

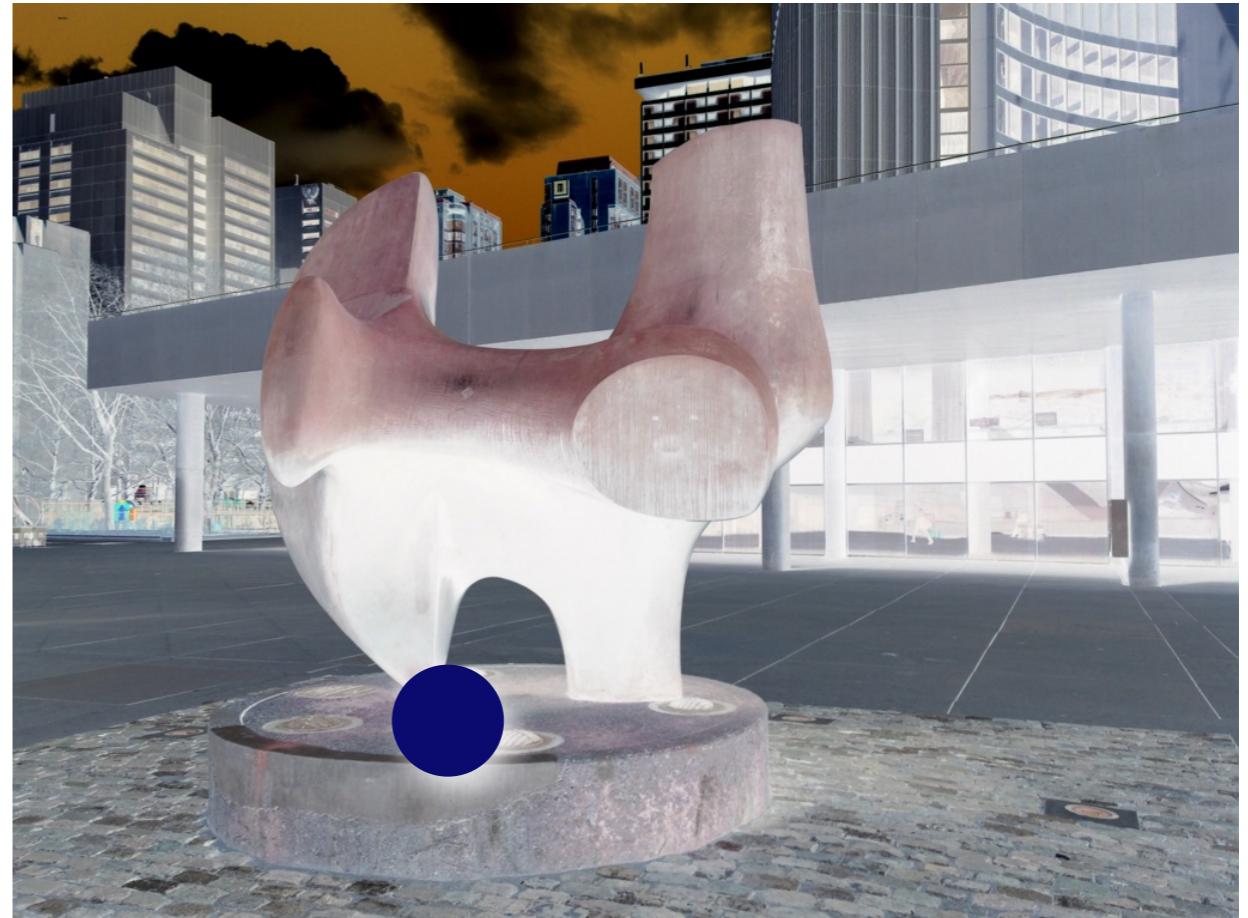
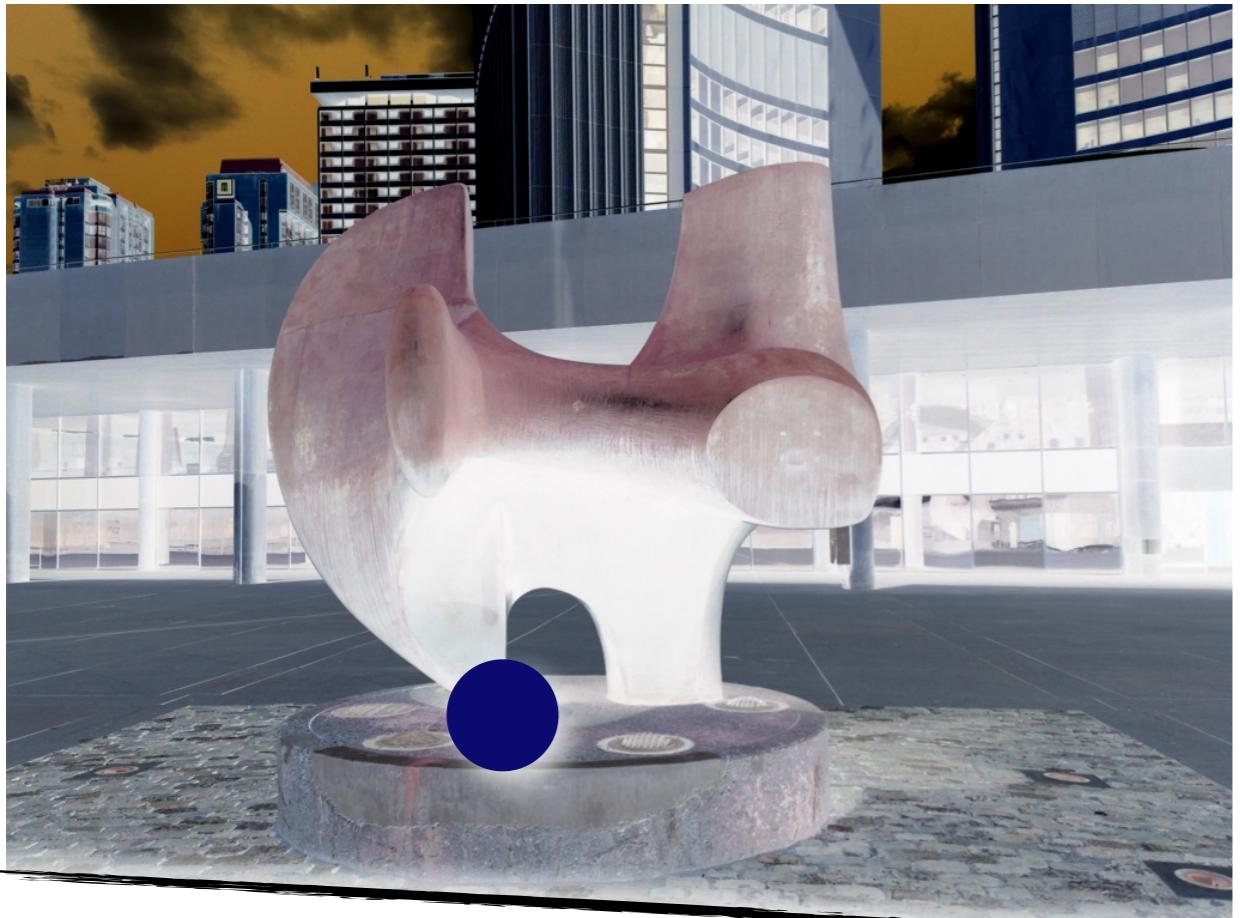
LECTURE TOPICS

Epipolar geometry

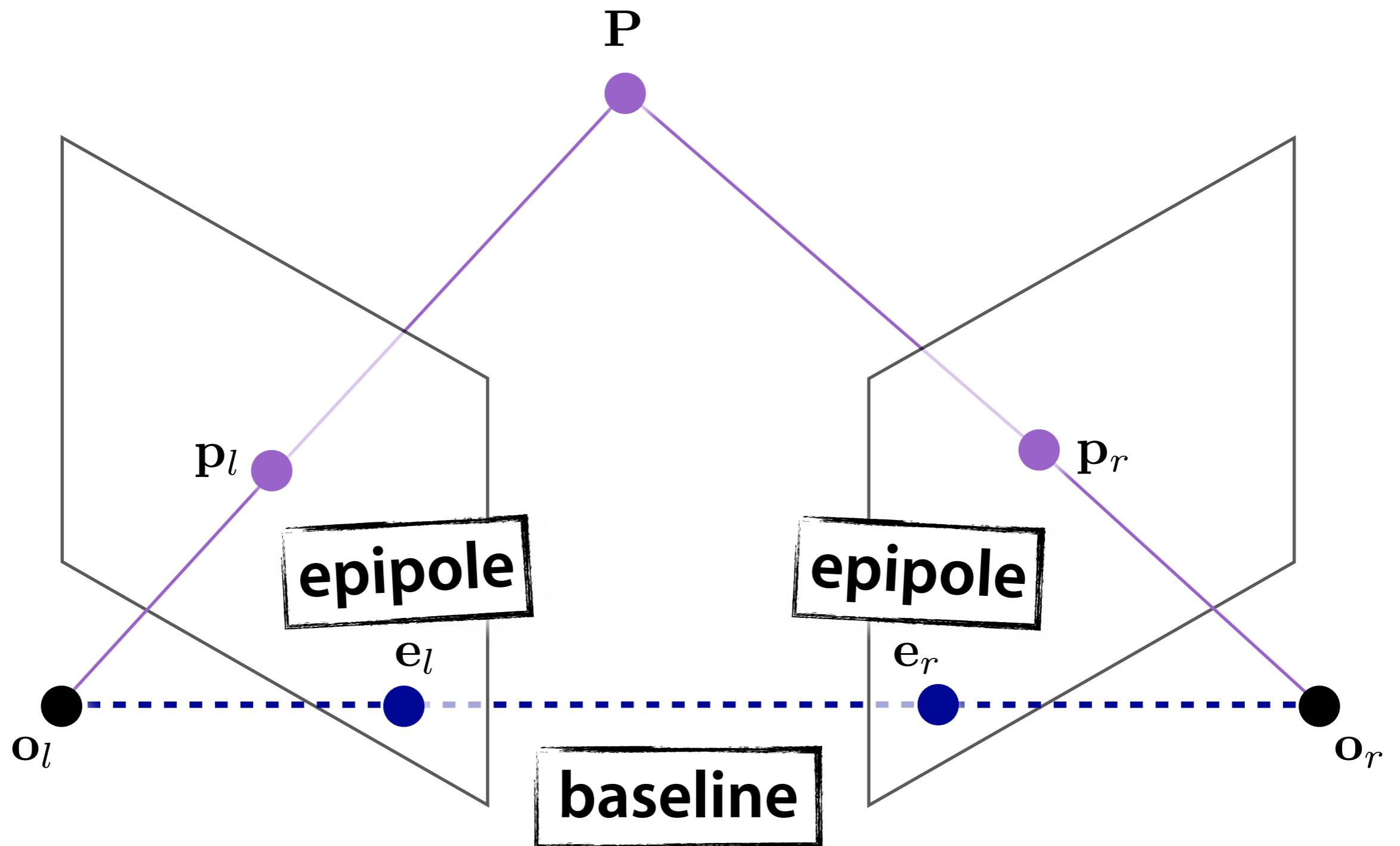
Fundamental matrix

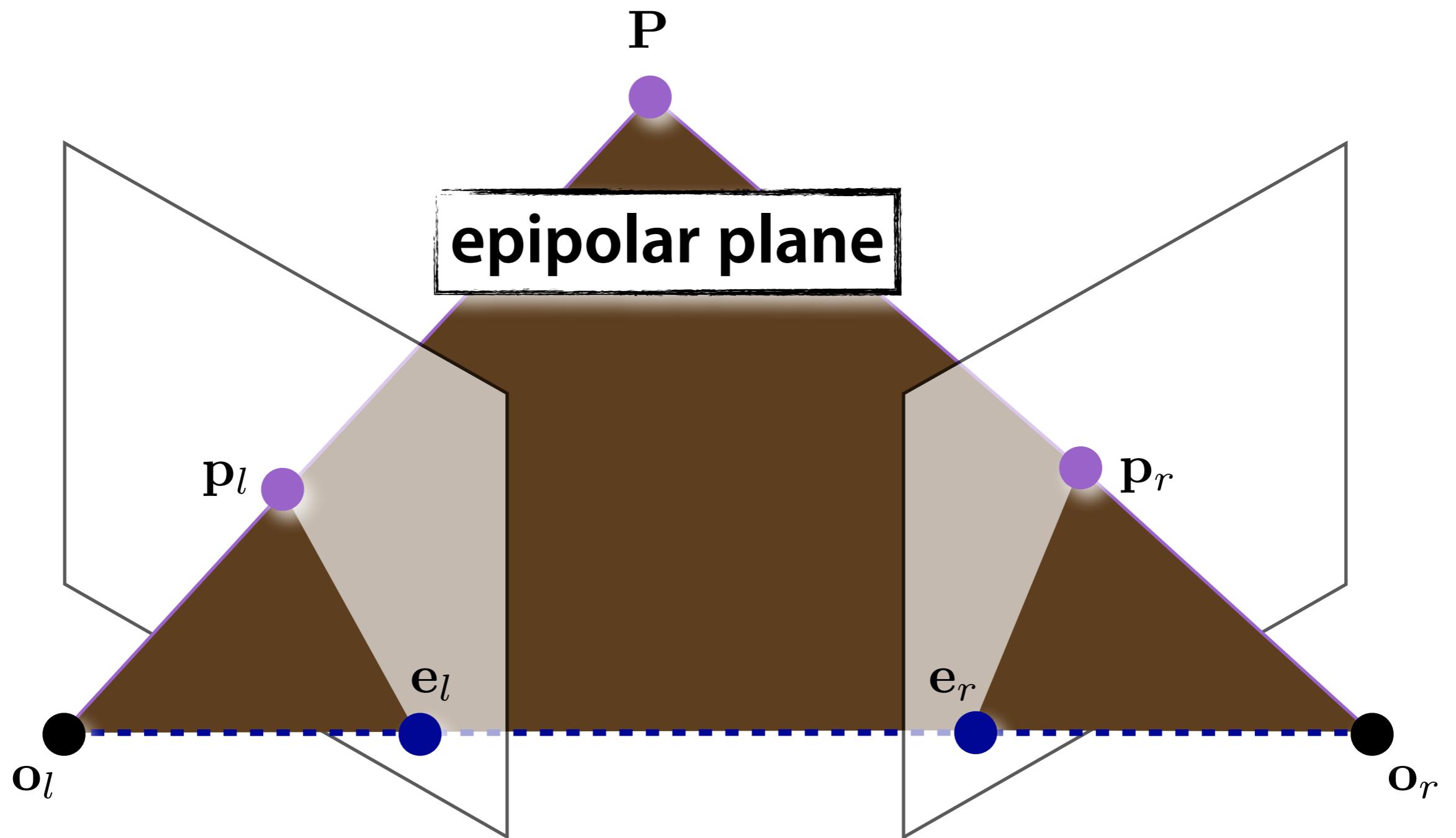
Stereo matching

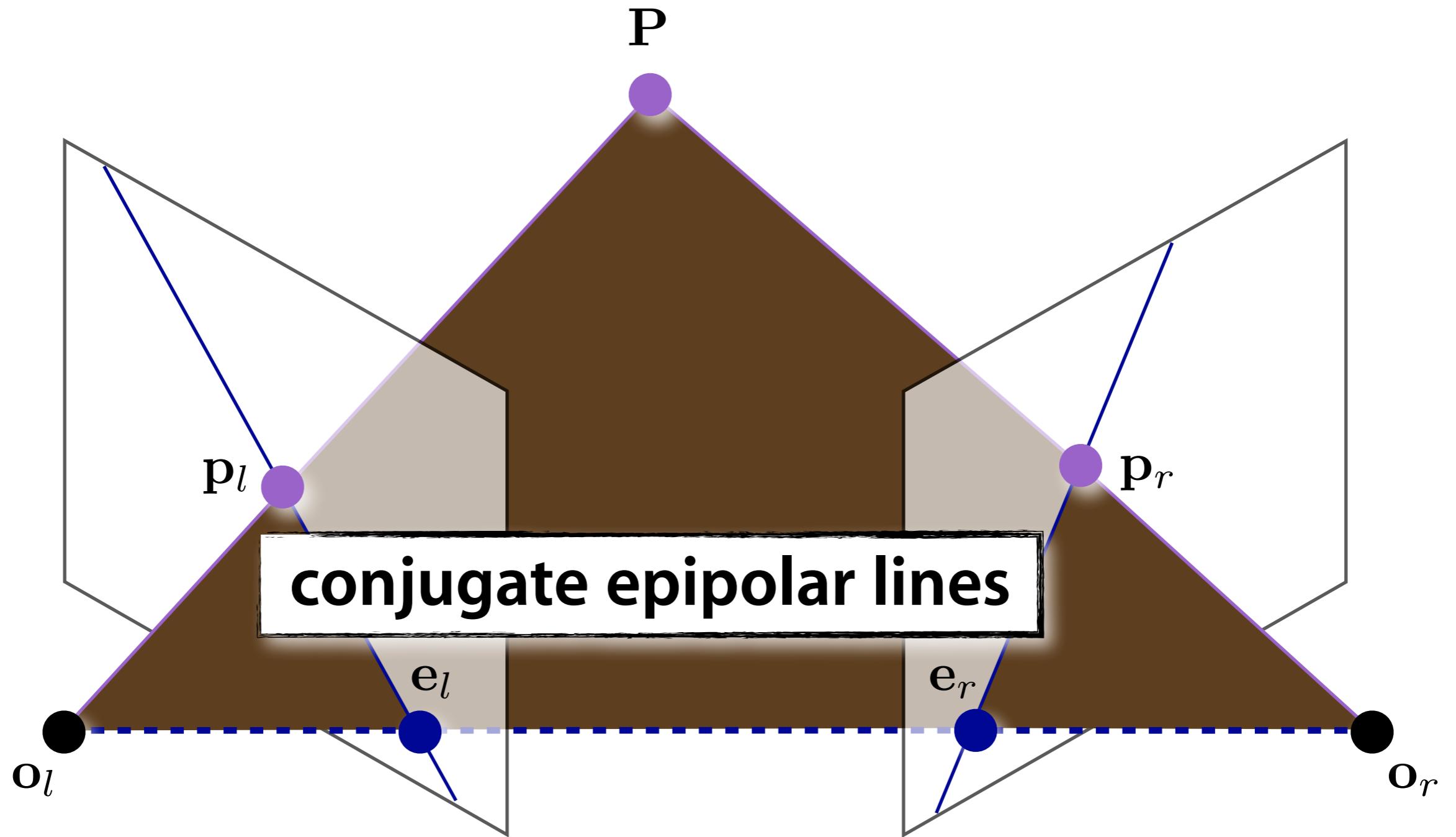
How can we recover 3D geometry from images?

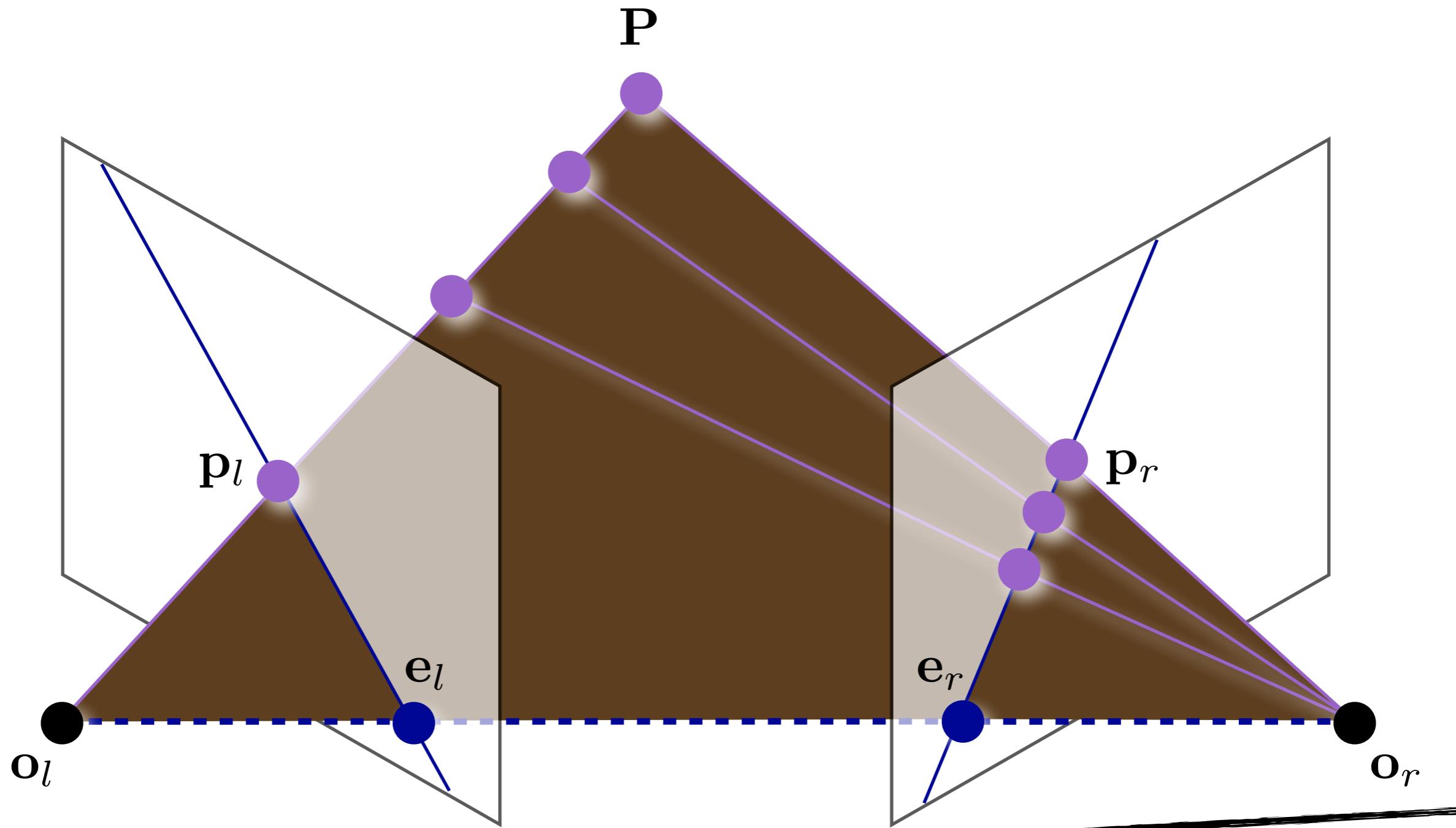


How can its match in the right image be found?

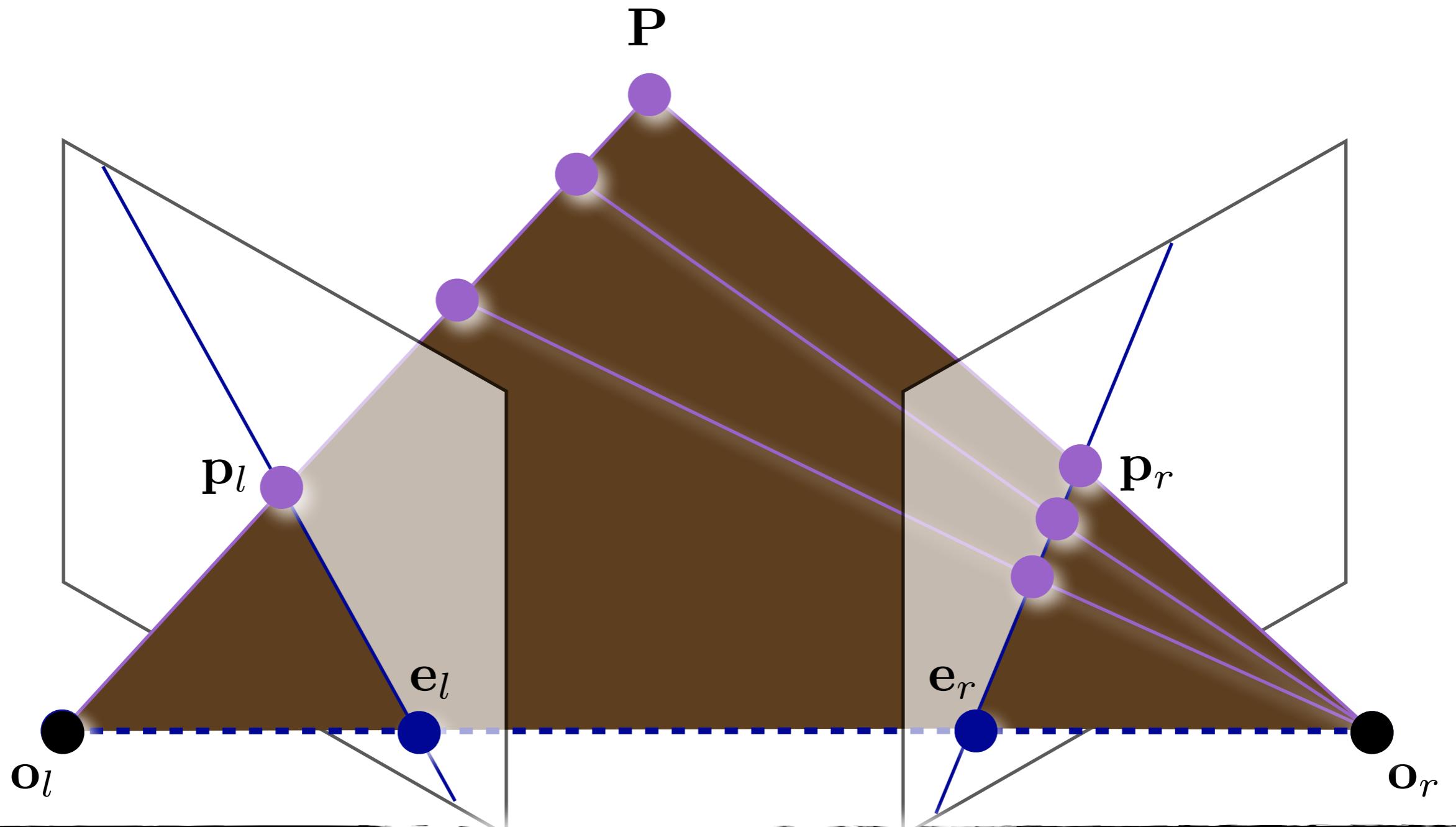






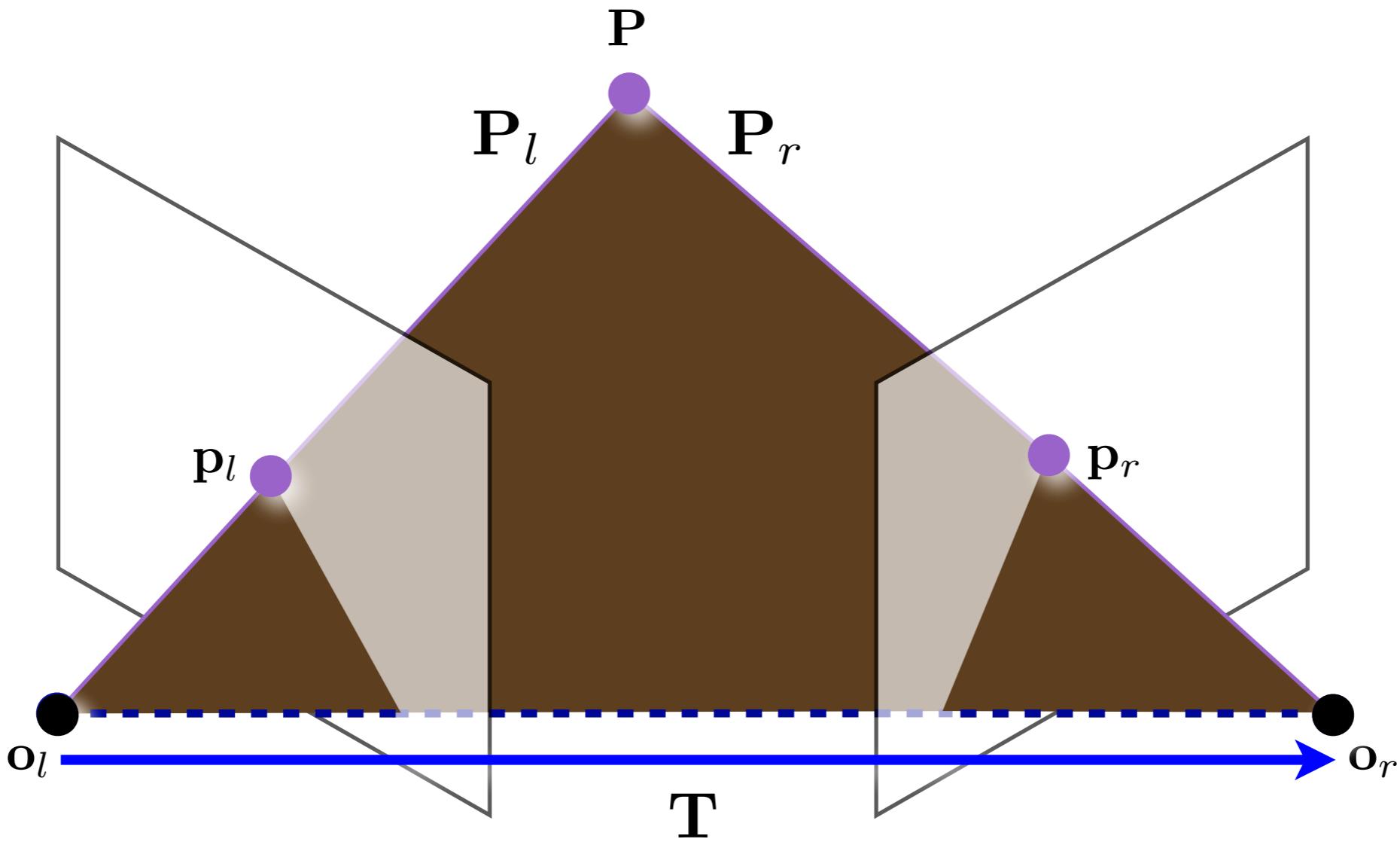


Given p_l , where does it project in the right image?



Epipolar constraint

Matching points must lie on conjugated epipolar lines



$T = O_r - O_l$ define the translation that shifts the left centre of projection to the right one

R define the rotation matrix that aligns the coordinate axes of the left and right views

$P_r = R(P_l - T)$ define the transformation between the coordinates of P in the left and right views

Definition: The **column rank** of an $m \times n$ matrix \mathbf{A} is the number of linearly independent columns of the matrix.

