Intro to

Computer Vision

with Prof. Kosta Derpanis

Frequency analysis

Ryerson University

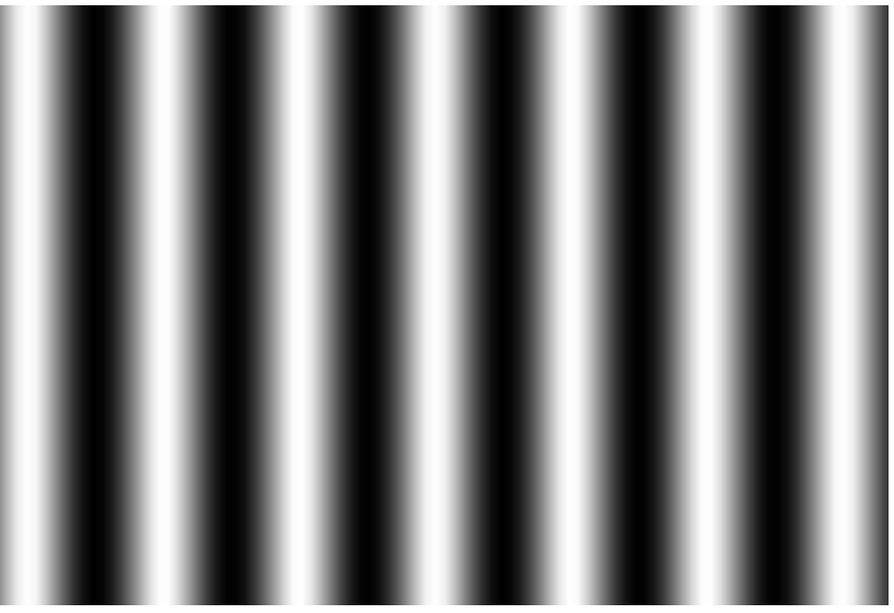
LECTURE TOPICS

Fourier transform

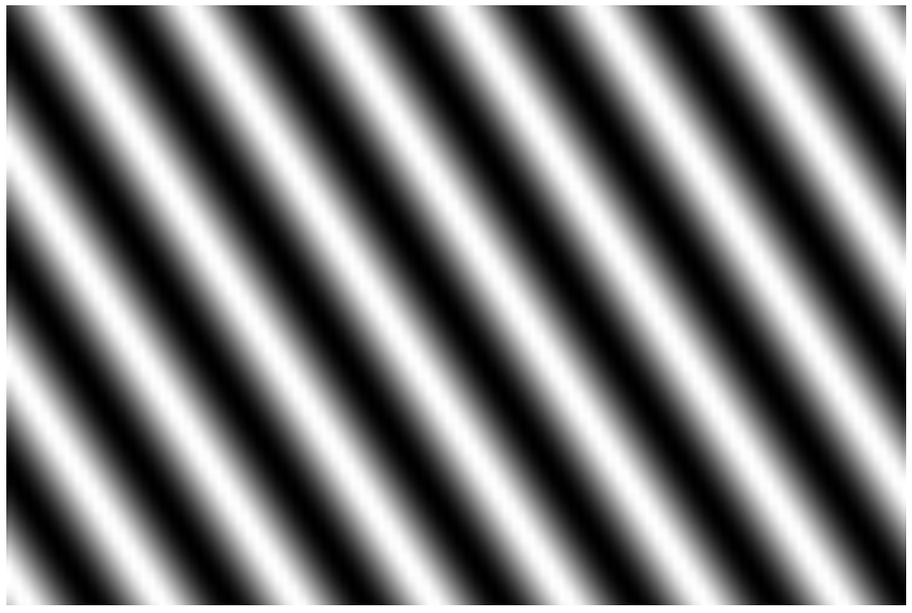
Fourier properties

Signal aliasing

$$-$$



$$+ \quad \alpha_3$$



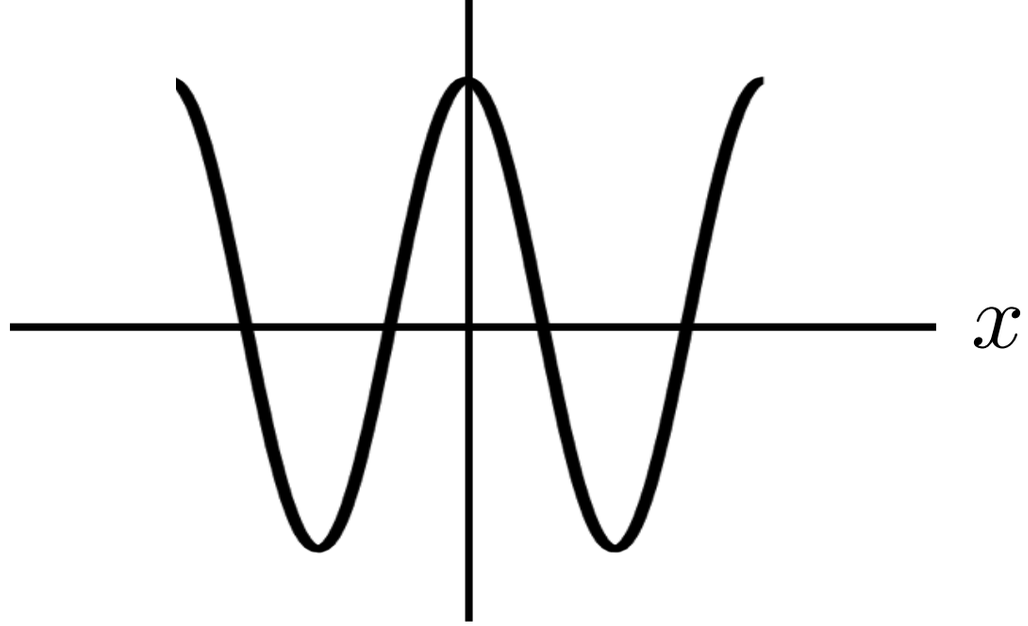
$$+ \quad \cdots$$

$$f[x,y] = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F[u,v] e^{2\pi i (x\frac{u}{M} + y\frac{v}{N})}$$

