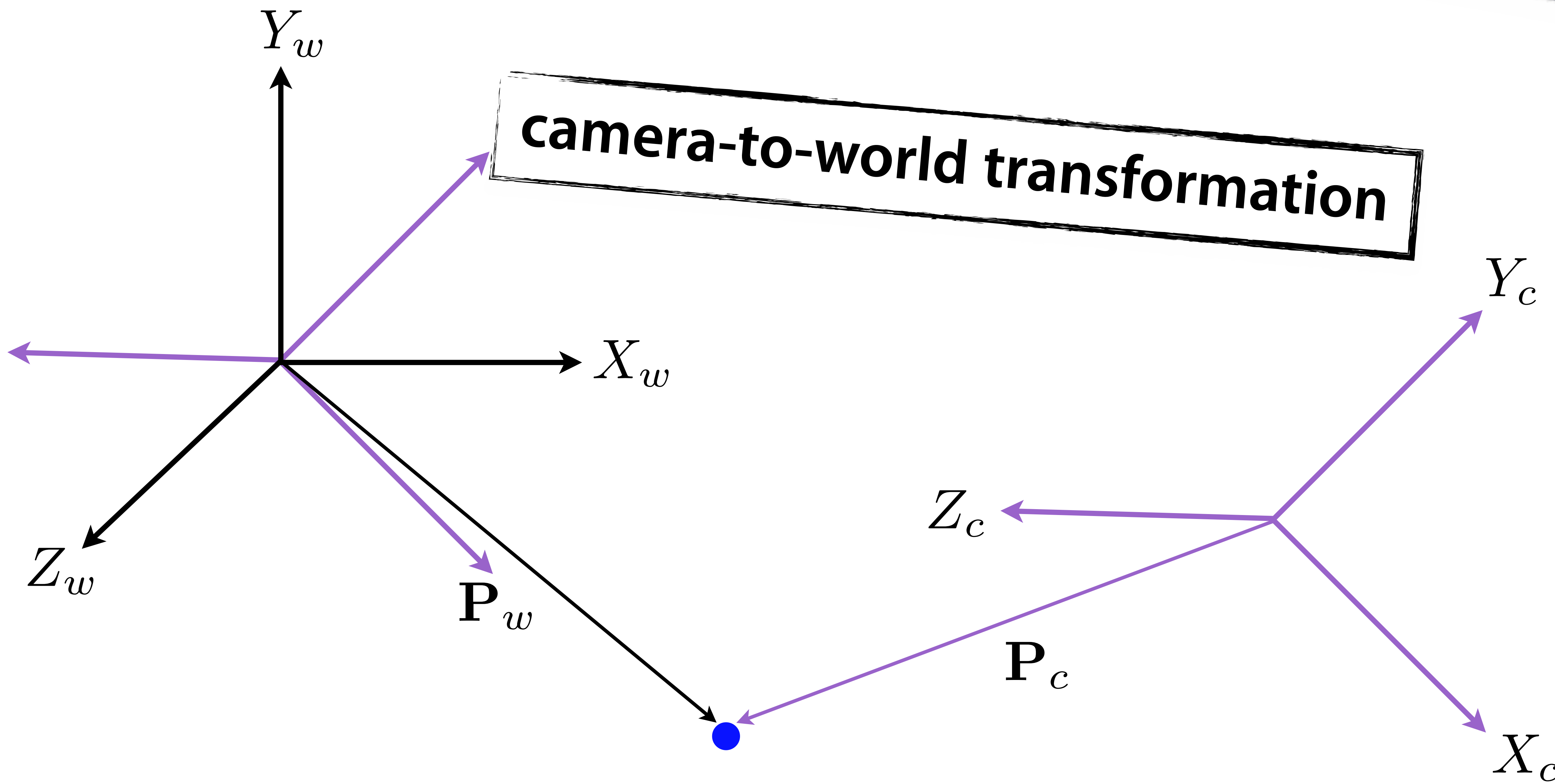
define the location and orientation of the camera reference frame with respect to a known world reference frame