The **row rank** of matrix \mathbf{A} is the number of linearly independent rows of \mathbf{A} .

row and column matrix ranks are always equal

Definition: The **column rank** of an $m \times n$ matrix \mathbf{A} is the number of linearly independent columns of the matrix.

The **row rank** of matrix \mathbf{A} is the number of linearly independent rows of \mathbf{A} .

row and column matrix ranks are always equal

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

What are the row and column ranks of this matrix?

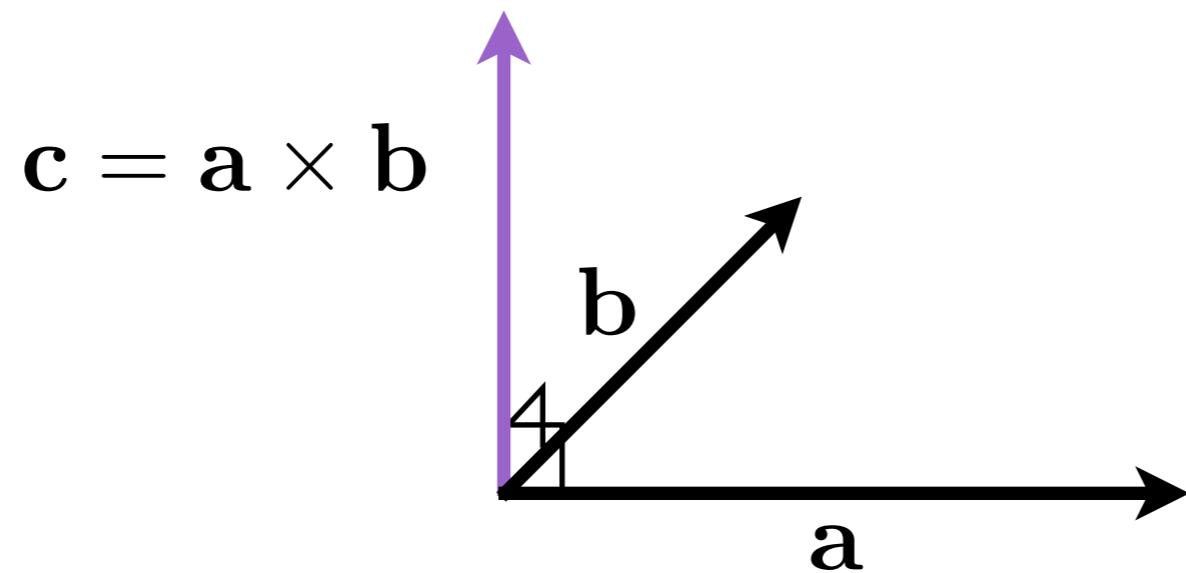
Definition: The **null space** of an $m \times n$ matrix \mathbf{A} is the set of all solutions to the homogeneous equation

$$\mathbf{Ax} = \mathbf{0}.$$

$$\text{Null}(\mathbf{A}) = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{Ax} = \mathbf{0}\}$$

**rank and nullity of a matrix
add up to the number of columns of the matrix**

Definition: The cross product of two vectors \mathbf{a} and \mathbf{b} is defined as a vector that is perpendicular to both \mathbf{a} and \mathbf{b} , i.e., $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$.



Given 3×1 **vectors** $\mathbf{a} = (a_1, a_2, a_3)^\top$ **and** $\mathbf{b} = (b_1, b_2, b_3)^\top$

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{pmatrix} -a_3b_2 + a_2b_3 \\ a_3b_1 - a_1b_3 \\ -a_2b_1 + a_1b_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\end{aligned}$$

matrix multiplication

Given 3×1 **vectors** $\mathbf{a} = (a_1, a_2, a_3)^\top$ **and** $\mathbf{b} = (b_1, b_2, b_3)^\top$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

skew symmetric matrix
 $\mathbf{S} = -\mathbf{S}^\top$

Given 3×1 **vectors** $\mathbf{a} = (a_1, a_2, a_3)^\top$ **and** $\mathbf{b} = (b_1, b_2, b_3)^\top$

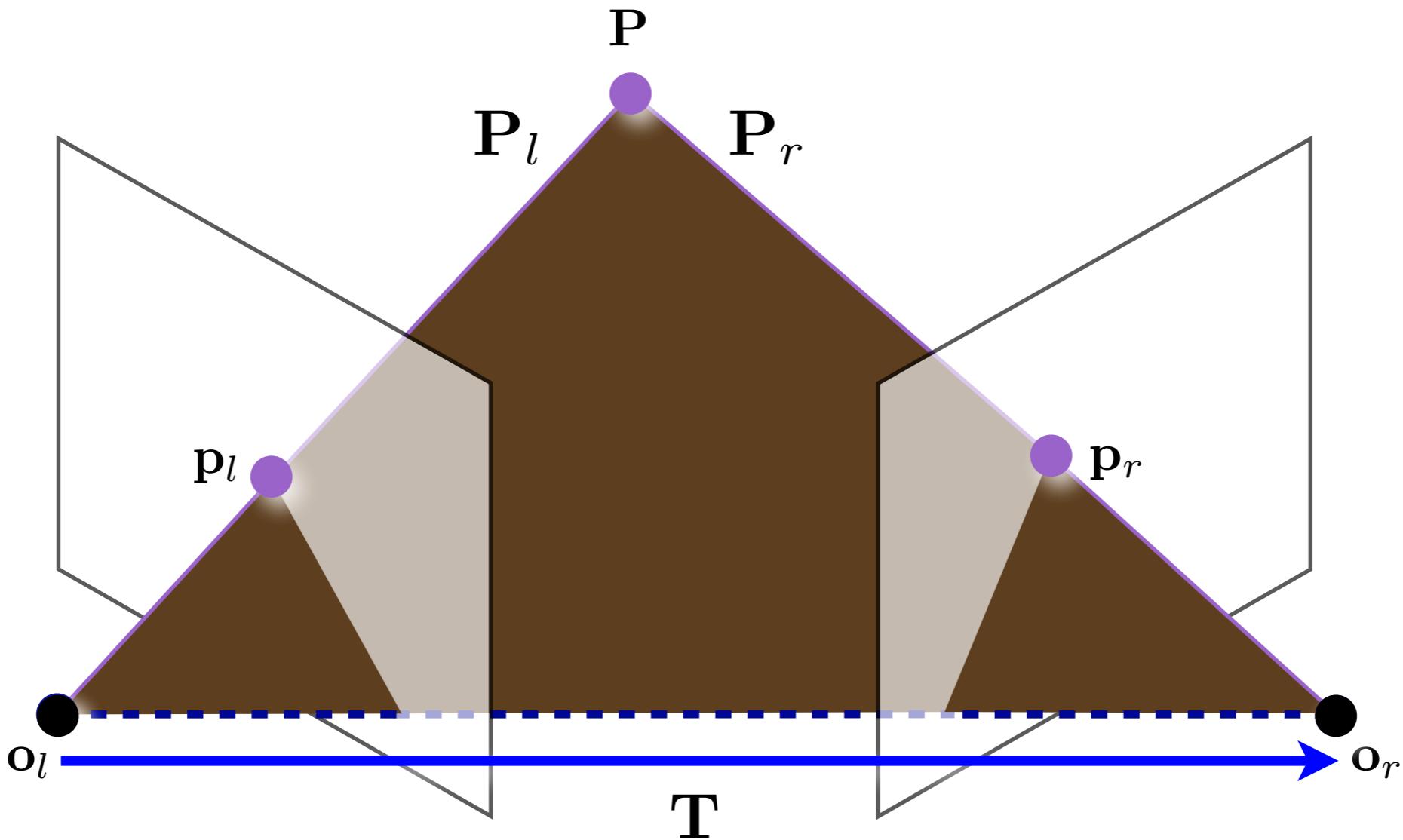
$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

rank 2 matrix

Given 3×1 **vectors** $\mathbf{a} = (a_1, a_2, a_3)^\top$ **and** $\mathbf{b} = (b_1, b_2, b_3)^\top$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= [\mathbf{a}_\times] \mathbf{b}$$

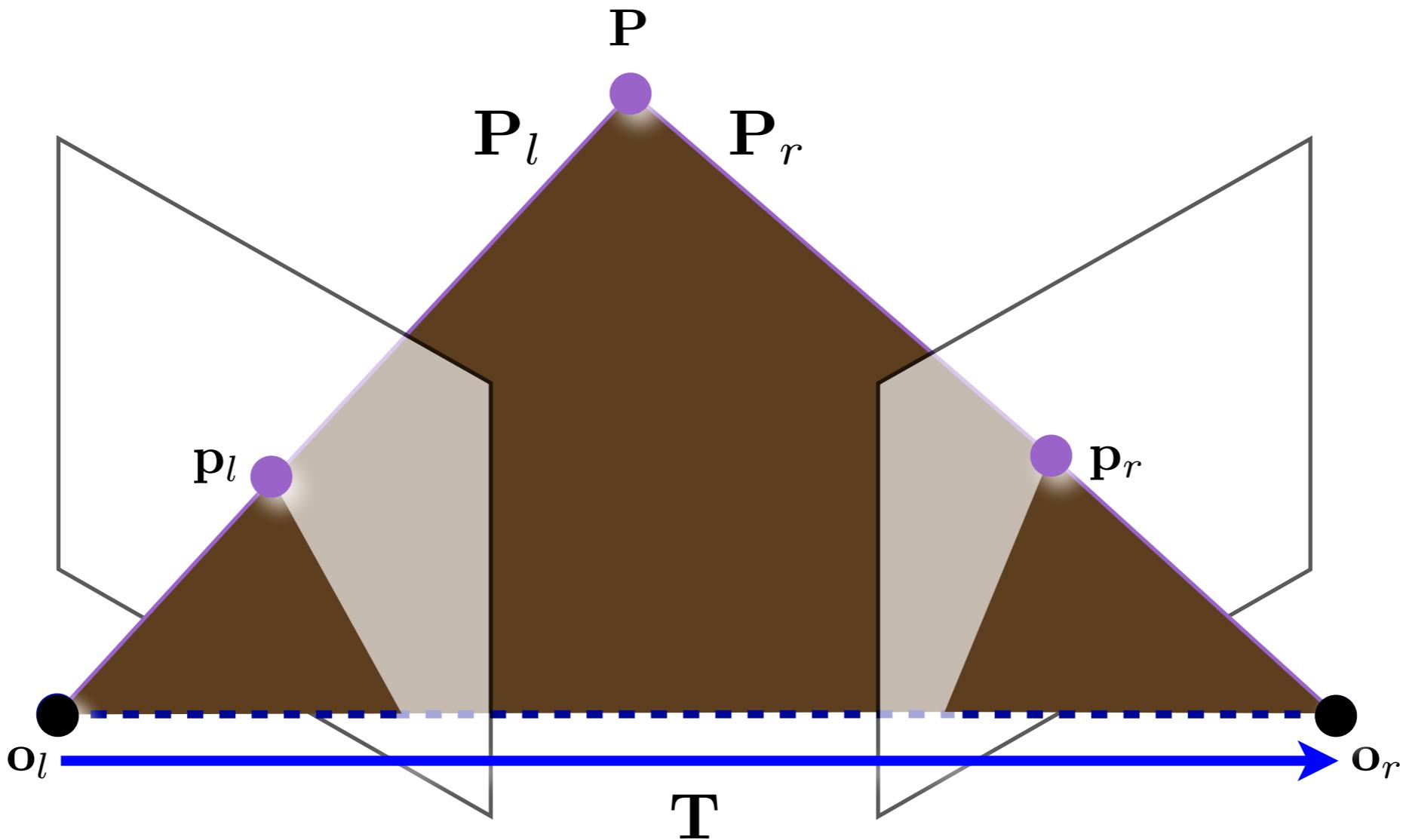


$P_r = R(P_l - T)$ define the transformation between the coordinates of P in the left and right views

Coplanarity condition:

$$(P_l - T) \cdot (T \times P_l) = 0$$

normal to the plane



$P_r = R(P_l - T)$ define the transformation between the coordinates of P in the left and right views

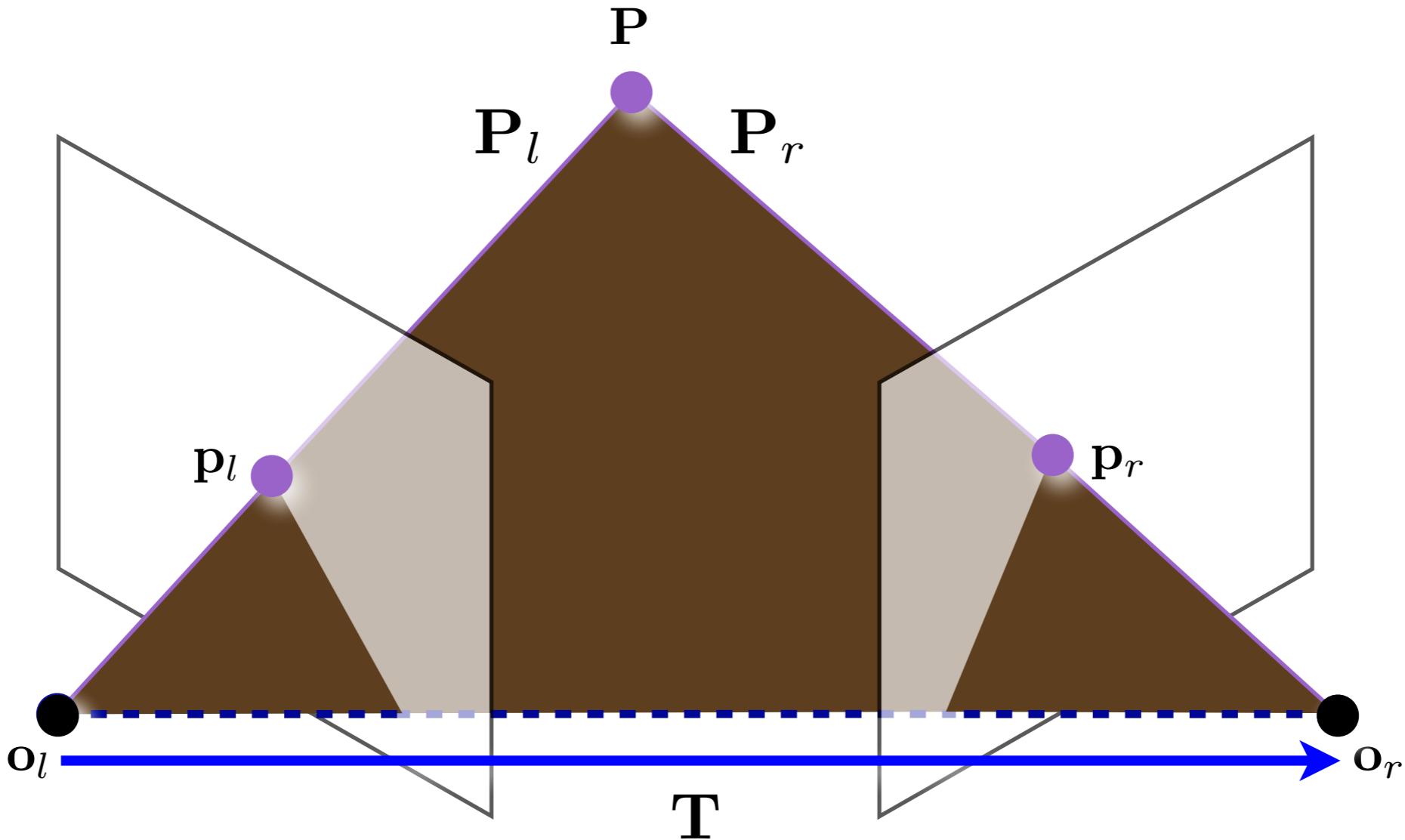
Coplanarity condition:

$$(P_l - T) \cdot (T \times P_l) = 0$$

substitute

$$(R^\top P_r) \cdot (T \times P_l) = 0$$

$$P_l = R^\top P_r + T$$



Coplanarity condition:

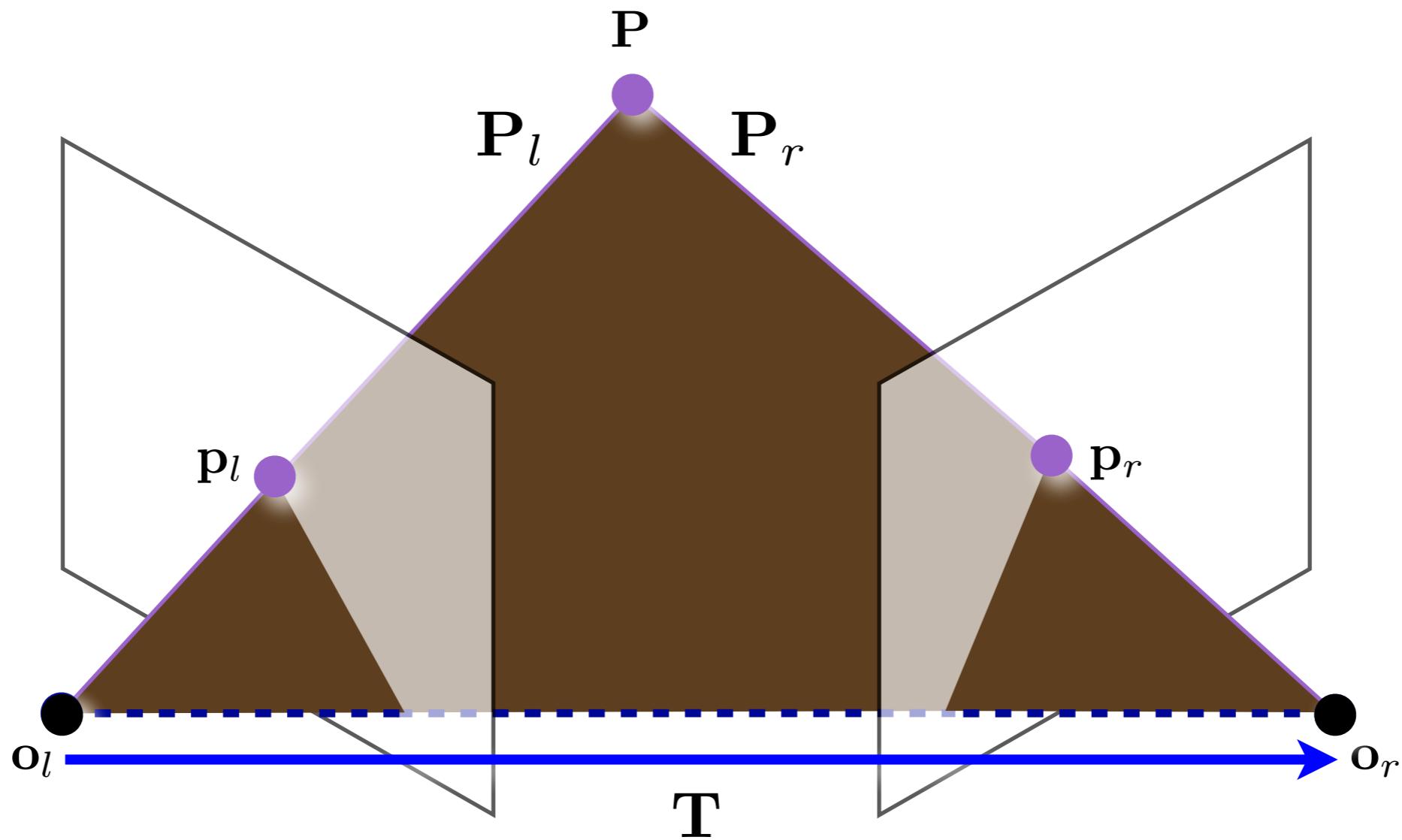
$$(\mathbf{R}^\top \mathbf{P}_r) \cdot (\mathbf{T} \times \mathbf{P}_l) = 0$$

cross product as matrix

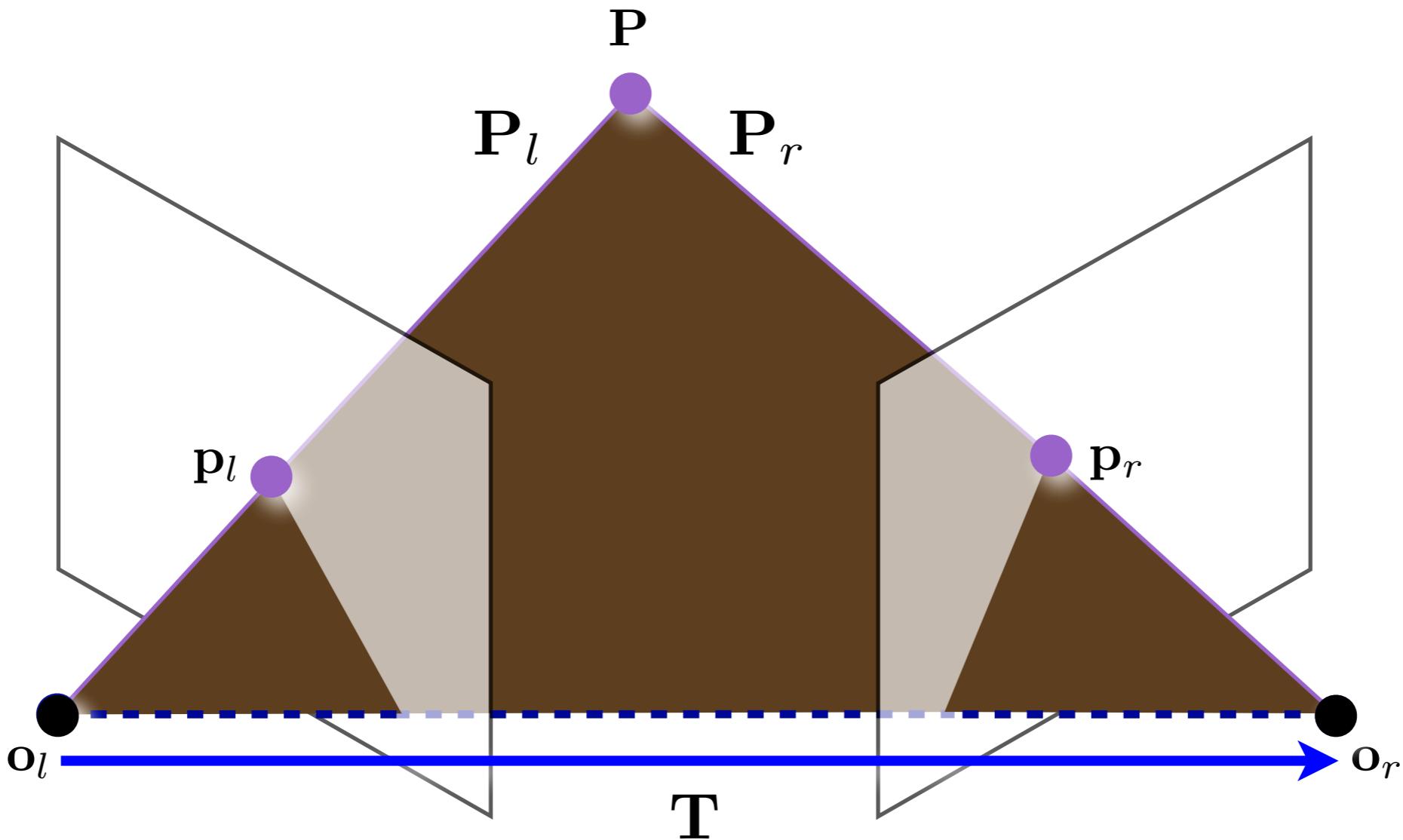
$$(\mathbf{R}^\top \mathbf{P}_r)^\top [\mathbf{T}_\times] \mathbf{P}_l = 0$$

rewrite

$$\mathbf{P}_r^\top \mathbf{R} [\mathbf{T}_\times] \mathbf{P}_l = 0$$



$$\mathbf{P}_r^\top \mathbf{R}[\mathbf{T}_\times] \mathbf{P}_l = 0$$

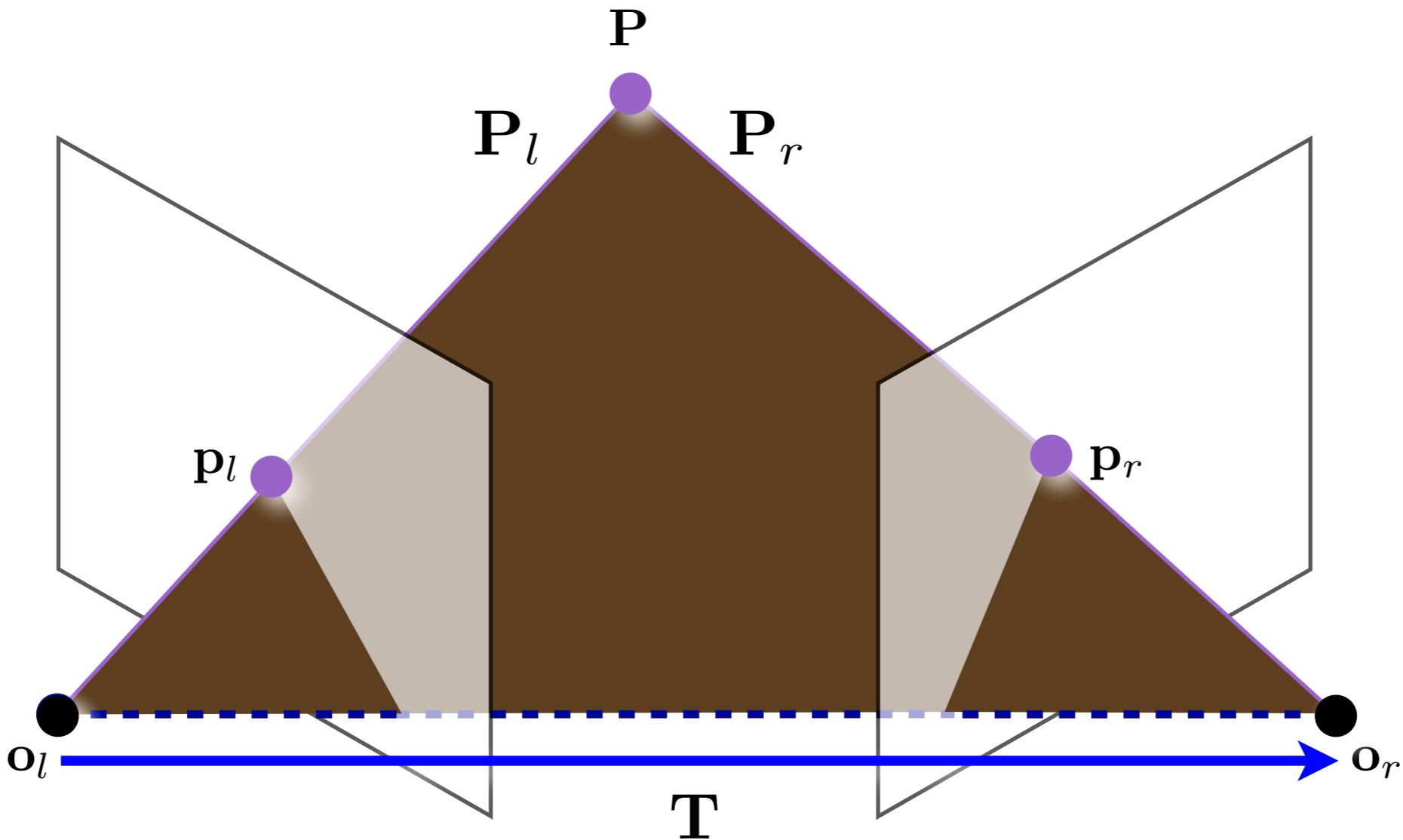


$$\mathbf{P}_r^\top \mathbf{R}[\mathbf{T}_\times] \mathbf{P}_l = 0$$

$$\mathbf{P}_r^\top \mathbf{E} \mathbf{P}_l = 0$$

Let $\mathbf{E} = \mathbf{R}[\mathbf{T}_\times]$

Essential Matrix

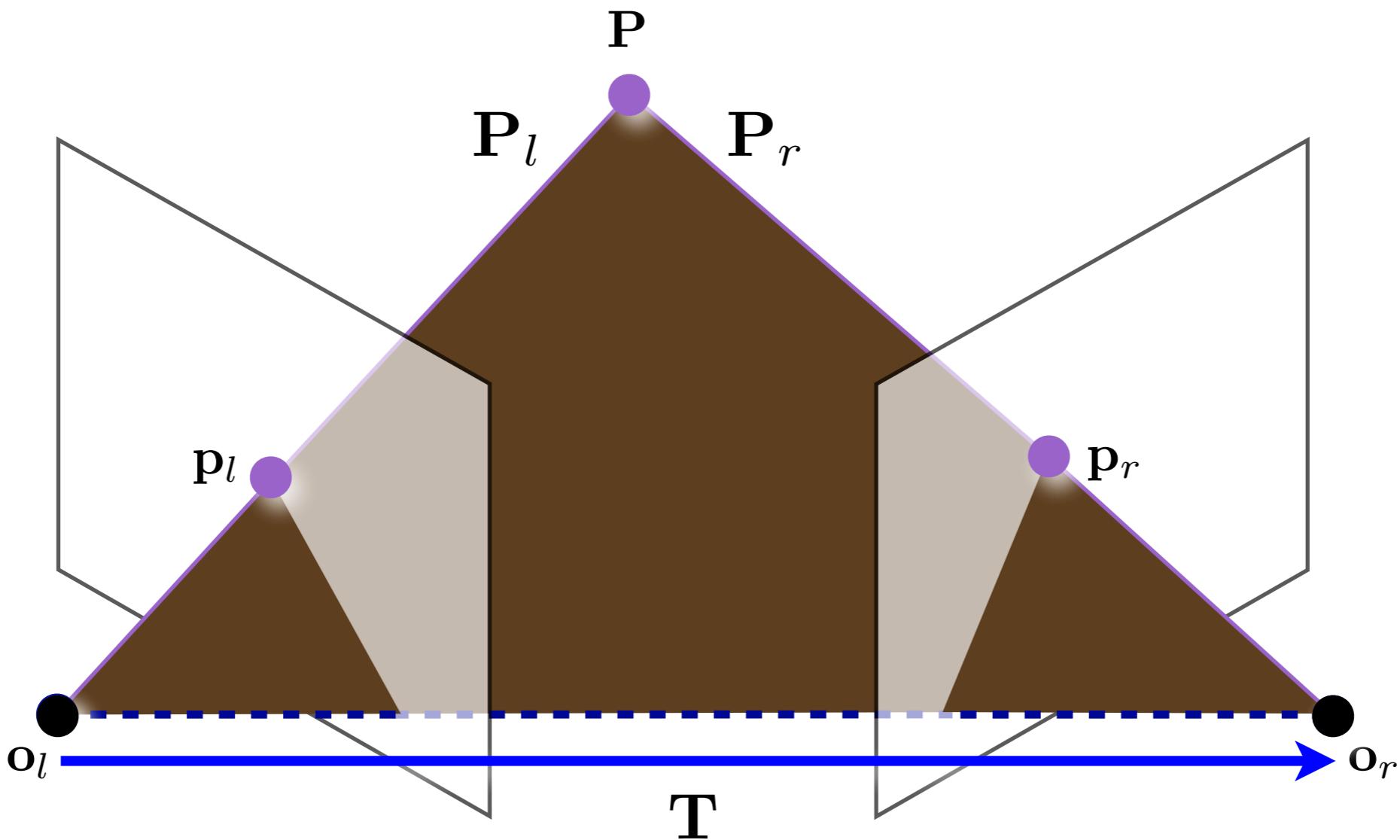


$$\mathbf{P}_r^\top \mathbf{R}[\mathbf{T}_\times] \mathbf{P}_l = 0$$

Let $\mathbf{E} = \mathbf{R}[\mathbf{T}_\times]$

$$\mathbf{P}_r^\top \mathbf{E} \mathbf{P}_l = 0$$

How do we revise constraint to relate
image points (in camera coordinates)?

