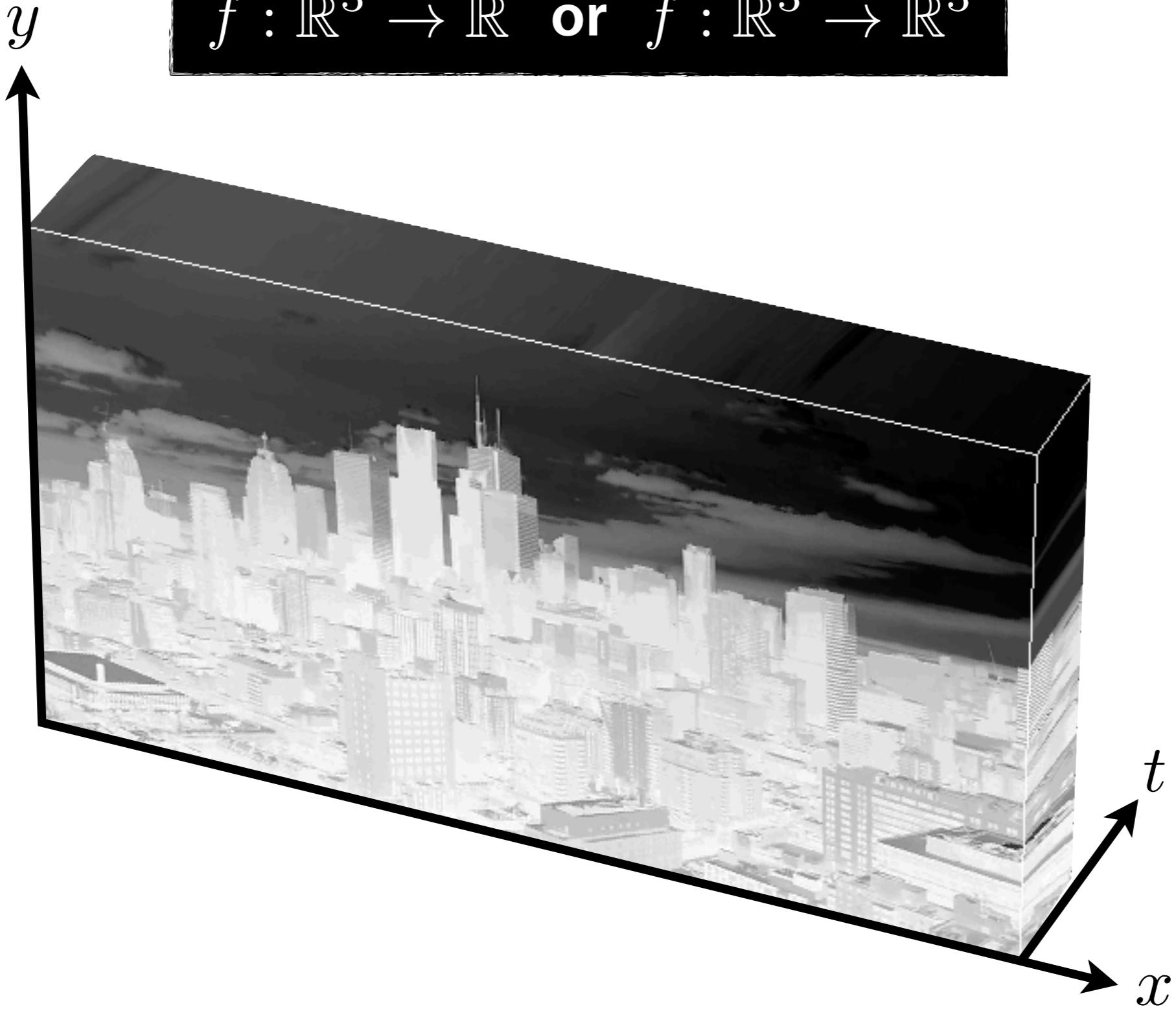
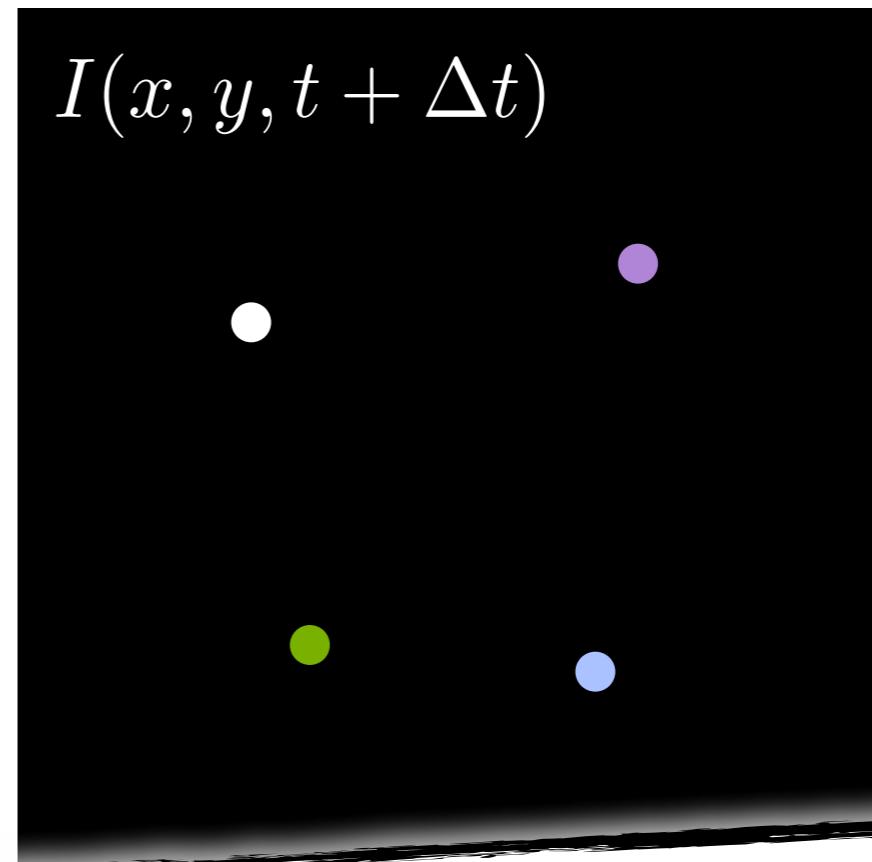
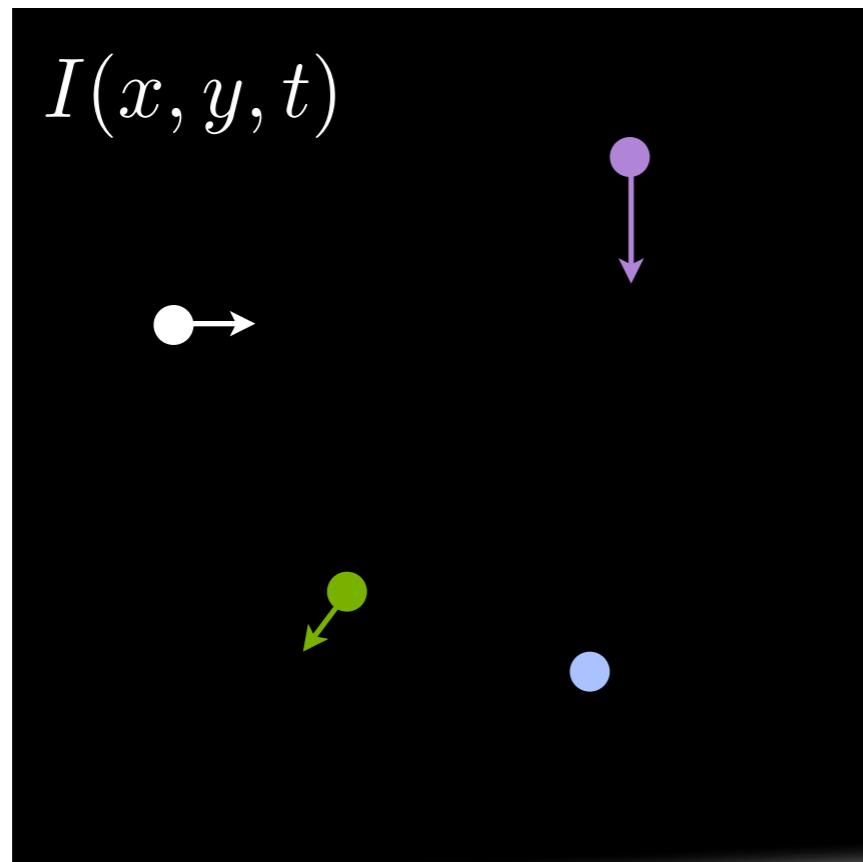


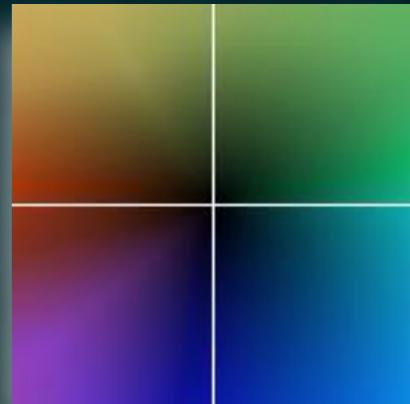
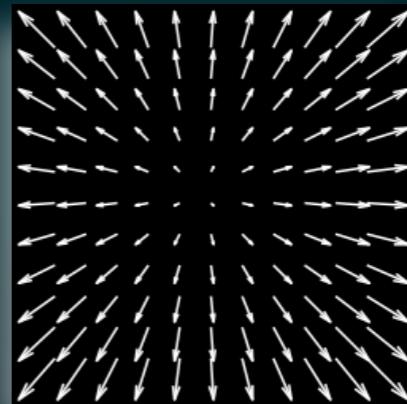
$$f : \mathbb{R}^3 \rightarrow \mathbb{R} \text{ or } f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$