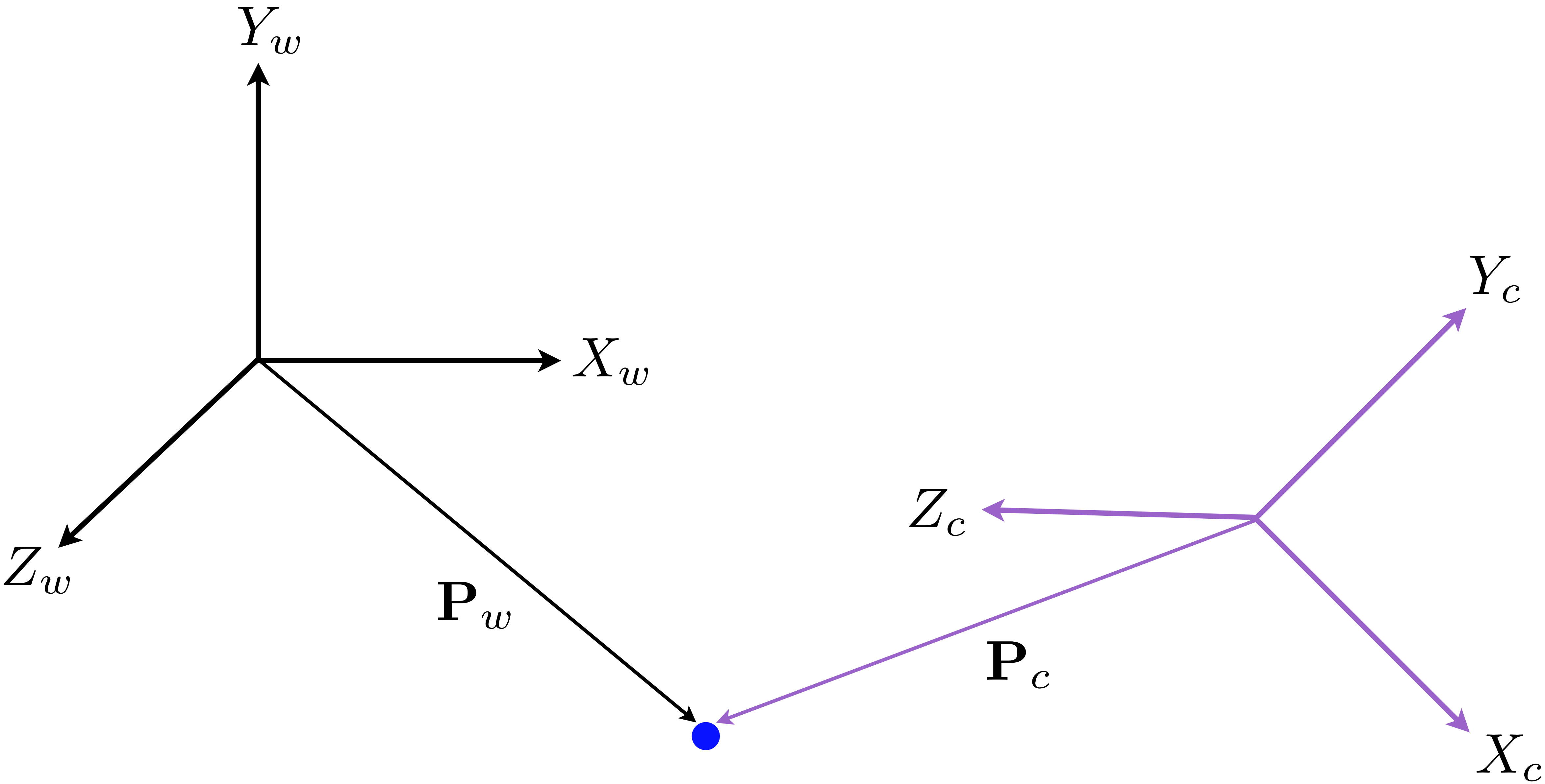
Definition

intrinsic camera parameters