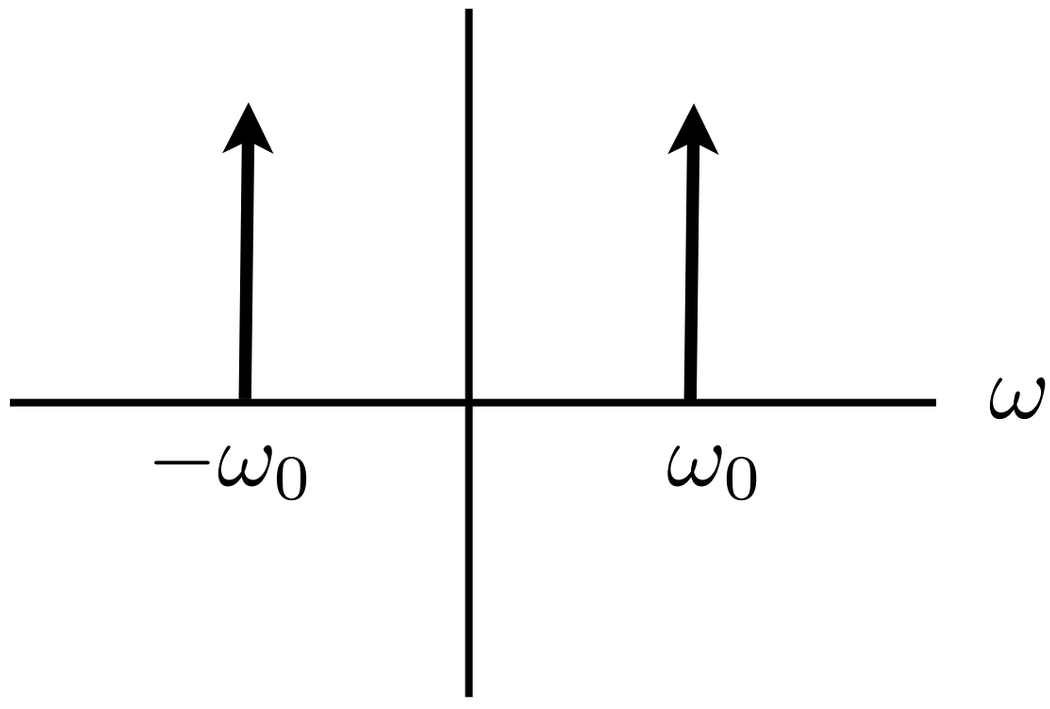
where

$$u = 0, \ldots, M-1$$
$$v = 0, \ldots, N-1$$

$$\cos(2\pi\omega_0 x)$$

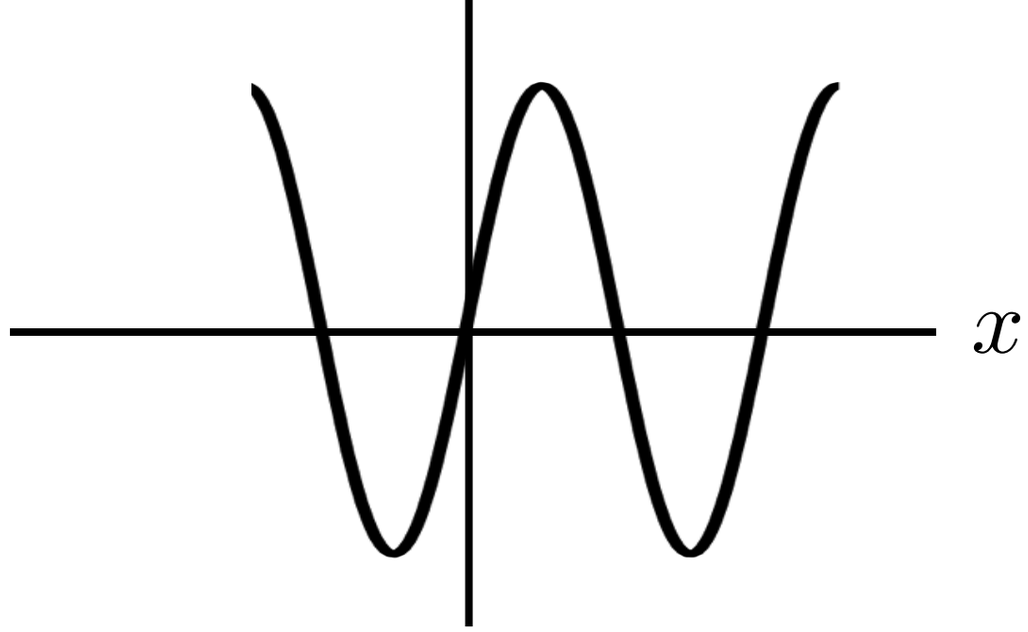


$$\frac{1}{2}\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right)$$

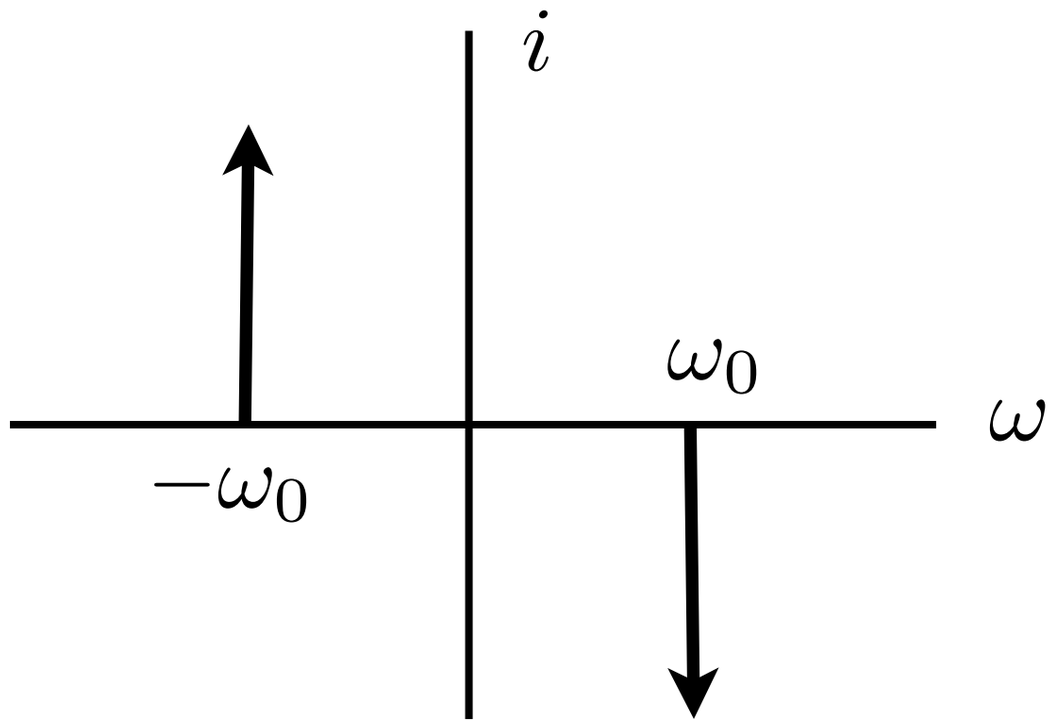


$$\cos(2\pi\omega_0 x) = \frac{1}{2}\left(e^{i2\pi\omega_0 x} + e^{-i2\pi\omega_0 x}\right)$$

$$\sin(2\pi\omega_0 x)$$

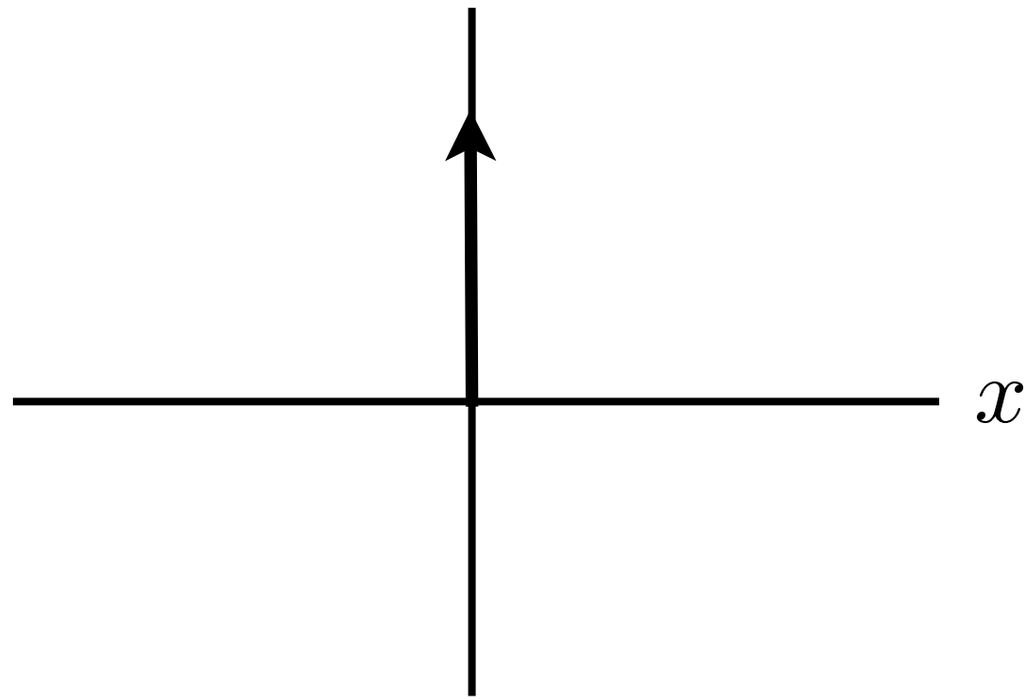


$$\frac{1}{2} i \left(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right)$$

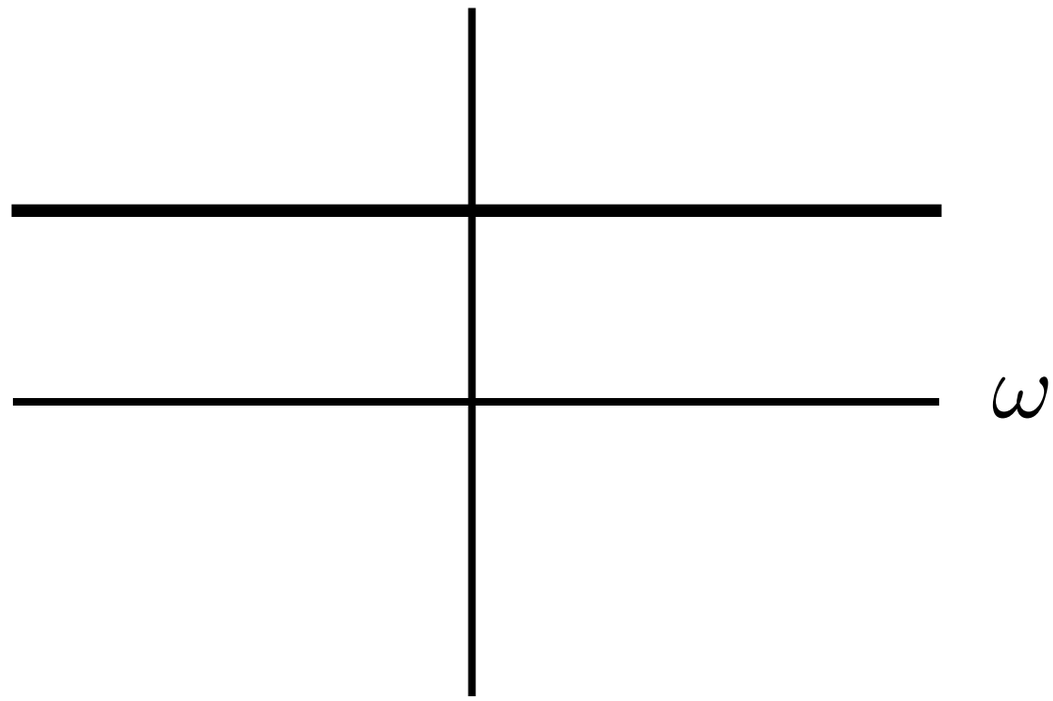


$$\sin(2\pi\omega_0 x) = \frac{1}{2i}\left(e^{i2\pi\omega_0 x} - e^{-i2\pi\omega_0 x}\right)$$