Problem Definition



GOAL: Estimate pixel displacement between images

optical flow



y

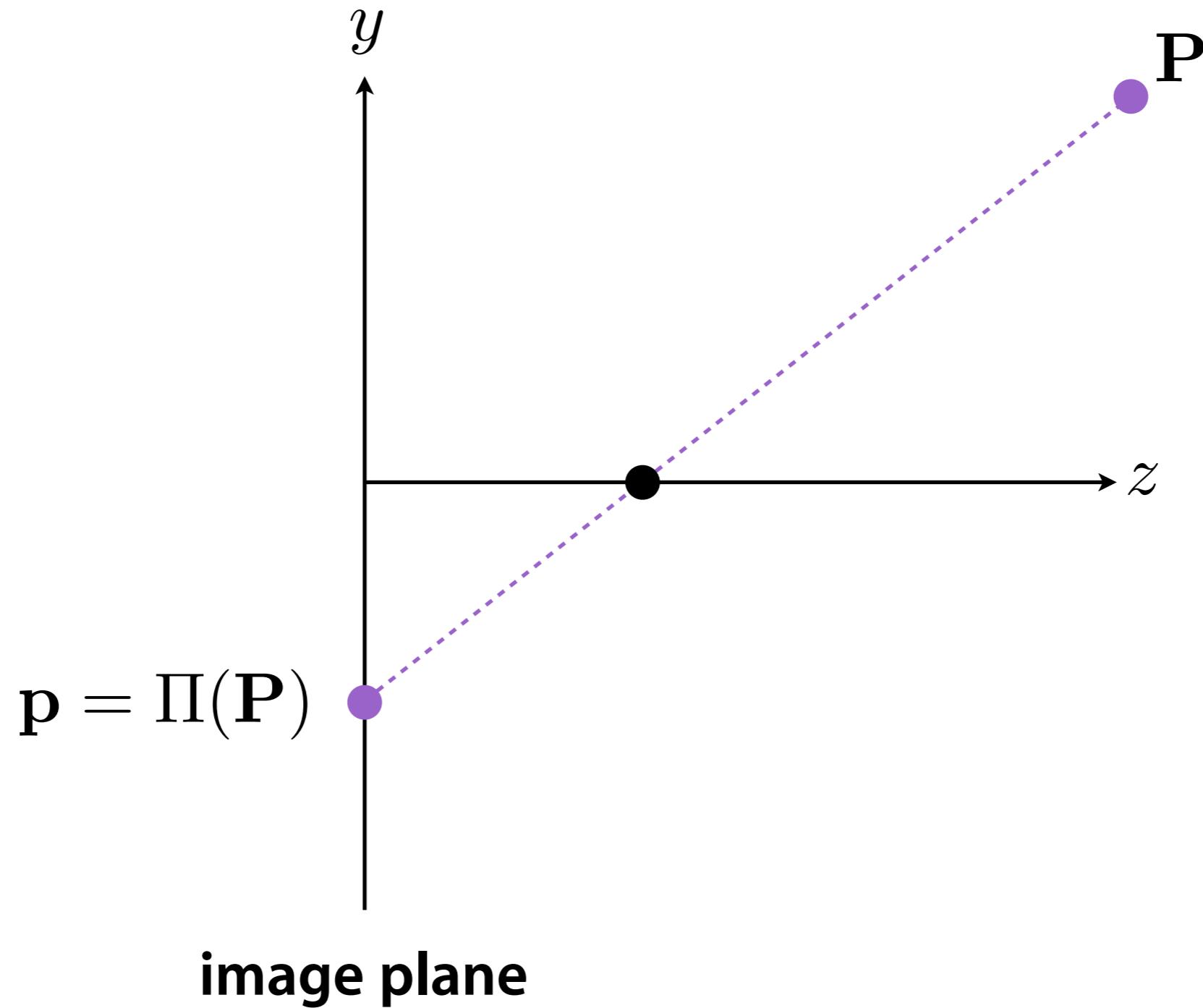


P

z

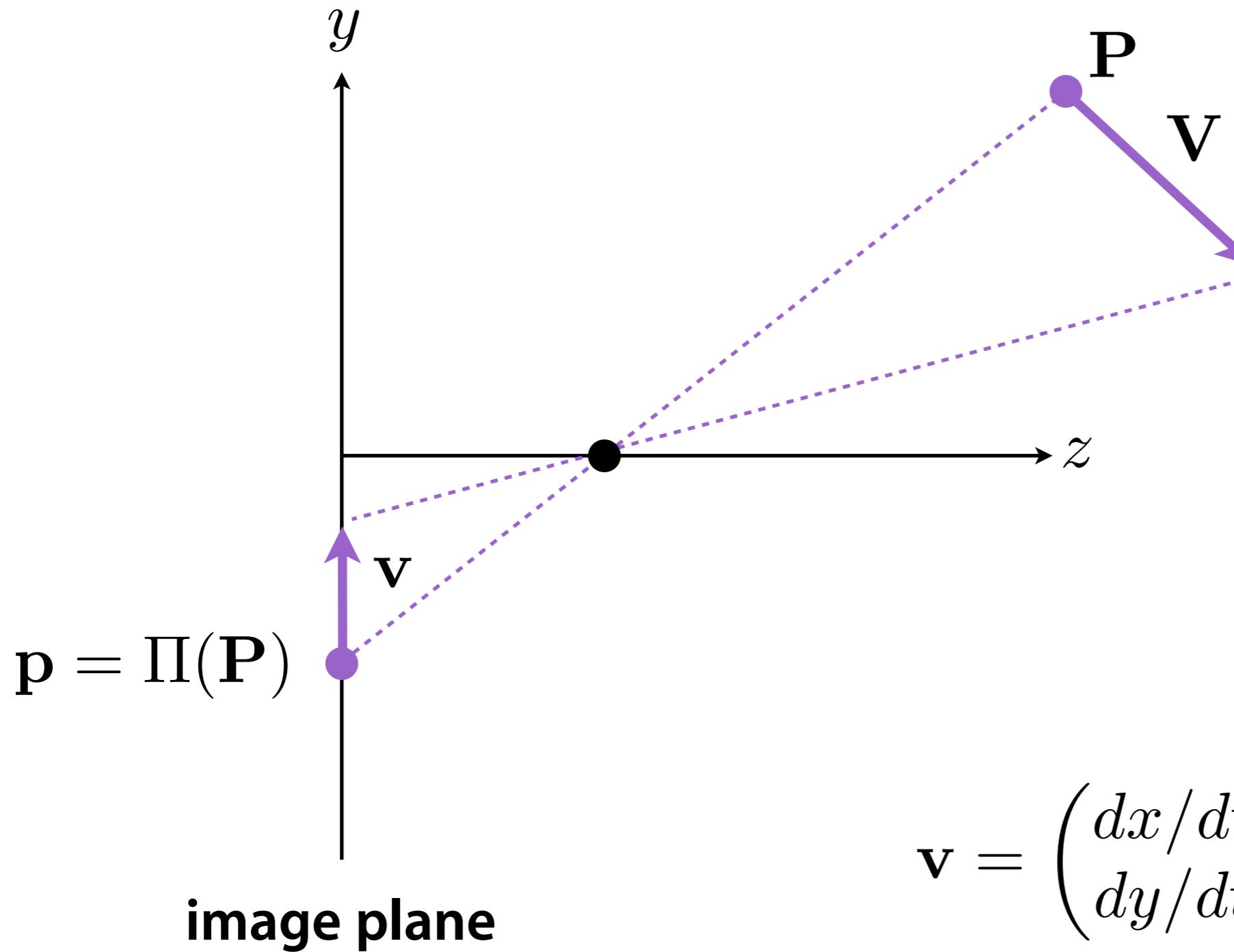


image plane



Motion Field

**projection of the 3D scene velocities onto
the image plane**



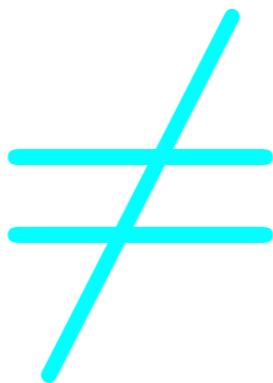
$$\mathbf{v} = \begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \frac{d\mathbf{p}}{dt}$$
$$= \frac{d\Pi(\mathbf{P})}{dt}$$

Optical Flow

**distribution of apparent velocities of
movement of brightness patterns in the image**

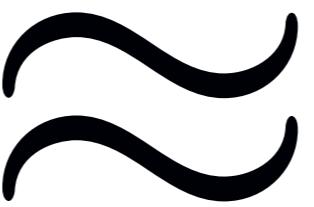
motion derived from image measurements

Motion Field



Optical Flow

Motion Field



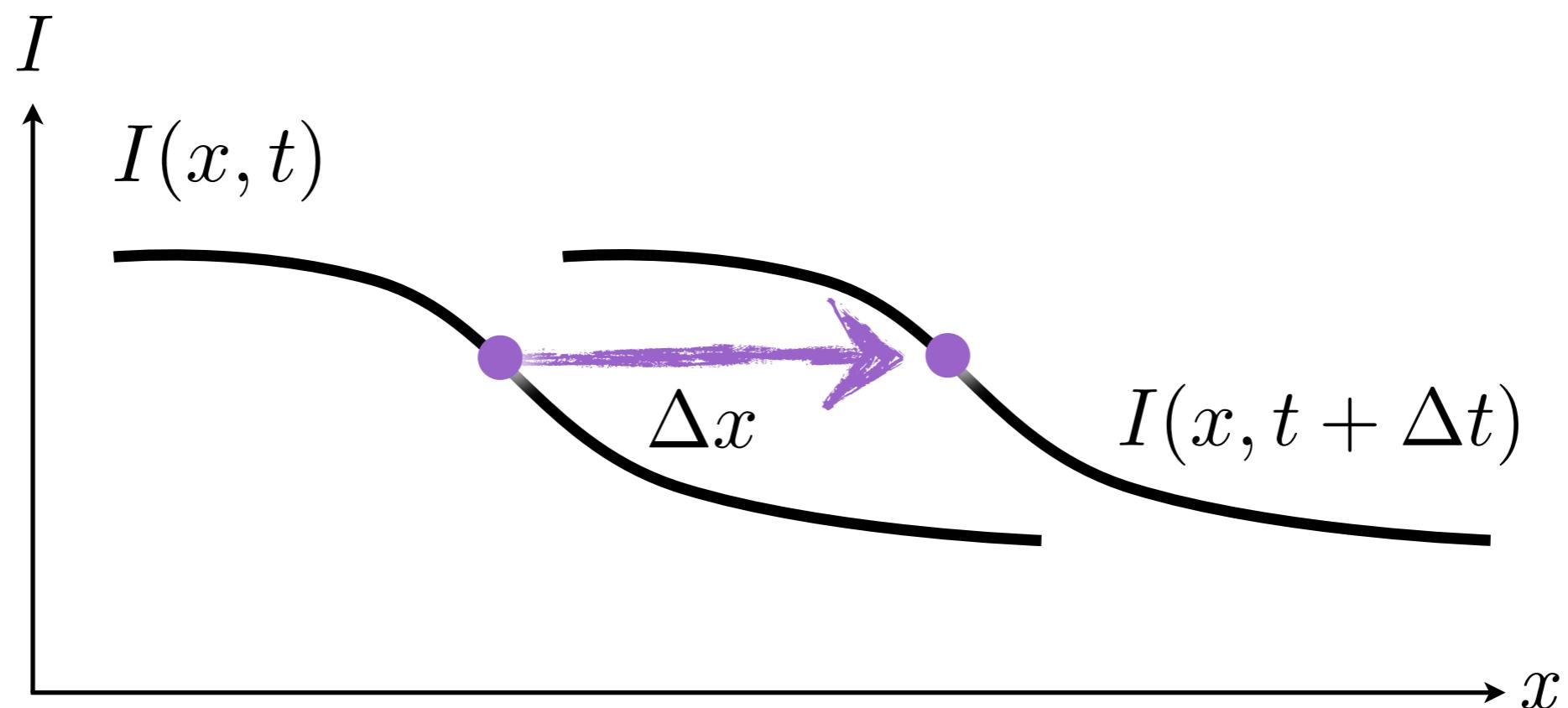
Optical Flow

Assumption

Image brightness of a point remains **constant over time**

other image properties are possible

Brightness
Constancy



Brightness
Constancy
Assumption

$$I(x, y, t) = I(x + u\Delta t, y + v\Delta t, t + \Delta t)$$

Brightness
Constancy
Assumption

$$I(x, y, t) = I(x + v\Delta t, y + v\Delta t, t + \Delta t)$$

assuming small duration

Review: Taylor series

Taylor Series Expansion

$$f(x + \Delta x, y + \Delta y, z + \Delta z) = f(x, y, z)$$

$$+ f_x(x, y, z)\Delta x + f_y(x, y, z)\Delta y + f_z(x, y, z)\Delta z$$

+ higher order terms

Brightness
Constancy
Assumption

$$I(x, y, t) = I(x + u\Delta t, y + v\Delta t, t + \Delta t)$$

since image is non-linear in flow, flow recovery is difficult

Brightness
Constancy
Assumption

$$I(x, y, t) = I(x + u\Delta t, y + v\Delta t, t + \Delta t)$$

Taylor series

$$\cancel{I(x, y, t)} = \cancel{I(x, y, t)} + \frac{\partial I}{\partial x} u \Delta t + \frac{\partial I}{\partial y} v \Delta t + \frac{\partial I}{\partial t} \Delta t + \text{h.o.t.}$$

simplify

$$\frac{\partial I}{\partial x} u \Delta t + \frac{\partial I}{\partial y} v \Delta t + \frac{\partial I}{\partial t} \Delta t + \text{h.o.t.} = 0$$

divide by Δt

$$\frac{\partial I}{\partial x} u + \frac{\partial I}{\partial y} v + \frac{\partial I}{\partial t} + \text{h.o.t.} = 0$$