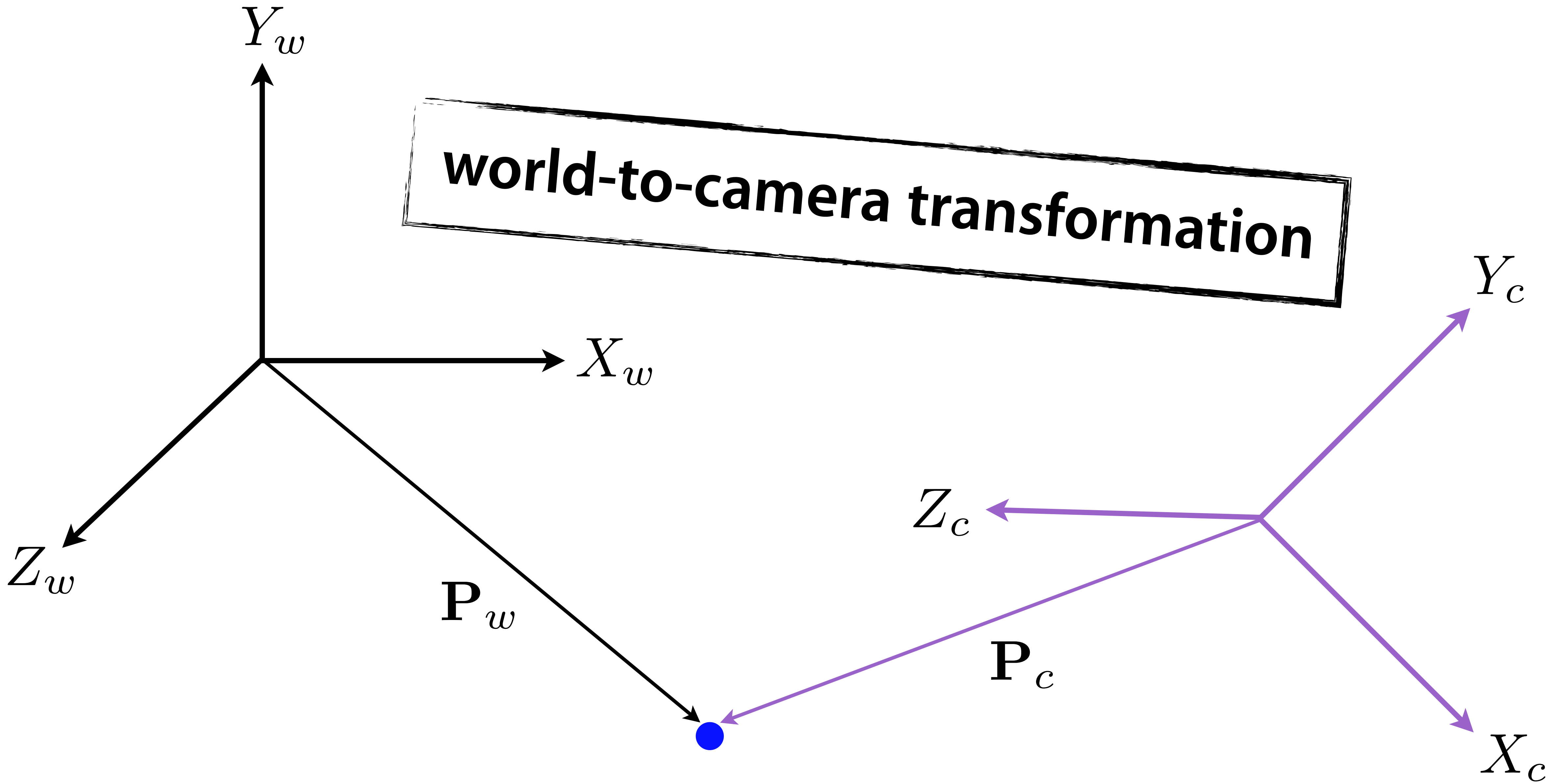
link pixel coordinates with the corresponding coordinates in the camera frame