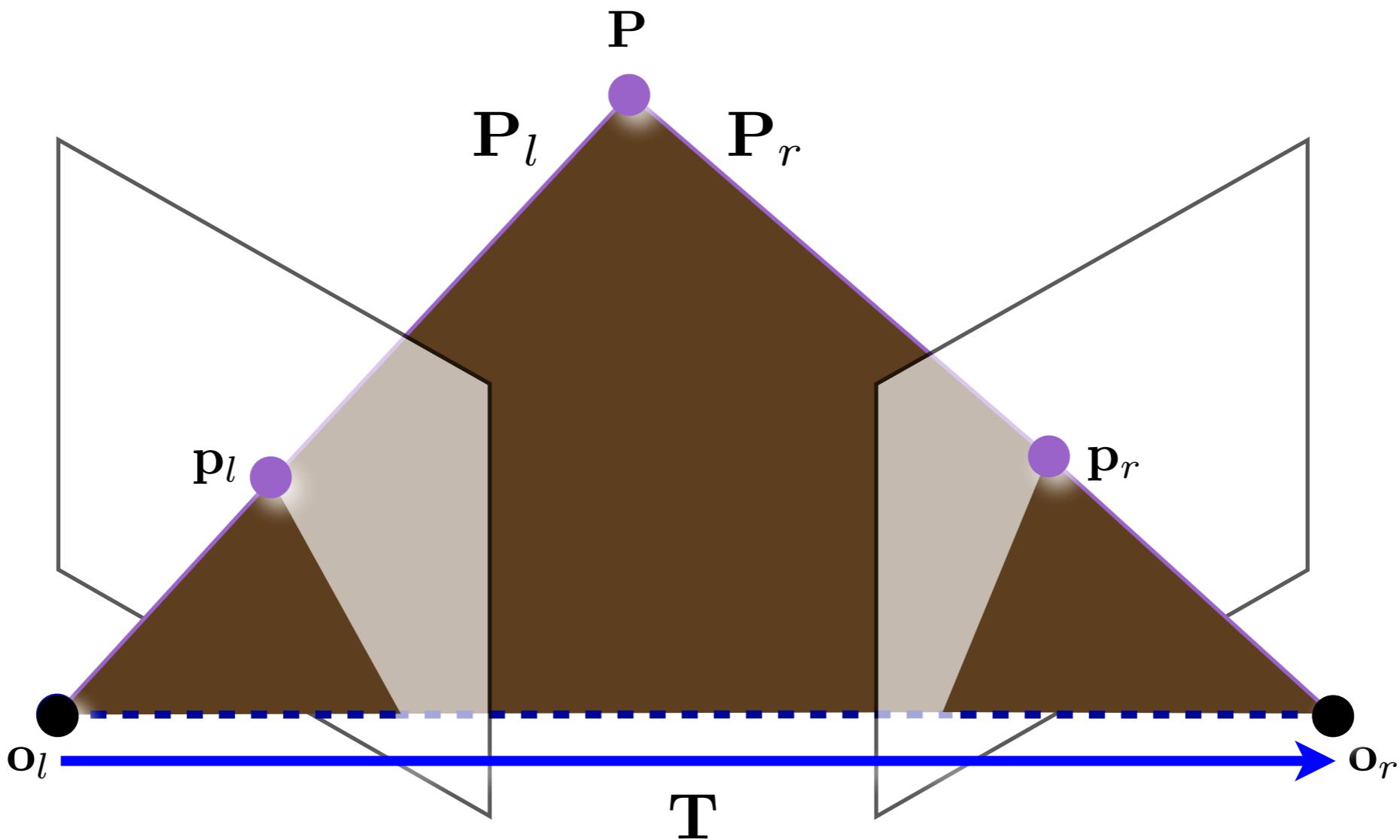


$$\mathbf{P}_r^\top \mathbf{E} \mathbf{P}_l = 0$$

$$\left(\frac{\mathbf{p}_r Z_r}{\mathcal{Z}_r} \right)^\top \mathbf{E} \left(\frac{\mathbf{p}_l Z_l}{\mathcal{Z}_l} \right) = 0$$

Recall: $\mathbf{p} = f \frac{\mathbf{P}}{Z}$

simplify



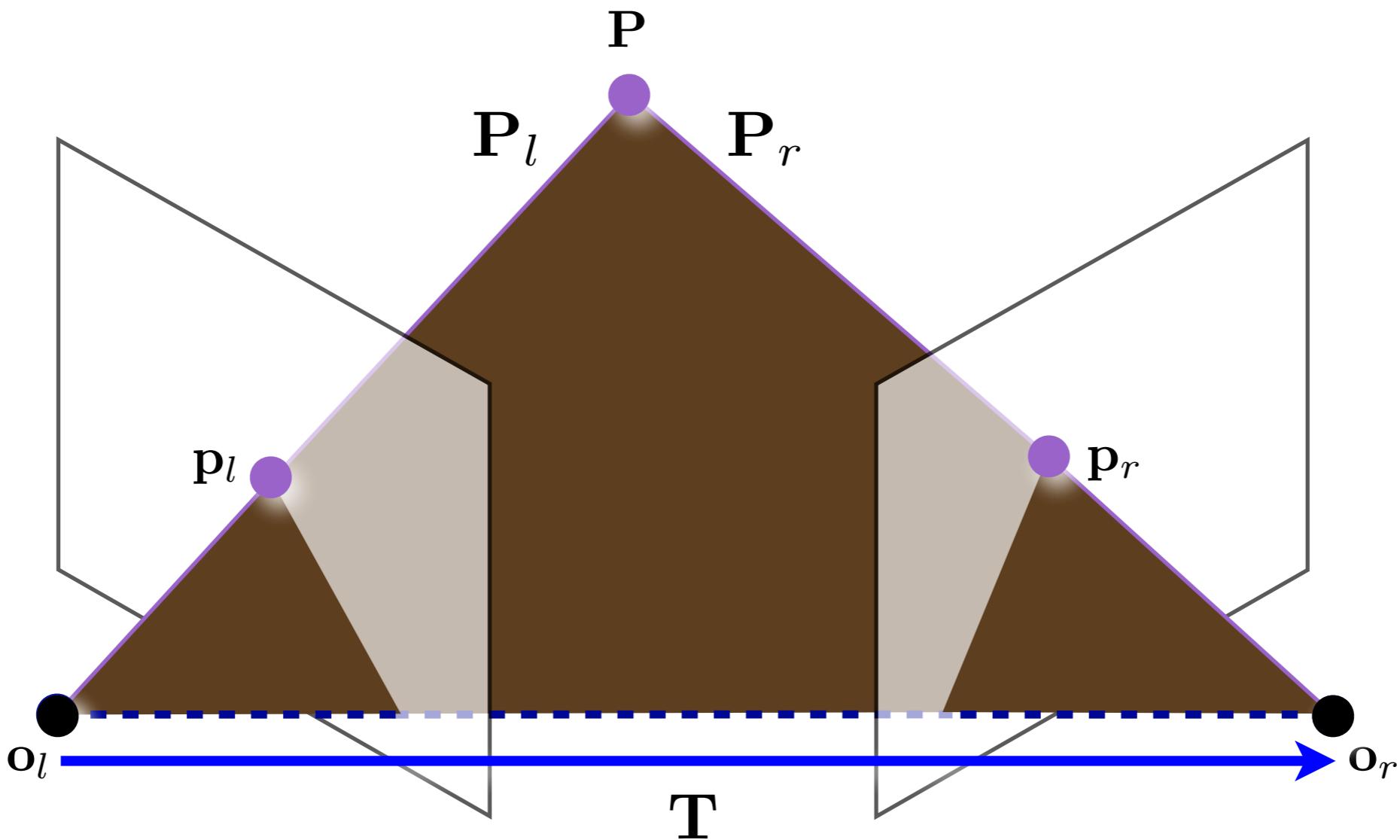
$$\mathbf{P}_r^\top \mathbf{E} \mathbf{P}_l = 0$$

$$\left(\frac{\mathbf{p}_r Z_r}{Z_r} \right)^\top \mathbf{E} \left(\frac{\mathbf{p}_l Z_l}{Z_l} \right) = 0$$

$$\mathbf{p}_r^\top \mathbf{E} \mathbf{p}_l = 0$$

Recall: $\mathbf{p} = f \frac{\mathbf{P}}{Z}$

simplify



$$p_r^T E p_l = 0$$

assumes that image points are measured in camera coordinates rather than pixel coordinates

$$\begin{pmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R & -R_1^\top T \\ & -R_2^\top T \\ & -R_3^\top T \end{pmatrix} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

intrinsic matrix

M_{int}

extrinsic matrix

M_{ext}

intrinsic matrix is invertible

$$\begin{pmatrix} \alpha x_{\text{im}} \\ \alpha y_{\text{im}} \\ \alpha \end{pmatrix} = \begin{pmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R} & -\mathbf{R}_1^\top \mathbf{T} \\ & -\mathbf{R}_2^\top \mathbf{T} \\ & -\mathbf{R}_3^\top \mathbf{T} \end{pmatrix} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

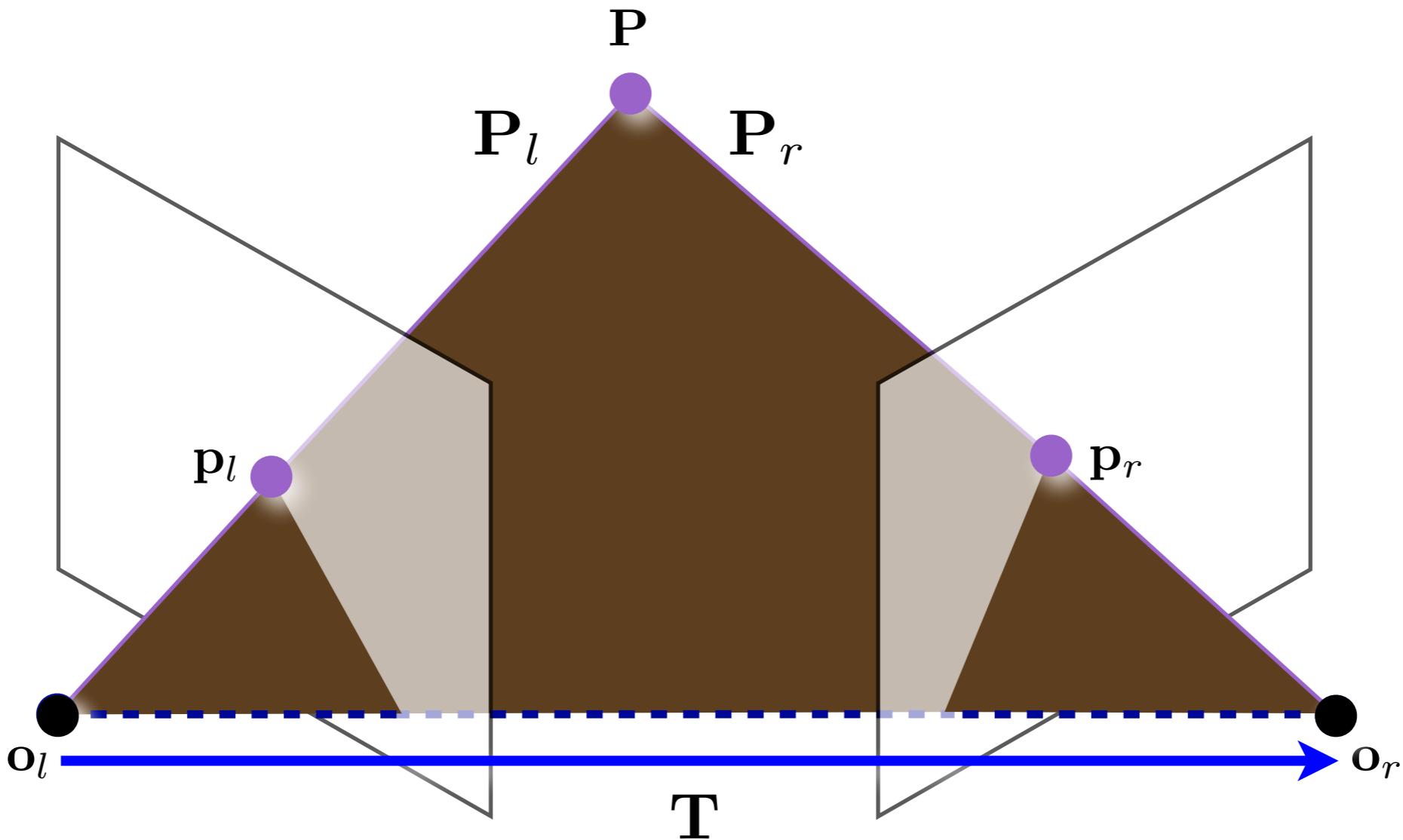
divide by α

intrinsic matrix

\mathbf{M}_{int}

extrinsic matrix

\mathbf{M}_{ext}

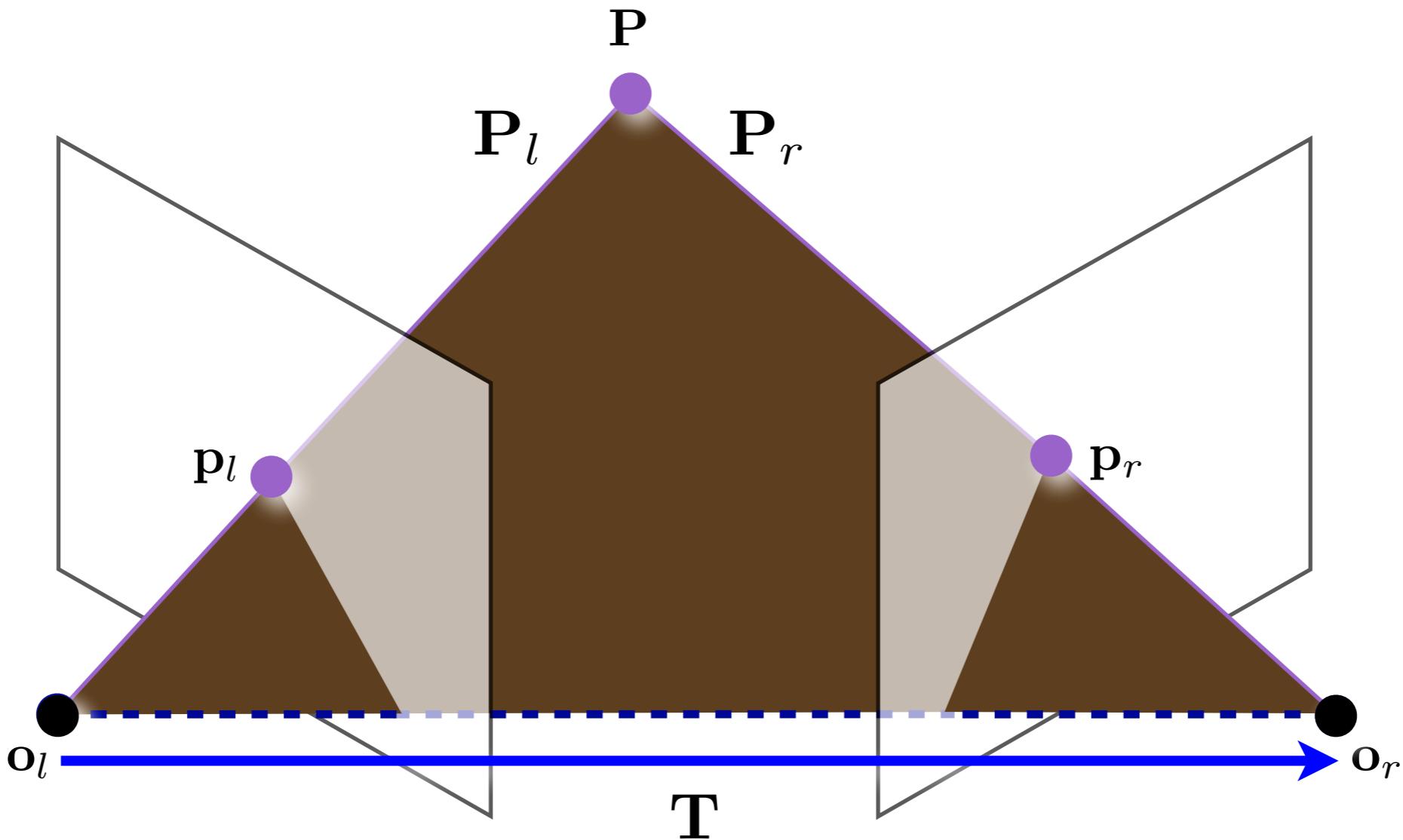


$$\mathbf{p}_r^\top \mathbf{E} \mathbf{p}_l = 0$$

We have:

$$\mathbf{p}_l = \mathbf{M}_{\text{int}, l}^{-1} \tilde{\mathbf{p}}_l \quad \mathbf{p}_r = \mathbf{M}_{\text{int}, r}^{-1} \tilde{\mathbf{p}}_r$$

pixel coordinates



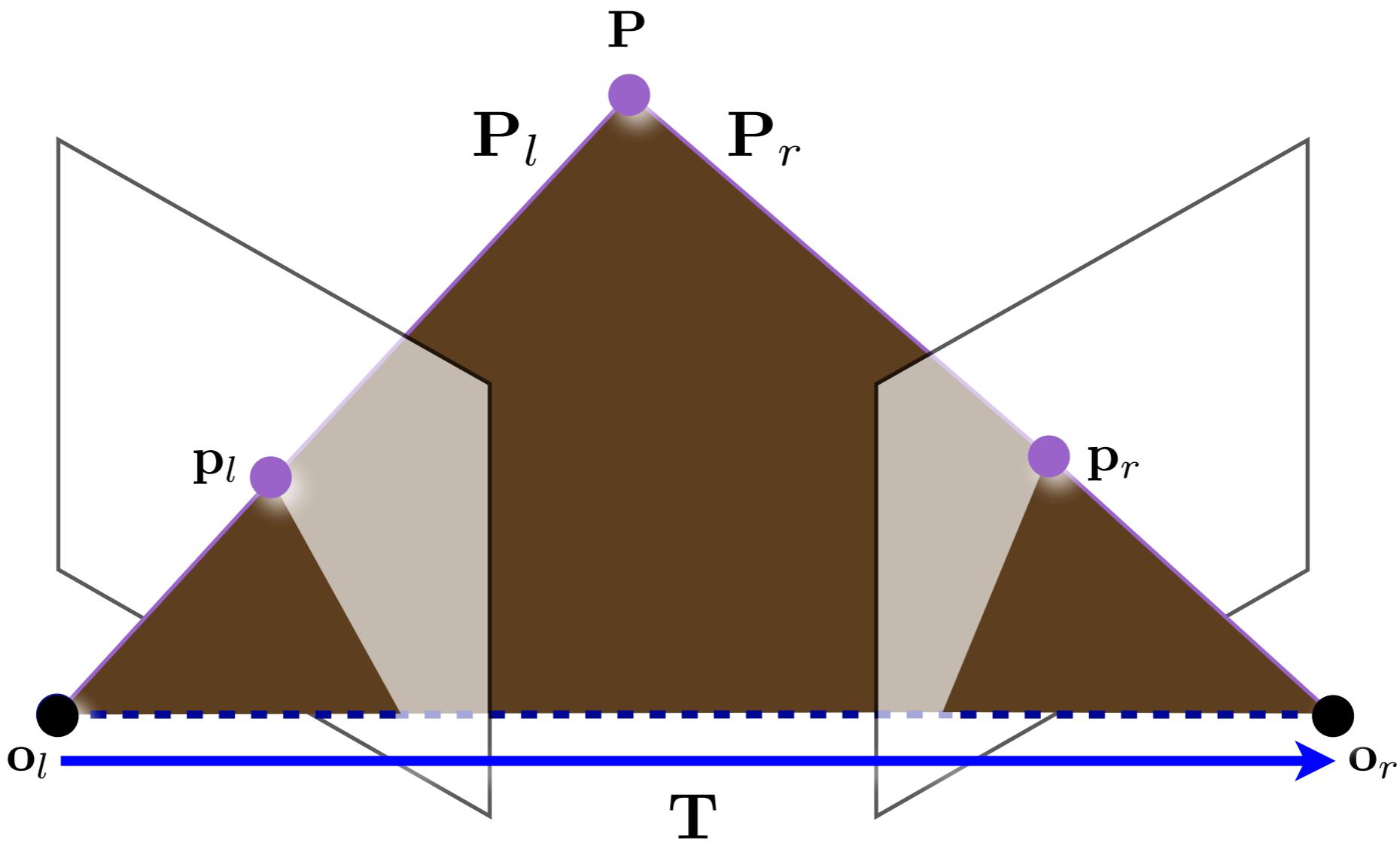
$$\mathbf{p}_r^\top \mathbf{E} \mathbf{p}_l = 0$$

We have:

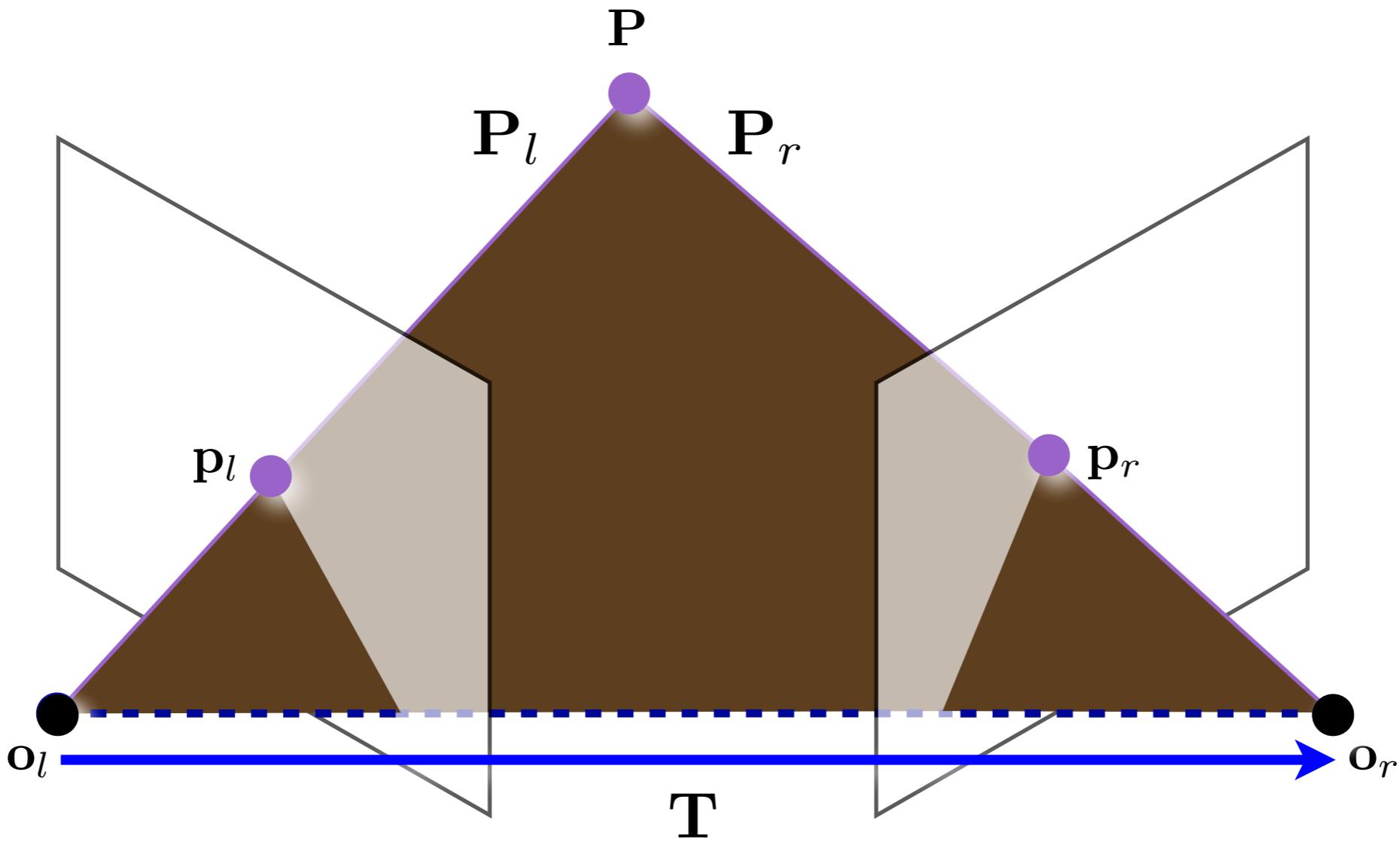
$$\mathbf{p}_l = \mathbf{M}_{\text{int},l}^{-1} \tilde{\mathbf{p}}_l \quad \mathbf{p}_r = \mathbf{M}_{\text{int},r}^{-1} \tilde{\mathbf{p}}_r$$

$$\tilde{\mathbf{p}}_r^\top \mathbf{M}_{\text{int},r}^{-\top} \mathbf{E} \mathbf{M}_{\text{int},l}^{-1} \tilde{\mathbf{p}}_l = 0$$

substitute



$$\tilde{\mathbf{p}}_r^\top \mathbf{M}_{\text{int},r}^{-\top} \mathbf{E} \mathbf{M}_{\text{int},l}^{-1} \tilde{\mathbf{p}}_l = 0$$



$$\tilde{\mathbf{p}}_r^\top \mathbf{M}_{\text{int},r}^{-\top} \mathbf{E} \mathbf{M}_{\text{int},l}^{-1} \tilde{\mathbf{p}}_l = 0$$

Let $\mathbf{F} = \mathbf{M}_{\text{int},r}^{-\top} \mathbf{E} \mathbf{M}_{\text{int},l}^{-1}$

Fundamental
Matrix Constraint

$$\tilde{\mathbf{p}}_r^\top \mathbf{F} \tilde{\mathbf{p}}_l = 0$$

Fundamental Matrix Constraint

$$\tilde{\mathbf{p}}_r^\top \mathbf{F} \tilde{\mathbf{p}}_l = 0$$

Captures the two view projective geometry

Independent of scene structure

Can be computed from correspondences
without knowledge of the
camera intrinsics and extrinsics

Defined up to a scalar factor

Rank is 2

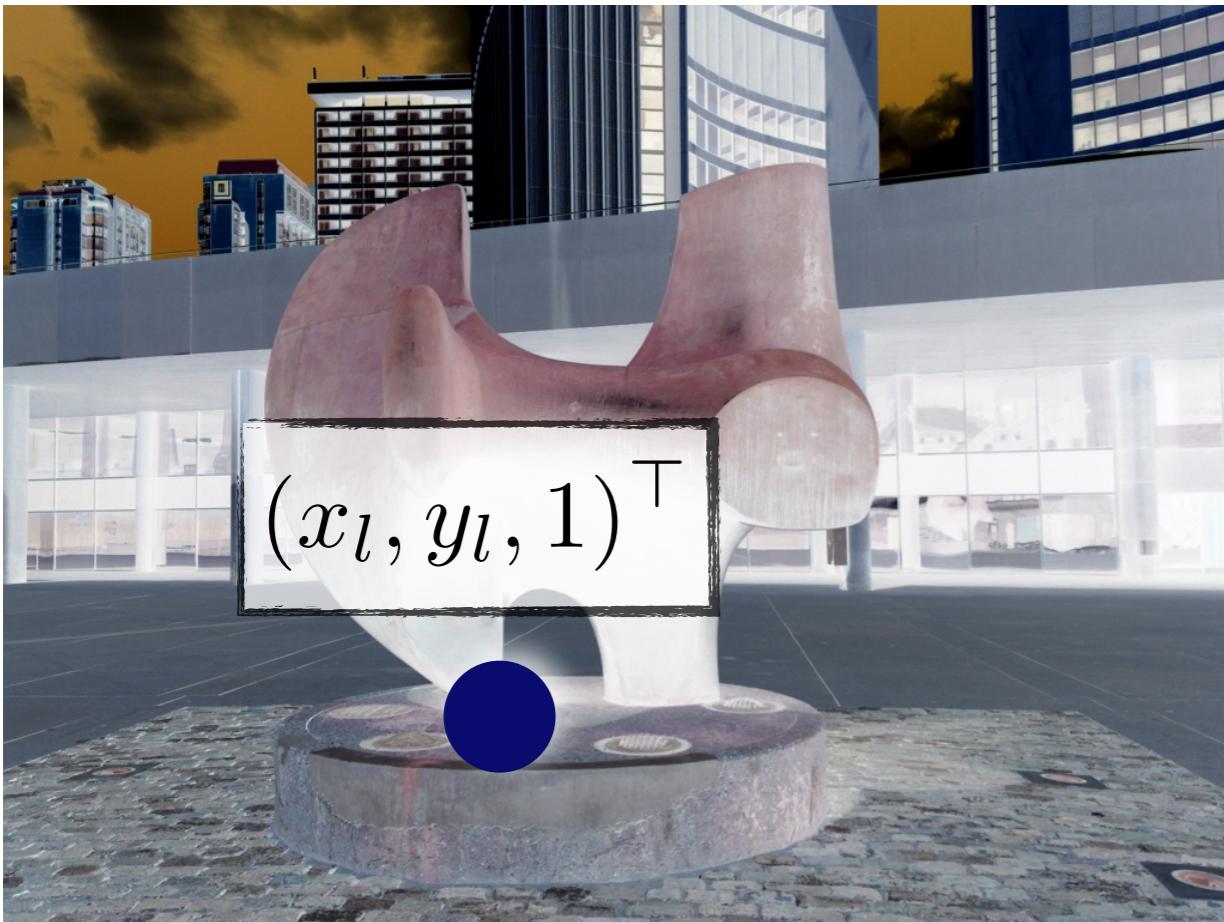
Fundamental Matrix Constraint

$$\tilde{\mathbf{p}}_r^\top \mathbf{F} \tilde{\mathbf{p}}_l = 0$$

Rank is 2

Epipoles are the left and right null vectors

$$\mathbf{F} \mathbf{e}_l = \mathbf{0} \text{ and } \mathbf{e}_r^\top \mathbf{F} = \mathbf{0}$$