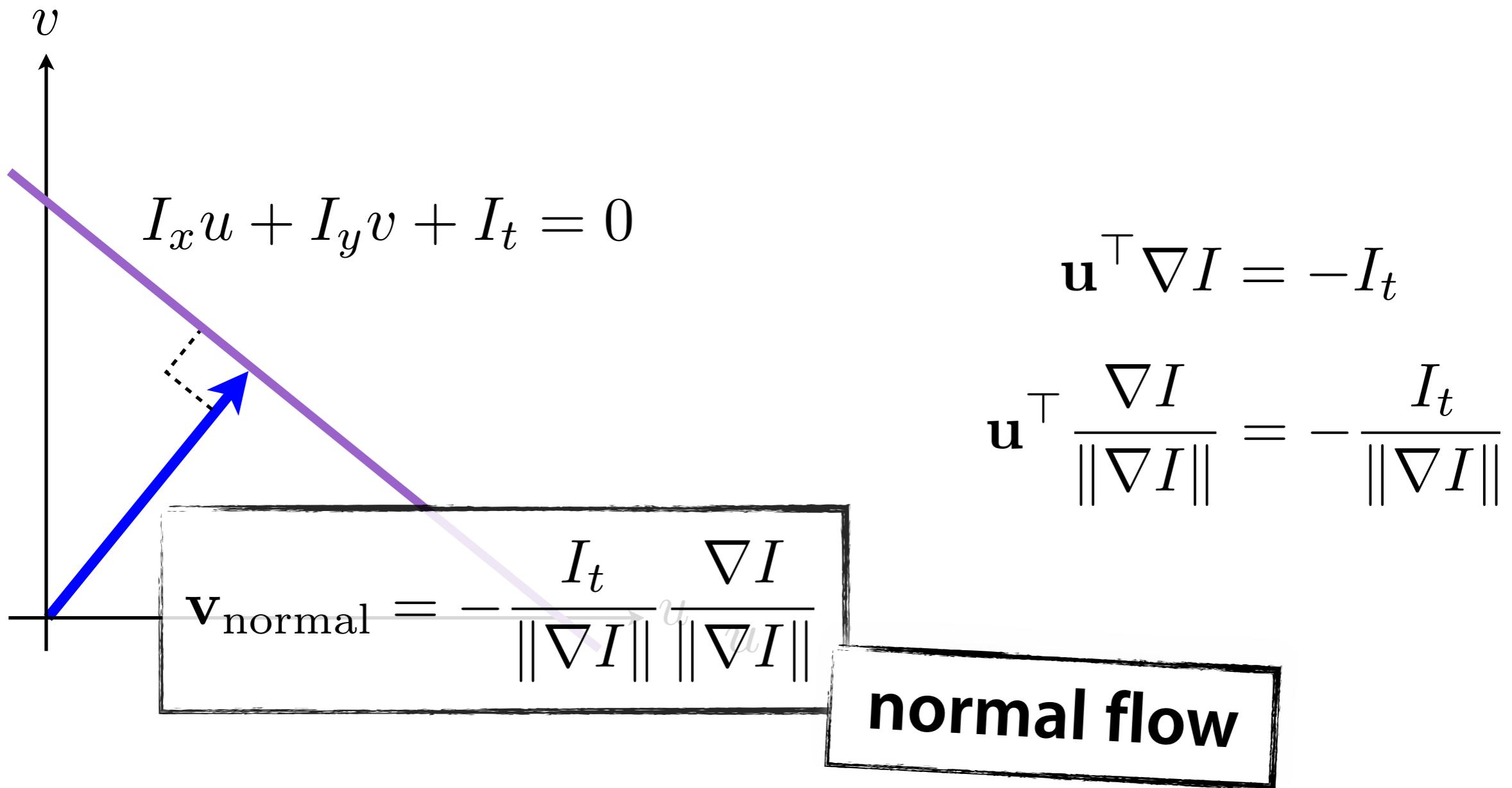
$$\Delta t \rightarrow 0$$

**Brightness
Constancy Constraint**

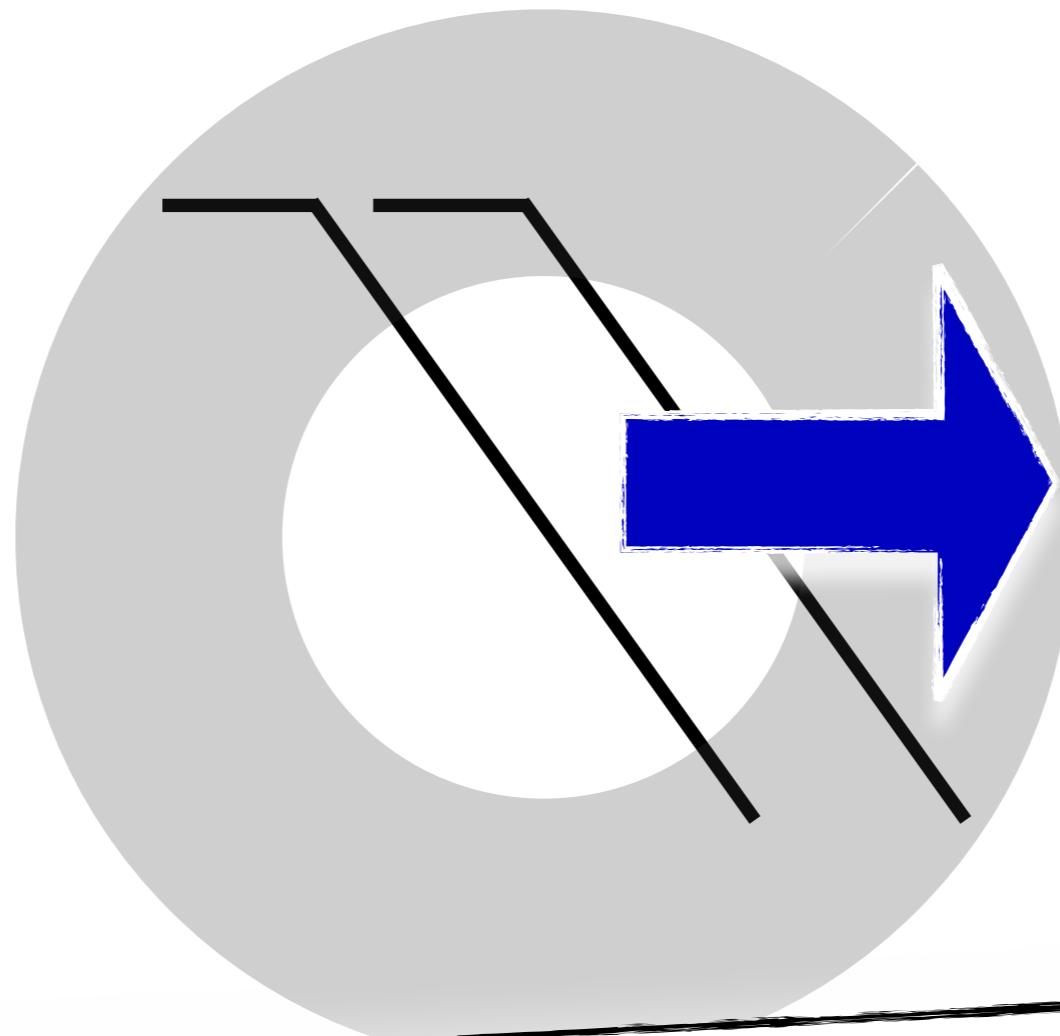
$$I_x u + I_y v + I_t = 0$$

Brightness Constancy Constraint

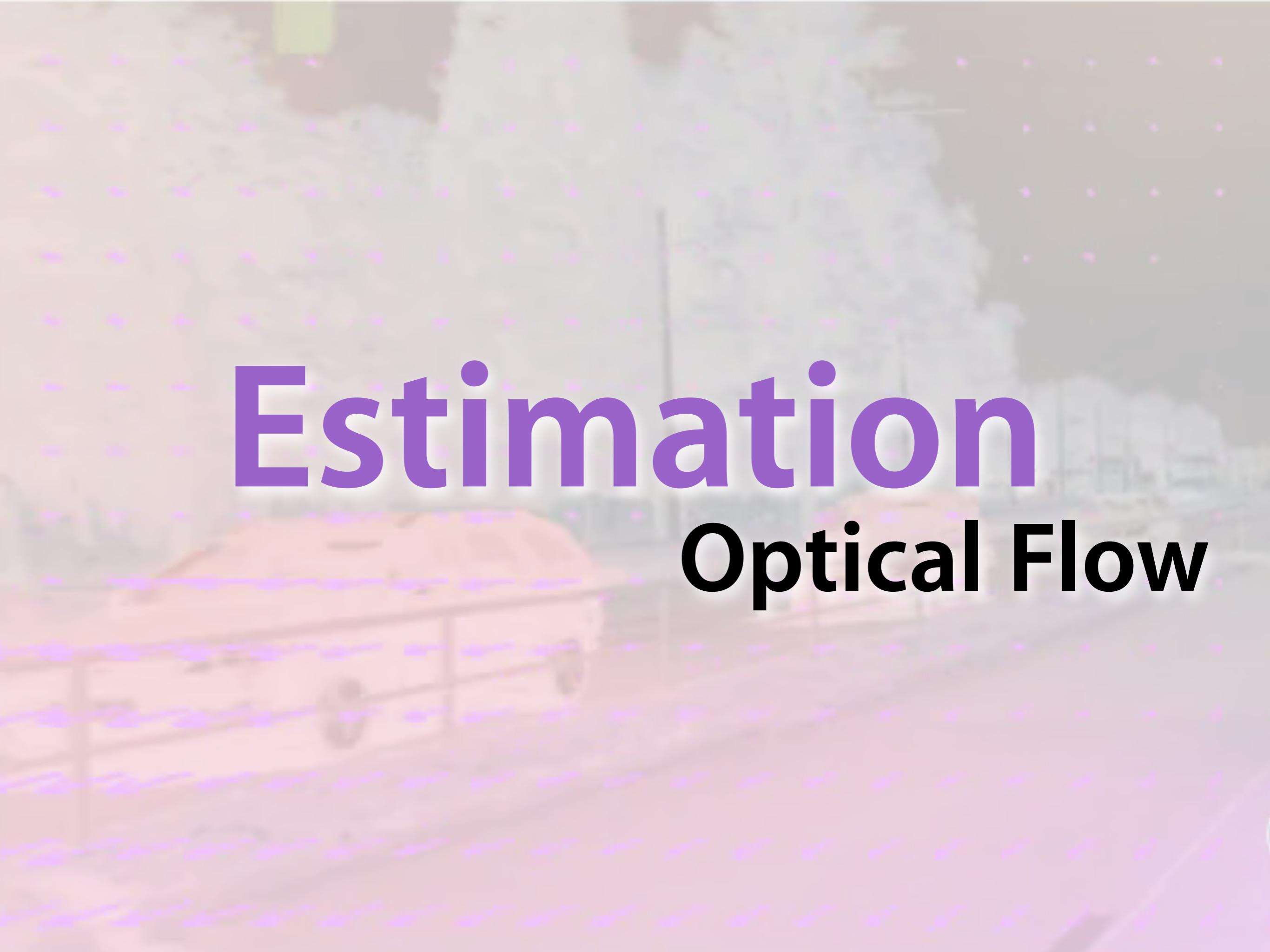
$$I_x u + I_y v + I_t = 0$$



Aperture Problem



Which way are the lines moving?



Estimation

Optical Flow

Local

A 3D point cloud visualization of a forest scene. The ground is a light brown color with some dark spots. There are several trees with white trunks and purple leaves. Two rectangular boxes with black borders and white backgrounds are overlaid on the image. The left box contains the word "Local" and the right box contains the word "Global".