Extrinsic Parameters





$$\mathbf{P}_w = {}^{w}\mathbf{R}_c\mathbf{P}_c + \mathbf{T}$$

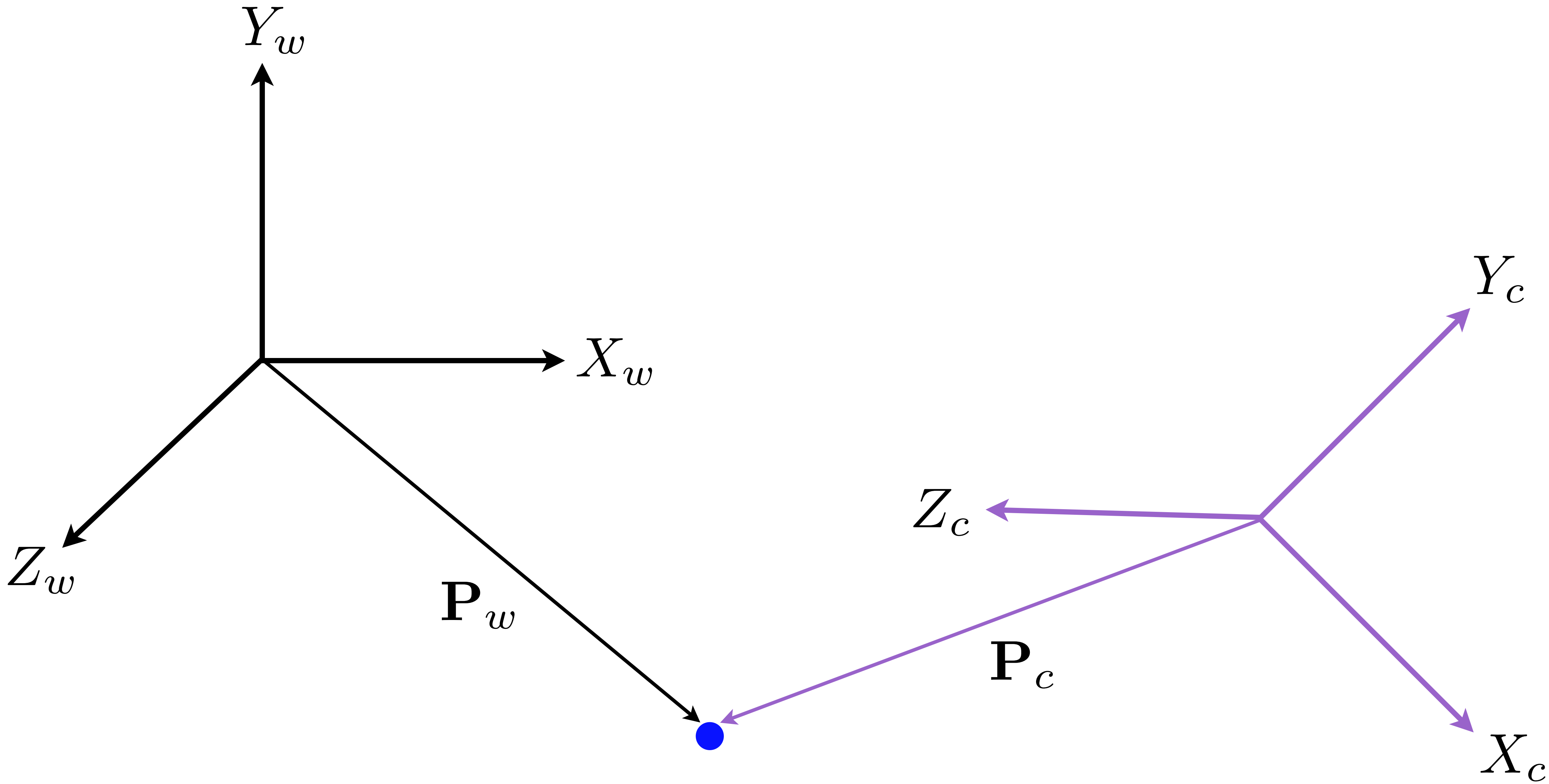


$$\mathbf{P}_w = {}^w\mathbf{R}_c \mathbf{P}_c + \mathbf{T}$$

$${}^w\mathbf{R}_c^\top \mathbf{P}_w = \cancel{{}^w\mathbf{R}_c^\top\, {}^w\mathbf{R}_c} \mathbf{P}_c + {}^w\mathbf{R}_c^\top \mathbf{T}$$

$${}^w\mathbf{R}_c^\top \mathbf{P}_w - {}^w\mathbf{R}_c^\top \mathbf{T} = \mathbf{P}_c$$