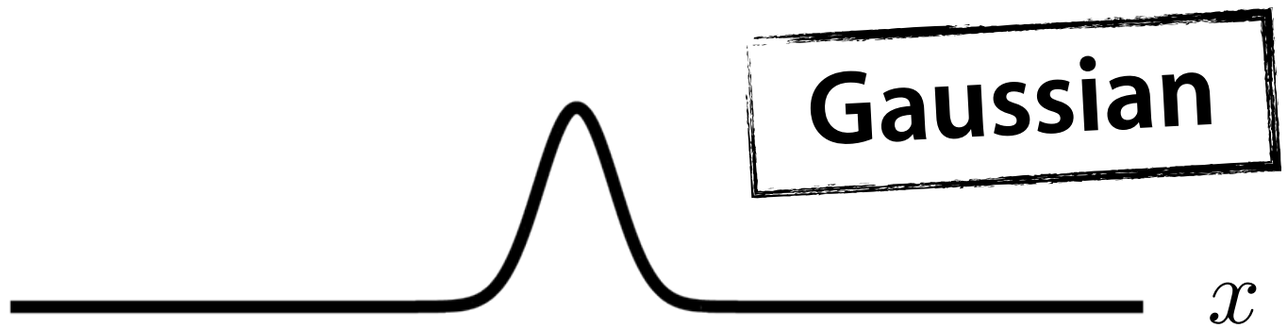
$$f(x) = \delta(x)$$



$$F(\omega) = 1$$

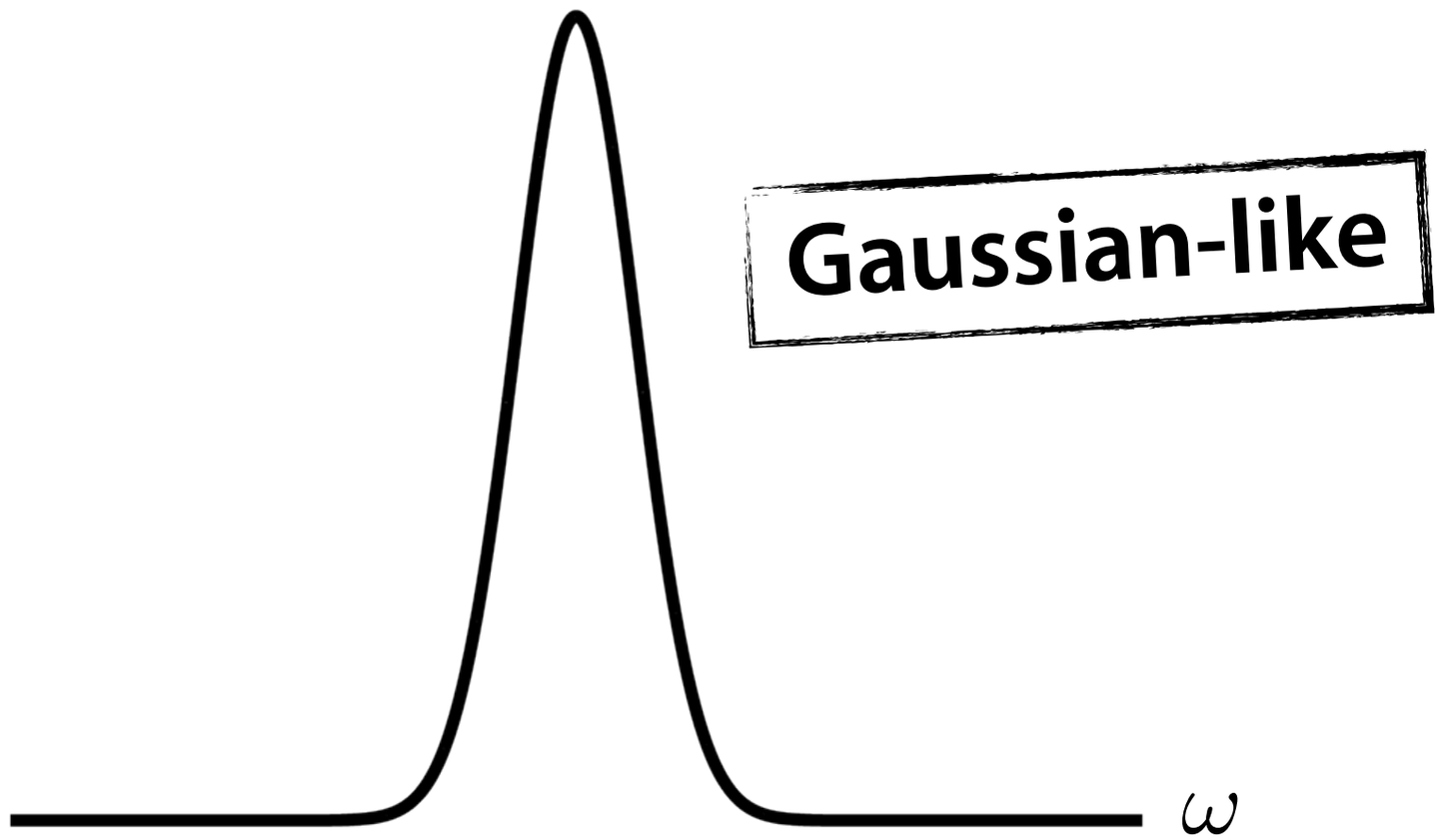


$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

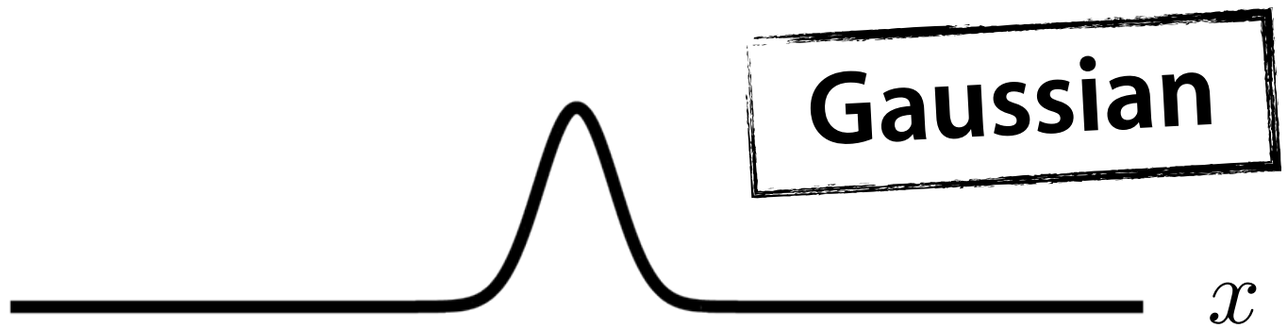


$$G(\omega) = e^{-\frac{(2\pi\omega)^2\sigma^2}{2}}$$

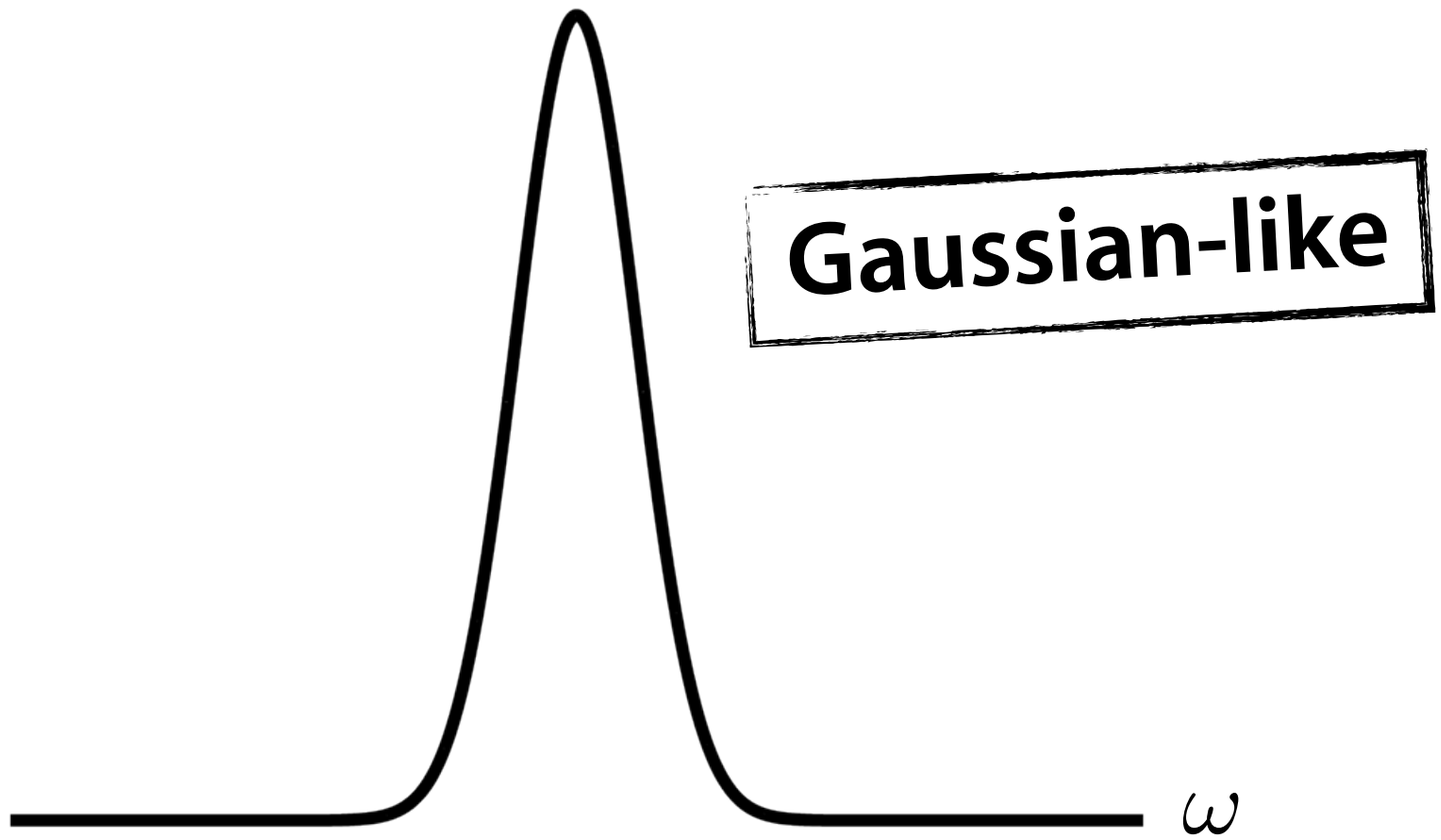
not normalized



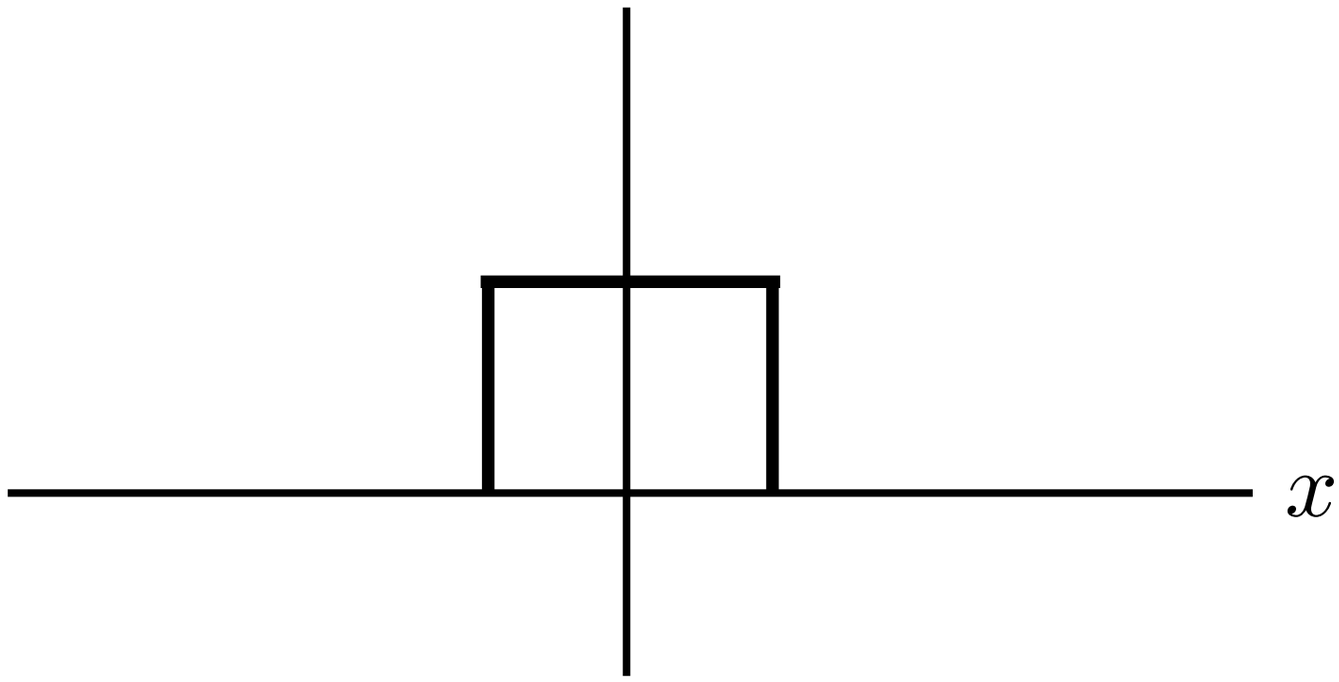
$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$