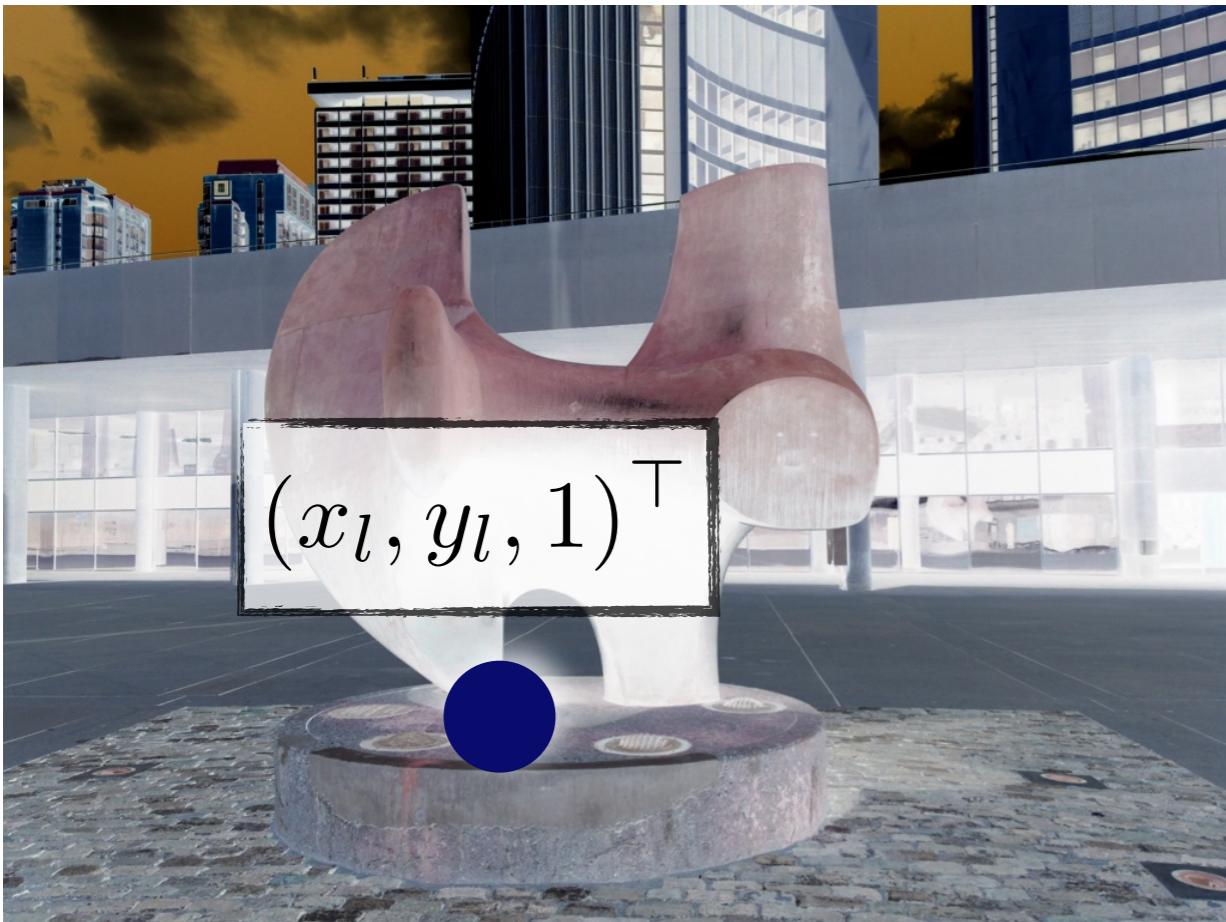
$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$(x_r \quad y_r \quad 1) \mathbf{F} (x_l \quad y_l \quad 1)^\top = 0$$

Let $\mathbf{k} = \mathbf{F} (x_l \quad y_l \quad 1)^\top$

$$(x_r \quad y_r \quad 1) \mathbf{k} = 0$$

What does this constraint mean geometrically?



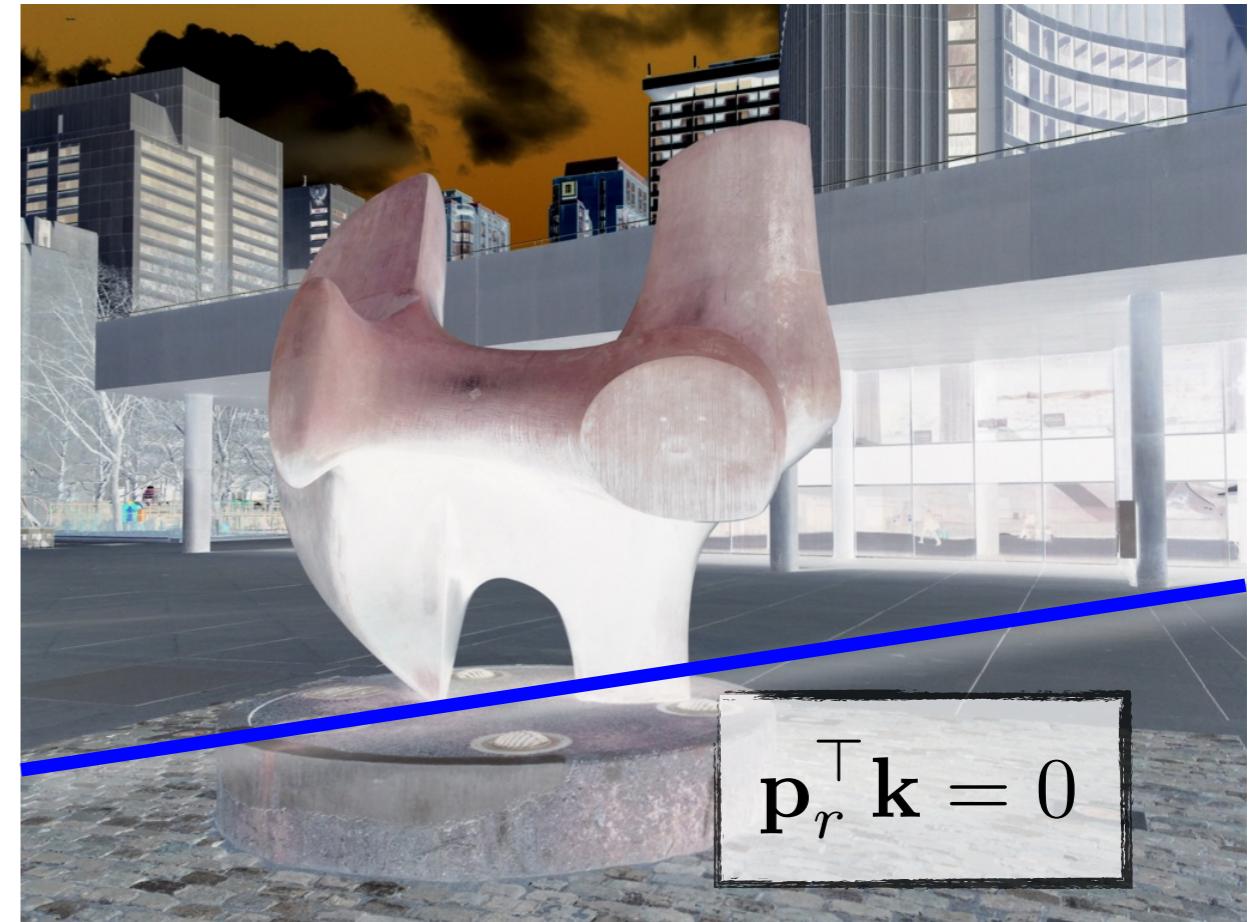
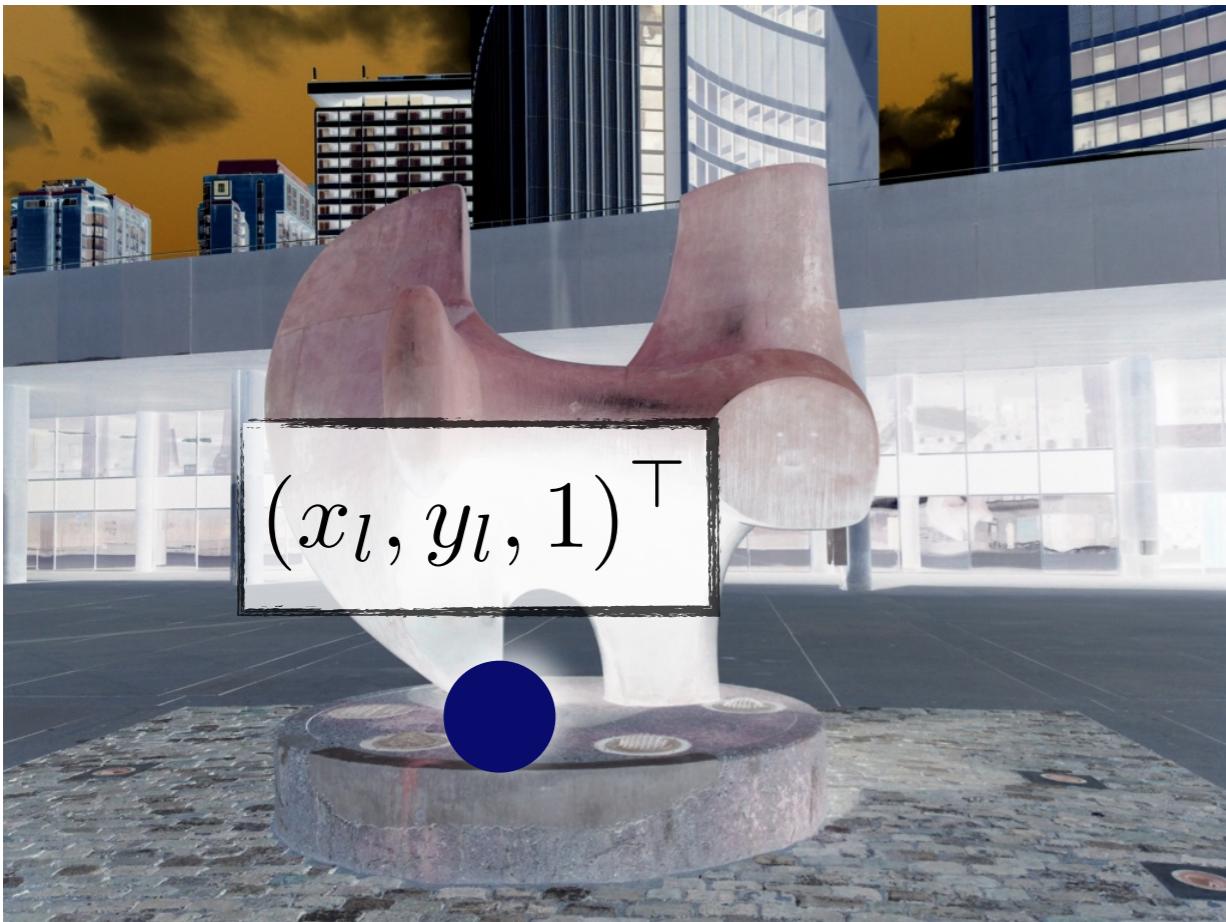
$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$(x_r \quad y_r \quad 1) \mathbf{F} (x_l \quad y_l \quad 1)^\top = 0$$

Let $\mathbf{k} = \mathbf{F} (x_l \quad y_l \quad 1)^\top$

$$(x_r \quad y_r \quad 1) \mathbf{k} = 0$$

search space reduced to a line



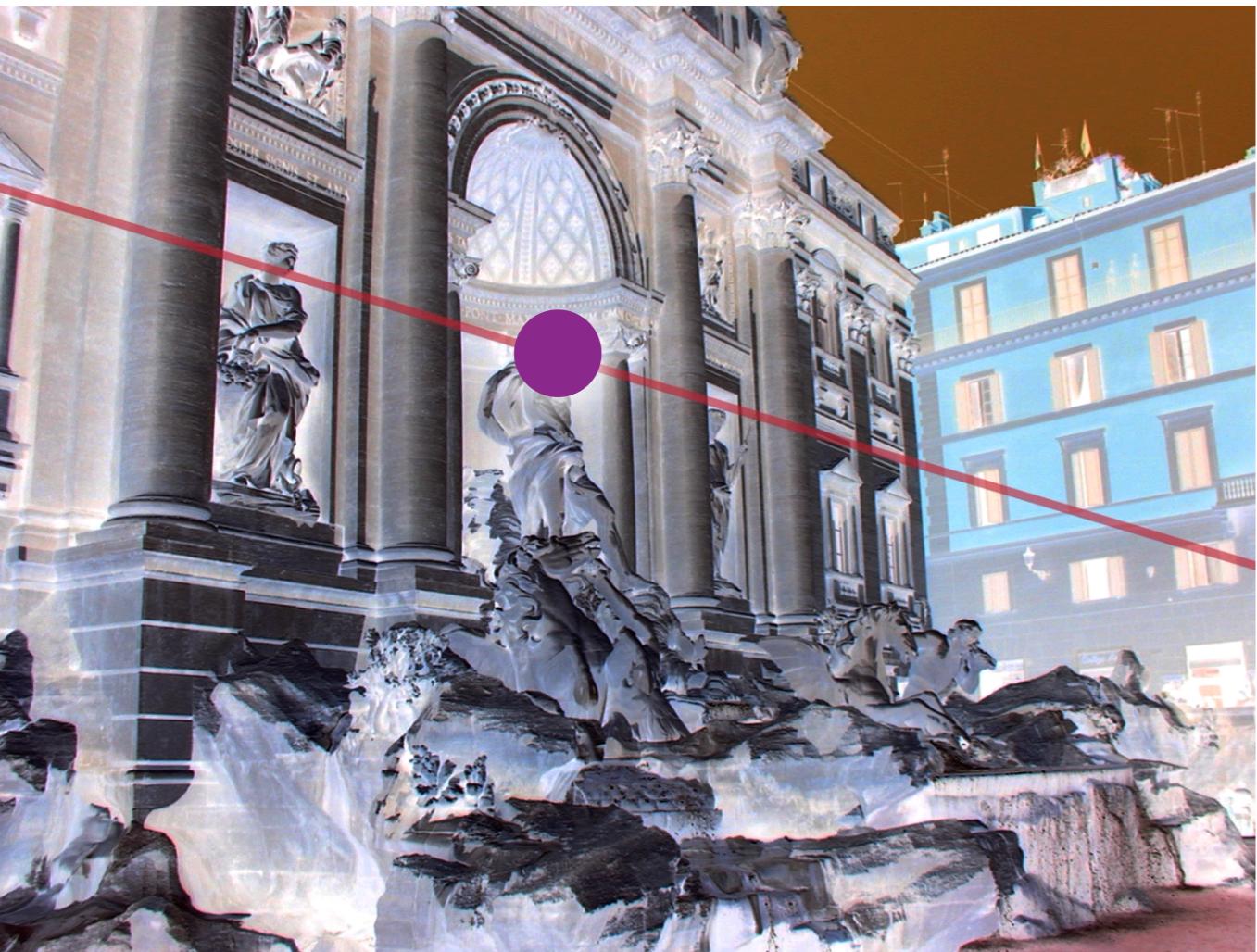
$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$(x_r \quad y_r \quad 1) \mathbf{F} (x_l \quad y_l \quad 1)^\top = 0$$

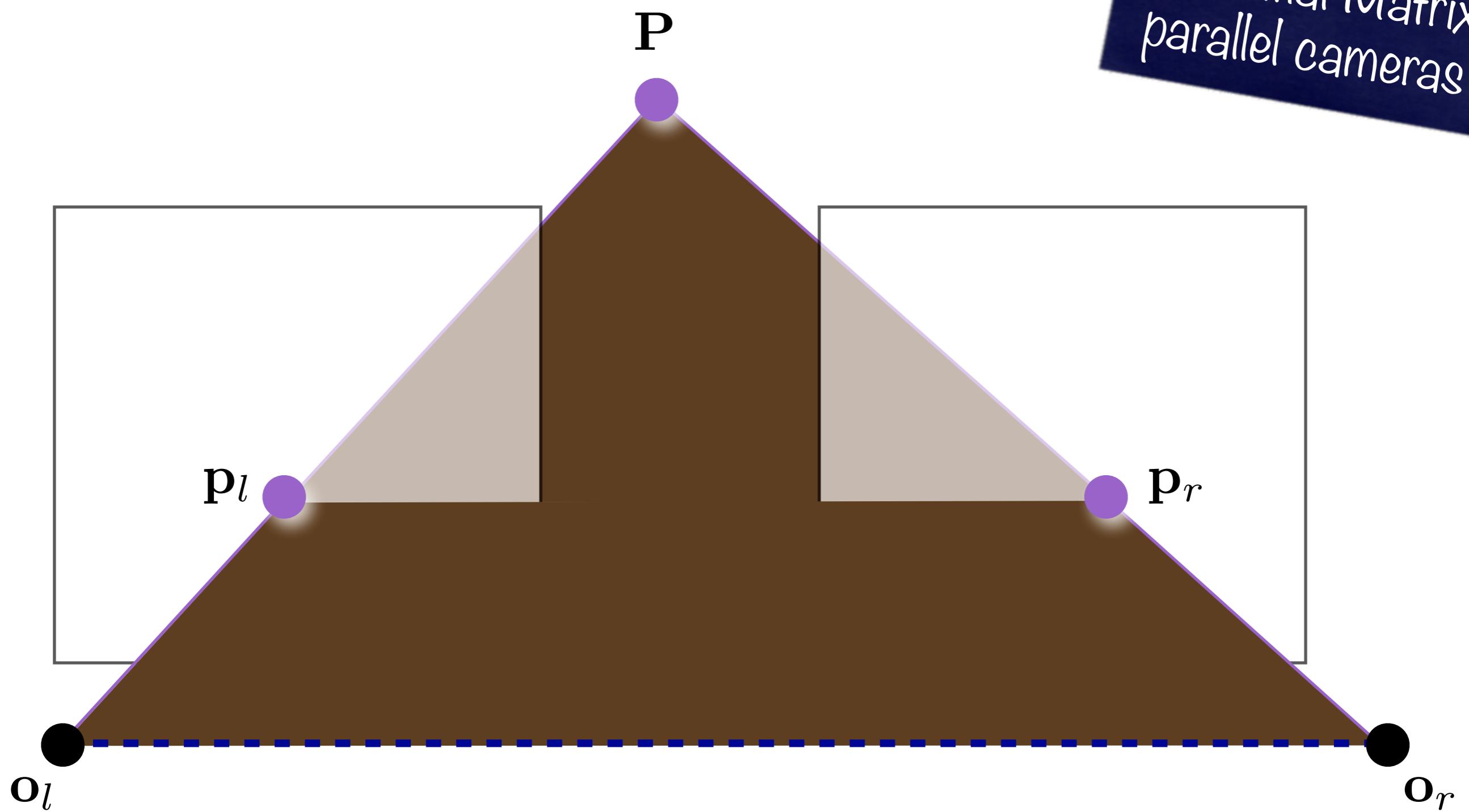
Let $\mathbf{k} = \mathbf{F} (x_l \quad y_l \quad 1)^\top$

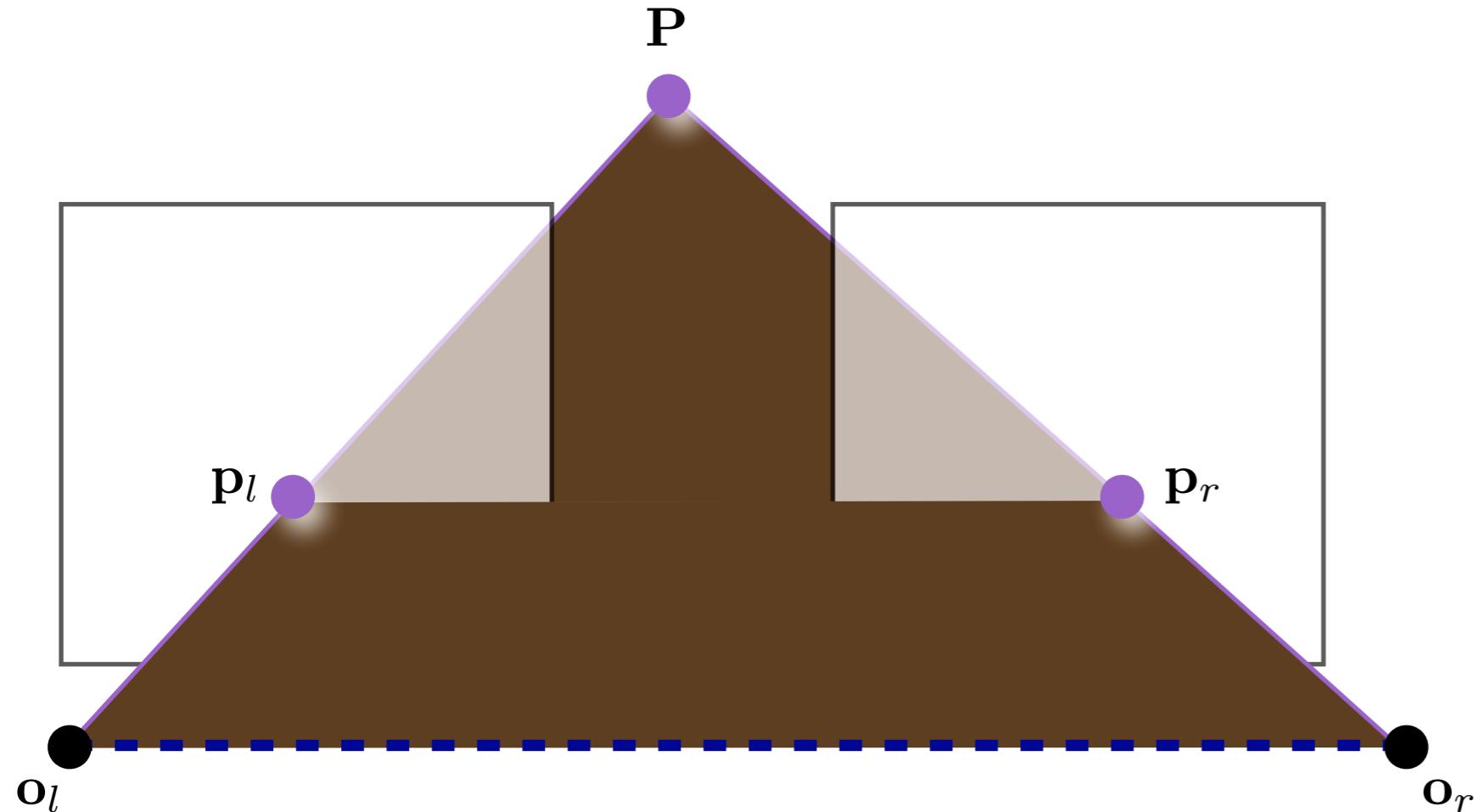
$$(x_r \quad y_r \quad 1) \mathbf{k} = 0$$

search space reduced to a line



Special case:
Essential Matrix
parallel cameras





$$\mathbf{R} = \mathbf{I}_{3 \times 3}$$

$$\mathbf{T} = (B \quad 0 \quad 0)^\top$$

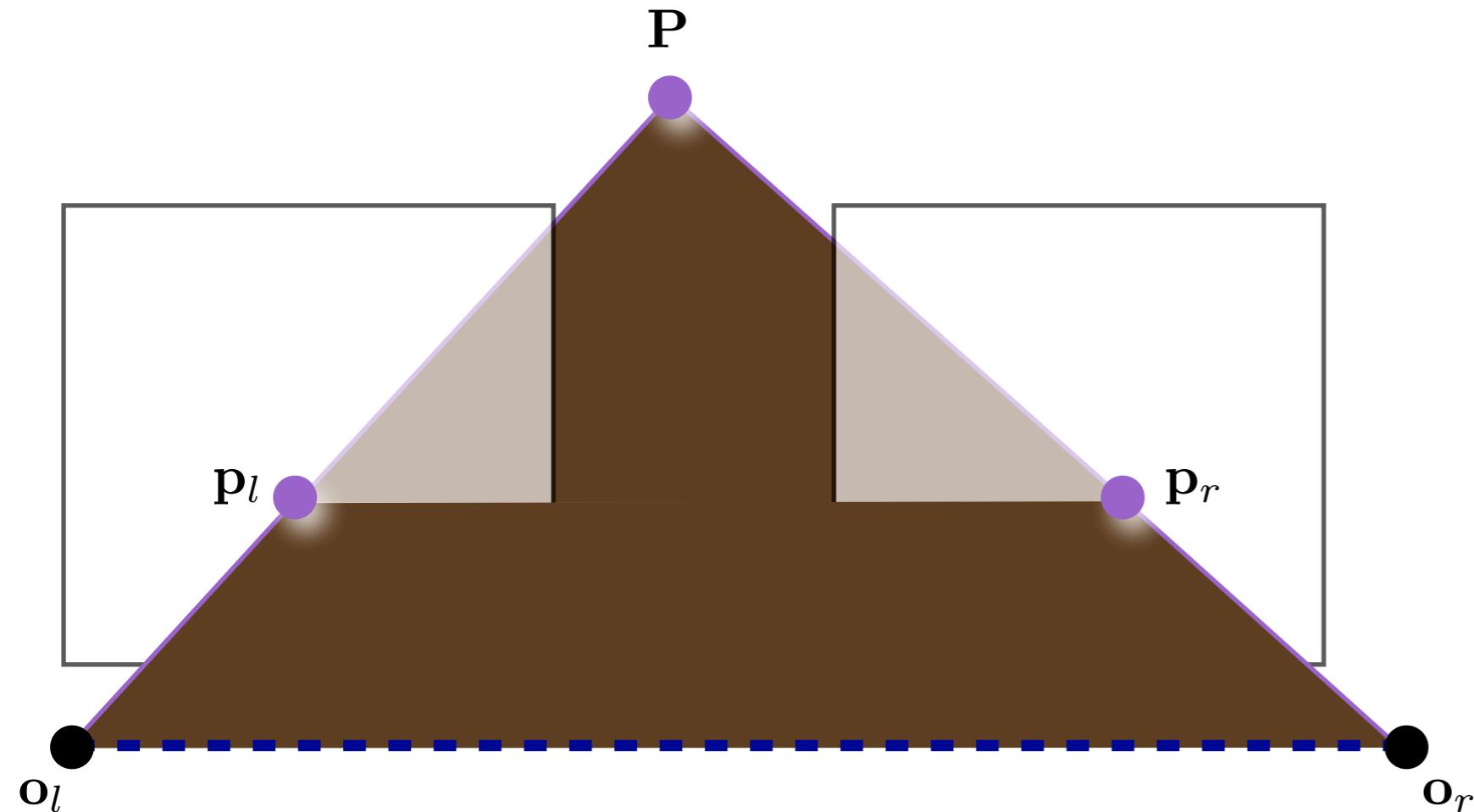
$$\mathbf{p}_r^\top \mathbf{E} \mathbf{p}_l = 0$$

$$\mathbf{p}_r^\top \mathbf{R}[\mathbf{T}_\times] \mathbf{p}_l = 0$$

$$(x_r \quad y_r \quad 1)^\top \cancel{\mathbf{R}}[\mathbf{T}_\times] (x_l \quad y_l \quad 1) = 0$$

$$(x_r \quad y_r \quad 1)^\top [\mathbf{T}_\times] (x_l \quad y_l \quad 1) = 0$$

simplify



$$\mathbf{R} = \mathbf{I}_{3 \times 3}$$

$$\mathbf{T} = (B \ 0 \ 0)^\top$$

$$\mathbf{p}_r^\top \mathbf{E} \mathbf{p}_l = 0$$

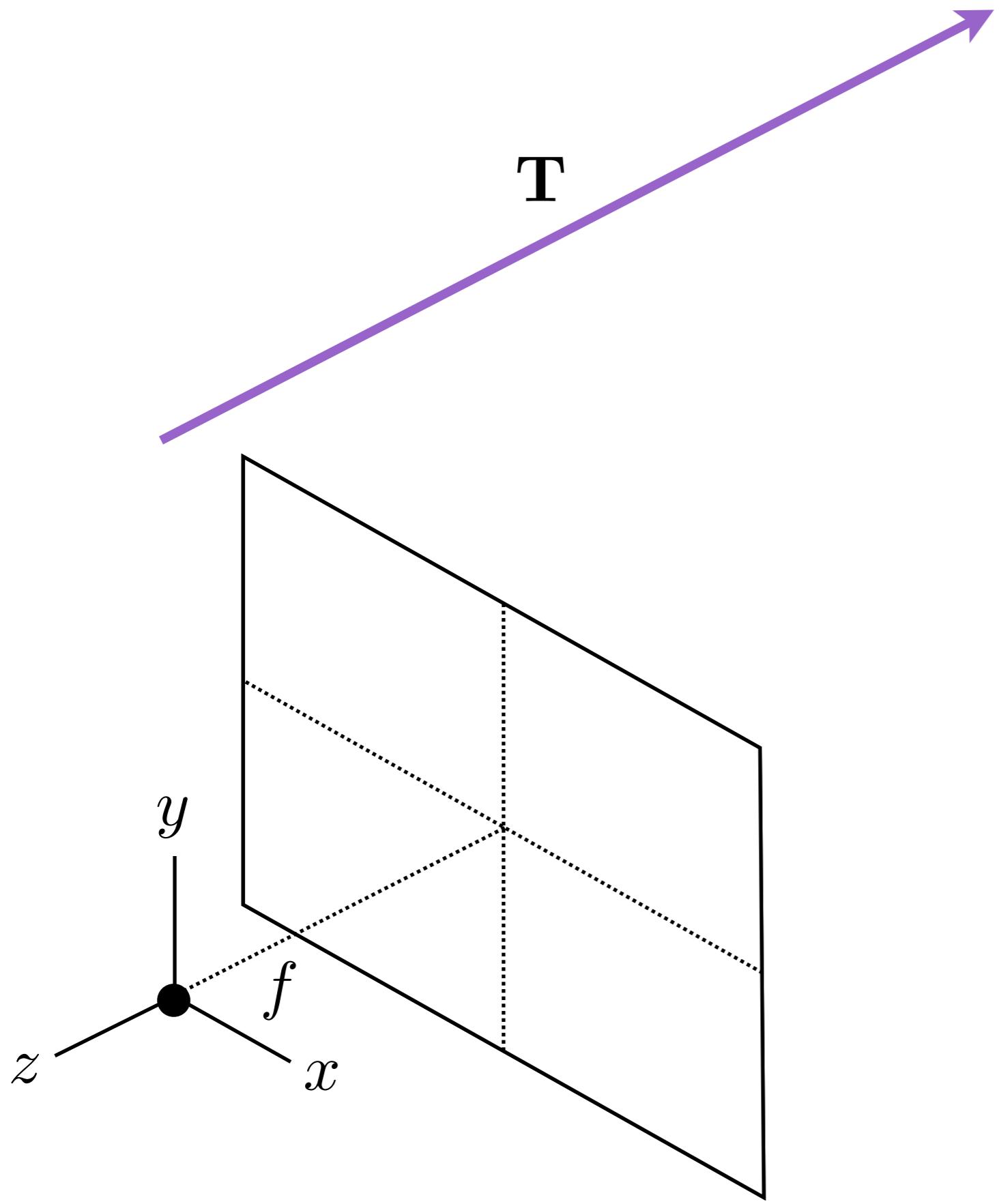
$$(x_r \ y_r \ 1)^\top [\mathbf{T}_\times] (x_l \ y_l \ 1) = 0$$