$${}^w\mathbf{R}_c^\top (\mathbf{P}_w - \mathbf{T}) = \mathbf{P}_c$$



$$ {}^{w}\mathbf{R}_c^{\top}(\mathbf{P}_w - \mathbf{T}) = \mathbf{P}_c $$

World-to-Camera Transformation

$$\mathbf{P}_c = \mathbf{R}(\mathbf{P}_w - \mathbf{T})$$

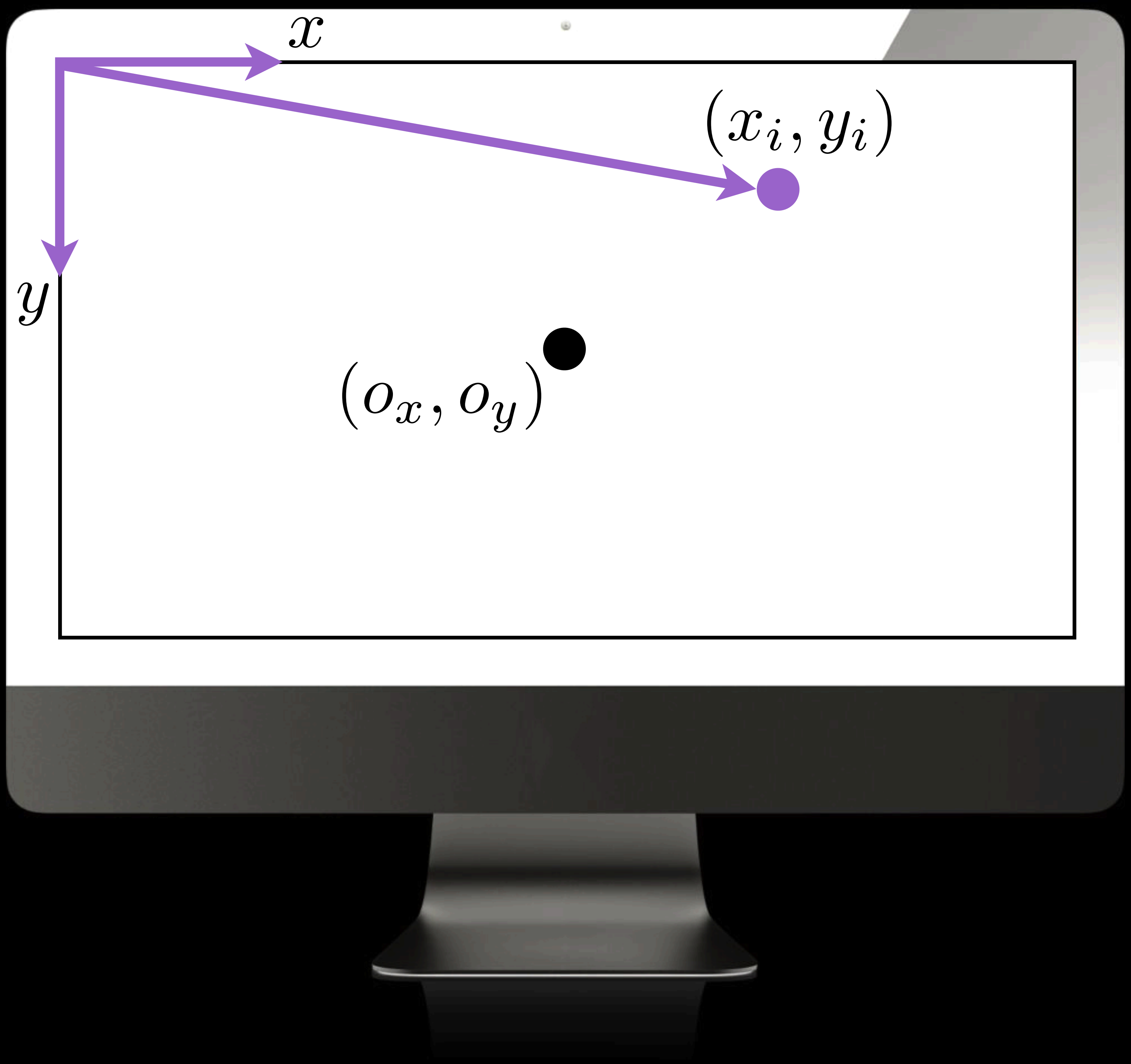
World-to-Camera Transformation

$$\mathbf{P}_c = \mathbf{R}(\mathbf{P}_w - \mathbf{T})$$

What is the total number of extrinsic parameters?