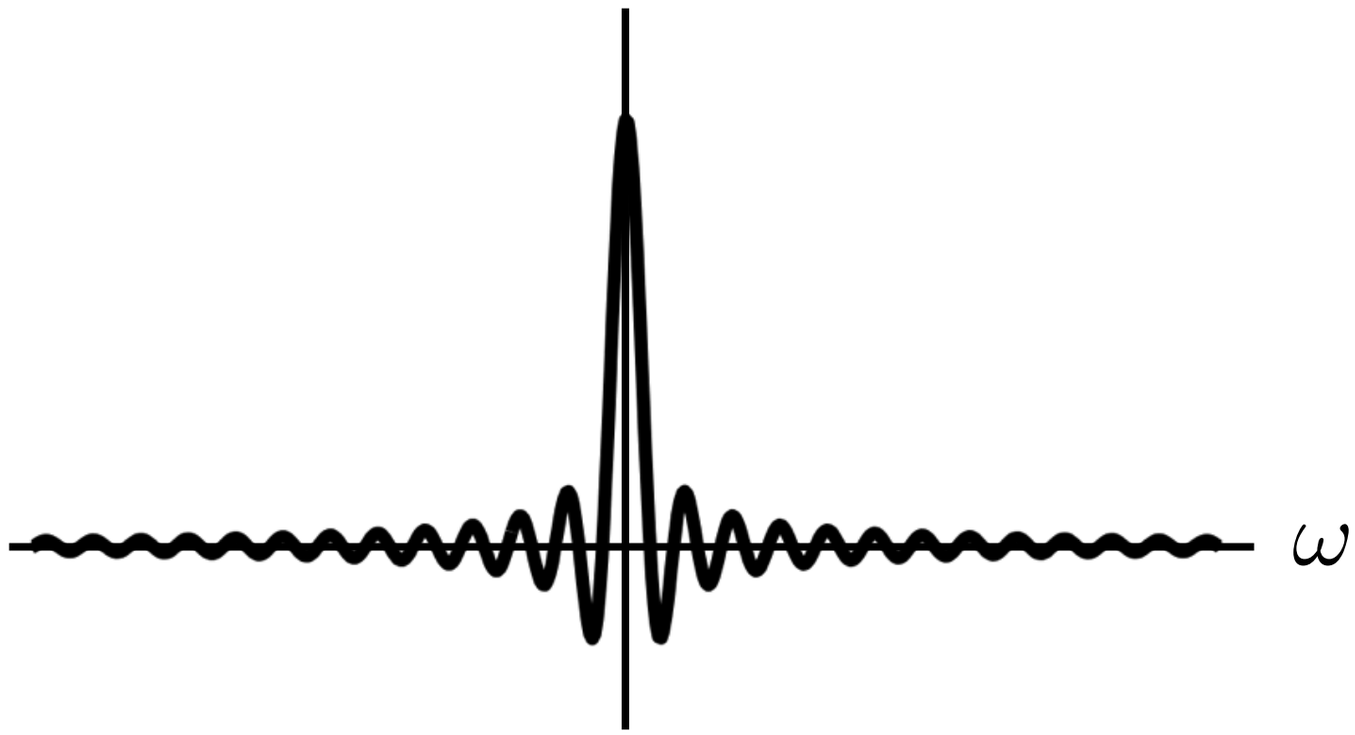
$$G(\omega) = e^{-\frac{(2\pi\omega)^2\sigma^2}{2}}$$