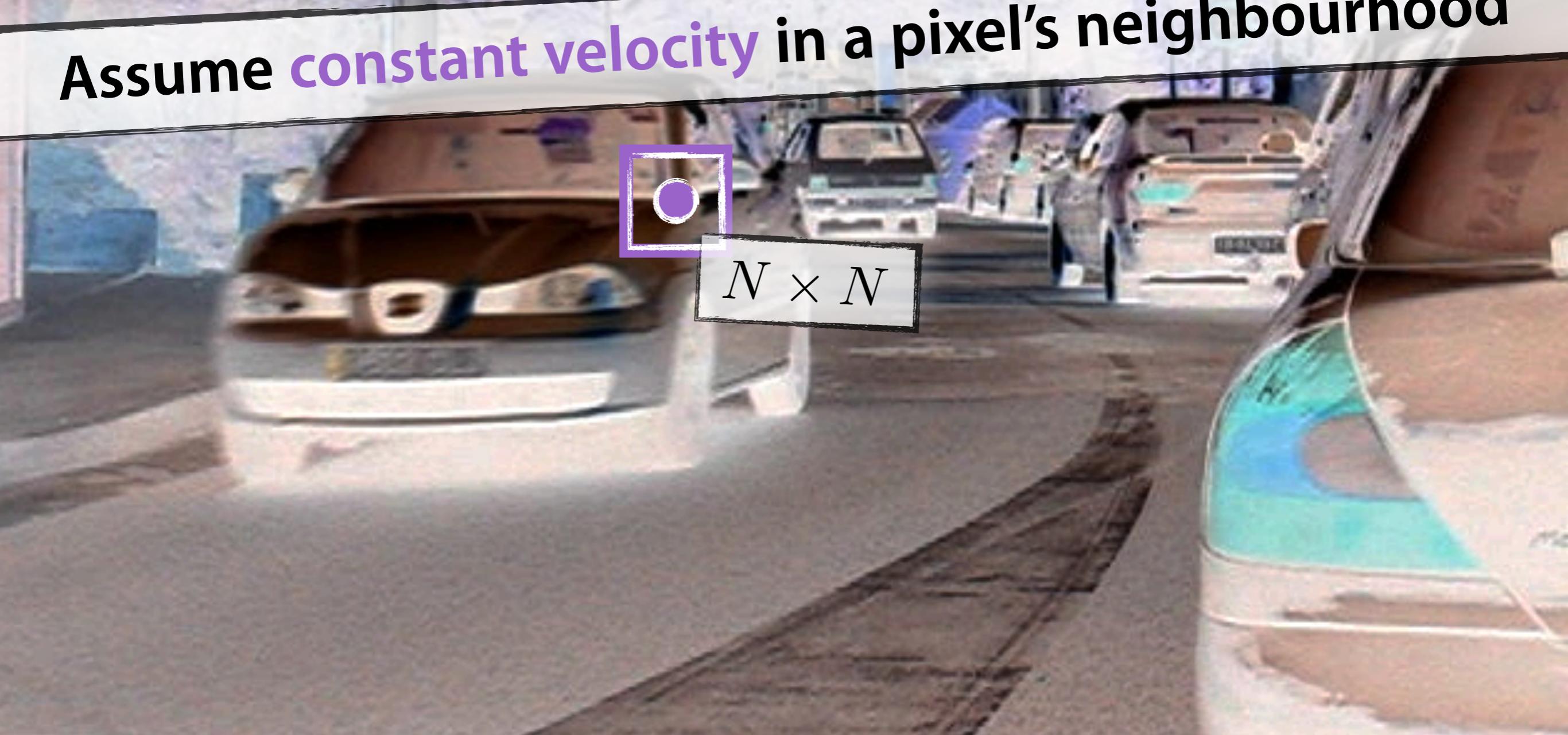
Global



Assume **constant velocity** in a pixel's neighbourhood



$N \times N$



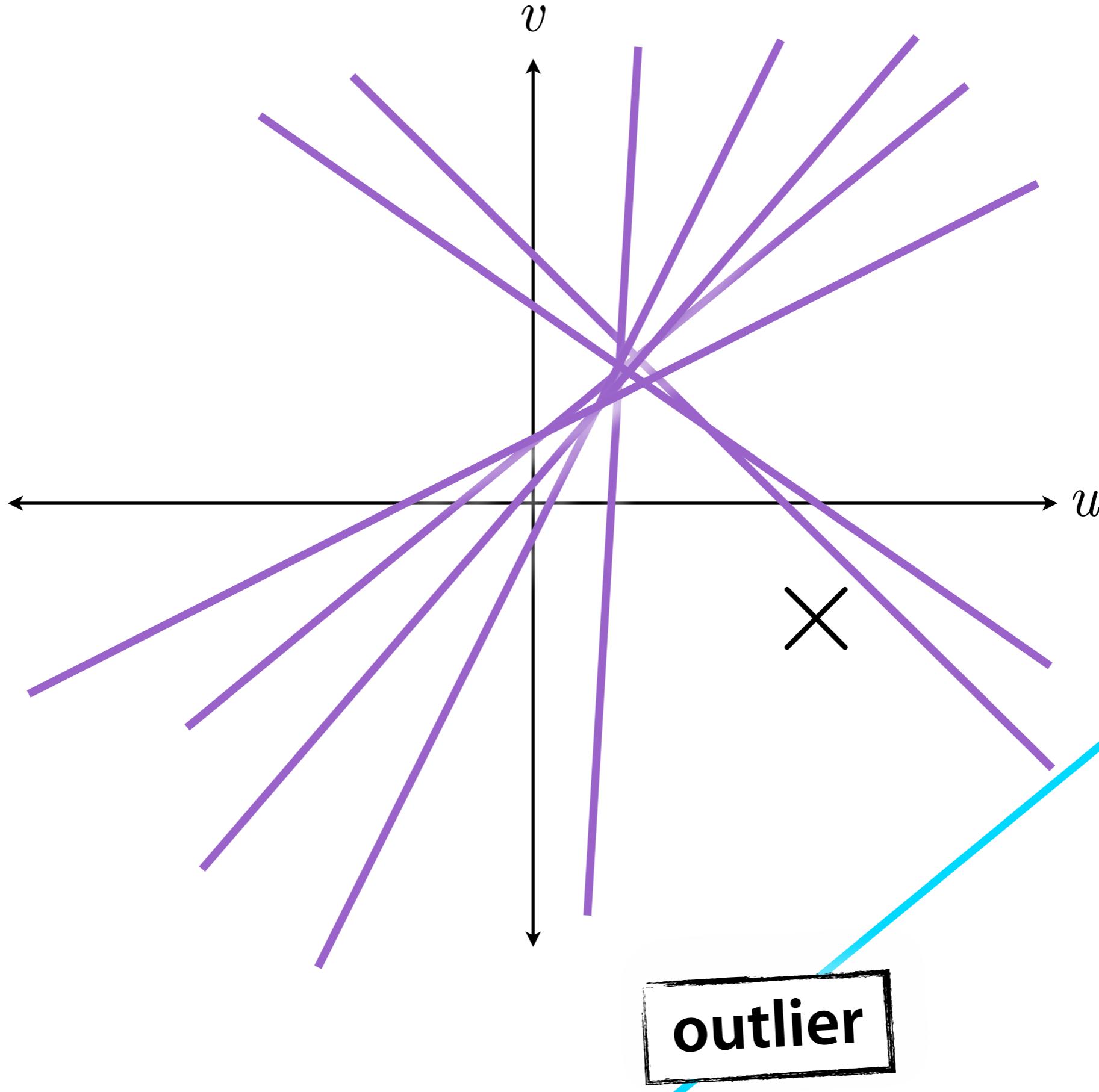
Lucas-Kanade

$$\begin{pmatrix} I_x(\mathbf{p}_1) & I_y(\mathbf{p}_1) \\ I_x(\mathbf{p}_2) & I_y(\mathbf{p}_2) \\ \vdots & \vdots \\ I_x(\mathbf{p}_{N^2}) & I_y(\mathbf{p}_{N^2}) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = - \begin{pmatrix} I_t(\mathbf{p}_1) \\ I_t(\mathbf{p}_2) \\ \vdots \\ I_t(\mathbf{p}_{N^2}) \end{pmatrix}$$

$\mathbf{A}_{N^2 \times 2}$ $\mathbf{b}_{N^2 \times 1}$

How can we solve this system?

$$\arg \min_{\mathbf{v}} \|\mathbf{A}\mathbf{v} - \mathbf{b}\|^2$$



$$\arg \min_{\mathbf{v}} \|\mathbf{A}\mathbf{v} - \mathbf{b}\|^2$$

$$\arg\min_{\mathbf{v}} \|\mathbf{A}\mathbf{v}-\mathbf{b}\|^2$$

$$\mathbf{A}^\top \mathbf{A} \mathbf{v} = \mathbf{A}^\top \mathbf{b}$$

$$\begin{pmatrix} \sum I_x^2 & \sum I_xI_y \\ \sum I_xI_y & \sum I_y^2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = - \begin{pmatrix} \sum I_xI_t \\ \sum I_yI_t \end{pmatrix}$$

$$\mathbf{A}^\top \mathbf{A}_{2\times 2} \qquad\qquad\qquad \mathbf{v}_{2\times 1} \qquad\qquad\qquad \mathbf{A}^\top \mathbf{b}_{2\times 1}$$

$$\arg \min_{\mathbf{v}} \|\mathbf{A}\mathbf{v} - \mathbf{b}\|^2$$

$$\mathbf{A}^\top \mathbf{A}\mathbf{v} = \mathbf{A}^\top \mathbf{b}$$

$$\begin{pmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_x I_y & \sum I_y^2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = - \begin{pmatrix} \sum I_x I_t \\ \sum I_y I_t \end{pmatrix}$$

$\mathbf{A}^\top \mathbf{A}_{2 \times 2} \qquad \qquad \qquad \mathbf{v}_{2 \times 1} \qquad \qquad \qquad \mathbf{A}^\top \mathbf{b}_{2 \times 1}$

Does this look familiar?

Harris Corners

$$E(\Delta x, \Delta y) = (\Delta x \quad \Delta y) \mathbf{M} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$$\mathbf{M} = \sum_{x,y} w(x,y) \begin{pmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{pmatrix}$$

M captures the structure within the local patch

$$\arg\min_{\mathbf{v}} \|\mathbf{A}\mathbf{v}-\mathbf{b}\|^2$$

$$\mathbf{A}^\top\mathbf{A}\mathbf{v}=\mathbf{A}^\top\mathbf{b}$$

$$\begin{pmatrix} \sum I_x^2 & \sum I_xI_y \\ \sum I_xI_y & \sum I_y^2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = - \begin{pmatrix} \sum I_xI_t \\ \sum I_yI_t \end{pmatrix}$$

$$\mathbf{A}^\top\mathbf{A}_{2\times 2} \qquad\qquad\qquad \mathbf{v}_{2\times 1} \qquad\qquad\qquad \mathbf{A}^\top\mathbf{b}_{2\times 1}$$

$$\mathbf{v}=(\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\mathbf{b}$$

$$\mathbf{v} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b}$$

Conditions for solvability?

$\mathbf{A}^\top \mathbf{A}$ is invertible

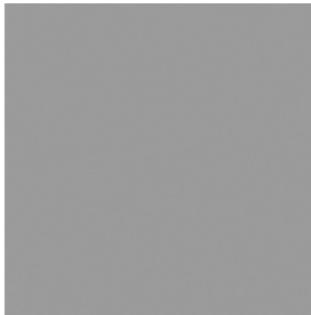
$\mathbf{A}^\top \mathbf{A}$ eigenvalues $\lambda_1, \lambda_2 \gg 0$

Conditions for solvability?