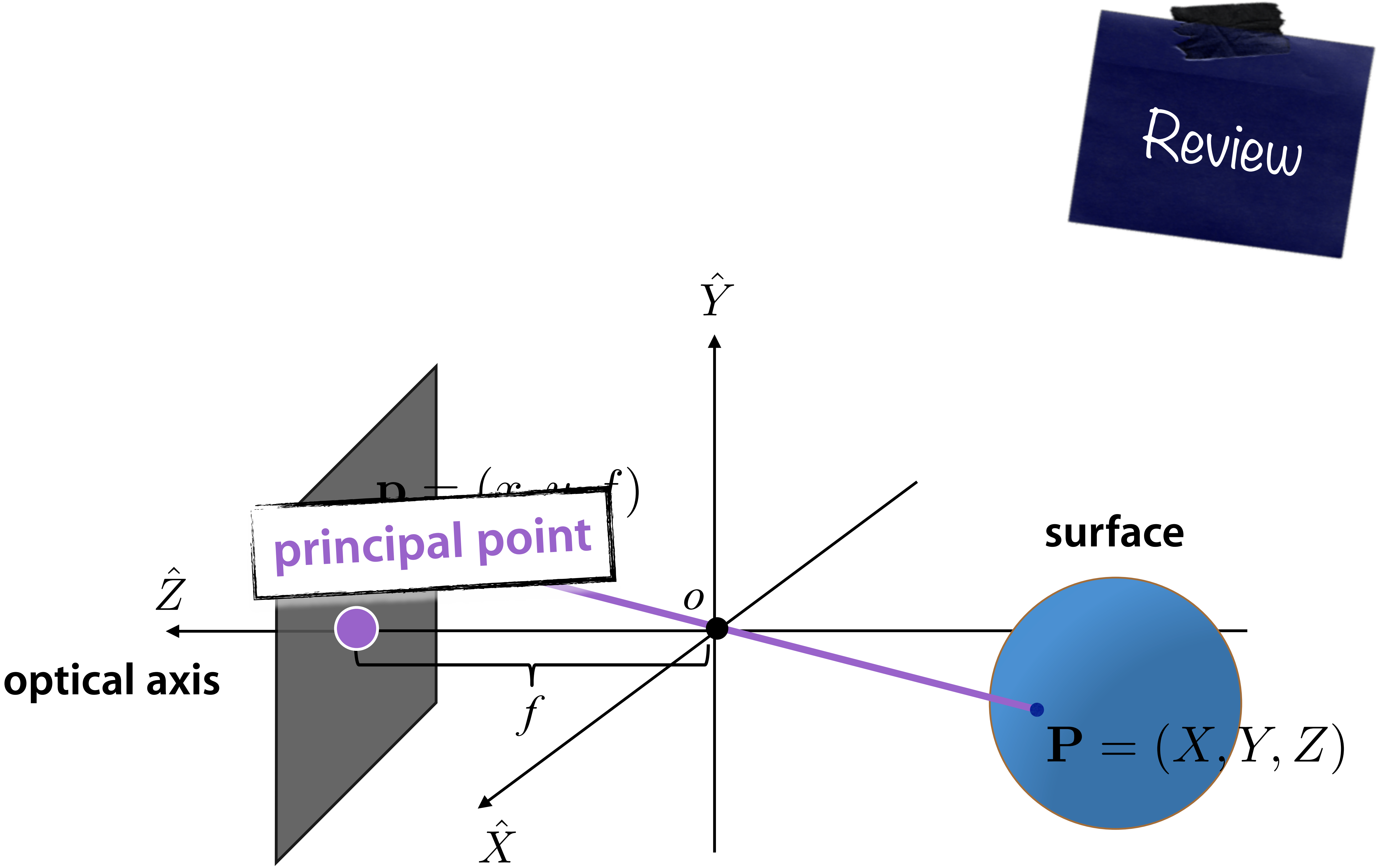
World-to-Camera Transformation $\mathbf{P}_c = \mathbf{R}(\mathbf{P}_w - \mathbf{T})$