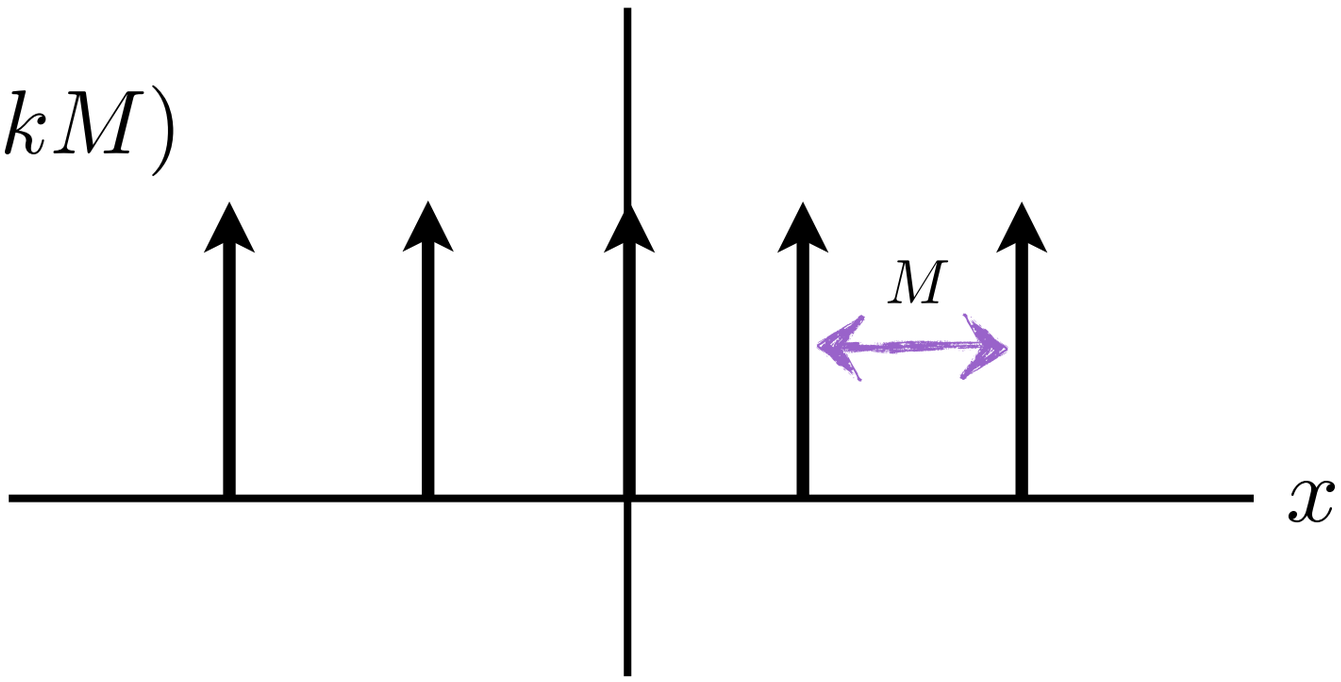
$\mathrm{box}(x)$



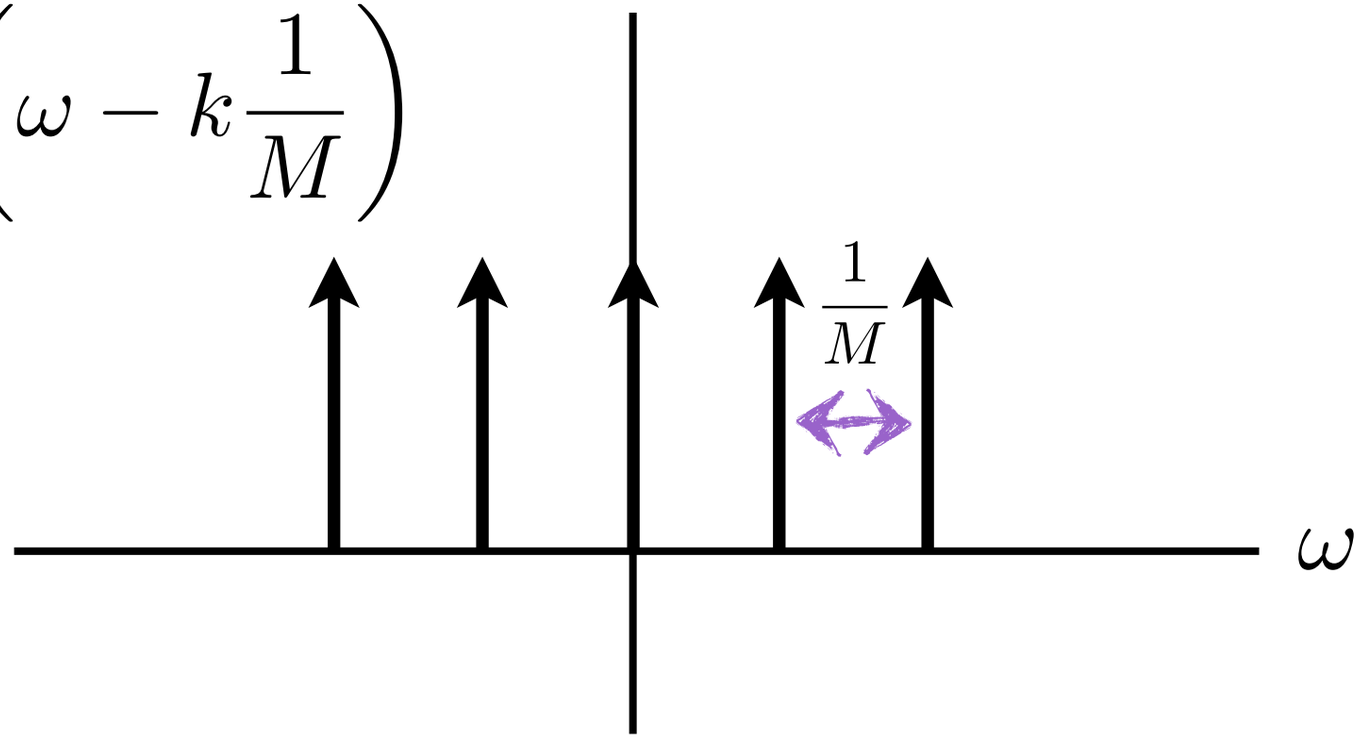
$\mathrm{sinc}(\omega)$



$$\mathrm{comb}(x) = \sum_{k=-\infty}^{\infty} \delta(x - kM)$$



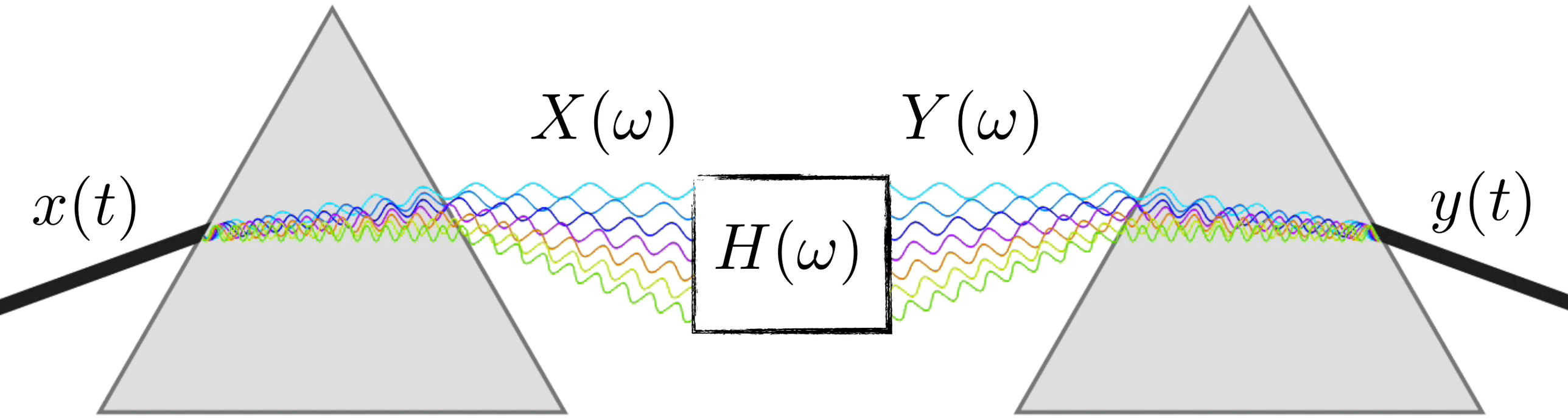
$$\mathrm{comb}(\omega) = \frac{1}{M} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{1}{M}\right)$$



Fourier transform properties

	Spatial domain	Frequency domain
Linearity	$c_1 f(x) + c_2 g(x)$	$c_1 F(\omega) + c_2 G(\omega)$
Scaling	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{\omega}{a}\right)$
Shifting	$f(x - x_0)$	$e^{-i2\pi\omega x_0} F(\omega)$
Differentiation	$\frac{d^n f(x)}{dx^n}$	$(i2\pi\omega)^n F(\omega)$
Convolution	$f(x) * g(x)$	$F(\omega)G(\omega)$

Convolution Theorem



$$Y(\omega) = H(\omega)X(\omega)$$

$O(N^2)$ **vs.** $O(N \log N)$

convolution **FFT-based convolution**

$$\mathcal{F}\{f(x) * g(x)\}(\omega) = F(\omega)G(\omega)$$

Proof: Convolution Theorem

$$\mathcal{F}\{f * g\}(\omega)$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u)g(t-u)du \right] e^{-i2\pi\omega t} dt$$

convolution

$$\mathcal{F}\{f(x) * g(x)\}(\omega) = F(\omega)G(\omega)$$

Proof:
Convolution
Theorem

$$\mathcal{F}\{f * g\}(\omega)$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u)g(t-u)du \right] e^{-i2\pi\omega t} dt$$

Fourier transform

$$\mathcal{F}\{f(x) * g(x)\}(\omega) = F(\omega)G(\omega)$$

Proof: Convolution Theorem

$$\mathcal{F}\{f * g\}(\omega)$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u)g(t-u)du \right] e^{-i2\pi\omega t} dt$$

change order of integration

$$= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g(t-u)e^{-i2\pi\omega t} dt \right] du$$

Does this look familiar?

Fourier Transform Properties

	Spatial domain	Frequency domain
Shifting	$f(x - x_0)$	$e^{-i2\pi\omega x_0} F(\omega)$

Prove: $\mathcal{F}\{f(x) * g(x)\}(\omega) = F(\omega)G(\omega)$

Proof: Convolution Theorem

$$\mathcal{F}\{f * g\}(\omega)$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u)g(t-u)du \right] e^{-i2\pi\omega t} dt$$

change order of integration

$$= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g(t-u)e^{-i2\pi\omega t} dt \right] du$$

Does this look familiar?

$$e^{-i2\pi\omega u}G(\omega)$$

Prove: $\mathcal{F}\{f(x) * g(x)\}(\omega) = F(\omega)G(\omega)$

Proof: Convolution Theorem

$$\mathcal{F}\{f * g\}(\omega)$$

$$= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g(t-u) e^{-i2\pi\omega t} dt \right] du$$

shift theorem

$$e^{-i2\pi\omega u} G(\omega)$$

$$= \int_{-\infty}^{\infty} f(u) e^{-i2\pi\omega u} G(\omega) du$$

factor out

$$= G(\omega) \int_{-\infty}^{\infty} f(u) e^{-i2\pi\omega u} du$$

Prove: $\mathcal{F}\{f(x) * g(x)\}(\omega) = F(\omega)G(\omega)$

Proof: Convolution Theorem

$$\mathcal{F}\{f * g\}(\omega)$$

$$= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g(t-u) e^{-i2\pi\omega t} dt \right] du$$

$$= G(\omega) \int_{-\infty}^{\infty} f(u) e^{-i2\pi\omega u} du$$

Does this look familiar?

Fourier transform

Prove: $\mathcal{F}\{f(x) * g(x)\}(\omega) = F(\omega)G(\omega)$

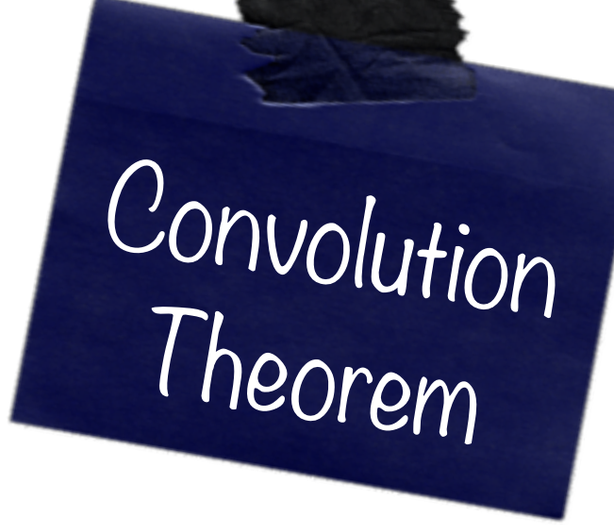
Proof: Convolution Theorem

$$\mathcal{F}\{f * g\}(\omega)$$

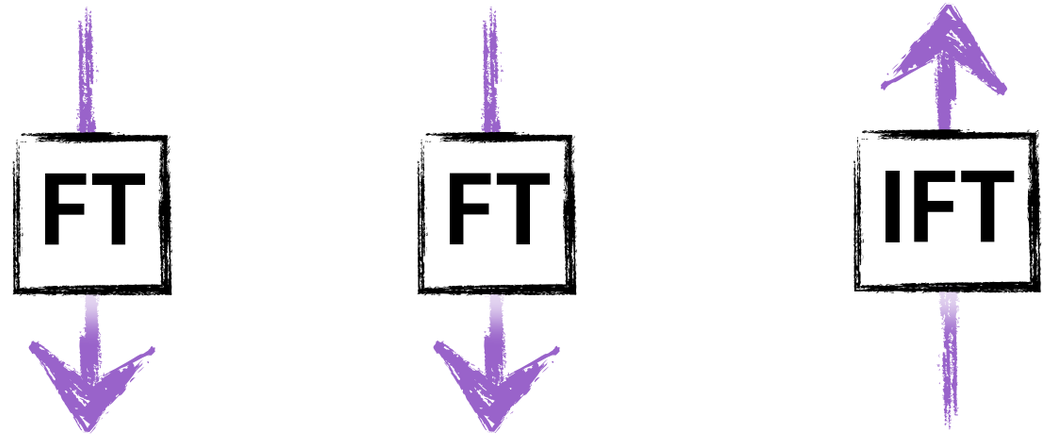
$$= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g(t-u) e^{-i2\pi\omega t} dt \right] du$$

$$= G(\omega) \int_{-\infty}^{\infty} f(u) e^{-i2\pi\omega u} du$$

$$= F(\omega)G(\omega)$$



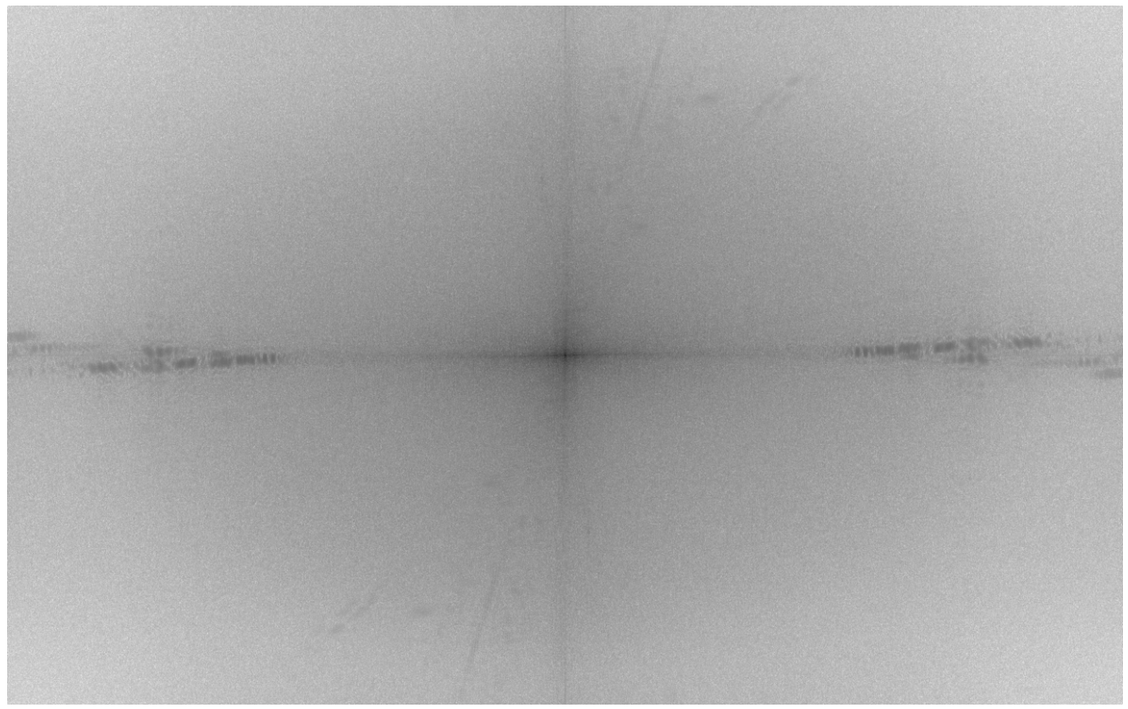
Spatial domain $f \quad * \quad g \quad = \quad h$