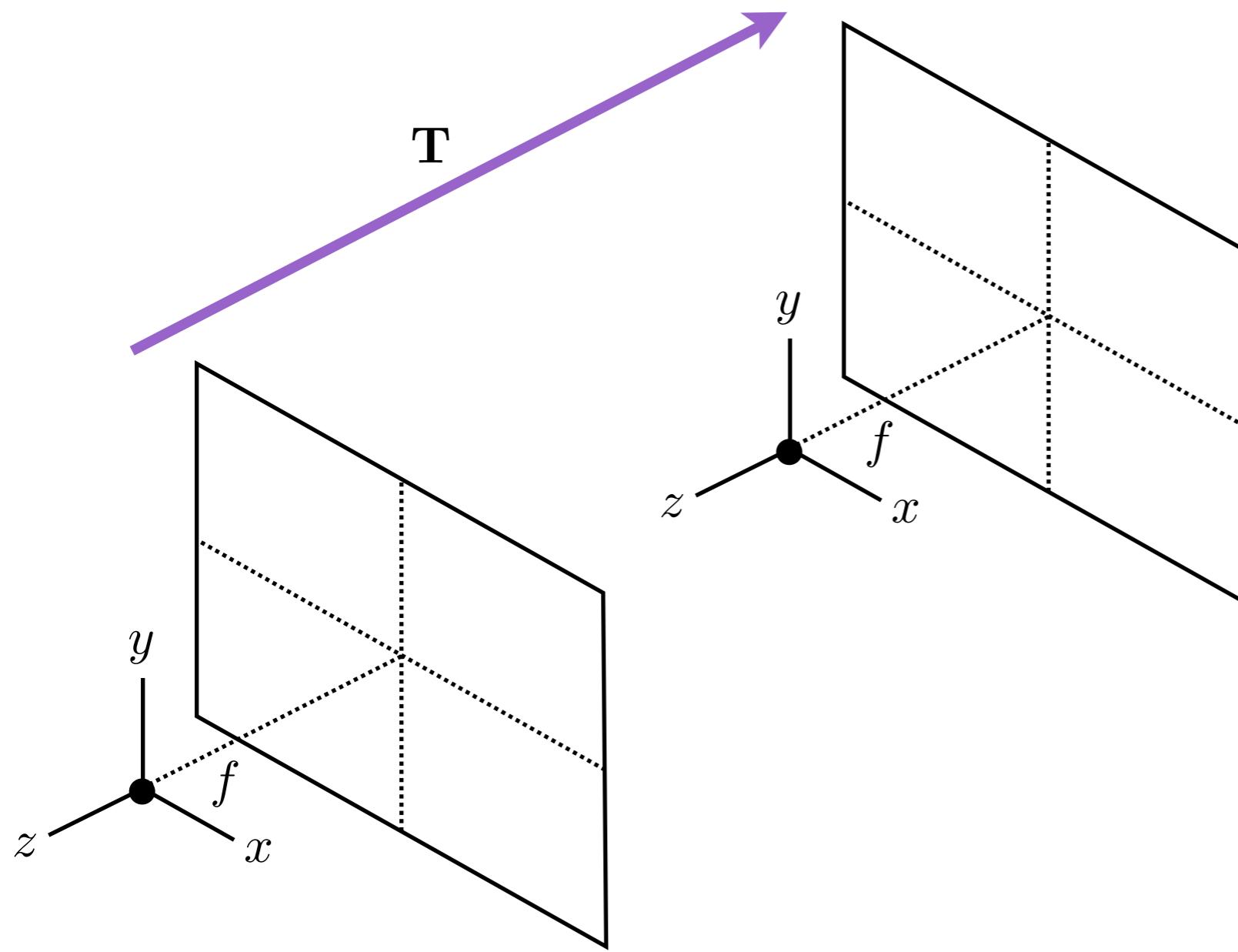
expand and simplify

$$y_r = y_l$$

corresponding points lie on same horizontal line

Special case:
Essential Matrix
forward motion



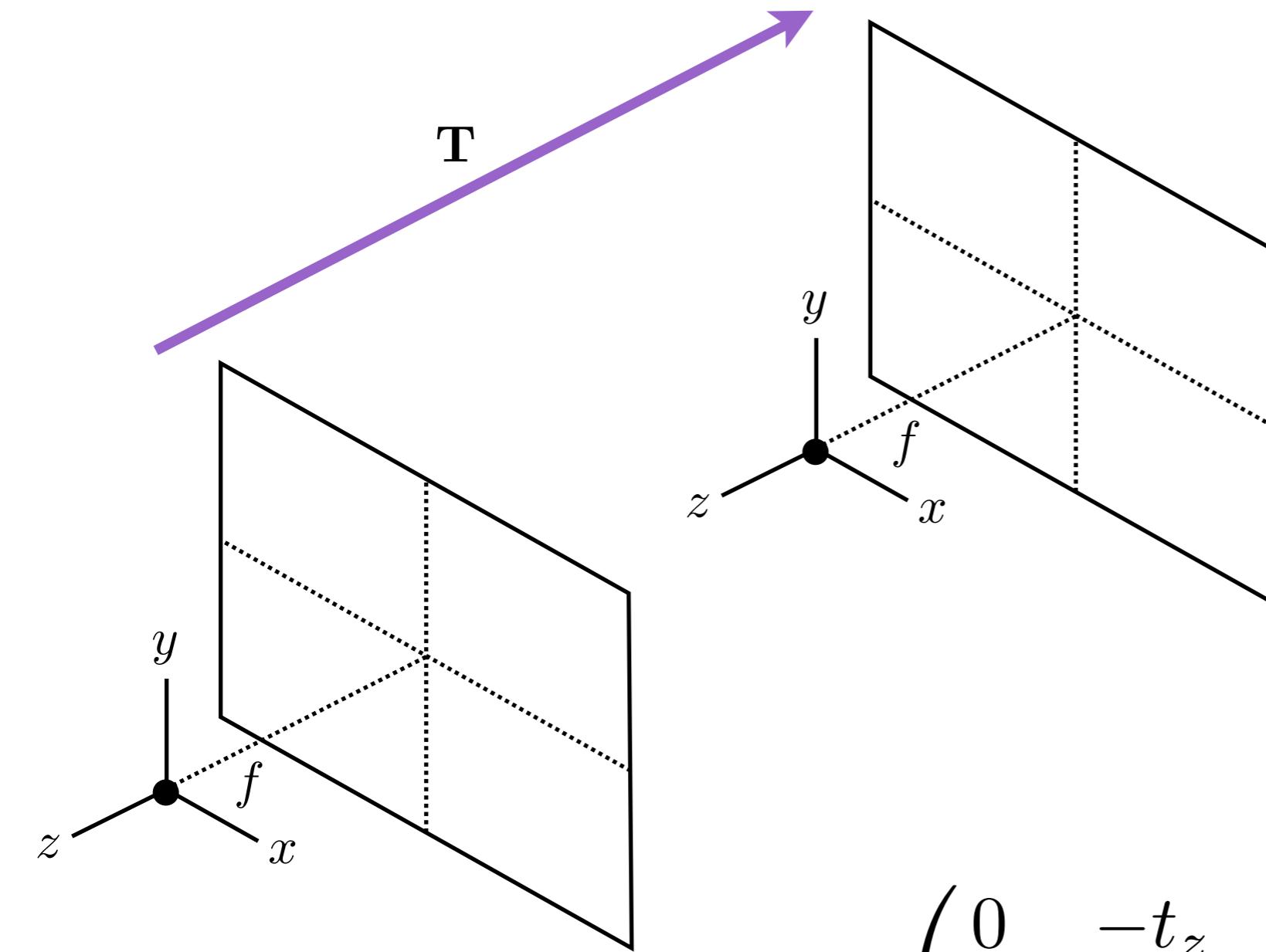


$$\mathbf{R} = \mathbf{I}_{3 \times 3}$$

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & t_z \end{pmatrix}^\top$$

$$\mathbf{E} = [\mathbf{T}_\times] \mathbf{R}$$

$$= \begin{pmatrix} 0 & -t_z & 0 \\ t_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{I}_{3 \times 3}$$

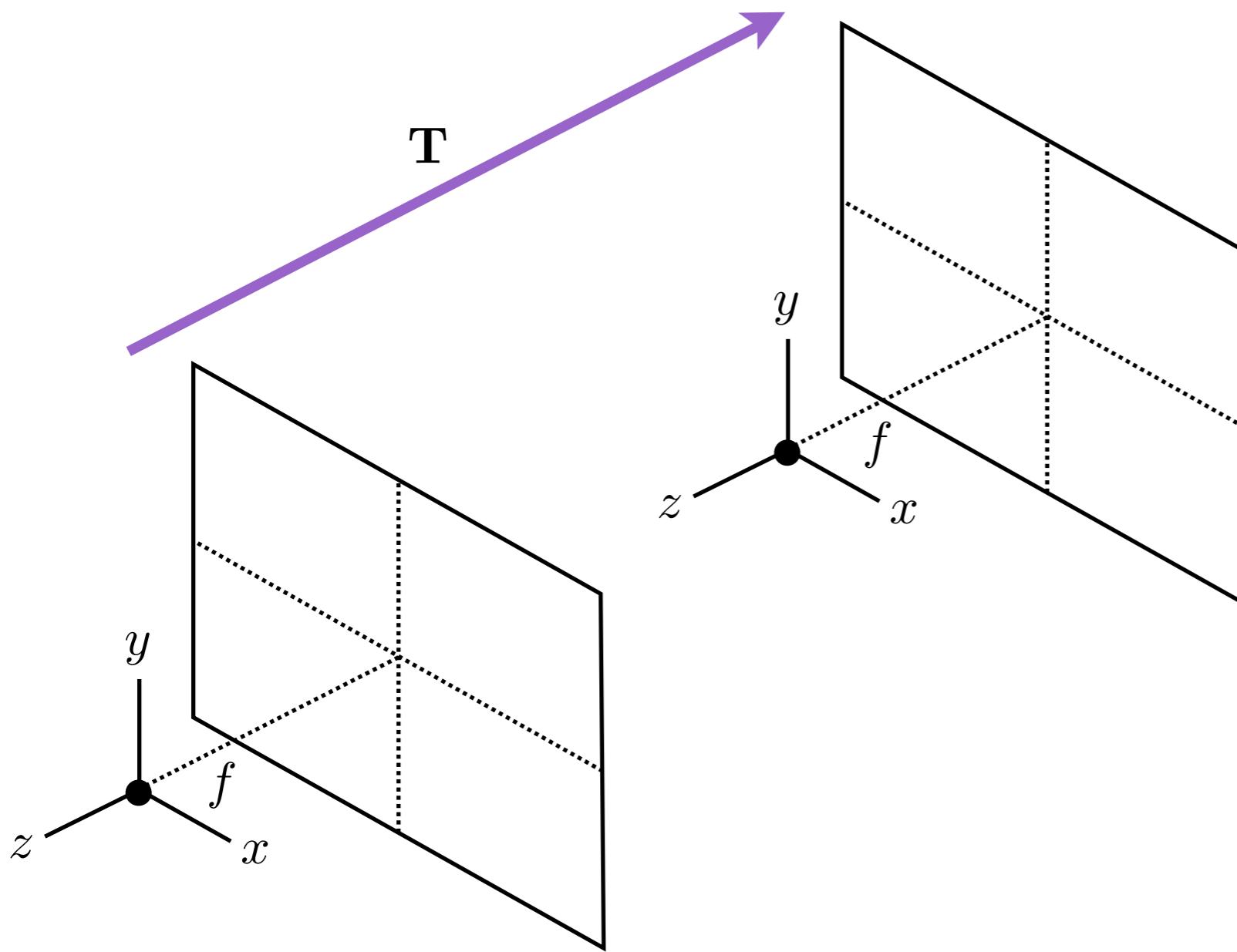


$$\mathbf{R} = \mathbf{I}_{3 \times 3}$$

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & t_z \end{pmatrix}^\top$$

$$\mathbf{E} = \begin{pmatrix} 0 & -t_z & 0 \\ t_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Essential Matrix is defined up to a scalar multiple

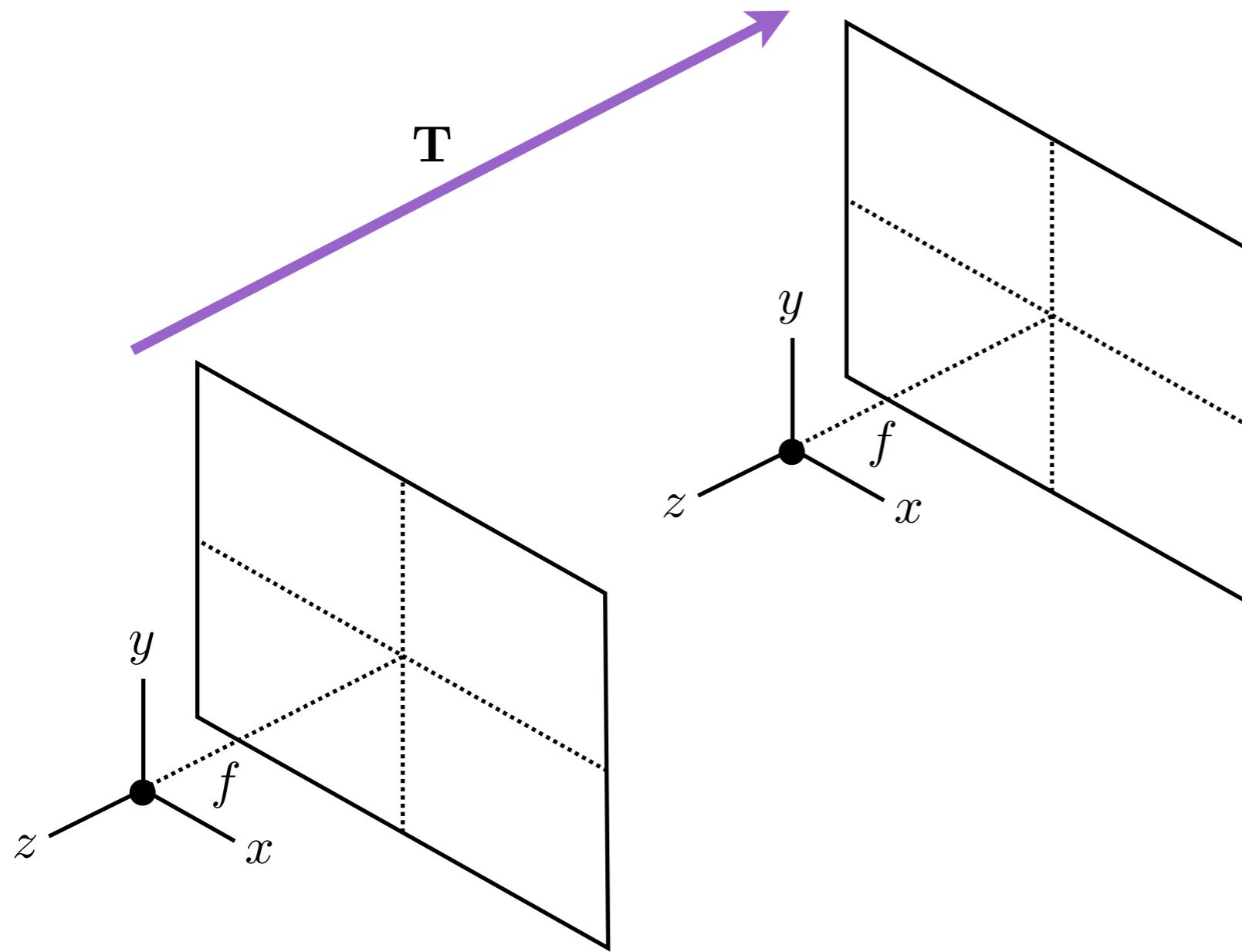


$$\mathbf{l}' = \mathbf{E}\mathbf{x}$$

$$\mathbf{R} = \mathbf{I}_{3 \times 3}$$

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & t_z \end{pmatrix}^\top$$

$$\mathbf{E} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



$$\mathbf{R} = \mathbf{I}_{3 \times 3}$$

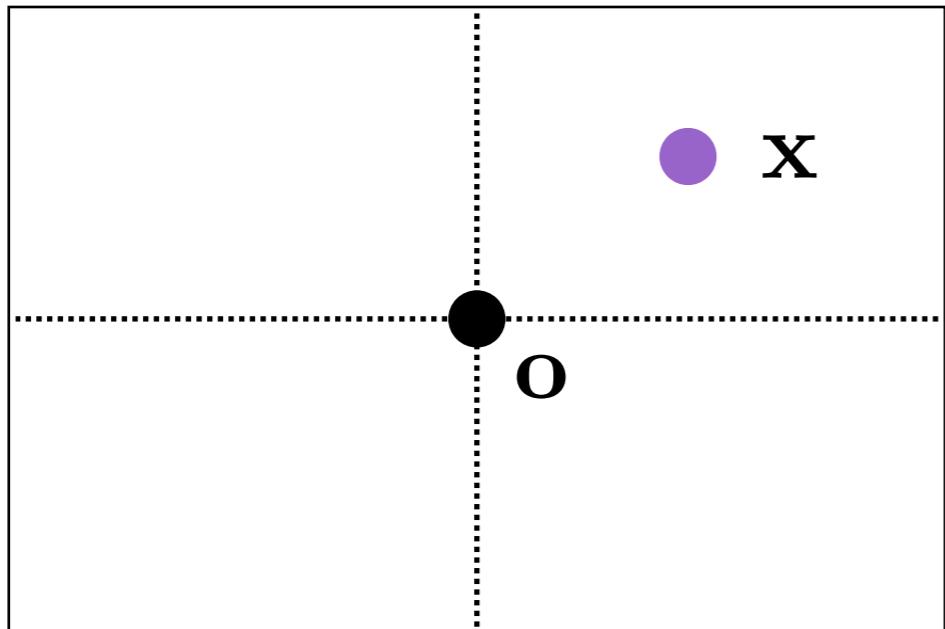
$$\mathbf{T} = \begin{pmatrix} 0 & 0 & t_z \end{pmatrix}^\top$$

$$\mathbf{E} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

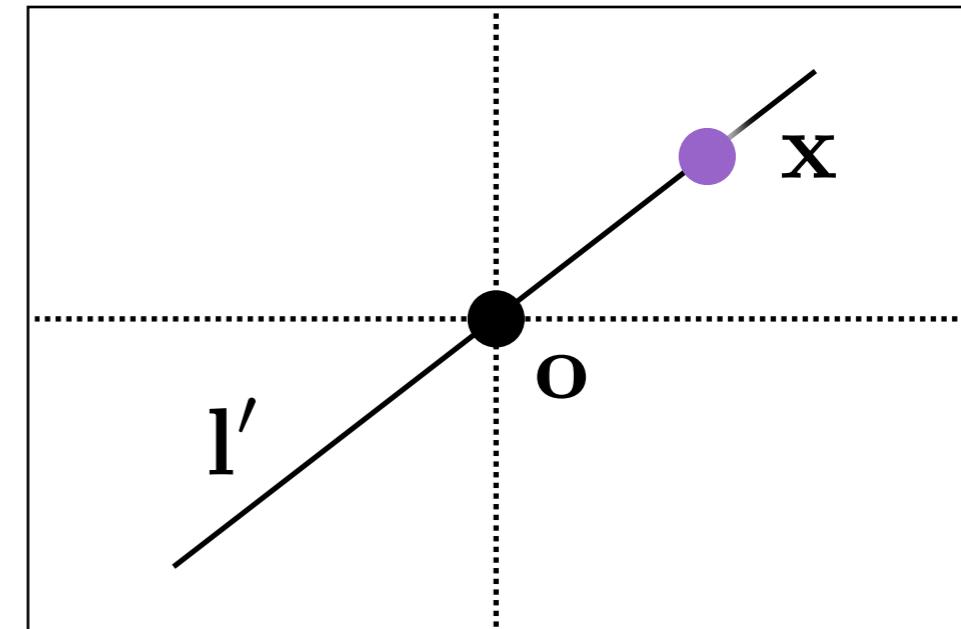
$$\mathbf{l}' = \mathbf{Ex} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

image origin is the epipole in both images

$$\mathbf{l}' = \mathbf{Ex} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$



initial image



second camera

points move along lines radiating from epipole

H

vs.

R

Homography

Fundamental

$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

Homography matrix maps a point to a point

$$l' = Fx$$

Fundamental Matrix maps a point to a line

Estimation Fundamental Matrix

8-point
algorithm

$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$(x_r \quad y_r \quad 1) \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} x_l \\ y_l \\ 1 \end{pmatrix} = 0$$

8-point algorithm

$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$(x_r x_l \quad x_r y_l \quad x_r \quad y_r x_l \quad y_r y_l \quad y_r \quad x_l \quad y_l \quad 1) \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

one point correspondence across two views

8-point algorithm

$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$\begin{pmatrix} x_{r1}x_{l1} & x_{r1}y_{l1} & x_{r1} & y_{r1}x_{l1} & y_{r1}y_{l1} & y_{r1} & x_{l1} & y_{l1} & 1 \\ & & \vdots & & & & & & \\ x_{rn}x_{ln} & x_{rn}y_{ln} & x_{rn} & y_{rn}x_{ln} & y_{rn}y_{ln} & y_{rn} & x_{ln} & y_{ln} & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

***n* point correspondences**

$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$\begin{pmatrix} x_{r1}x_{l1} & x_{r1}y_{l1} & x_{r1} & y_{r1}x_{l1} & y_{r1}y_{l1} & y_{r1} & x_{l1} & y_{l1} & 1 \\ & & \vdots & & & & & & \\ x_{rn}x_{ln} & x_{rn}y_{ln} & x_{rn} & y_{rn}x_{ln} & y_{rn}y_{ln} & y_{rn} & x_{ln} & y_{ln} & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

matrix rank is 8

eight point correspondences yield a non-zero solution

solution is null vector

$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$\begin{pmatrix} x_{r1}x_{l1} & x_{r1}y_{l1} & x_{r1} & y_{r1}x_{l1} & y_{r1}y_{l1} & y_{r1} & x_{l1} & y_{l1} & 1 \\ & & \vdots & & & & & & \\ x_{rn}x_{ln} & x_{rn}y_{ln} & x_{rn} & y_{rn}x_{ln} & y_{rn}y_{ln} & y_{rn} & x_{ln} & y_{ln} & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

matrix rank is 8

use more than eight points and solve with
homogeneous least-squares

2

common solutions

1

Lagrange Multipliers

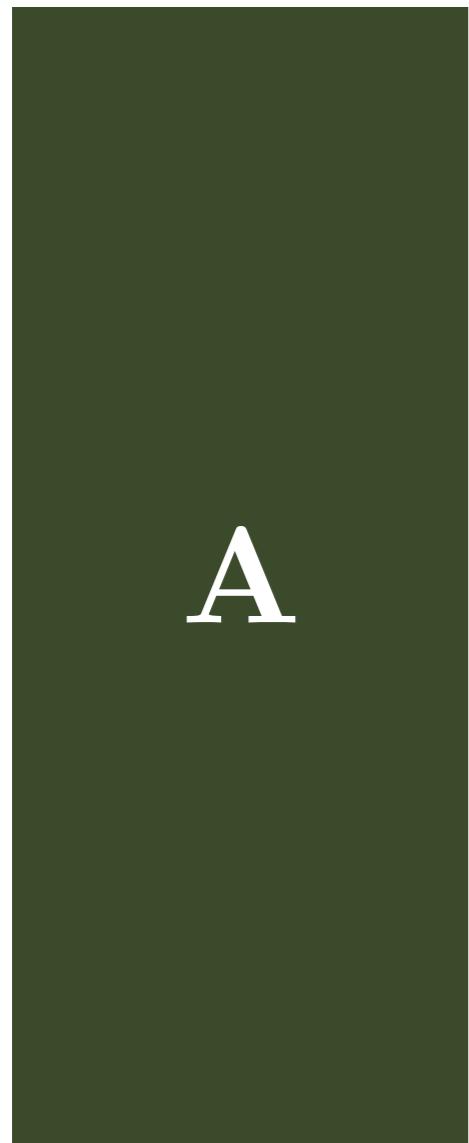
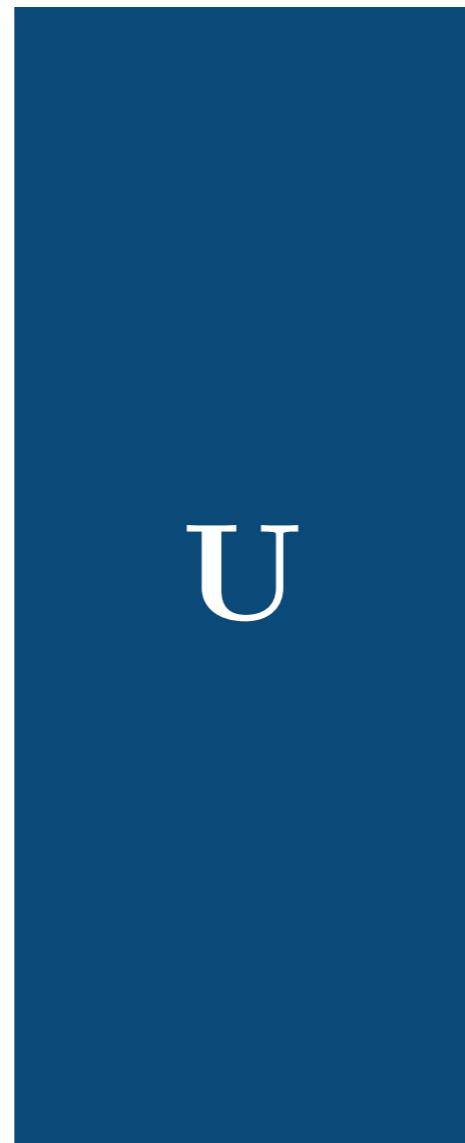
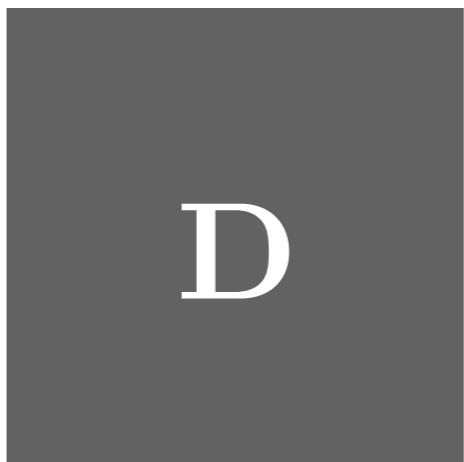
2

Singular Value Decomposition

SVD

Singular Value Decomposition

Review: SVD

 $=$  $m \times n$  $n \times n$  $n \times n$

SVD

Definition: For any given matrix $A \in \mathbb{R}^{m \times n}$ its Singular Value Decomposition (SVD) is defined as

$$A = UDV^\top$$

such that

U is an $m \times n$ matrix with orthogonal columns

V^\top is an $n \times n$ orthogonal matrix

D is an $n \times n$ matrix with non-negative entries,
termed the **singular values**

assume diagonal values of D are sorted in descending order,
assume diagonal values of D are sorted in descending order,

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$$

SVD for Optimization

Find a vector x such that it minimizes

$$\|Ax\|^2$$

SVD for Optimization

Find a vector x such that it minimizes

$$\|Ax\|^2$$

subject to the constraint

$$\|x\| = 1$$

Homogeneous least squares

SVD for Optimization

Find a vector x such that it minimizes

$$\|Ax\|^2$$

subject to the constraint

$$\|x\| = 1$$

Why do we need the unit vector constraint?

avoid **trivial** solution

Find a vector x such that it minimizes

$$\|Ax\|^2$$

subject to the constraint

$$\|x\| = 1$$

$$\begin{aligned}\|Ax\|^2 &= \|UDV^\top x\|^2 \\ &= (UDV^\top x)^\top (UDV^\top x) \\ &= (x^\top VDU^\top)(UDV^\top x)\end{aligned}$$

What does this product evaluate to?