three translation and three rotation parameters

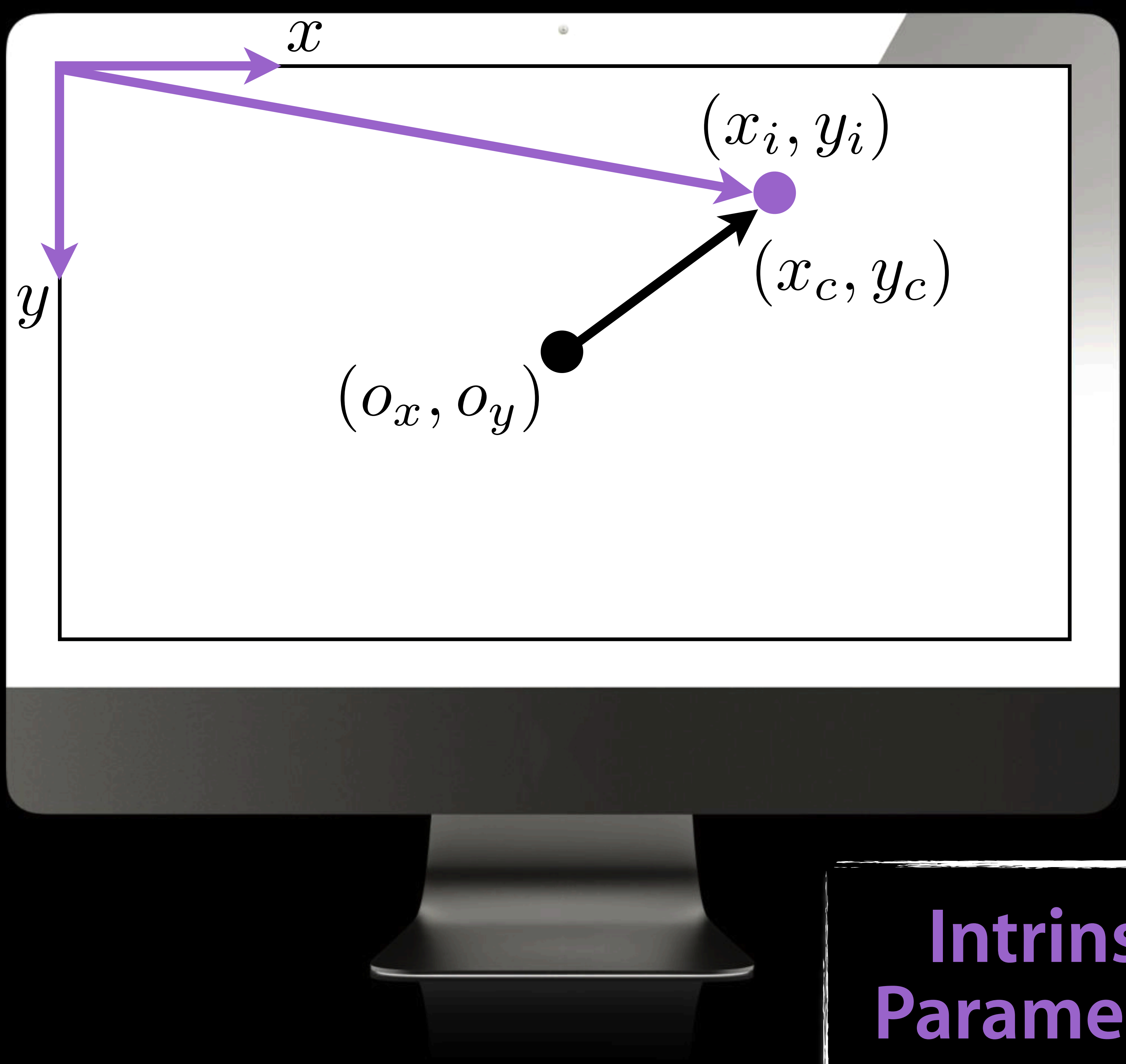


pixel-to-camera transformation

$$x_c = -s_x(x_i - o_x)$$
$$y_c = -s_y(y_i - o_y)$$



Review
$\hat{Y}$
principal point
surface
$\hat{Z}$
o
optical axis
f
$\mathbf{P} = (X, Y, Z)$
$\hat{X}$



pixel-to-camera transformation

$$x_c = -s_x(x_i - o_x)$$
$$y_c = -s_y(y_i - o_y)$$

Intrinsic Parameters (o_x, o_y, s_x, s_y, f)

Intrinsic Parameters (o_x, o_y, s_x, s_y, f)