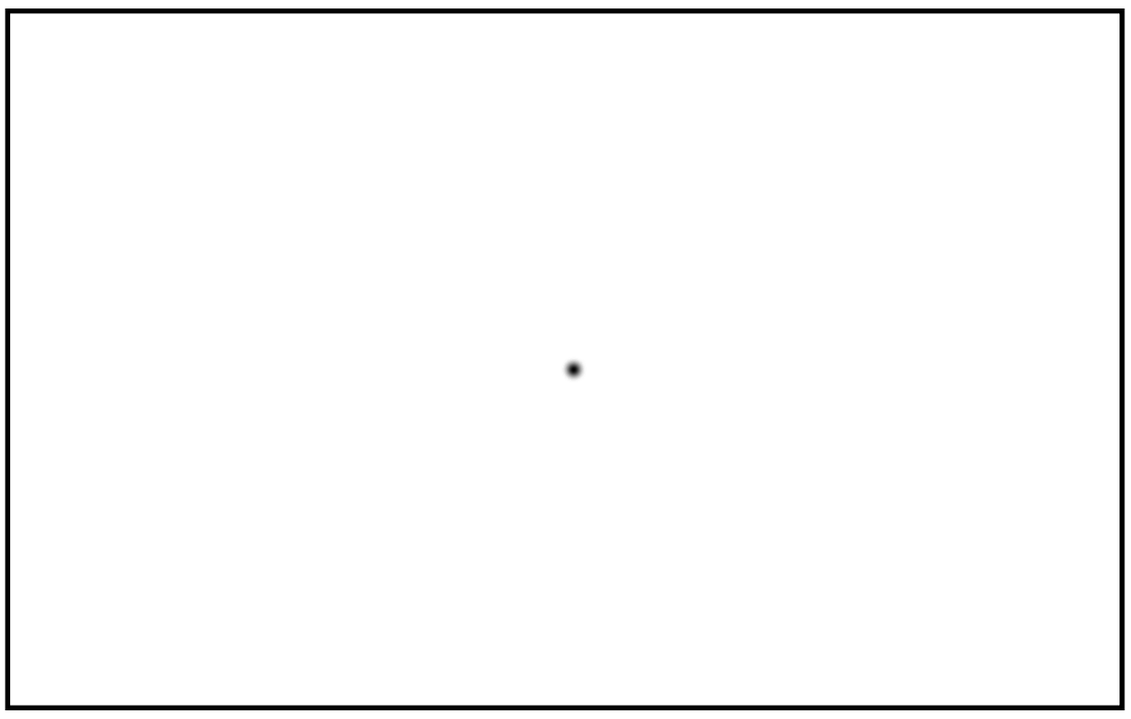


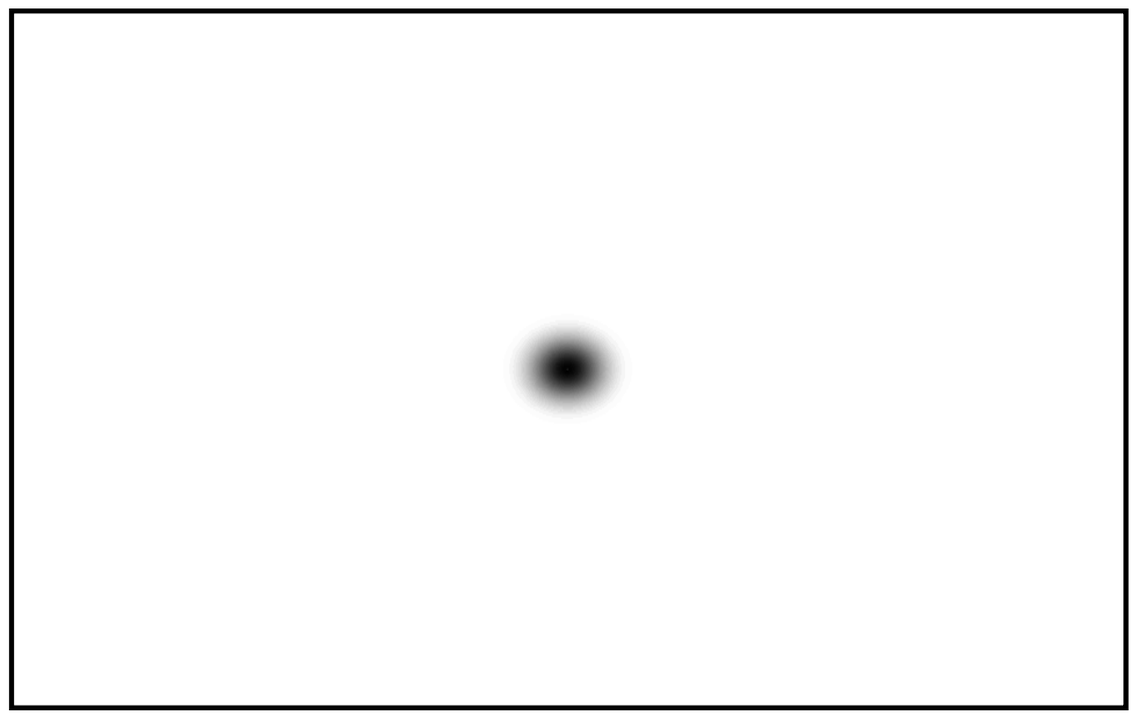
Frequency domain $F \quad \times \quad G \quad = \quad H$

Spatial domain

Frequency domain

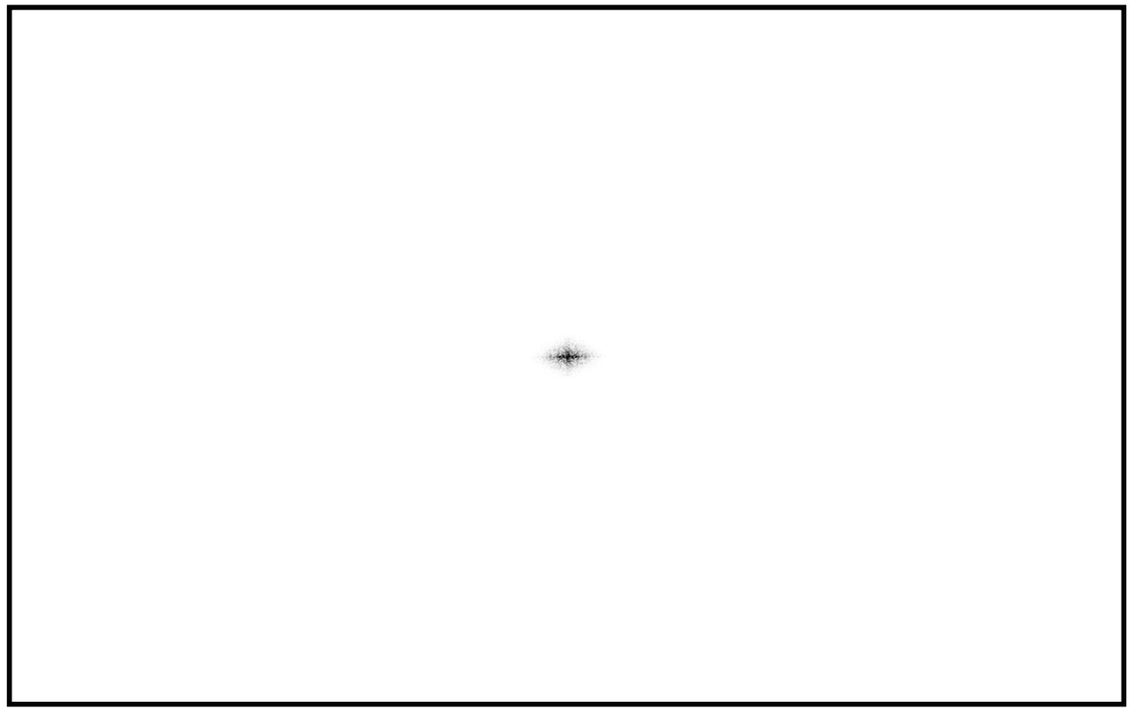






Spatial domain

Frequency domain



Frequency Domain Filtering

```
im = imread('toronto.jpg');
im = double(rgb2gray(im))/255;
[imh, imw] = size(im);

sigma = 10;
f = fspecial('gaussian', 2*sigma*3+1, sigma);
f = padarray(f, [imh imw]-(2*sigma*3+1), 'post');
f = circshift(f, -3*[sigma sigma]);

im_dft = fft2(im, size(im,1), size(im,2));
f_dft  = fft2(f,  size(im,1), size(im,2));

im_f_dft = im_dft .* f_dft;
im_f     = ifft2(im_f_dft);

imshow(im_f, [])
```

```
im = imread('toronto.jpg');
im = double(rgb2gray(im))/255;
[imh, imw] = size(im);

sigma = 10;
f = fspecial('gaussian', 2*sigma*3+1, sigma);
f = padarray(f, [imh imw]-(2*sigma*3+1), 'post');
f = circshift(f, -3*[sigma sigma]);
f_dft = fft2(f,  size(im,1), size(im,2));

im_f_dft = im_dft .* f_dft;
im_f     = ifft2(im_f_dft);

imshow(im_f, [])
```