identity matrix

Find a vector x such that it minimizes

$$\|\mathbf{A}\mathbf{x}\|^2$$

subject to the constraint

$$\|\mathbf{x}\| = 1$$

$$\|\mathbf{A}\mathbf{x}\|^2 = \mathbf{x}^\top \mathbf{V} \mathbf{D} \mathbf{V}^\top \mathbf{x}$$

Find a vector x such that it minimizes

$$\|Ax\|^2$$

subject to the constraint

$$\|x\| = 1$$

$$\|Ax\|^2 = x^\top V D D^\top V^\top x$$

Let $y = V^\top x$

Now we minimize

$$y^\top D D^\top y$$

subject to the constraint

$$\|y\| = 1$$

$$\boxed{\|y\| = \|V^\top x\| = \|x\| = 1}$$

Find a vector x such that it minimizes

$$\|\mathbf{A}\mathbf{x}\|^2$$

subject to the constraint

$$\|\mathbf{x}\| = 1$$

Now we minimize

$$\mathbf{y}^\top \mathbf{D} \mathbf{D} \mathbf{y}$$

subject to the constraint

$$\|\mathbf{y}\| = 1$$

Find a vector x such that it minimizes

$$\|\mathbf{A}\mathbf{x}\|^2$$

subject to the constraint

$$\|\mathbf{x}\| = 1$$

Now we minimize

$$\mathbf{y}^\top \mathbf{D} \mathbf{D} \mathbf{y}$$

subject to the constraint

$$\|\mathbf{y}\| = 1$$

$$d_1^2 y_1^2 + d_2^2 y_2^2 + \cdots + d_n^2 y_n^2$$

What is the value of y that yields the minimum?

$$\mathbf{y} = (0, 0, \dots, 1)^\top$$

tl;di

$$\mathbf{A}$$

$m \times n$

$$\mathbf{x}$$

$n \times 1$

$$=$$

$$\mathbf{0}$$

$m \times 1$

$$\mathbf{x}^* = \arg \min_{\mathbf{x}} \|\mathbf{Ax}\|^2 \text{ subject to } \|\mathbf{x}\| = 1$$

take SVD of \mathbf{A} , solution is last column of \mathbf{V}

Fundamental Matrix via SVD

$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$\begin{pmatrix}
 x_{r1}x_{l1} & x_{r1}y_{l1} & x_{r1} & y_{r1}x_{l1} & y_{r1}y_{l1} & y_{r1} & x_{l1} & y_{l1} & 1 \\
 & & & \vdots & & & & & \\
 x_{rn}x_{ln} & x_{rn}y_{ln} & x_{rn} & y_{rn}x_{ln} & y_{rn}y_{ln} & y_{rn} & x_{ln} & y_{ln} & 1
 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = \mathbf{0}$$

A

f

Fundamental Matrix must be rank 2

estimated F matrix may be FULL RANK

Enforce
rank 2

$$\tilde{\mathbf{F}}^* = \operatorname*{argmin}_{\tilde{\mathbf{F}}} \|\tilde{\mathbf{F}} - \hat{\mathbf{F}}\|^2 \text{ subject to } \det(\tilde{\mathbf{F}}) = 0$$

Enforce
rank 2

3

major steps

Step 1

take SVD

$$\mathbf{F} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top$$

Step 2

adjust singular value

$$\Sigma' = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Step 3

reassemble Fundamental Matrix

$$\mathbf{F}' = \mathbf{U} \boldsymbol{\Sigma}' \mathbf{V}^\top$$

Enforce
rank 2
summary

1. Take SVD of F
2. Set smallest singular value of F to 0
3. Recompute F

$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$\begin{pmatrix} x_{r1}x_{l1} & x_{r1}y_{l1} & x_{r1} & y_{r1}x_{l1} & y_{r1}y_{l1} & y_{r1} & x_{l1} & y_{l1} & 1 \\ & & & \vdots & & & & & \\ x_{rn}x_{ln} & x_{rn}y_{ln} & x_{rn} & y_{rn}x_{ln} & y_{rn}y_{ln} & y_{rn} & x_{ln} & y_{ln} & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \end{pmatrix} = 0$$

\mathbf{A}

A is ill-conditioned

\mathbf{f}

$$\mathbf{p}_r^\top \mathbf{F} \mathbf{p}_l = 0$$

$$\begin{pmatrix} x_{r1}x_{l1} & x_{r1}y_{l1} & x_{r1} & y_{r1}x_{l1} & y_{r1}y_{l1} & y_{r1} & x_{l1} & y_{l1} & 1 \\ & & & \vdots & & & & & \\ x_{rn}x_{ln} & x_{rn}y_{ln} & x_{rn} & y_{rn}x_{ln} & y_{rn}y_{ln} & y_{rn} & x_{ln} & y_{ln} & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \end{pmatrix} = 0$$

A

transform points such that A is better conditioned

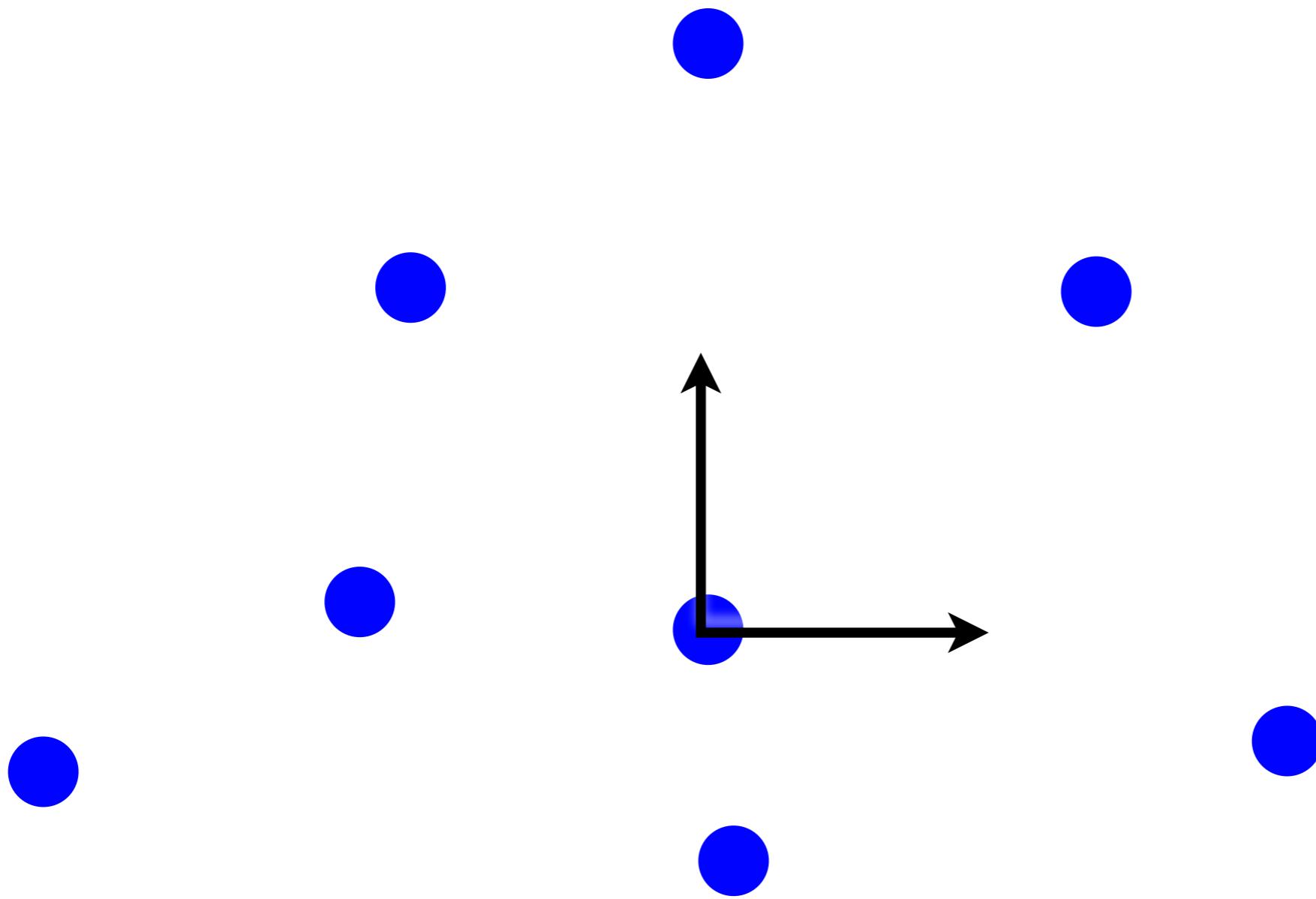
Normalized
8-point
algorithm

4

major steps

Step 1

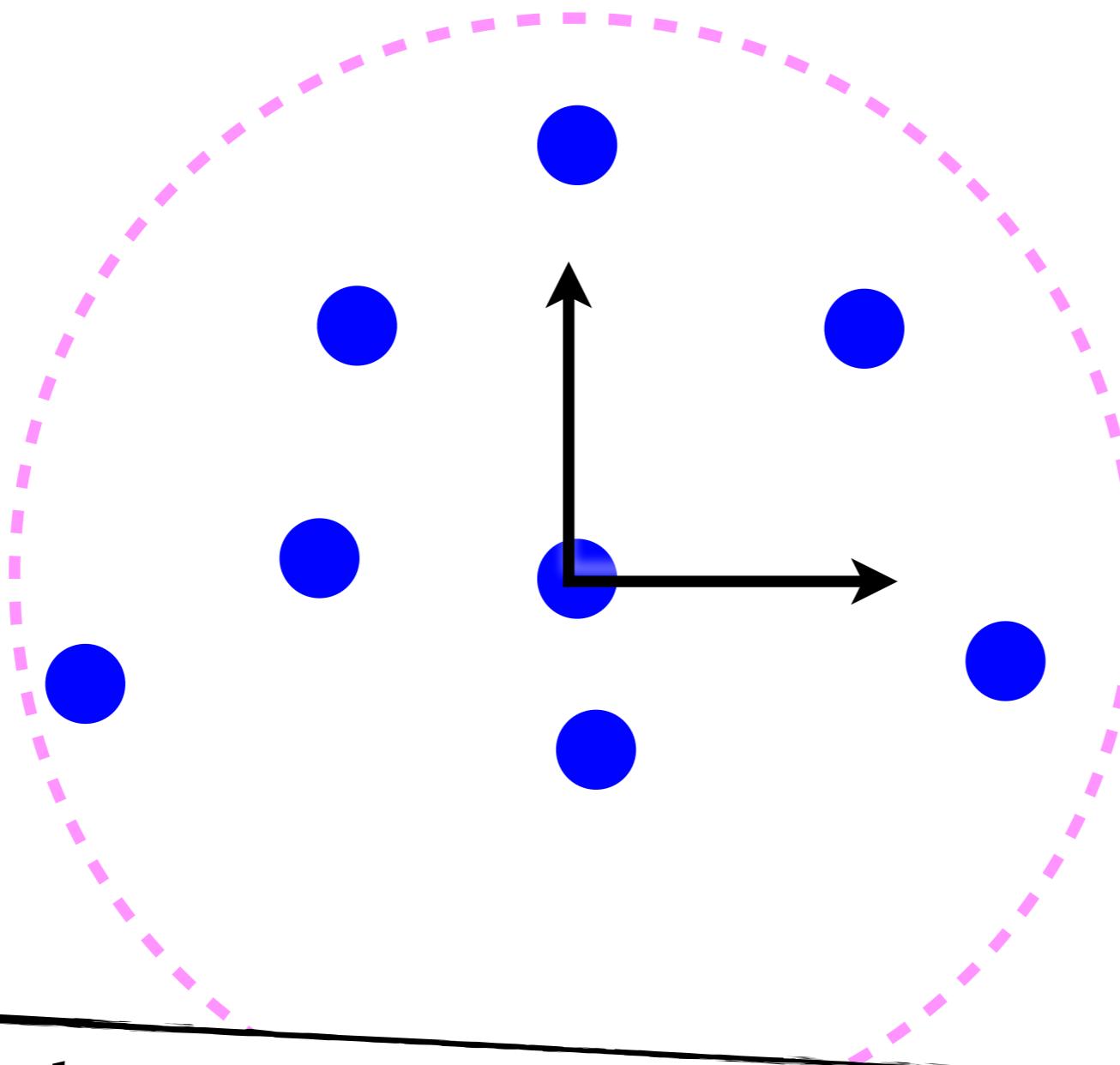
centre data



translate centroid of points to origin

Step 2

scale data



scale points to make mean square distance ~2 pixels

Step 3

compute Fundamental Matrix

$$(\mathbf{T}_r \mathbf{p}_r)^\top \mathbf{F}(\mathbf{T}_l \mathbf{p}_l) = 0$$

enforce rank 2 constraint on estimated \mathbf{F}

Step 4

de-normalize Fundamental Matrix

$$\mathbf{F} = \mathbf{T}_r^\top \hat{\mathbf{F}} \mathbf{T}_l$$

Normalized
8-point algorithm
summary

1. Translate points with centroid centred at origin
2. Apply scaling to image coordinates
3. Estimate Fundamental Matrix
4. De-normalize estimated Fundamental Matrix

8-point algorithm

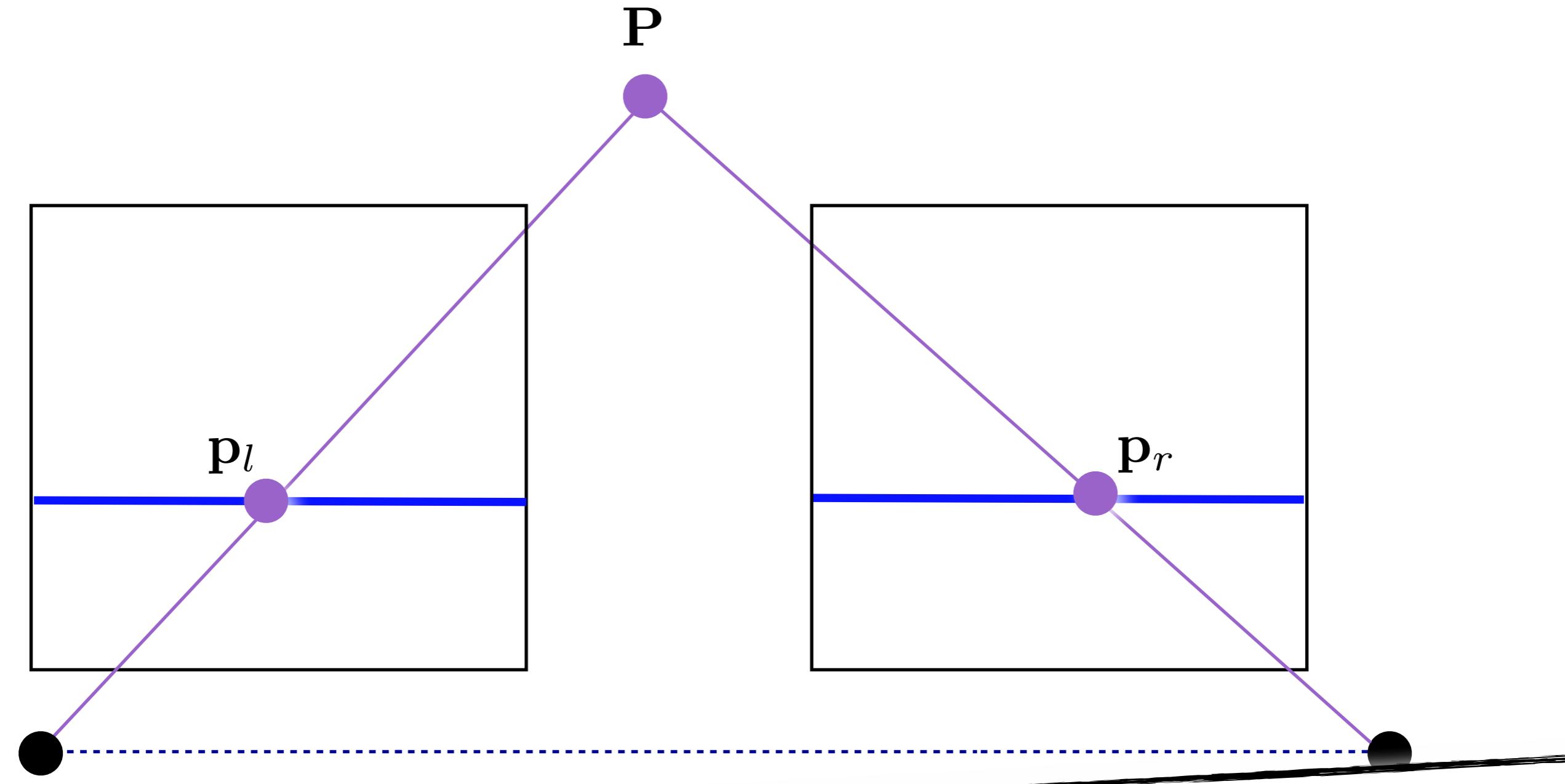
```
% build constraint matrix
A = [x2(1,:)'.*x1(1,:)' x2(1,:)'.*x1(2,:)' x2(1,:)' ...
      x2(2,:)'.*x1(1,:)' x2(2,:)'.*x1(2,:)' x2(2,:)' ...
      x1(1,:)' x1(2,:)' ones(npts,1) ];

% compute SVD matrix factorization
[U,D,V] = svd(A);

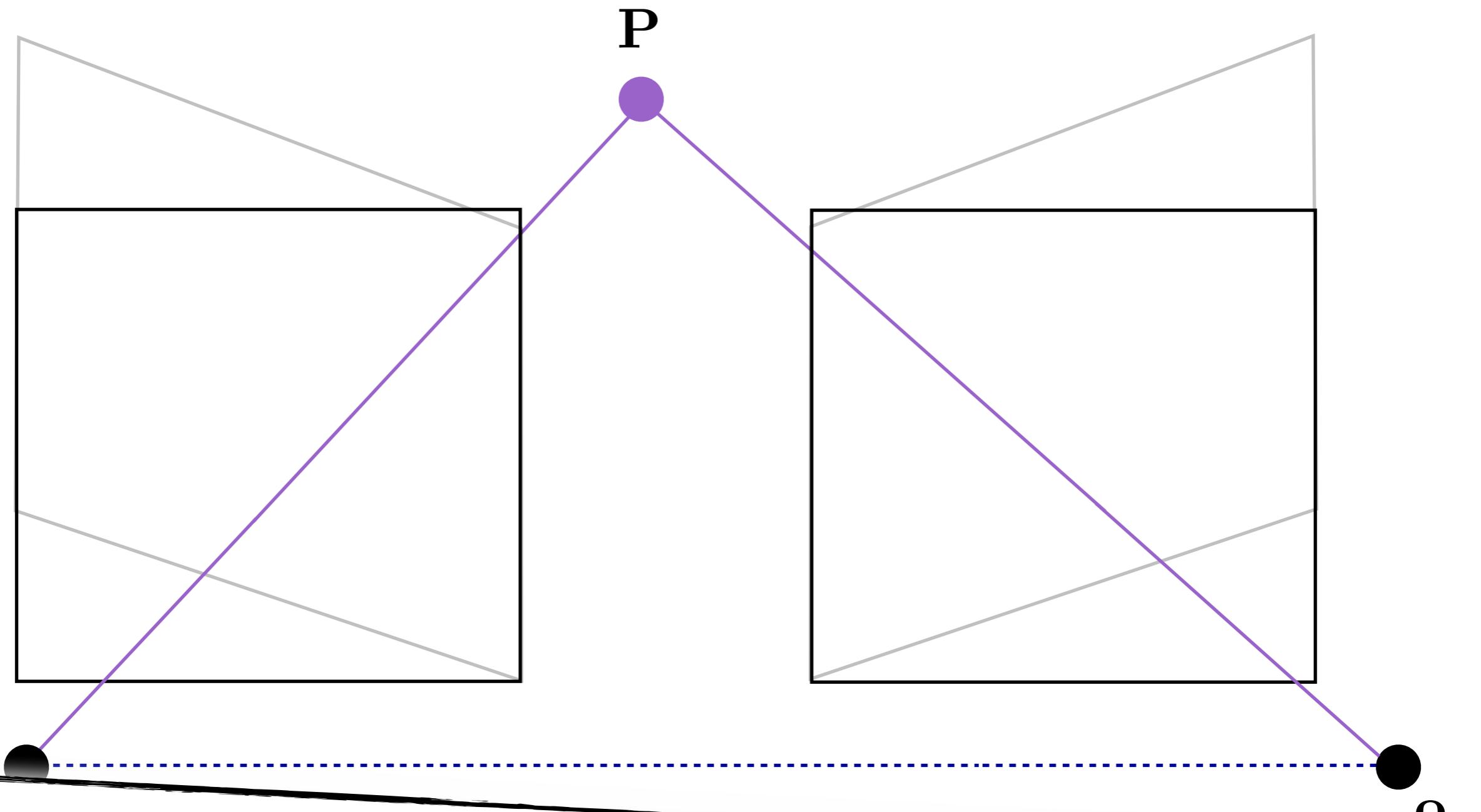
% extract Fundamental Matrix from the column of V
% corresponding to the smallest singular value
F = V(:,9);
F = reshape(F,3,3)';

% Enforce rank 2 constraint
[U,D,V] = svd(F);
F = U*diag([D(1,1) D(2,2) 0])*V';
```

Image Rectification



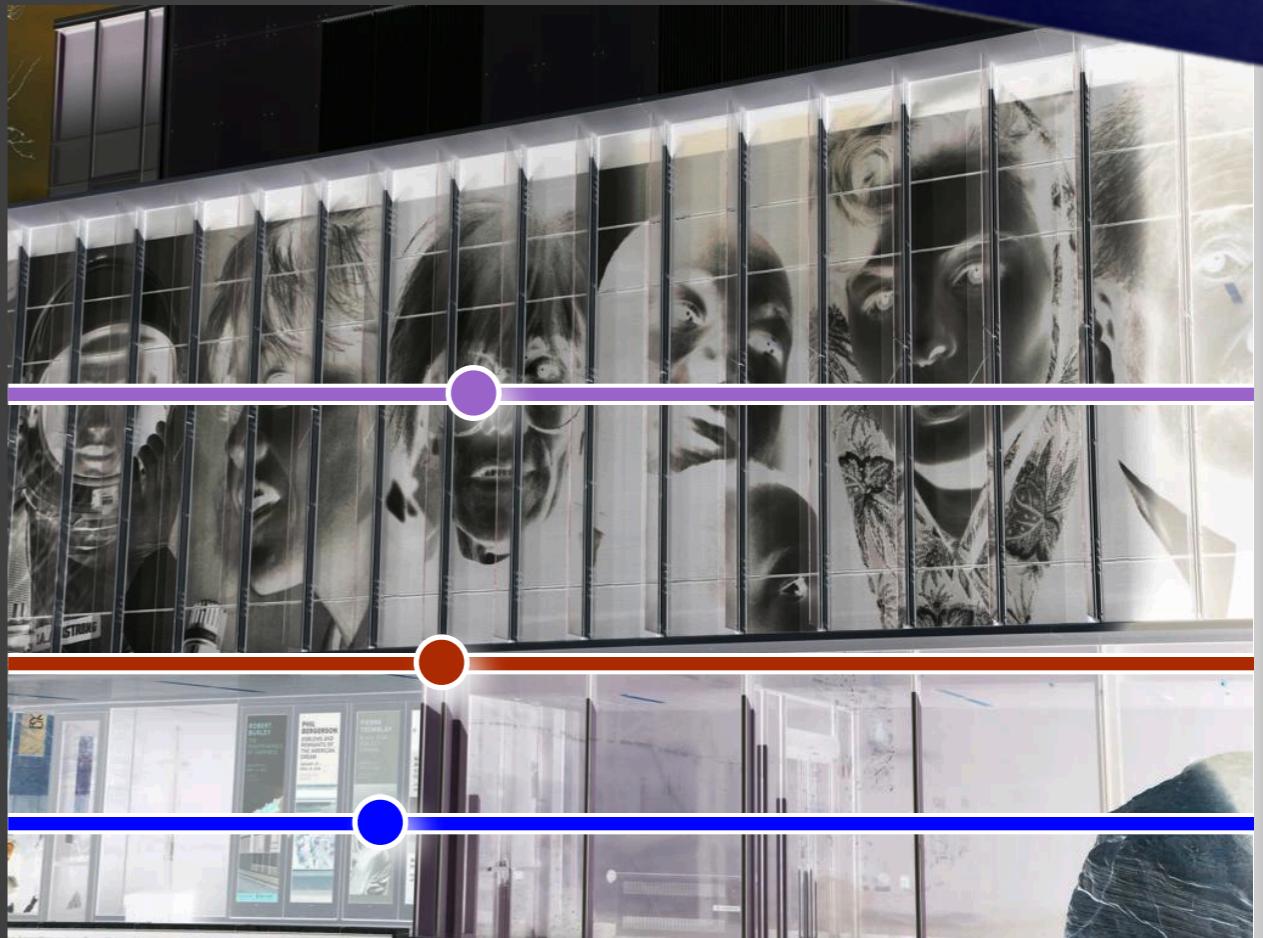
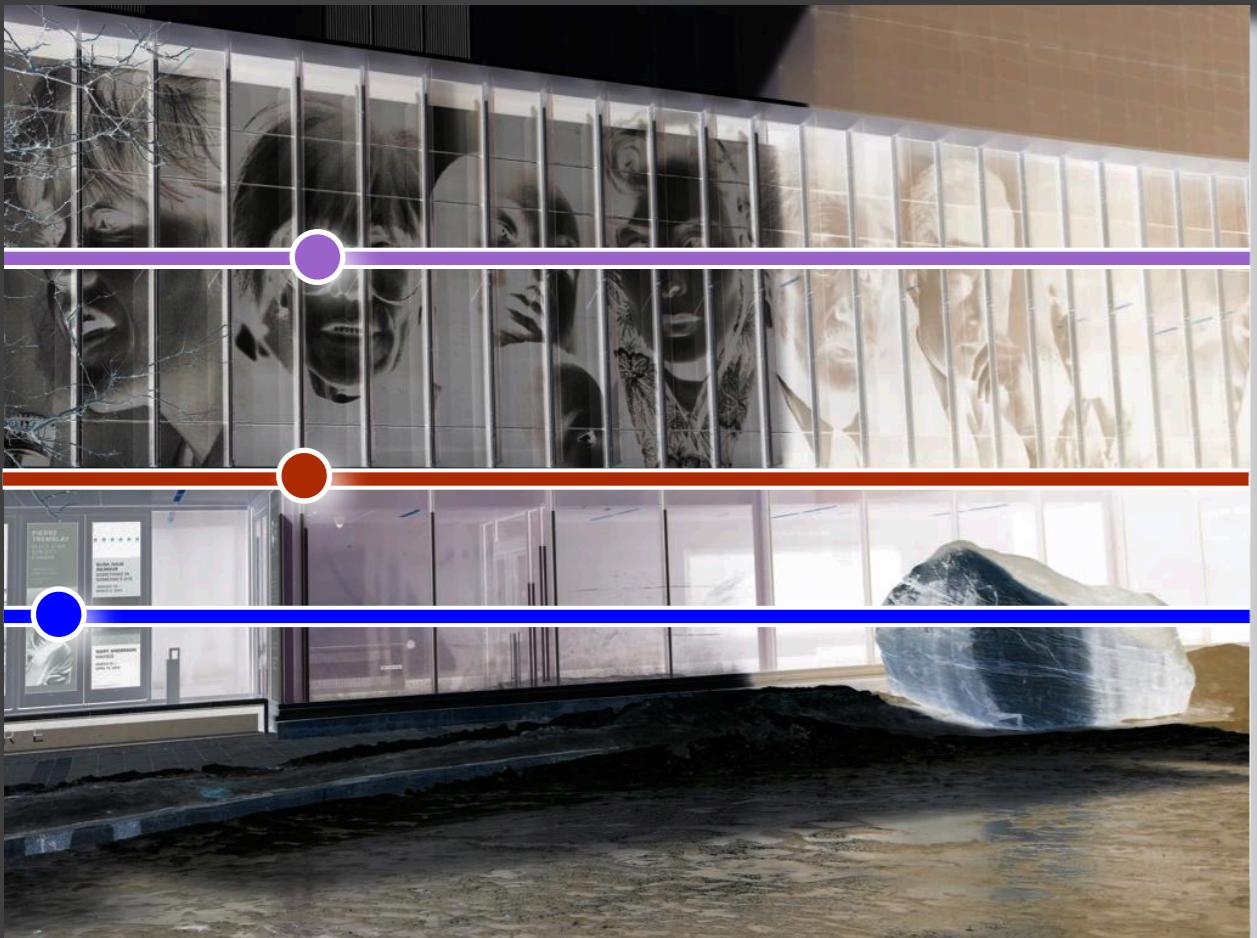
parallel image planes simplifies stereo matching