Simple analysis yields five intrinsic parameters

Additional parameters possible, e.g., lens distortion

World-to-Camera Transformation

$$\mathbf{P}_c = \mathbf{R}(\mathbf{P}_w - \mathbf{T})$$

$$\mathbf{P}_c = \begin{pmatrix} X_c \\ Y_c \\ Z_c \end{pmatrix} = \begin{pmatrix} \mathbf{R}_1^\top(\mathbf{P}_w - \mathbf{T}) \\ \mathbf{R}_2^\top(\mathbf{P}_w - \mathbf{T}) \\ \mathbf{R}_3^\top(\mathbf{P}_w - \mathbf{T}) \end{pmatrix}$$

rows of the rotation matrix

$$x_c = f\frac{X_c}{Z_c}$$

$$y_c = f\frac{Y_c}{Z_c}$$

substitute intrinsic parameterization

$$-(x_i - o_x)s_x = f\frac{X_c}{Z_c}$$

$$-(y_i - o_y)s_y = f\frac{Y_c}{Z_c}$$

substitute extrinsic parameterization

$$-(x_i - o_x)s_x = f\frac{\mathbf{R}_1^\top(\mathbf{P}_w - \mathbf{T})}{\mathbf{R}_3^\top(\mathbf{P}_w - \mathbf{T})}$$

$$-(y_i - o_y)s_y = f\frac{\mathbf{R}_2^\top(\mathbf{P}_w - \mathbf{T})}{\mathbf{R}_3^\top(\mathbf{P}_w - \mathbf{T})}$$

$$-(x_i - o_x)s_x = f\frac{\mathbf{R}_1^\top(\mathbf{P}_w - \mathbf{T})}{\mathbf{R}_3^\top(\mathbf{P}_w - \mathbf{T})}$$

$$-(y_i - o_y)s_y = f\frac{\mathbf{R}_2^\top(\mathbf{P}_w - \mathbf{T})}{\mathbf{R}_3^\top(\mathbf{P}_w - \mathbf{T})}$$

rewrite as matrix

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} & -\mathbf{R}_1^\top\mathbf{T} \\ \mathbf{R} & -\mathbf{R}_2^\top\mathbf{T} \\ & -\mathbf{R}_3^\top\mathbf{T} \end{pmatrix} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

Homogeneous Coordinates

$$-(x_i - o_x)s_x = f\frac{\mathbf{R}_1^\top(\mathbf{P}_w - \mathbf{T})}{\mathbf{R}_3^\top(\mathbf{P}_w - \mathbf{T})}$$

$$-(y_i - o_y)s_y = f\frac{\mathbf{R}_2^\top(\mathbf{P}_w - \mathbf{T})}{\mathbf{R}_3^\top(\mathbf{P}_w - \mathbf{T})}$$

rewrite as matrix

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{pmatrix}}_{\substack{\textit{intrinsic matrix} \\ \mathbf{M}_{\text{int}}}} \underbrace{\begin{pmatrix} & -\mathbf{R}_1^\top\mathbf{T} \\ \mathbf{R} & -\mathbf{R}_2^\top\mathbf{T} \\ & -\mathbf{R}_3^\top\mathbf{T} \end{pmatrix}}_{\substack{\textit{extrinsic matrix} \\ \mathbf{M}_{\text{ext}}}} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{pmatrix}}_{\substack{\textit{intrinsic matrix} \\ \mathbf{M}_{\text{int}}}} \underbrace{\begin{pmatrix} & -\mathbf{R}_1^\top \mathbf{T} \\ \mathbf{R} & -\mathbf{R}_2^\top \mathbf{T} \\ & -\mathbf{R}_3^\top \mathbf{T} \end{pmatrix}}_{\substack{\textit{extrinsic matrix} \\ \mathbf{M}_{\text{ext}}}} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

Linear Matrix Equation of Perspective Projection

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{M}_{\text{int}} \mathbf{M}_{\text{ext}} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}}$$

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} x_1/x_3 \\ x_2/x_3 \end{pmatrix}$$

$$\lambda \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix} = \begin{pmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} & -\mathbf{R}_1^\top \mathbf{T} \\ \mathbf{R} & -\mathbf{R}_2^\top \mathbf{T} \\ & -\mathbf{R}_3^\top \mathbf{T} \end{pmatrix} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

$$\lambda \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix} = \mathbf{M}_{\text{int}} \mathbf{M}_{\text{ext}} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

$$\lambda \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix} = \mathbf{M} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

$$\lambda \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix} = \mathbf{M}\mathbf{Q}^{-1}\mathbf{Q} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

$$\lambda \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix} = \mathbf{M}\mathbf{Q}^{-1}\mathbf{Q} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

projective ambiguity

$$\mathbf{Q} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$