DFT assumption: image signal is periodic

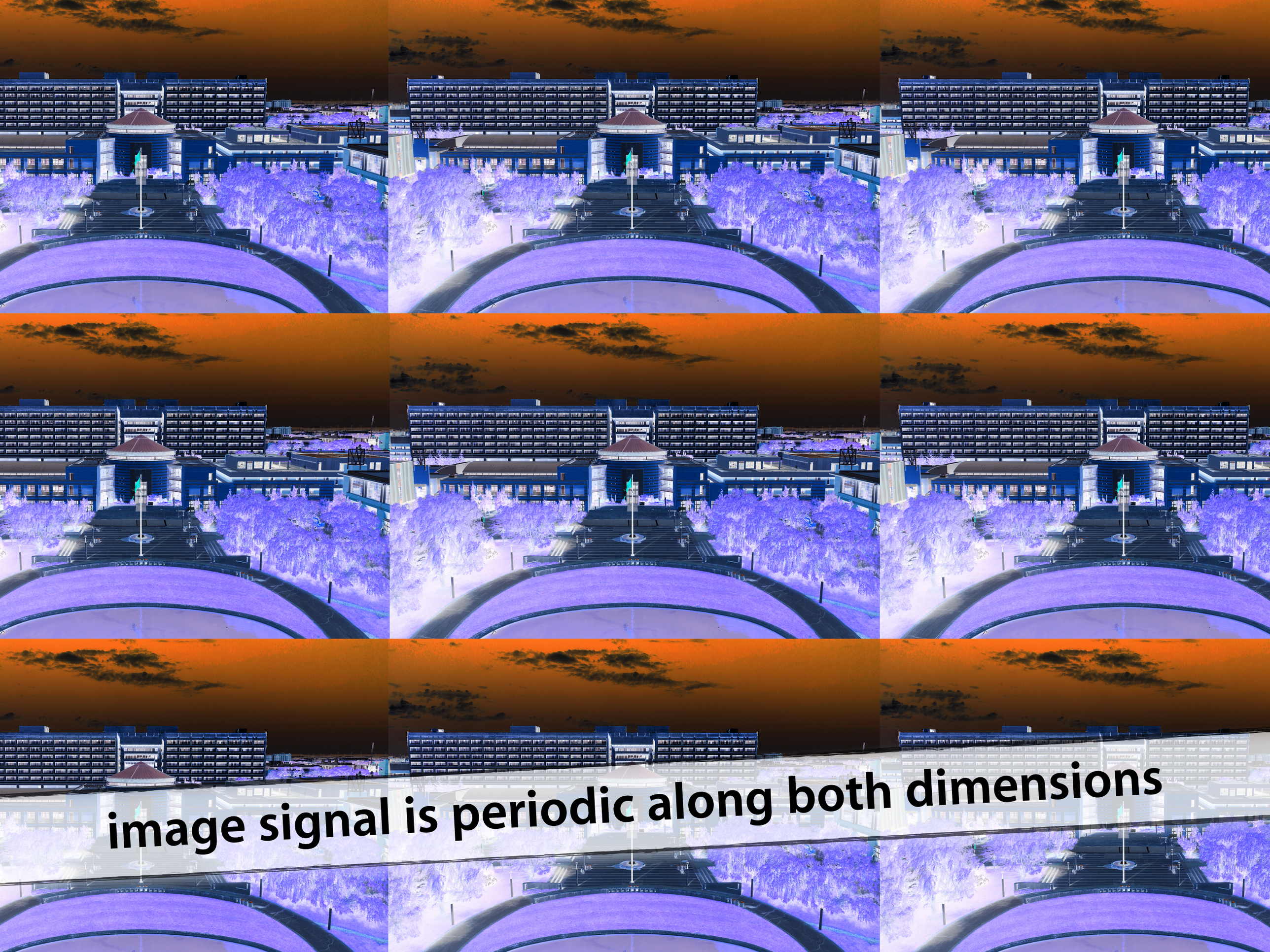


image signal is periodic along both dimensions

```
im = imread('toronto.jpg');
im = double(rgb2gray(im))/255;
[imh, imw] = size(im);

sigma = 10;
f = fspecial('gaussian', 2*sigma*3+1, sigma);
f = padarray(f, [imh imw]-(2*sigma*3+1), 'post');
f = circshift(f, -3*[sigma sigma]);

im_dft = fft2(im, size(im,1), size(im,2));
f_dft  = fft2(f,  size(im,1), size(im,2));

im_f_dft = im_dft .* f_dft;
im_f     = iff
```

pointwise multiplication

```
imshow(im_f, [])
```

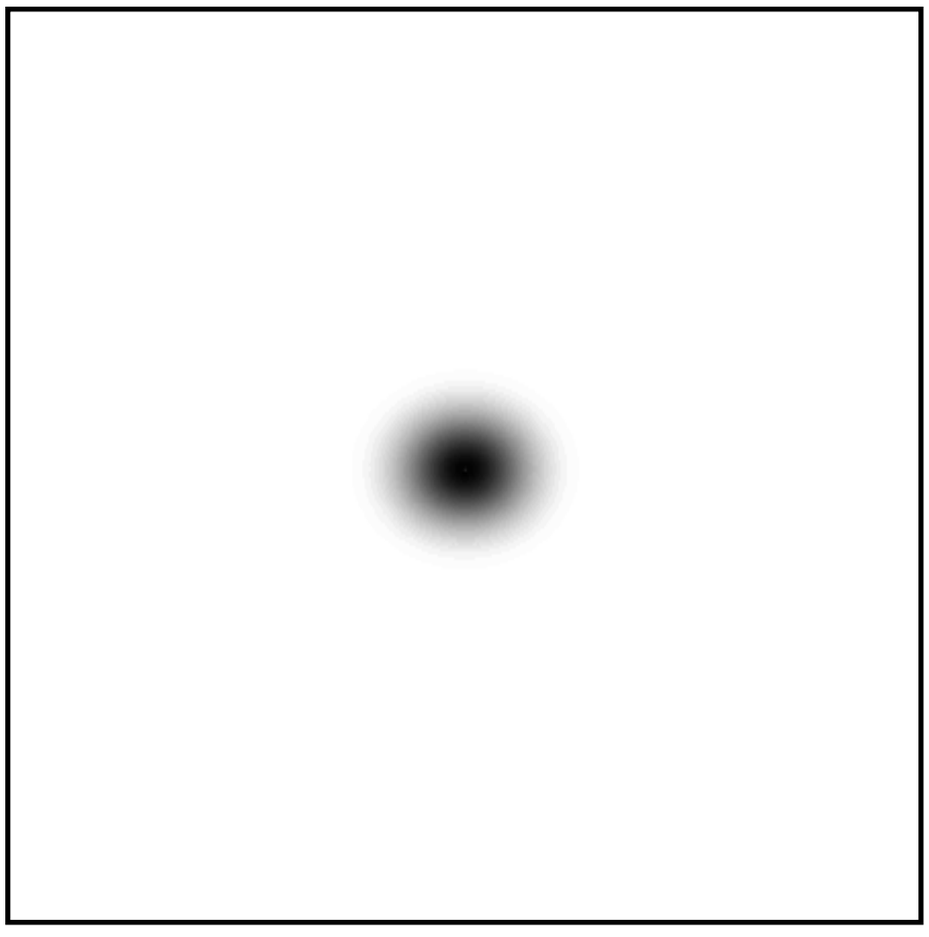
halftone image

DFT magnitude

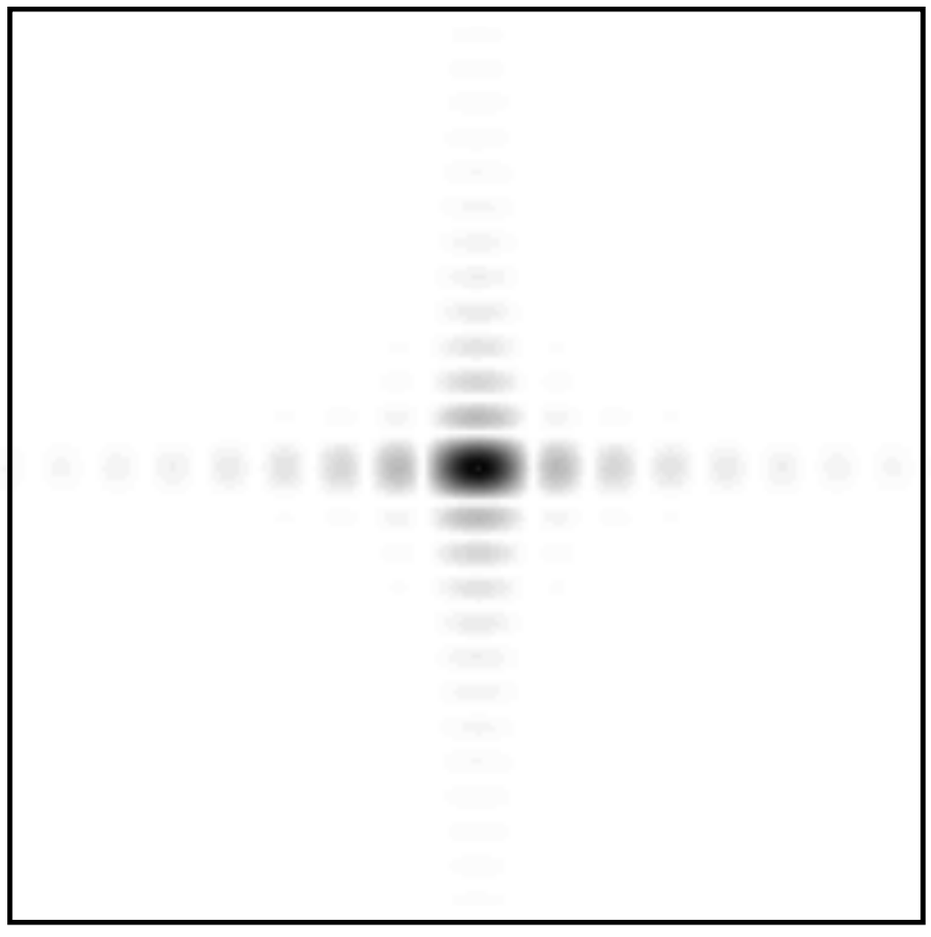
notch filtered spectrum

notch filtered image

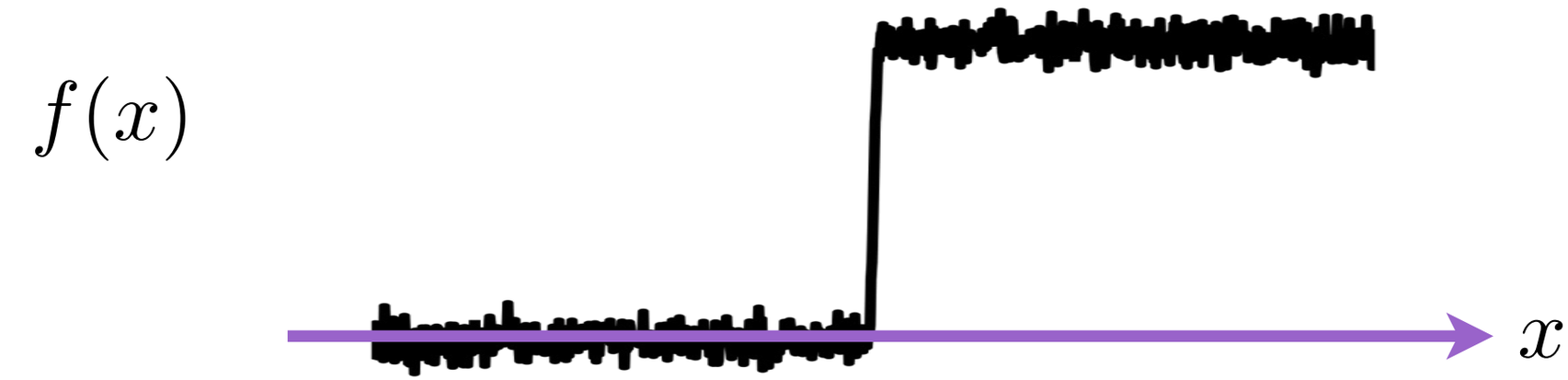
Why does the Gaussian filtered image appear smoother than the box filtered one?