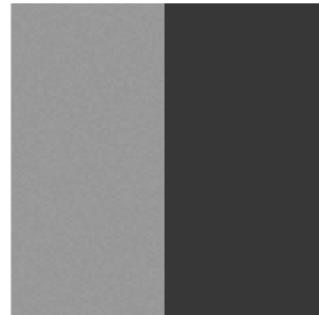
$\mathbf{A}^\top \mathbf{A}$ is invertible

$\mathbf{A}^\top \mathbf{A}$ eigenvalues $\lambda_1, \lambda_2 \gg 0$

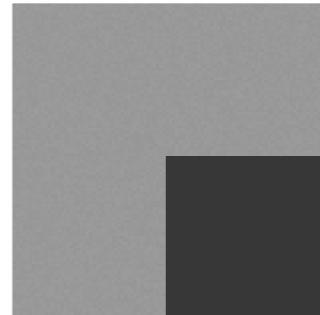
$\mathbf{A}^\top \mathbf{A}$ should be well conditioned, λ_1 / λ_2 not too large



“Flat”



“Edge”



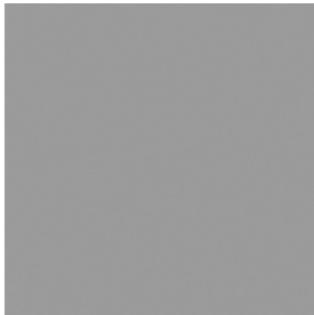
“Corner”

Conditions for solvability?

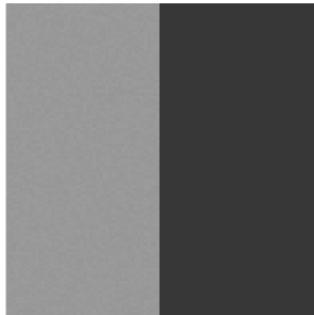
$\mathbf{A}^\top \mathbf{A}$ is invertible

$\mathbf{A}^\top \mathbf{A}$ eigenvalues $\lambda_1, \lambda_2 \gg 0$

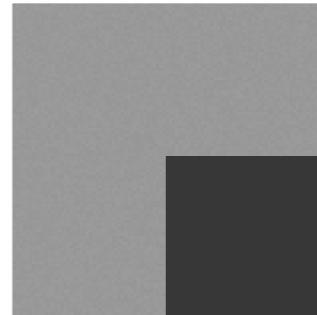
$\mathbf{A}^\top \mathbf{A}$ should be well conditioned, λ_1 / λ_2 not too large



“Flat”



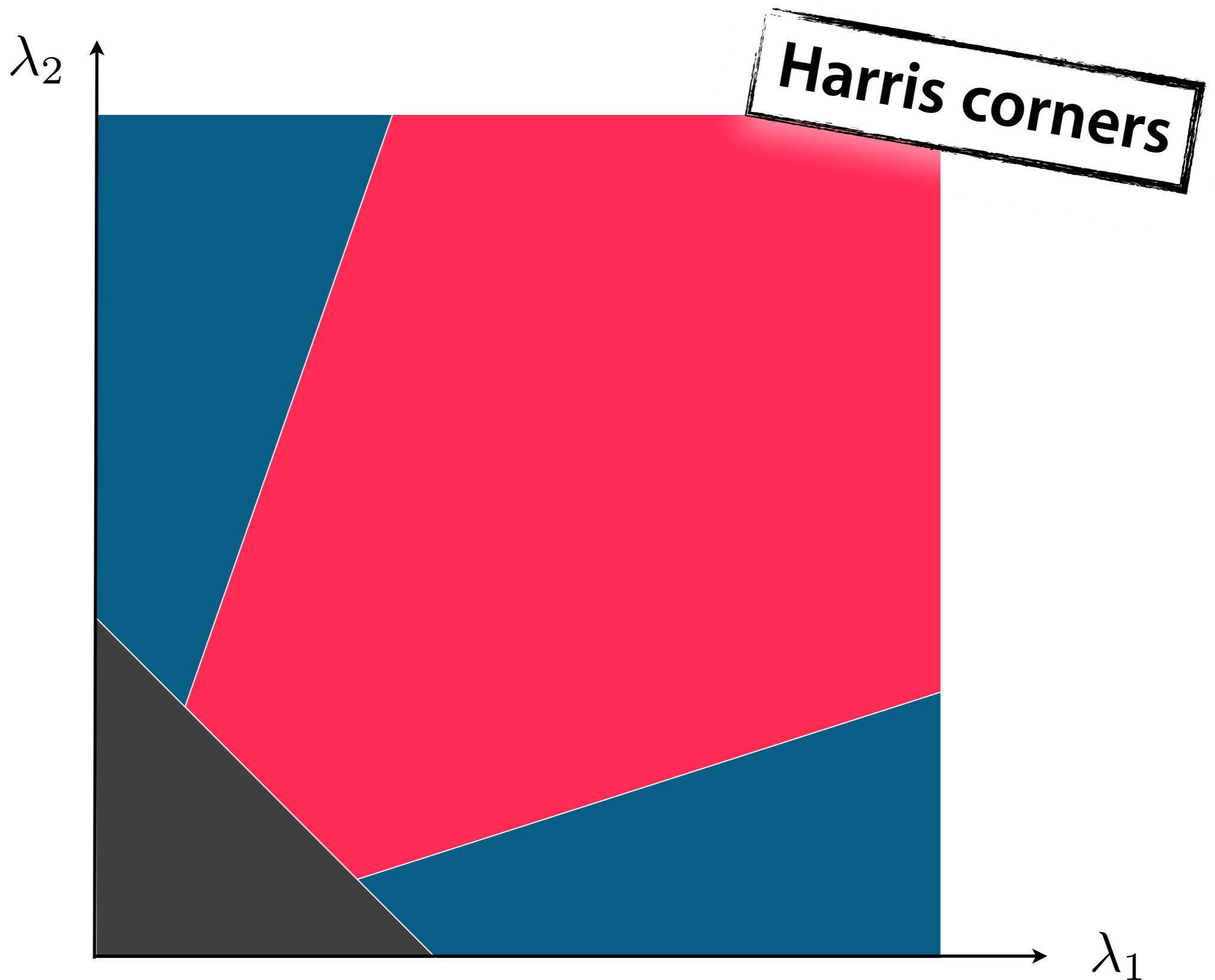
“Edge”



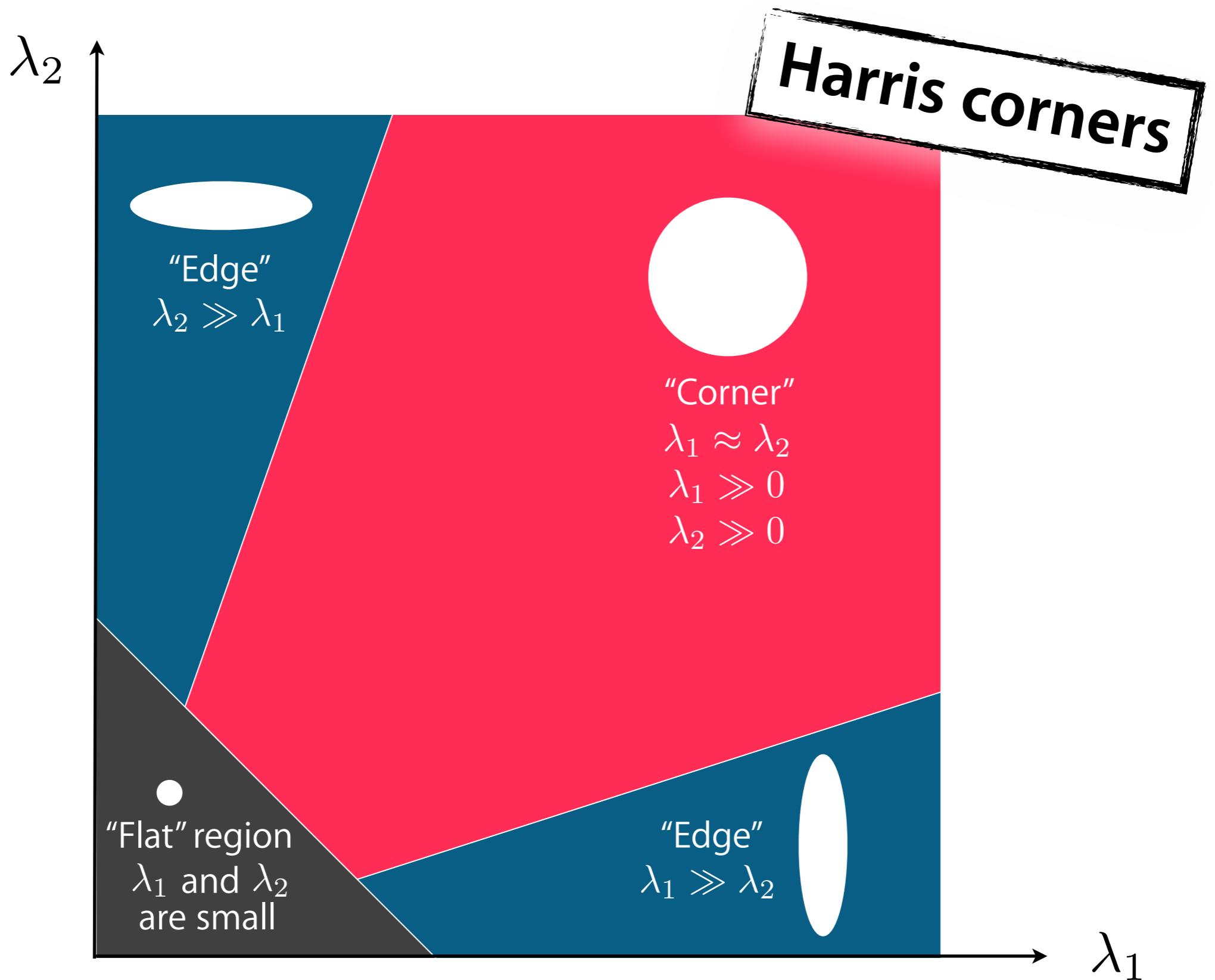
“Corner”

Which pattern satisfies these conditions?

Classification of image points using eigenvalues of M



Classification of image points using eigenvalues of M

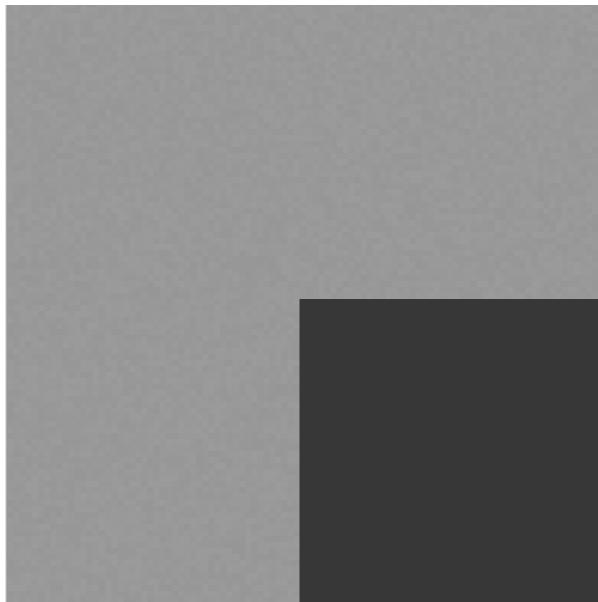


Conditions for solvability?