**reproject images onto a common plane
parallel to baseline**

apply a **homography** to each image

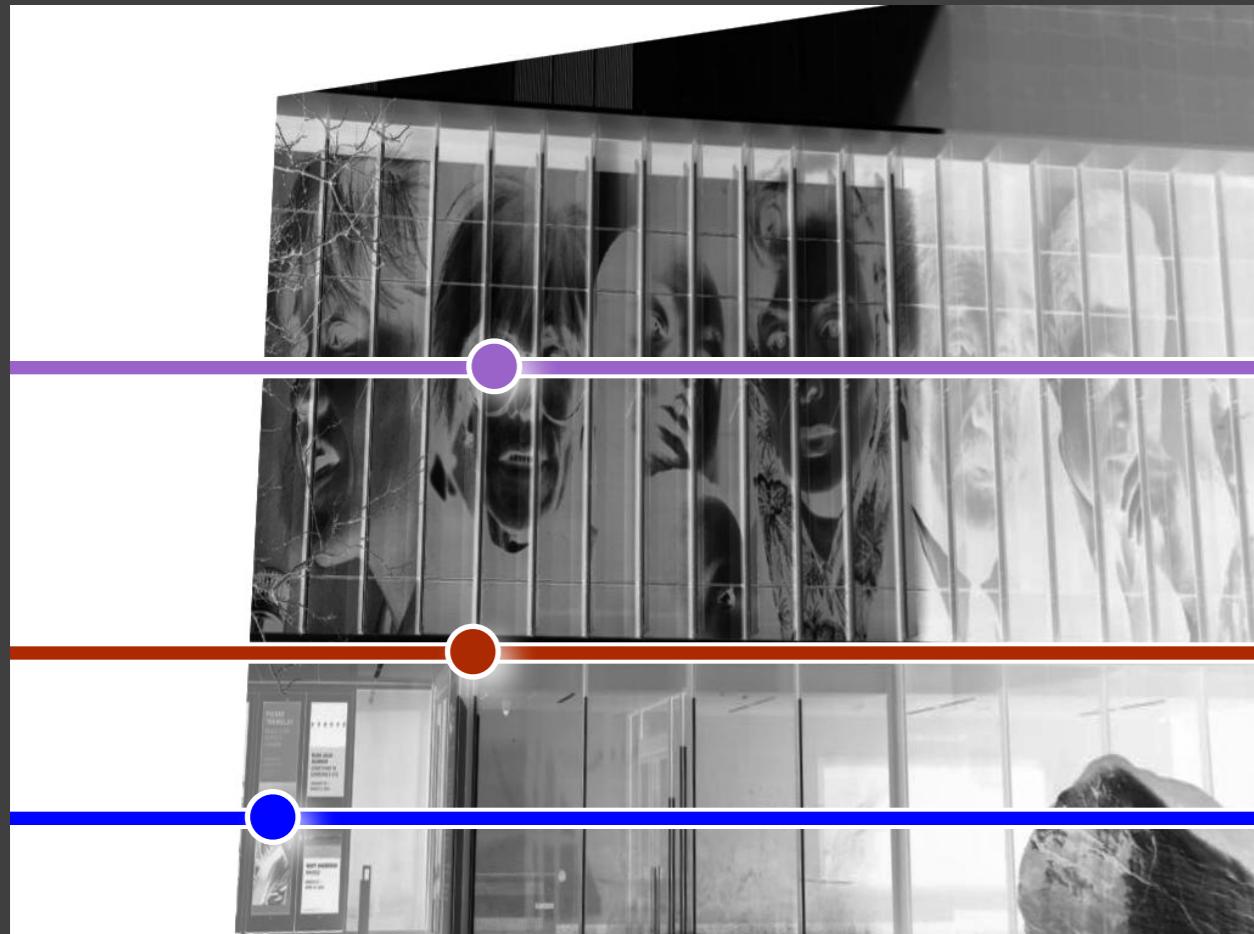
Original Image Pair



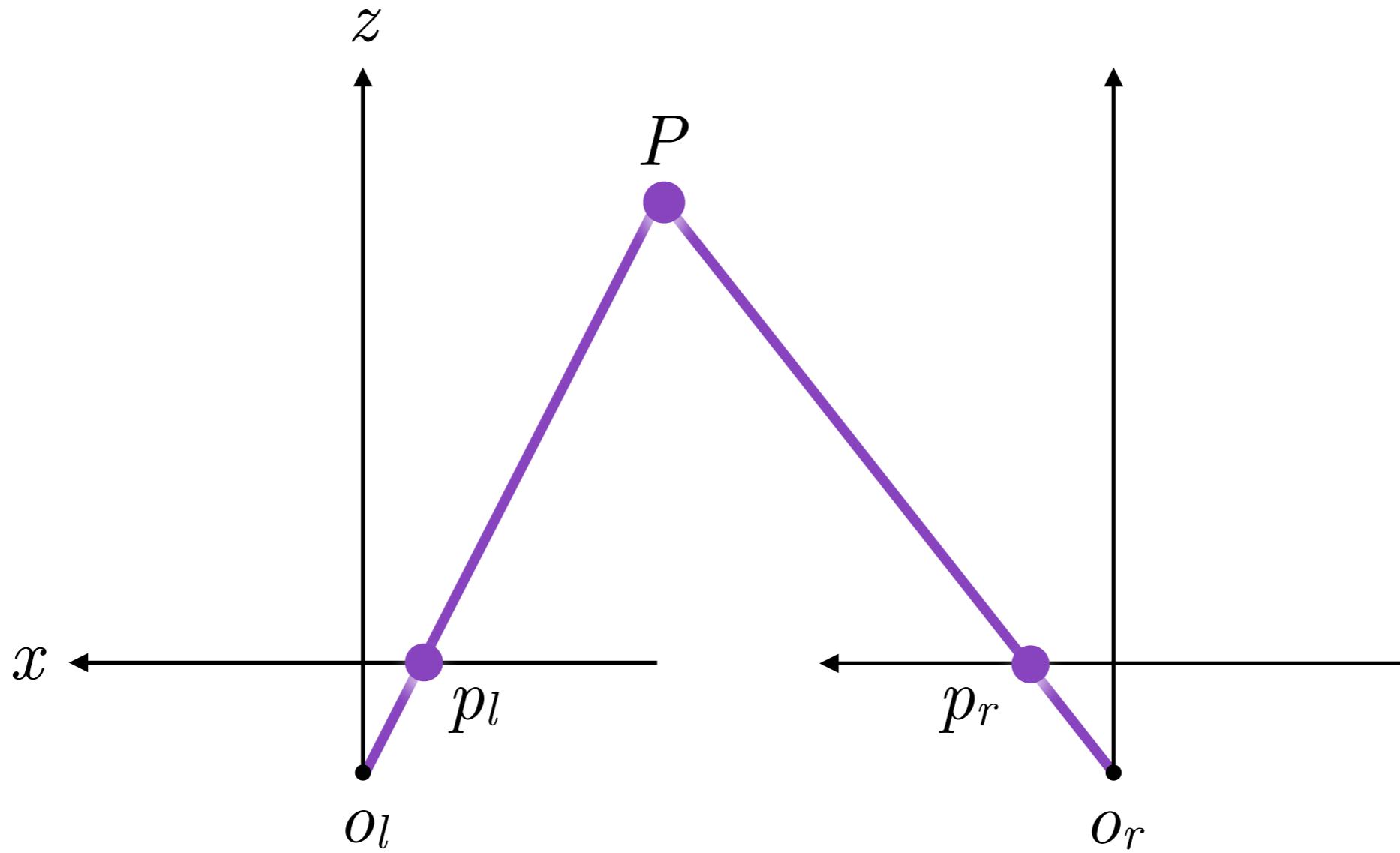
Rectified Image Pair



Rectified Image Pair

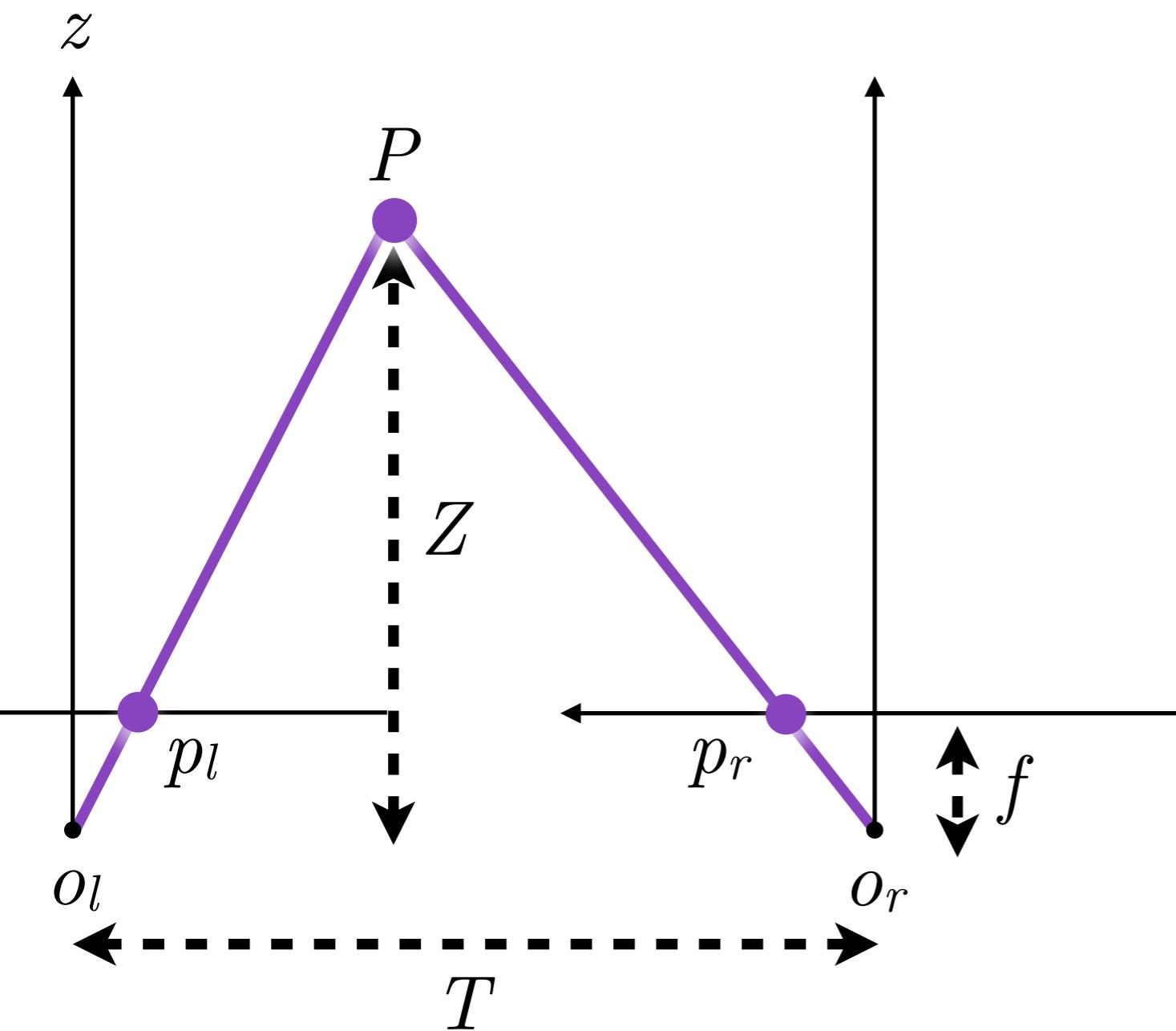


corresponding points are on the **same scan line**



3D position can be determined by the intersection of rays
process known as **triangulation**

Triangulation derivation



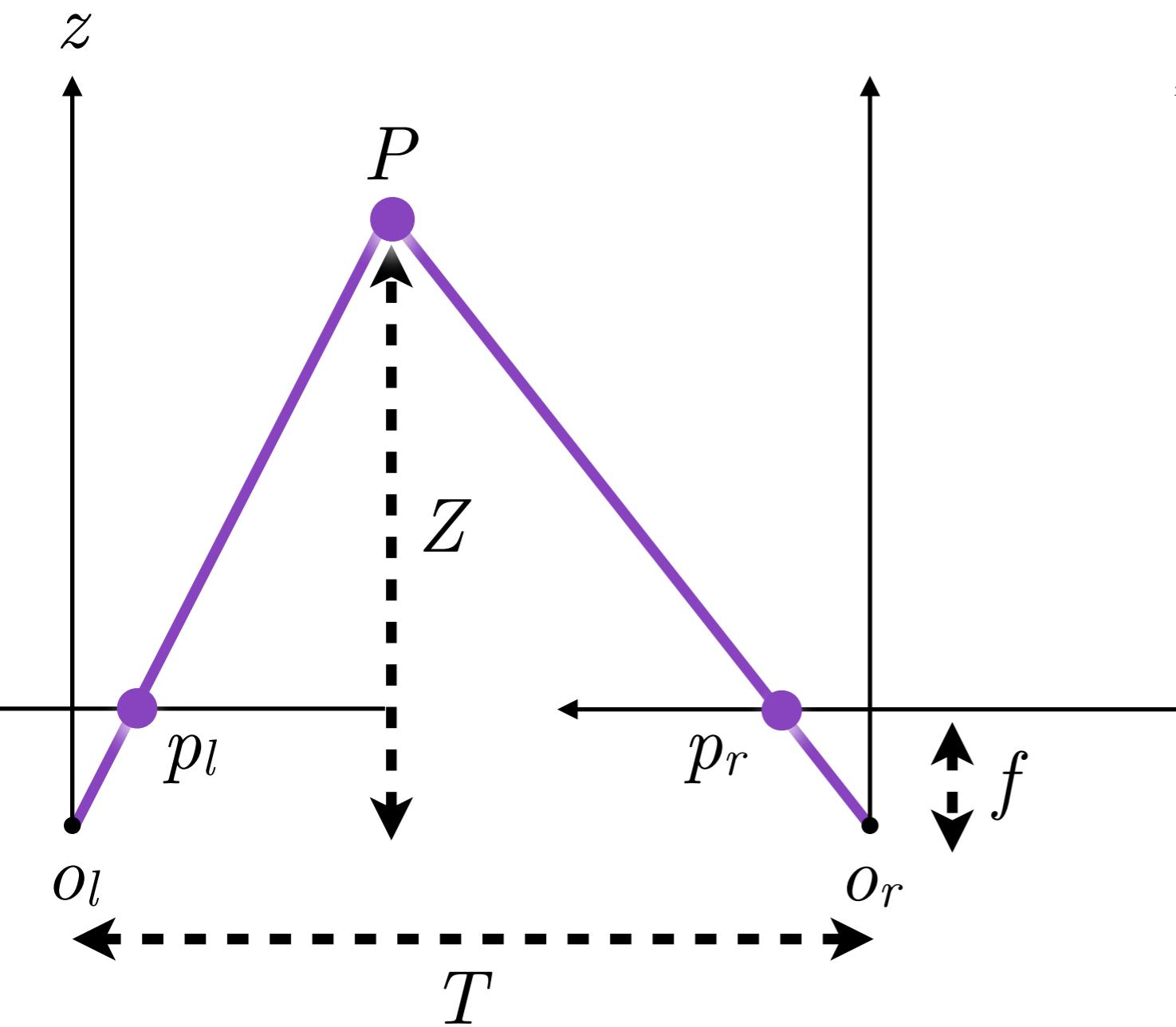
Definitions

T camera centre distance

Z distance to baseline

f common focal length

Triangulation derivation



$\triangle p_l P p_r$

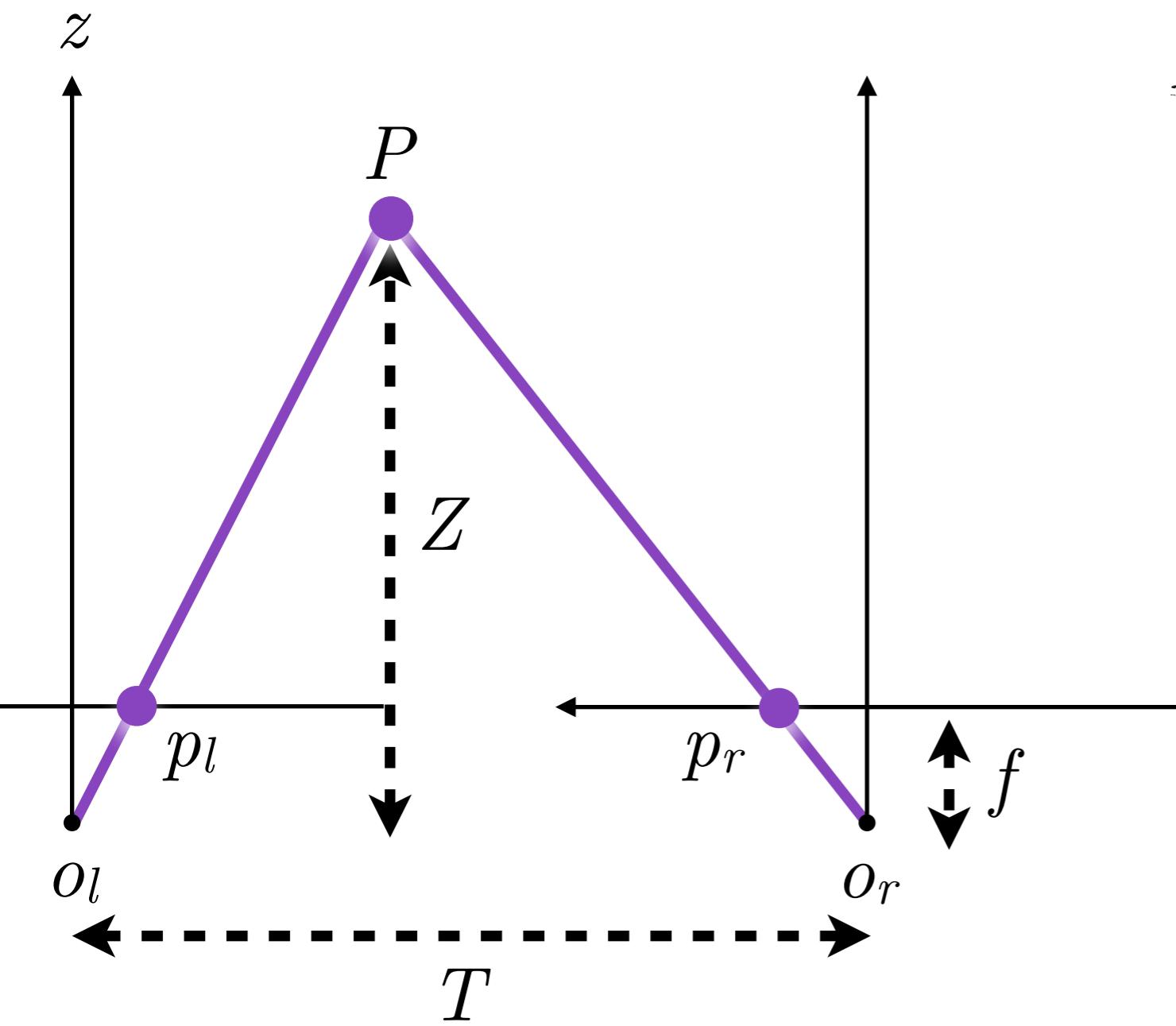
and

$\triangle o_l P o_r$

via similar triangles

$$\frac{T + x_l - x_r}{Z - f} = \frac{T}{Z}$$

Triangulation derivation



$$\frac{T + x_l - x_r}{Z - f} = \frac{T}{Z}$$

Let $x_r - x_l$ be the disparity

$$\frac{T - d}{Z - f} = \frac{T}{Z}$$

rearrange

Disparity
Equation

$$Z = f \frac{T}{d}$$

Disparity Equation

$$Z = f \frac{T}{d}$$

Assumes stereo cameras are exactly parallel

Assumes perfect correspondence

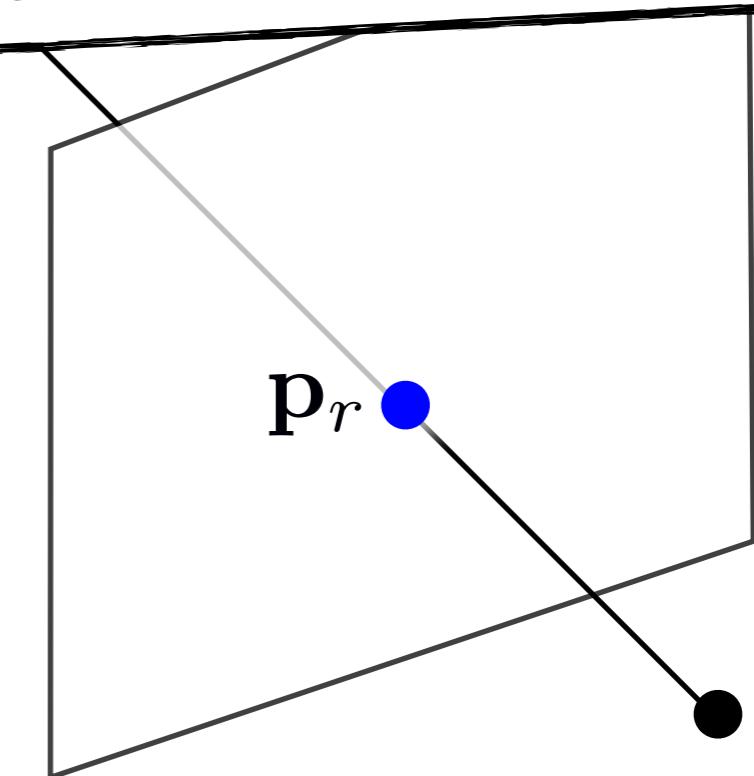
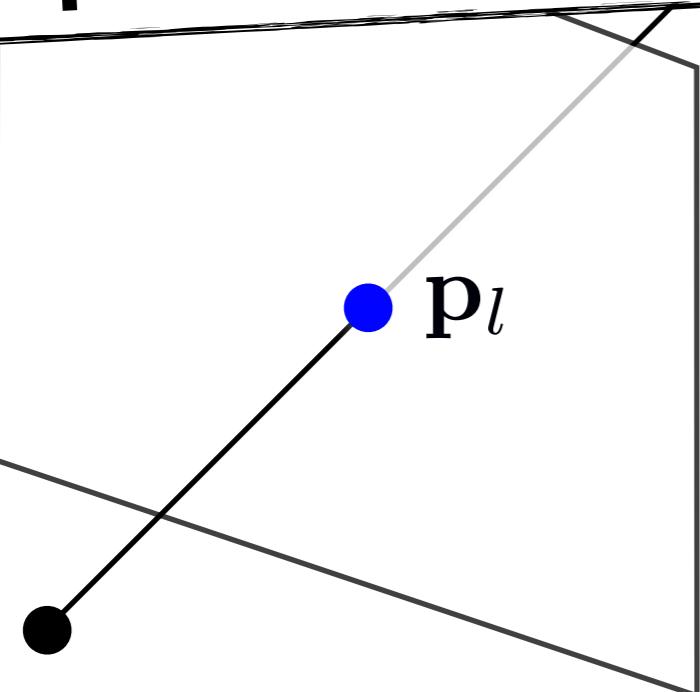
Depth is inversely proportional to disparity

Assume we have a fully calibrated stereo rig

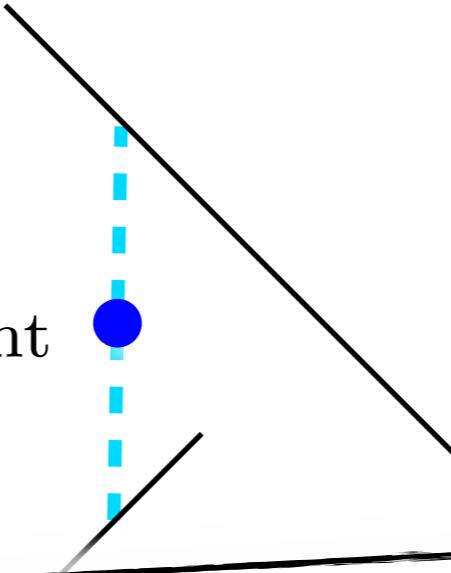
Assume we have a fully calibrated stereo rig

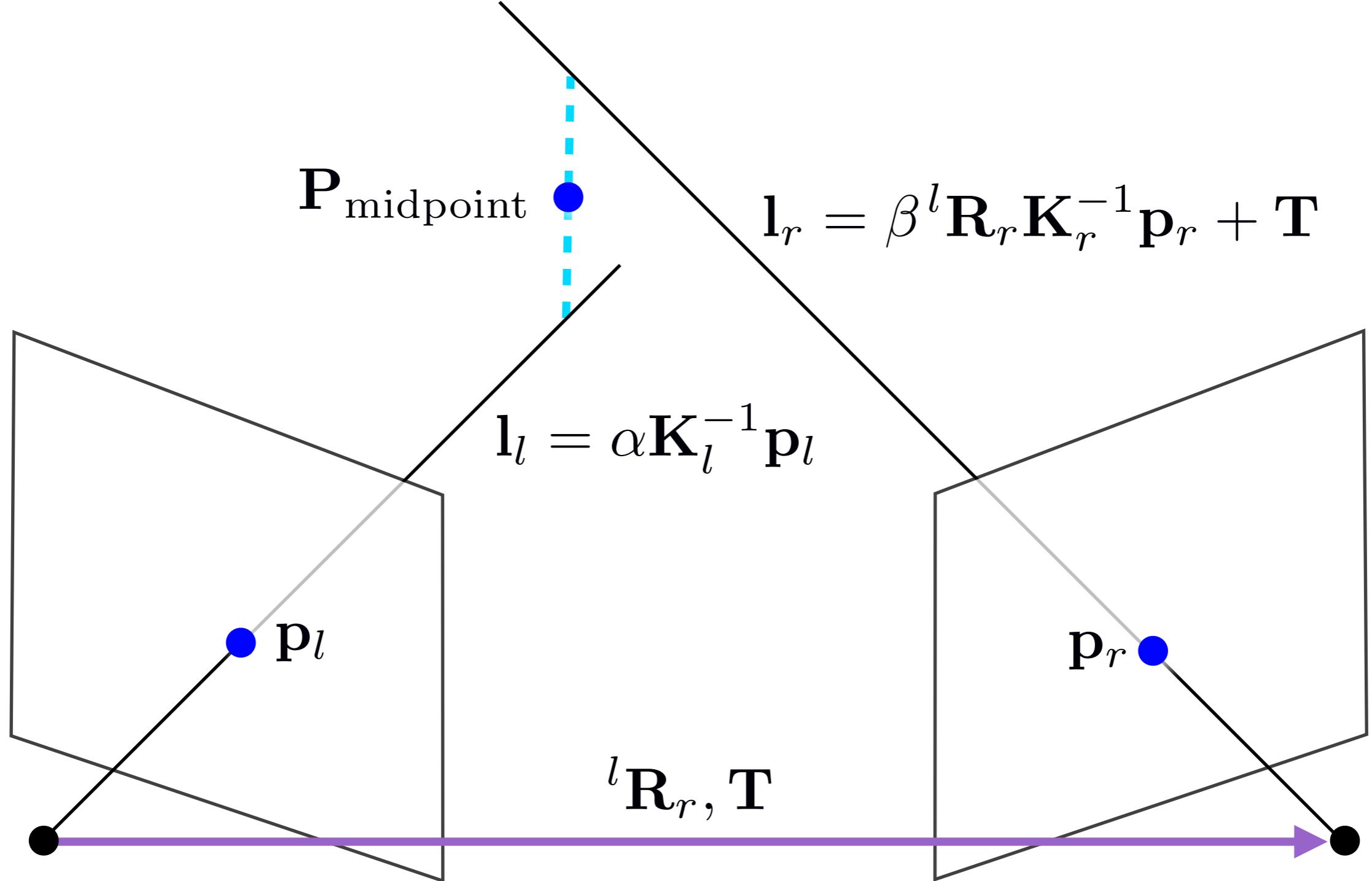
intrinsic and extrinsic parameters of cameras known

compute the midpoint of shortest line between rays



P_{midpoint}



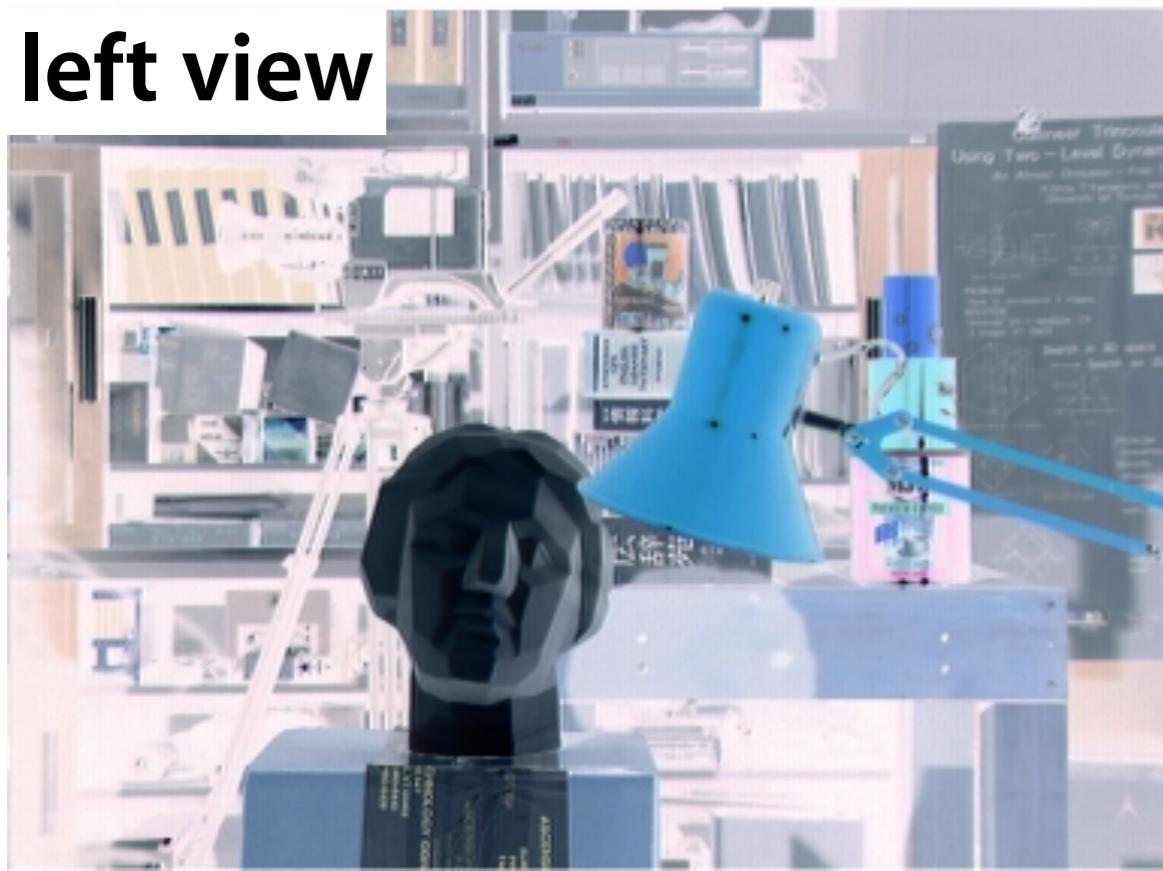


$$\arg \min_{\alpha, \beta} \| \alpha \mathbf{K}_l^{-1} \mathbf{p}_l - (\beta {}^l \mathbf{R}_r \mathbf{K}_r^{-1} \mathbf{p}_r + \mathbf{T}) \|$$