$\mathbf{A}^\top \mathbf{A}$ is invertible

$\mathbf{A}^\top \mathbf{A}$ eigenvalues $\lambda_1, \lambda_2 \gg 0$

$\mathbf{A}^\top \mathbf{A}$ should be well conditioned, λ_1 / λ_2 not too large



“Corner”

Pros

Easy to implement

Fast computations

Cons

Fails when brightness is not constant

Fails when intensity structure within window is poor

Fails along boundaries

Fails when the velocity is large

Locally constant translation is restrictive

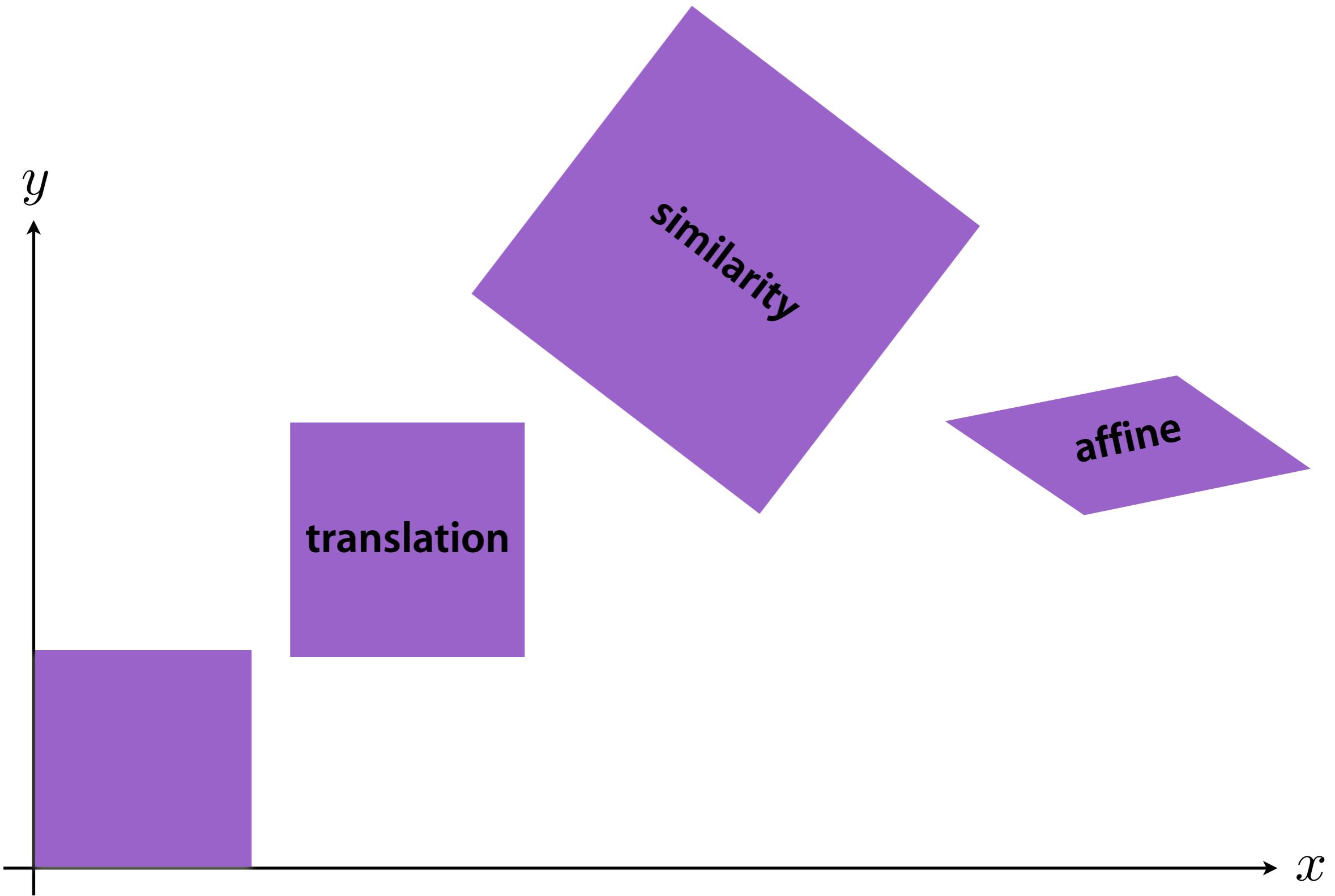
$$I(x, y, t) = I(x + \Delta x, y + \Delta y, t + \Delta t)$$

translational patch motion is restrictive

$$I(x, y, t) = I(x + \Delta x(\mathbf{x}; \mathbf{p}), y + \Delta y(\mathbf{x}; \mathbf{p}), t + \Delta t)$$

motion described by a parametric model

Parametric motion examples



Affine Motion

$$u(x, y) = a_1 + a_2x + a_3y$$

$$v(x, y) = a_4 + a_5x + a_6y$$

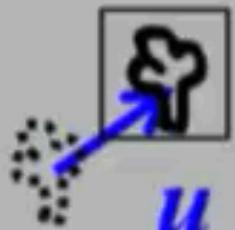
substitute into brightness constancy equation

$$I_x(a_1 + a_2x + a_3y) + I_y(a_4 + a_5x + a_6y) + I_t \approx 0$$

one linear constraint in six unknowns per pixel

solve over pixel neighbourhood using least-squares

Lucas-Kanade Optical Flow [Lucas, Kanade 81]



$$\text{minimize } E(\mathbf{u}) = \sum_{x \in W} (G(x) - F(x + \mathbf{u}))^2$$

Small \mathbf{u} \rightarrow Taylor-expand $F(x+u)$

$$F(x + \mathbf{u}) \approx F(x) + \frac{\partial F}{\partial x}^T \mathbf{u}$$

- $\rightarrow E(\mathbf{u})$ becomes quadratic of \mathbf{u}
- $\rightarrow \mathbf{u}$ can be obtained explicitly as

$$\mathbf{u} = \sum (G(x) - F(x)) \frac{\partial F}{\partial x} \left[\sum \frac{\partial F}{\partial x} \frac{\partial F}{\partial x}^T \right]^{-1}$$

KLT Tracking Summary

1. Find good features to track

Harris corners or threshold on smallest eigenvalue

2. Compute frame-to-frame feature displacement

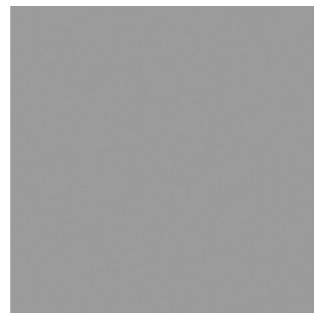
estimate patch translation using Lucas-Kanade

3. Verify appearance of patch

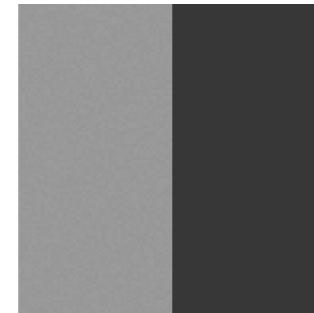
compute affine registration between current patch
and patch in first frame

Global

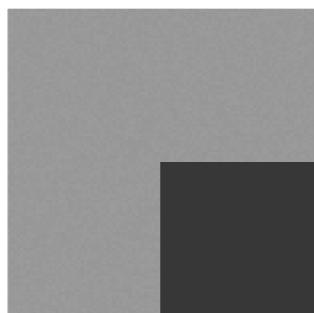
Local motion is inherently **ambiguous**



totally ambiguous



**definite along the normal,
ambiguous along the tangent**

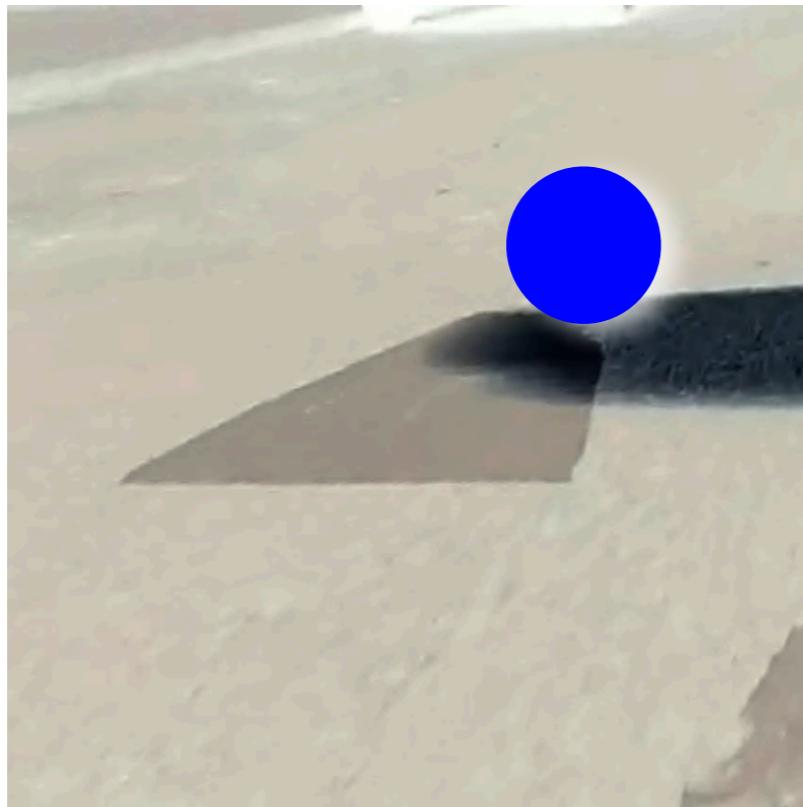


no ambiguity

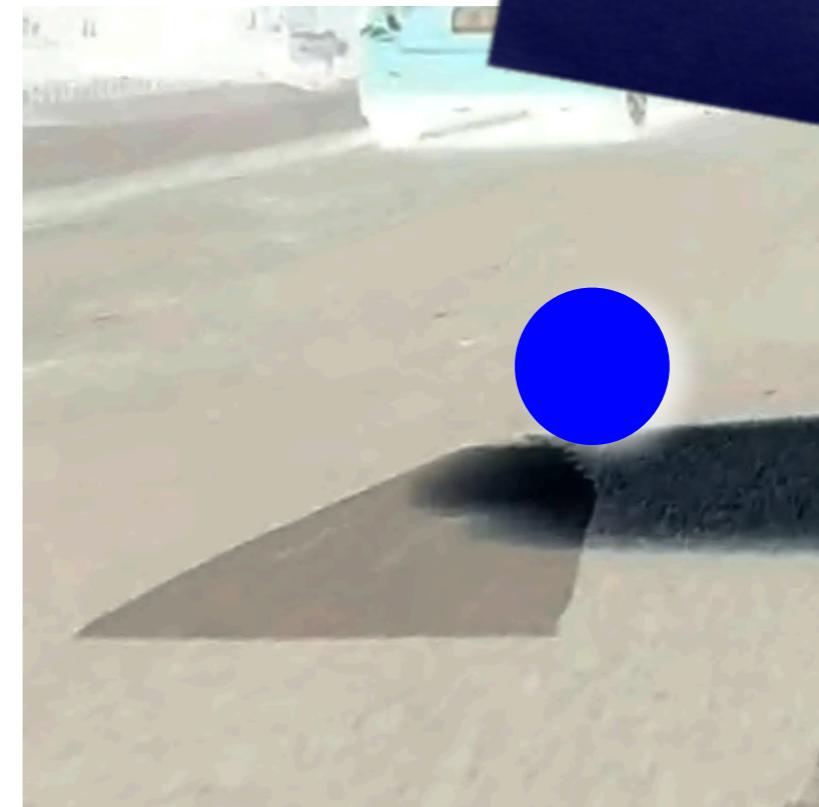
2

assumptions

Assumption
1



frame t

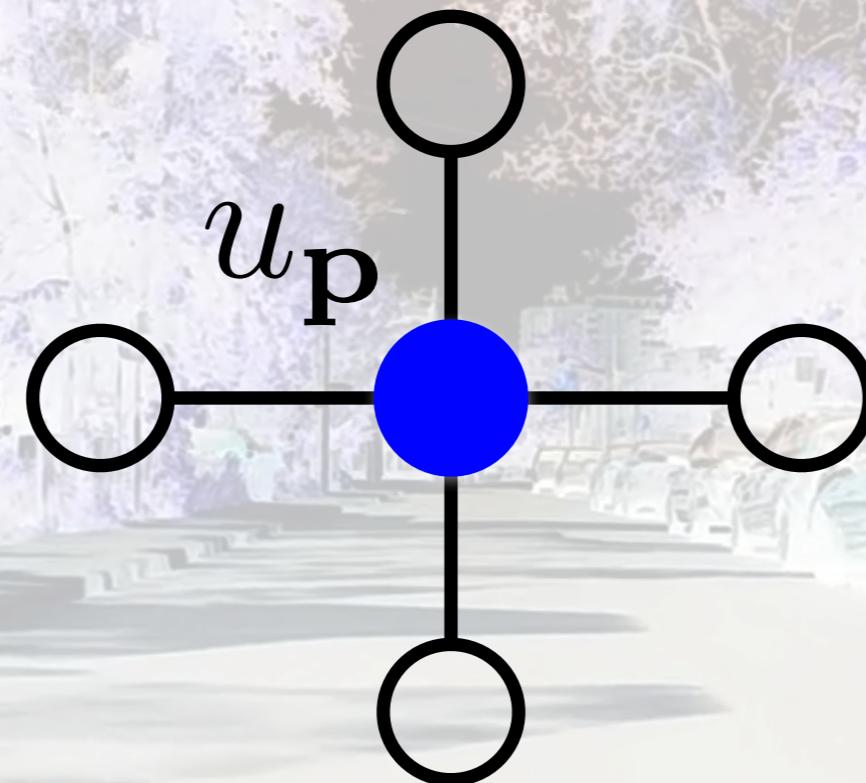


frame $t + 1$

$$I(x, y, t) = I(x + u, y + v, t + 1)$$

appearance constancy assumption

Assumption
2



$$u_p \approx u_n \text{ where } n \in \mathcal{N}(p)$$

velocity of neighbouring pixels vary slowly

Photometric Loss + Smoothness Loss

**minimize loss with respect to velocity field
over image pair**

Horn-Schunck

$$\begin{aligned} \arg \min_{\mathbf{u}, \mathbf{v}} & \sum_{i,j} \{ (I_x u_{i,j} + I_y v_{i,j} + I_t)^2 \\ & + \lambda [(u_{i,j} - u_{i+1,j})^2 + (u_{i,j} - u_{i,j+1})^2 \\ & + (v_{i,j} - v_{i+1,j})^2 + (v_{i,j} - v_{i,j+1})^2] \} \end{aligned}$$

each pixel is associated with its own velocity variable

data term

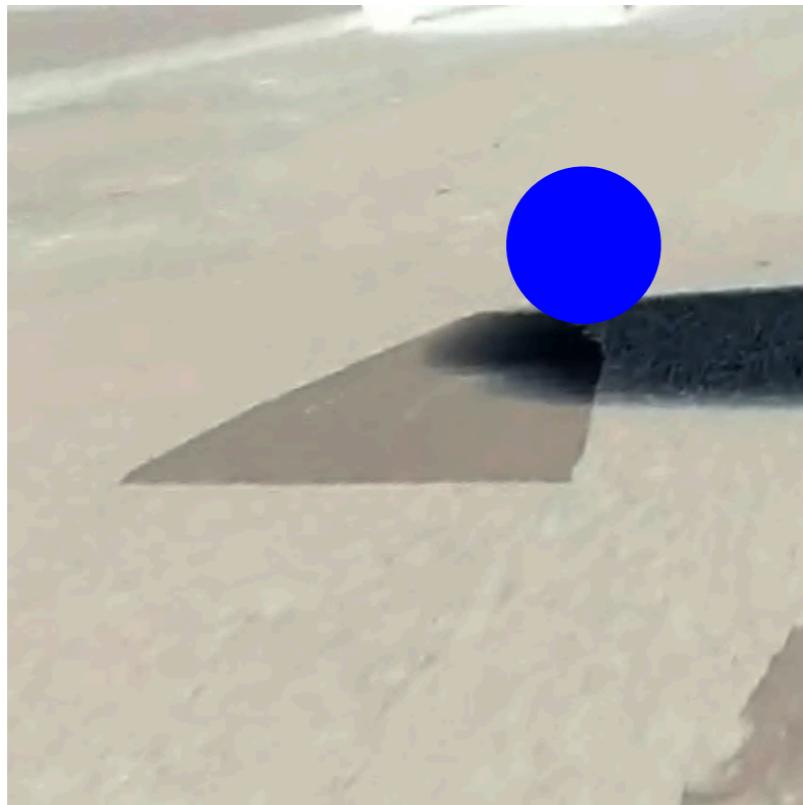
$$\sum \{(I_x u_{i,j} + I_y v_{i,j} + I_t)^2$$

measures departure from brightness constancy

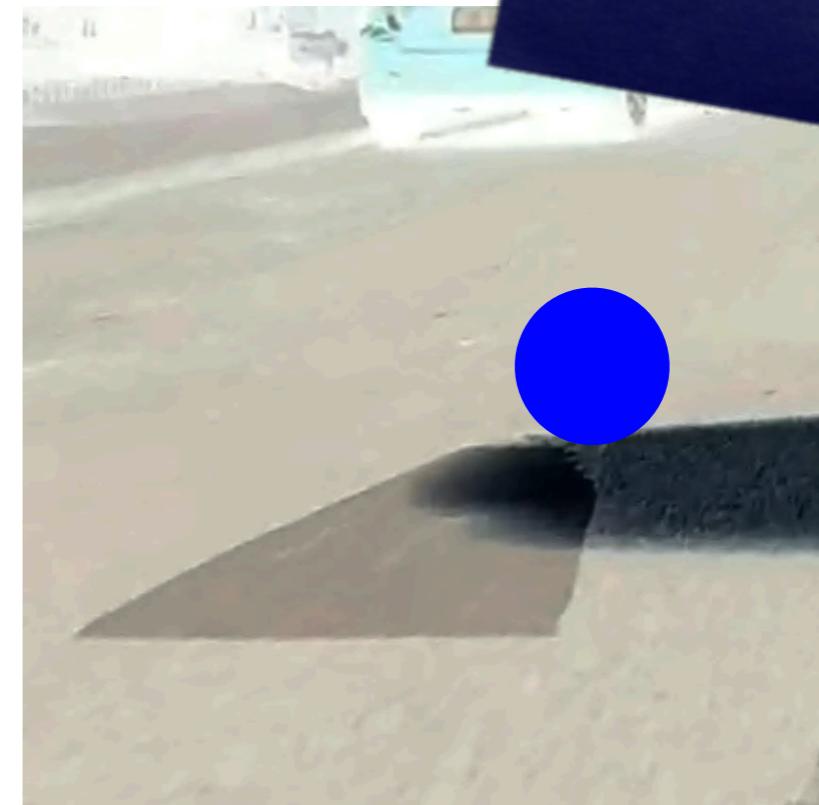
$$+ \lambda [(u_{i,j} - u_{i+1,j})^2 + (u_{i,j} -$$

$$+ (v_{i,j} - v_{i+1,j})^2 + (w_{i,j} - w_{i+1,j})^2]$$

Assumption
1



frame t



frame $t + 1$

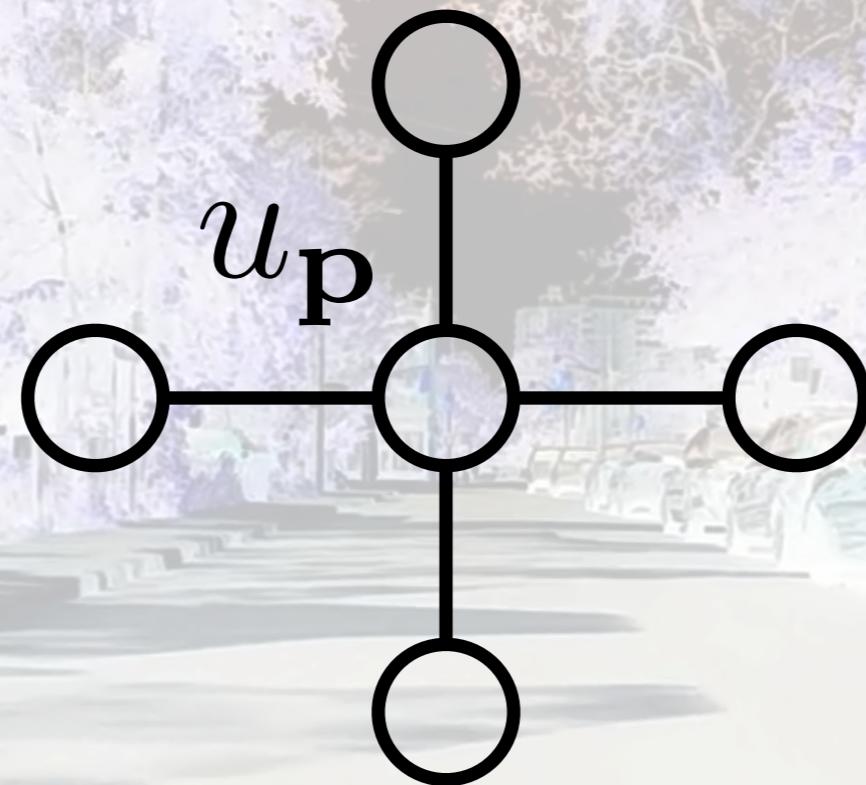
$$I(x, y, t) = I(x + u, y + v, t + 1)$$

appearance constancy assumption

$$\begin{aligned} & h \sum_{i,j} \left\{ (I_x u_{i,j} - I_y u_{i,j})^2 \right. \\ & + \lambda [(u_{i,j} - u_{i+1,j})^2 + (u_{i,j} - u_{i,j+1})^2 \right. \\ & \left. + (v_{i,j} - v_{i+1,j})^2 + (v_{i,j} - v_{i,j+1})^2] \right\} \end{aligned}$$

smoothness term

Assumption 2



$u_p \approx u_n$ where $n \in \mathcal{N}(p)$

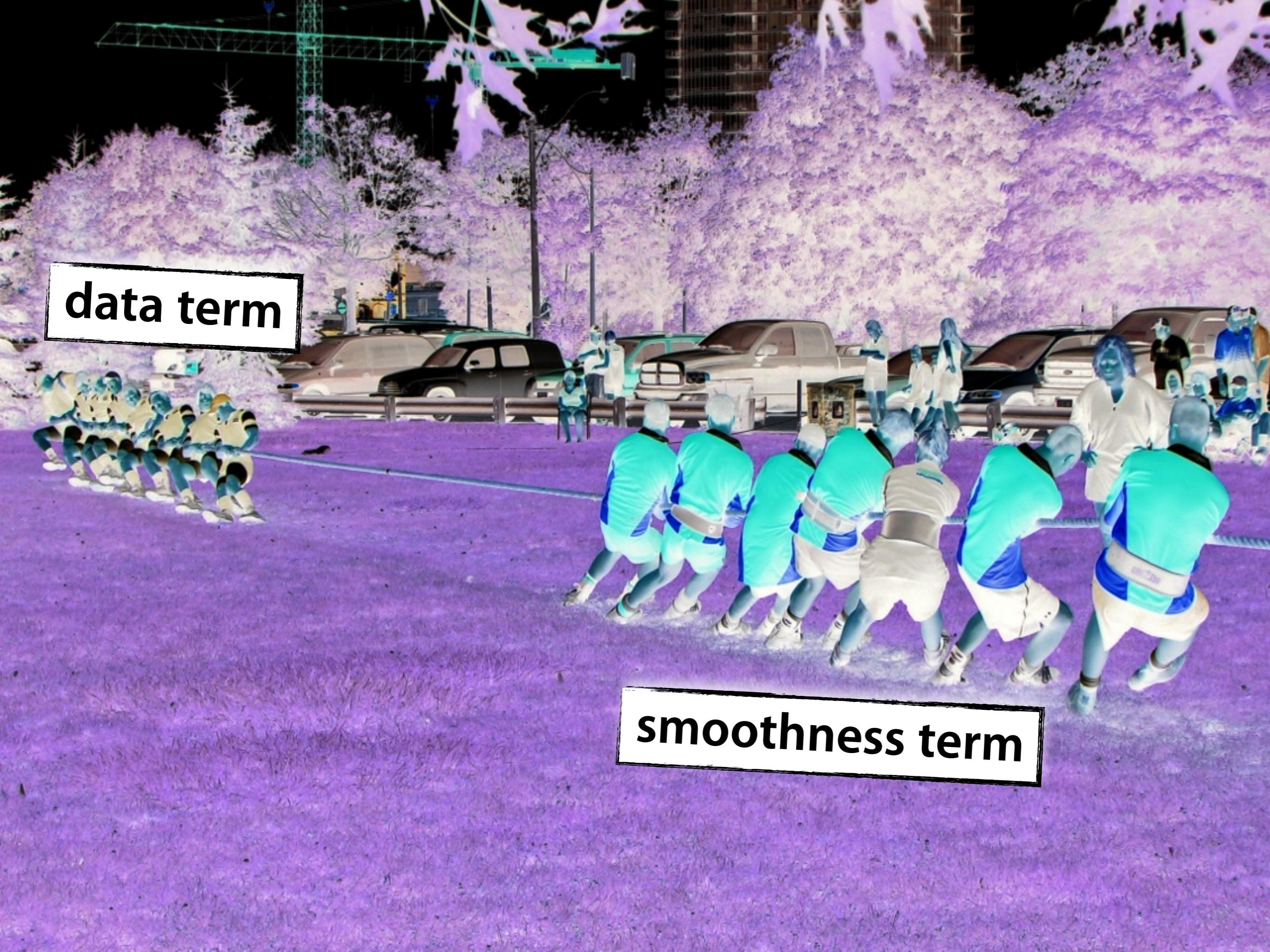
velocity of neighbouring pixels vary slowly