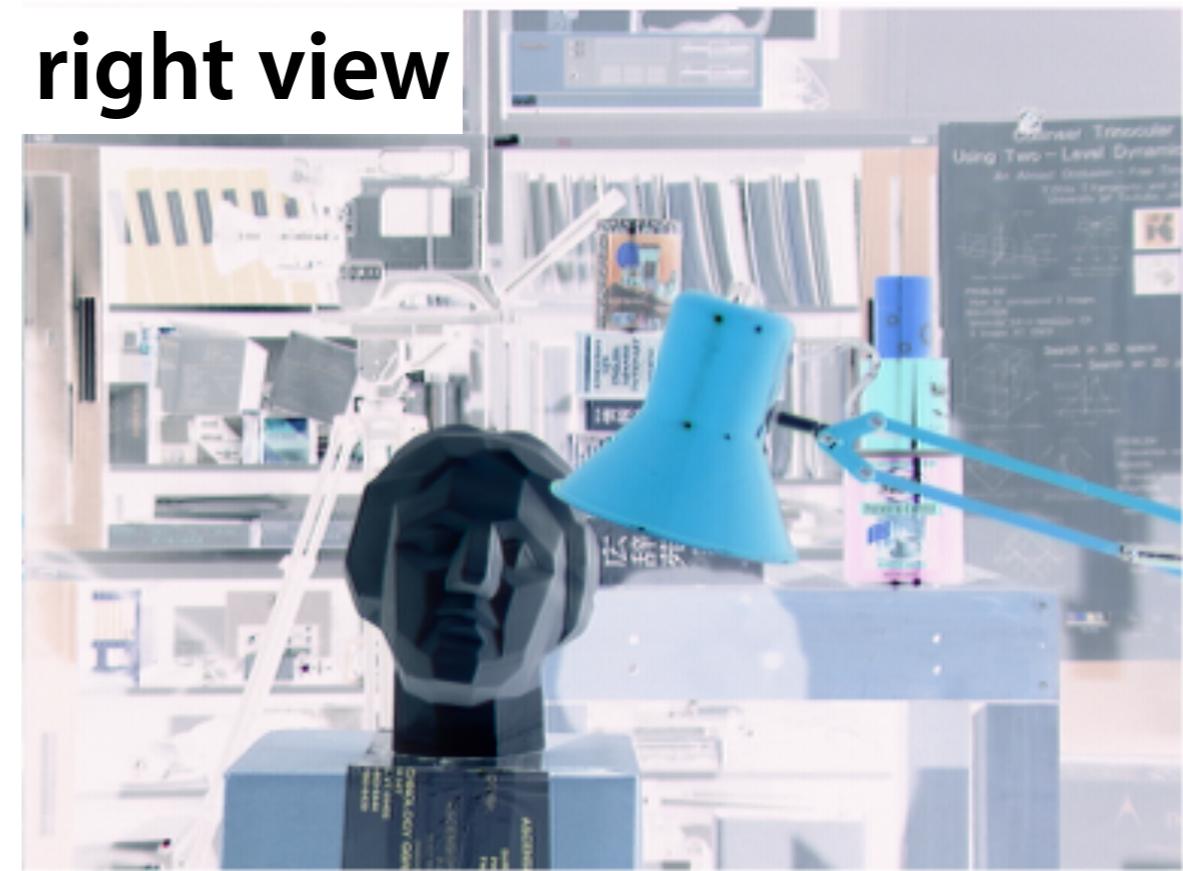
Stereo Matching

region-based methods

left view

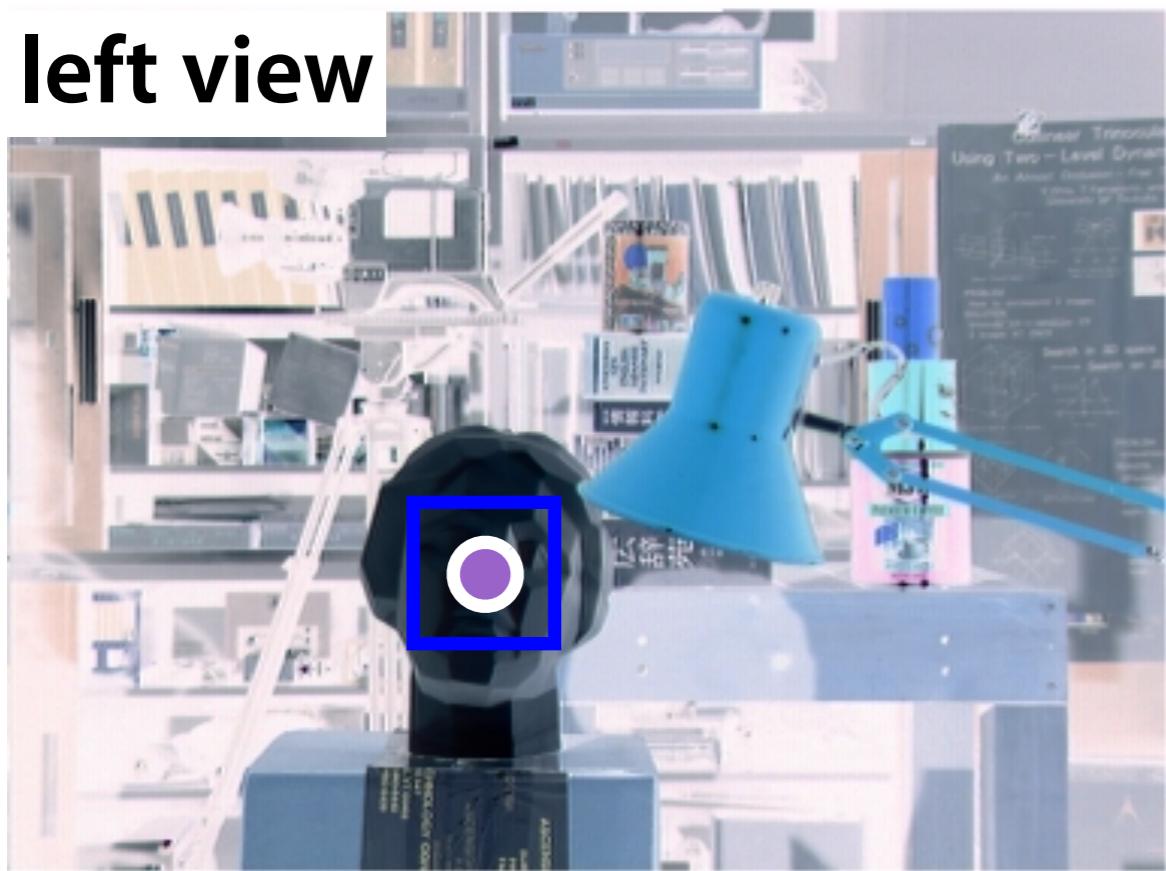


right view

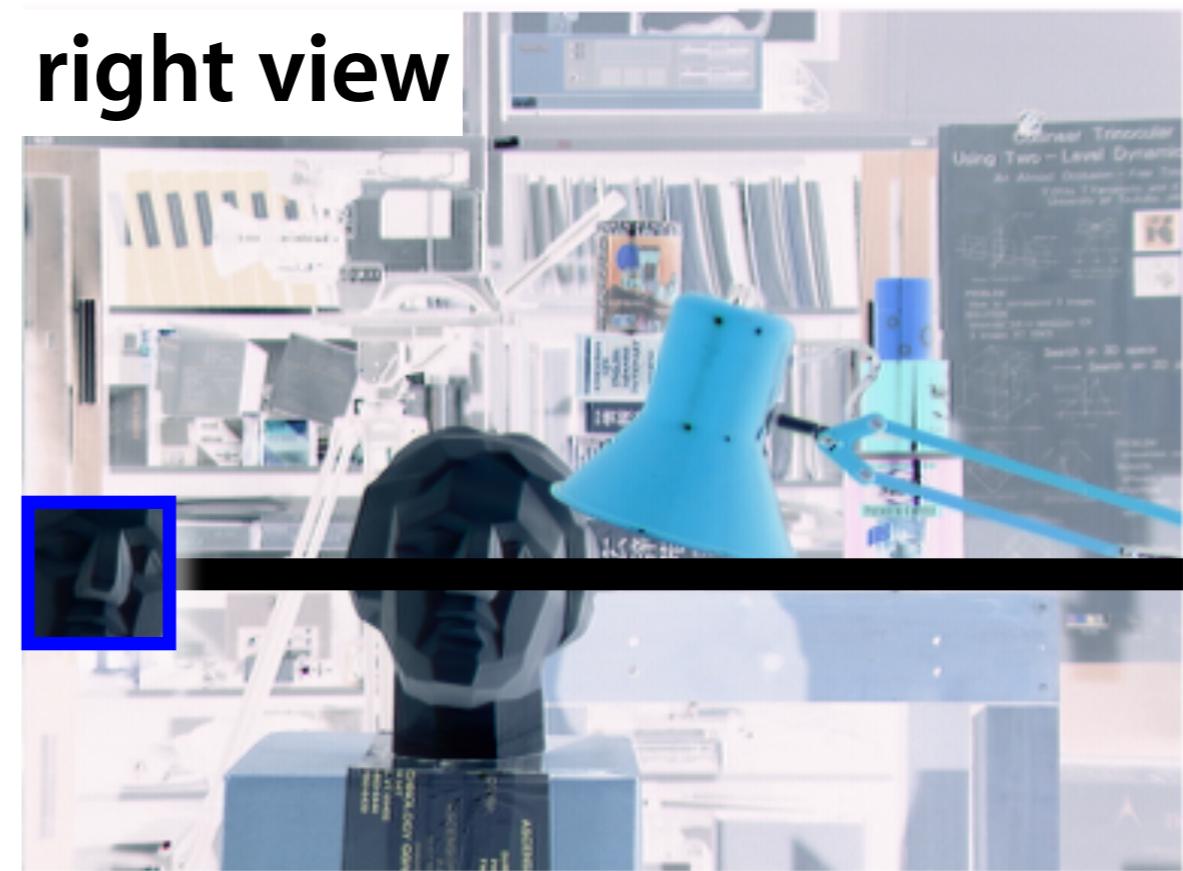


**for each point in left view, find “best” matching region
in right view**

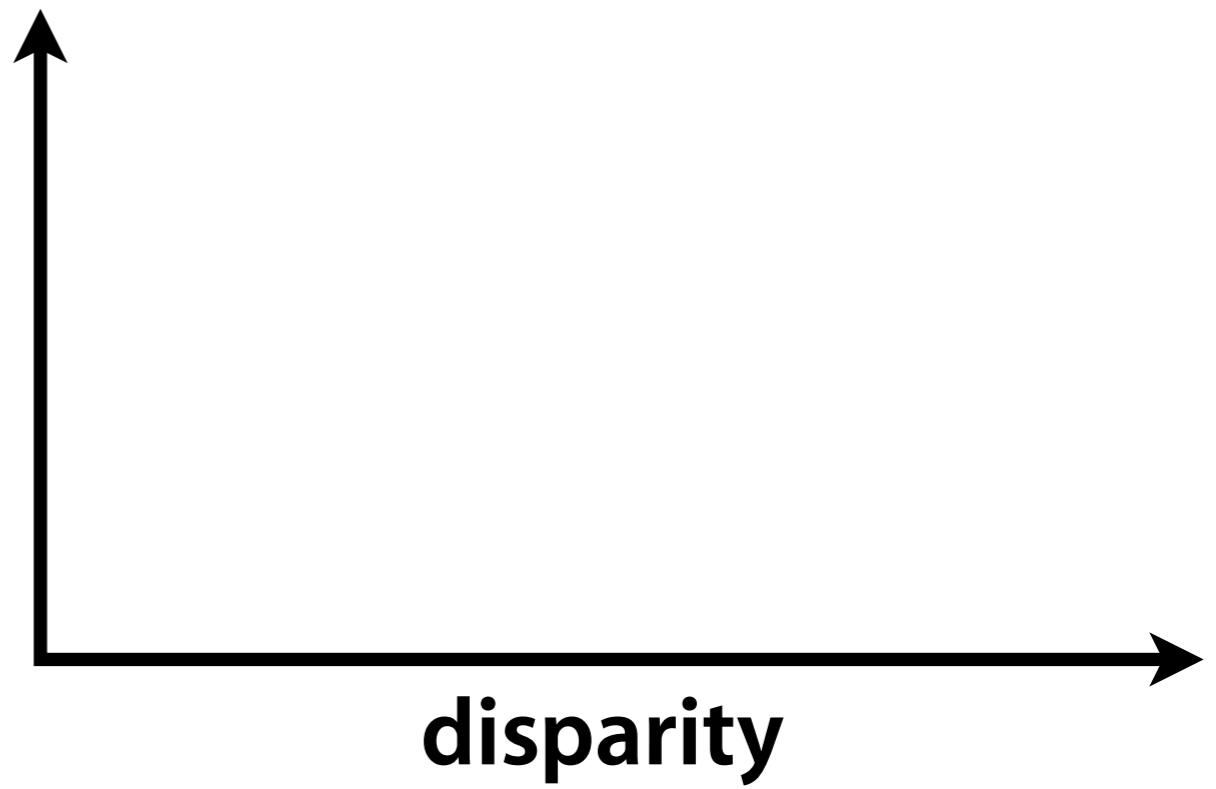
left view



right view

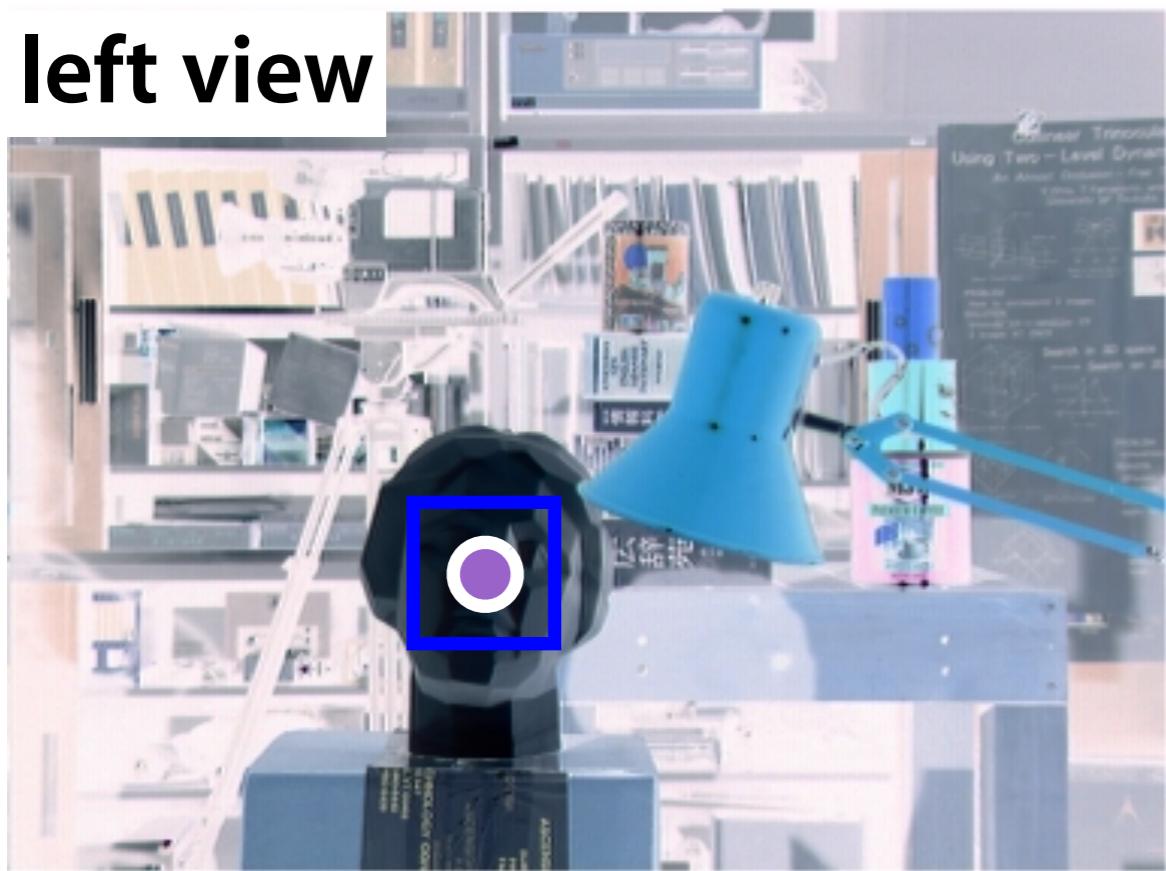


**matching cost
(SAD)**

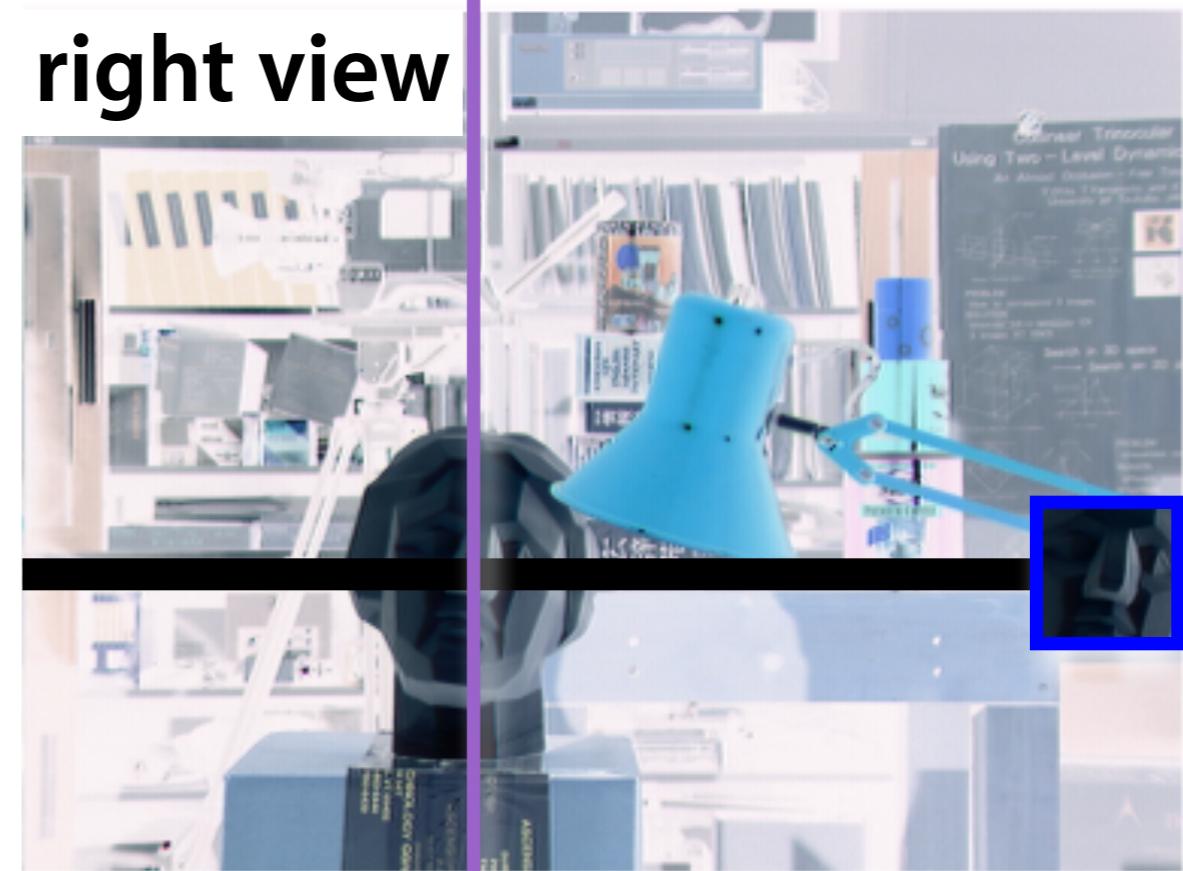


$$\text{SAD}(I,J) = \sum_{i,j} |I(i,j) - J(i,j)|$$

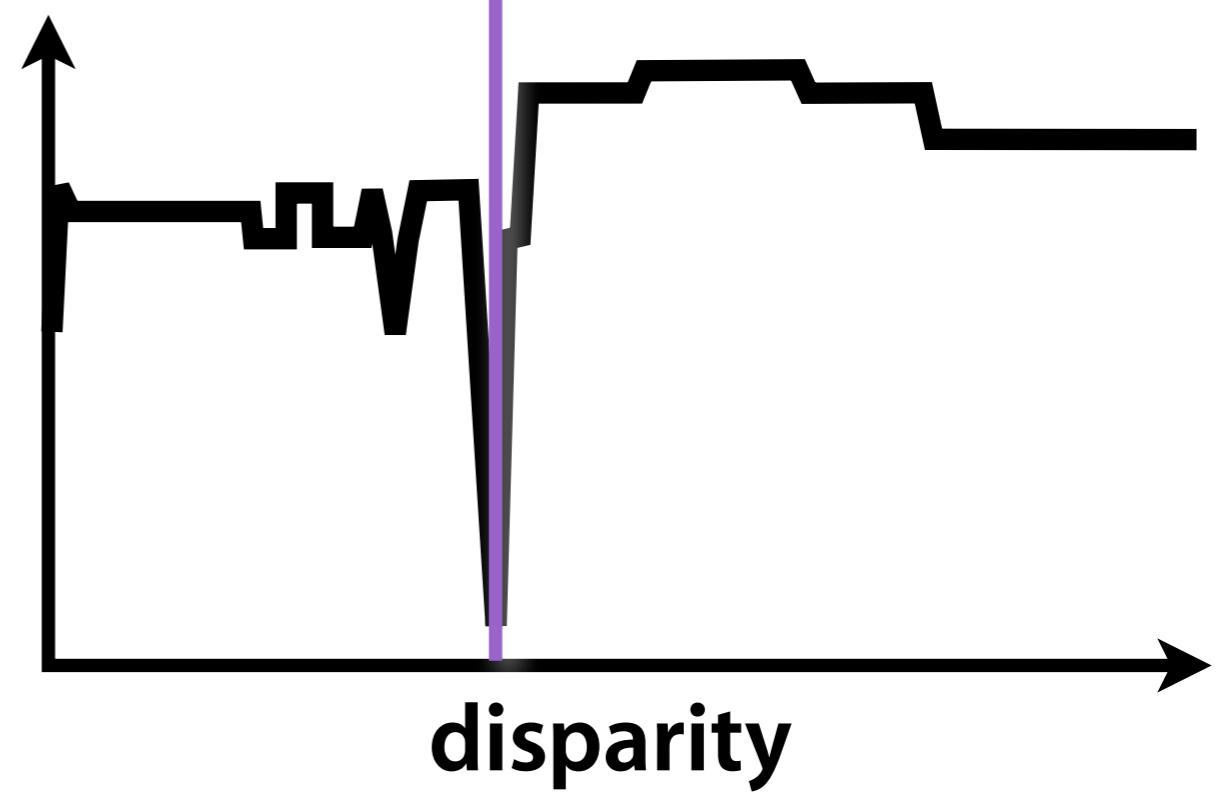
left view



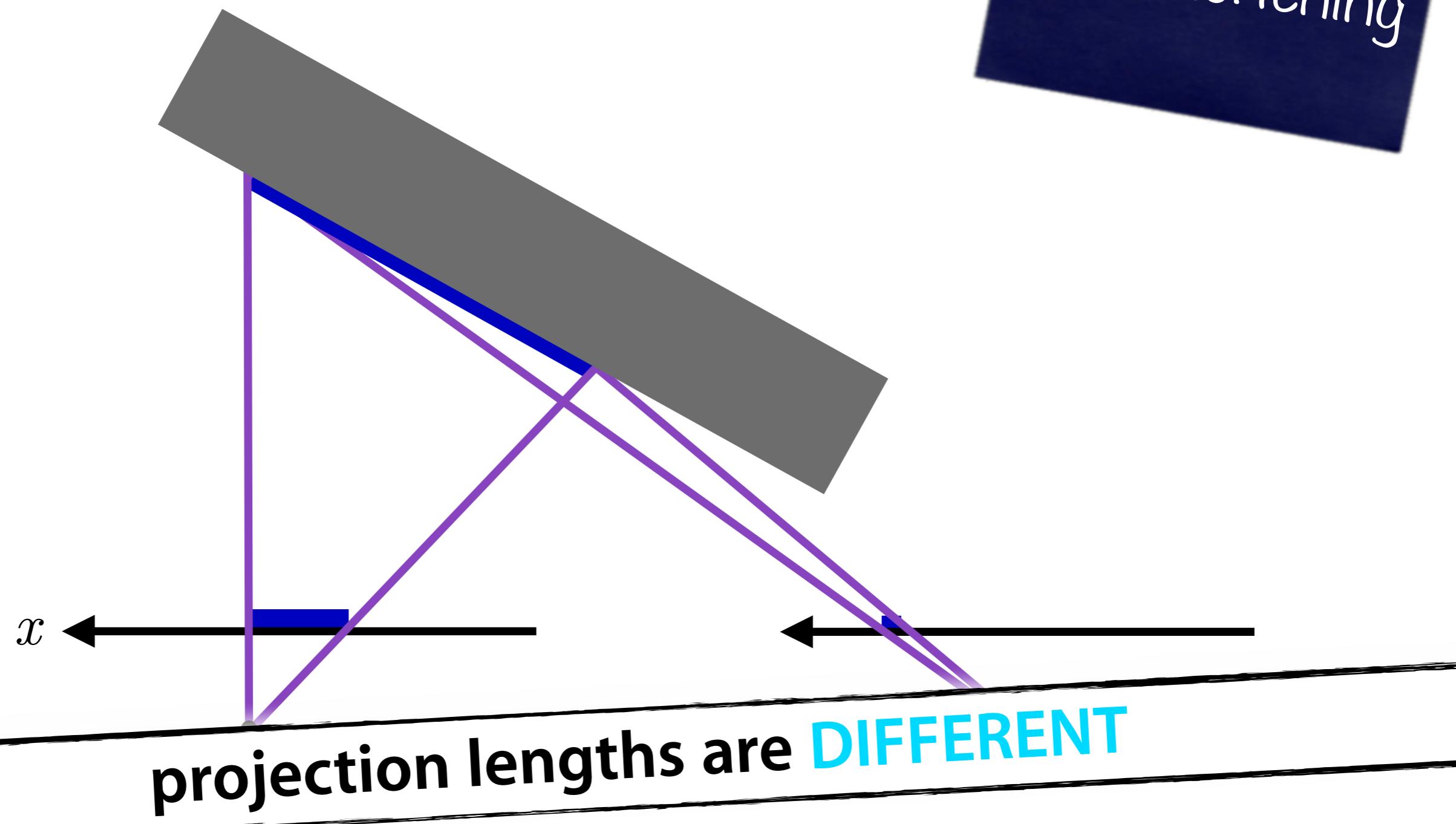
right view



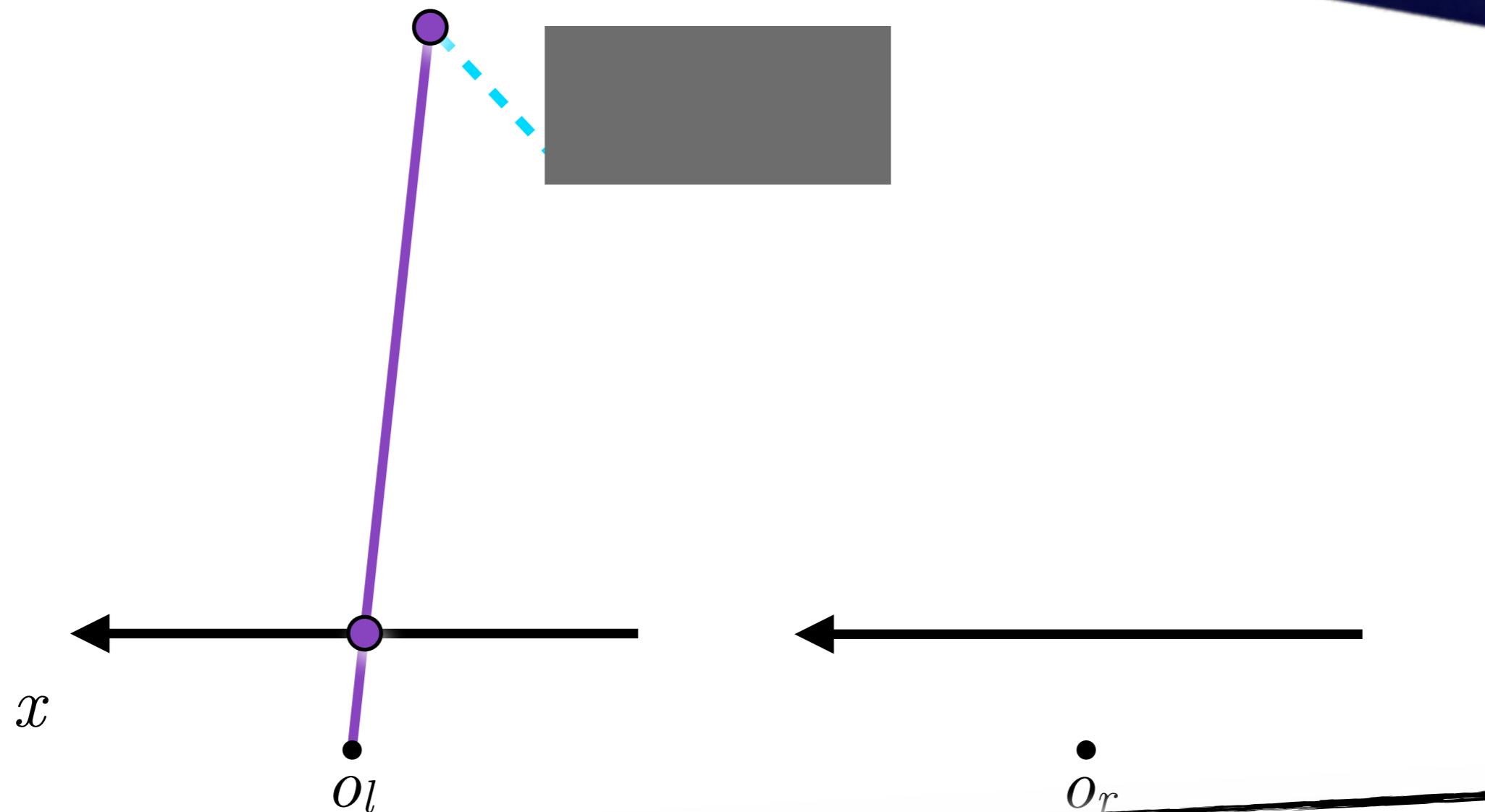
**matching cost
(SAD)**



Foreshortening



Binocular
half-occlusion



all points are not visible from both views

Repetitive structures



region matches are **ambiguous**