$$\begin{aligned} & h \sum_{i,j} \left\{ (I_x u_{i,j} - I_y u_{i,j})^2 \right. \\ & + \lambda [(u_{i,j} - u_{i+1,j})^2 + (u_{i,j} - u_{i,j+1})^2 \right. \\ & \left. + (v_{i,j} - v_{i+1,j})^2 + (v_{i,j} - v_{i,j+1})^2] \right\} \end{aligned}$$

smoothness term

$$\arg \min_{\mathbf{u}, \mathbf{v}} \sum_{i,j} \{(I_x u_{i,j} + I_y v_{i,j} + I_t)^2 + \lambda [(u_{i,j} - u_{i+1,j})^2 + (u_{i,j} - u_{i,j+1})^2 + (v_{i,j} - v_{i+1,j})^2 + (v_{i,j} - v_{i,j+1})^2]\}$$

relative weighting

A 3D point cloud visualization of a soccer field and stadium. The field is a light purple color, and the stadium seating in the background is dark purple. A large white banner with the text "data term" is positioned in the upper left corner. A white banner with the text "smoothness term" is positioned in the lower right corner.

data term

smoothness term

$$\begin{aligned}
& \arg \min_{\mathbf{u}, \mathbf{v}} \sum_{i,j} \{ (I_x u_{i,j} + I_y v_{i,j} + I_t)^2 \\
& \quad + \lambda [(u_{i,j} - u_{i+1,j})^2 + (u_{i,j} - u_{i,j+1})^2 \\
& \quad + (v_{i,j} - v_{i+1,j})^2 + (v_{i,j} - v_{i,j+1})^2] \}
\end{aligned}$$

Differentiate by unknowns and set to zero

$$\begin{aligned}
& (I_x u_{i,j} + I_y v_{i,j} + I_t) I_x + \\
& \lambda [4u_{i,j} - (u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1})] = 0
\end{aligned}$$

$$\begin{aligned}
& (I_x u_{i,j} + I_y v_{i,j} + I_t) I_y + \\
& \lambda [4v_{i,j} - (v_{i-1,j} + v_{i+1,j} + v_{i,j-1} + v_{i,j+1})] = 0
\end{aligned}$$

$$(I_x u_{i,j} + I_y v_{i,j} + I_t) I_x +$$

$$\lambda [4u_{i,j} - (u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1})] = 0$$

$$(I_x u_{i,j} + I_y v_{i,j} + I_t) I_y +$$

$$\lambda [4v_{i,j} - (v_{i-1,j} + v_{i+1,j} + v_{i,j-1} + v_{i,j+1})] = 0$$

each image point yields a pair of linear equations

$$(I_x u_{i,j} + I_y v_{i,j} + I_t) I_x +$$

$$\lambda[4u_{i,j} - (u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1})] = 0$$

$$(I_x u_{i,j} + I_y v_{i,j} + I_t) I_y +$$

$$\lambda[4v_{i,j} - (v_{i-1,j} + v_{i+1,j} + v_{i,j-1} + v_{i,j+1})] = 0$$

equations can be solved using Gaussian elimination

in theory

Pros

Integrates information globally

Dense flow

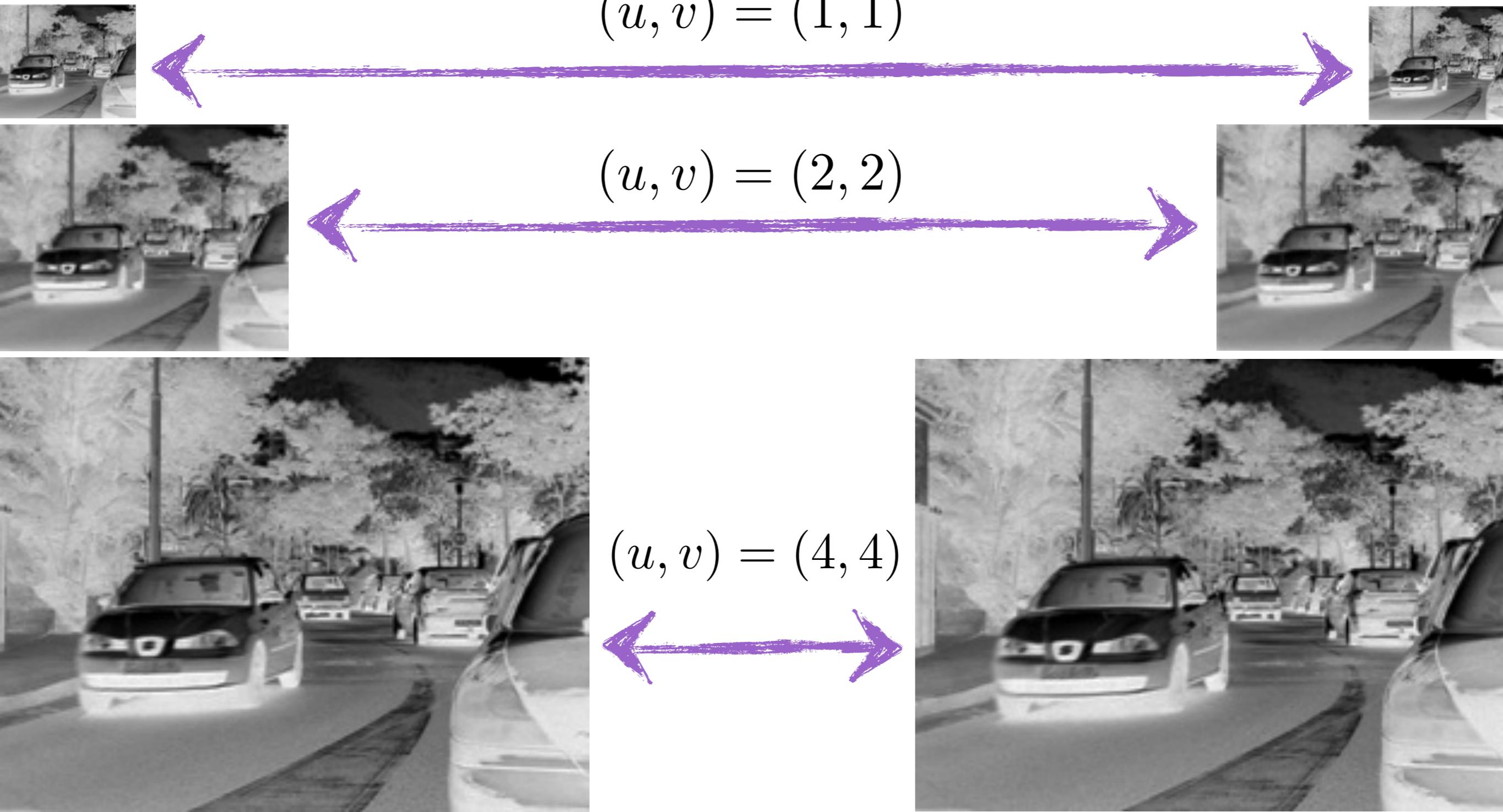
Cons

Fails when brightness is not constant

Slow iterative estimation

Smooths over boundaries

When the velocity is large the input is
temporally aliased
temporal derivatives not reliable

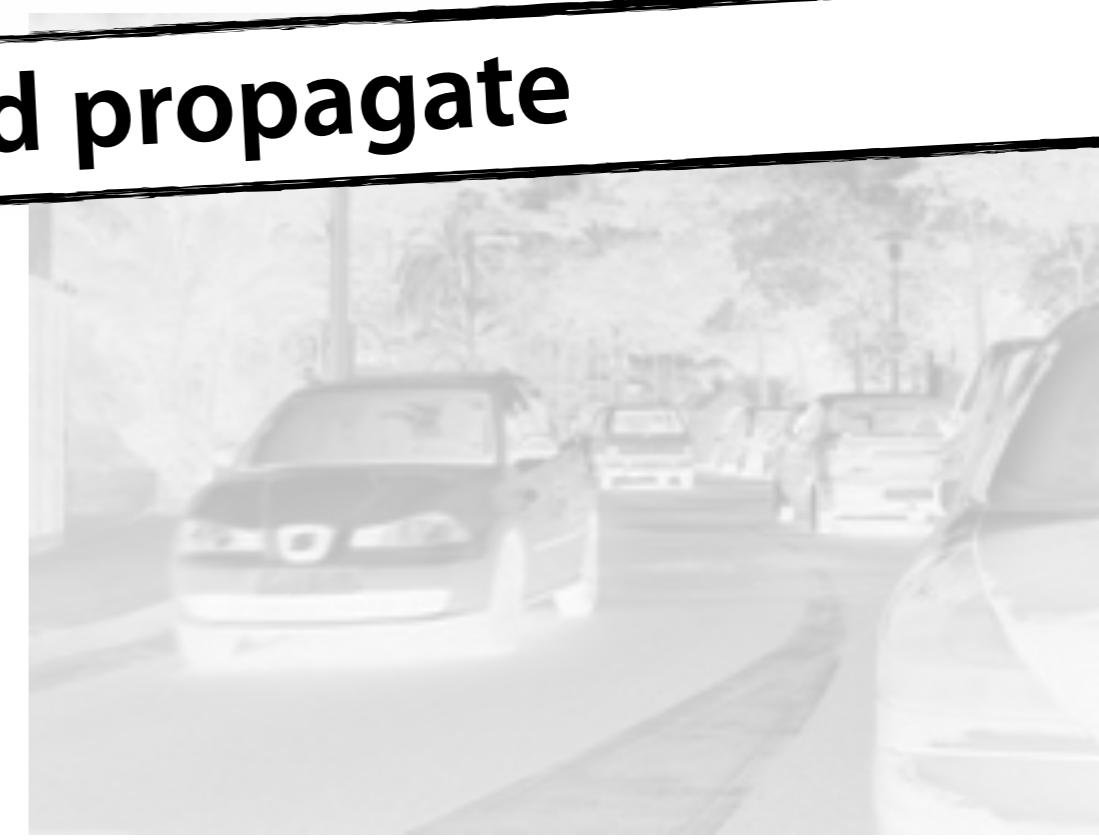

$$I(x, y, t)$$
$$I(x, y, t + 1)$$



warp, estimate flow and propagate



$I(x, y, t)$



$I(x, y, t + 1)$

Large displacements are **difficult to estimate**