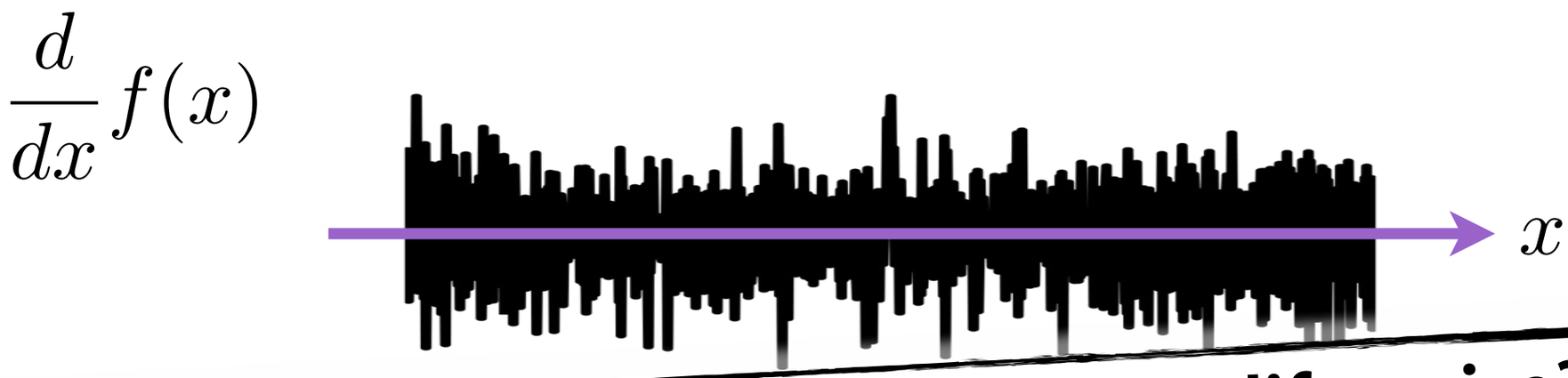


Gaussian Fourier magnitude



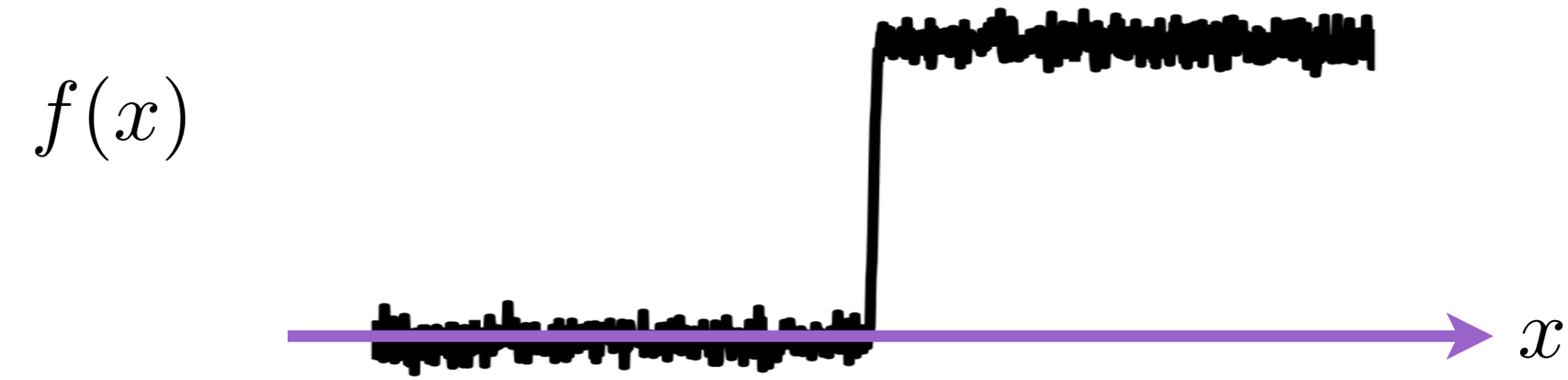
Box Fourier magnitude

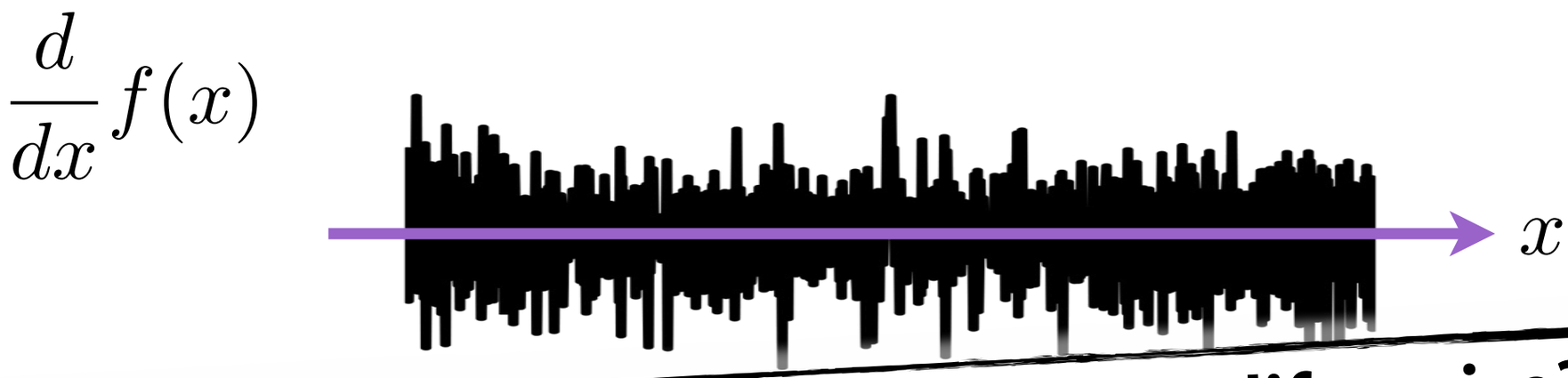




Why does differentiation amplify noise?

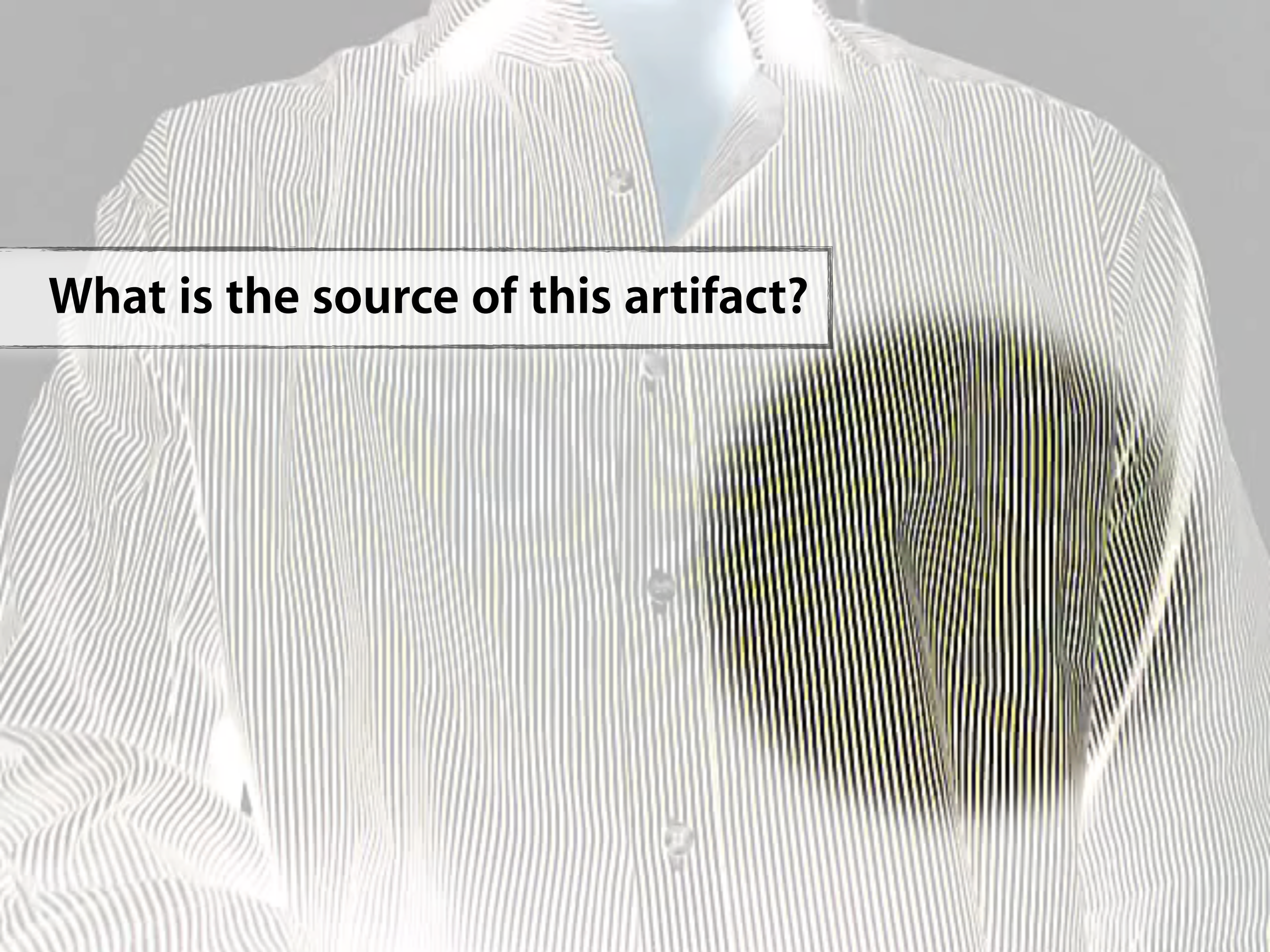
$$\mathcal{F}\left\{\frac{d^n f(x)}{dx^n}\right\} = (i2\pi\omega)^n F(\omega)$$



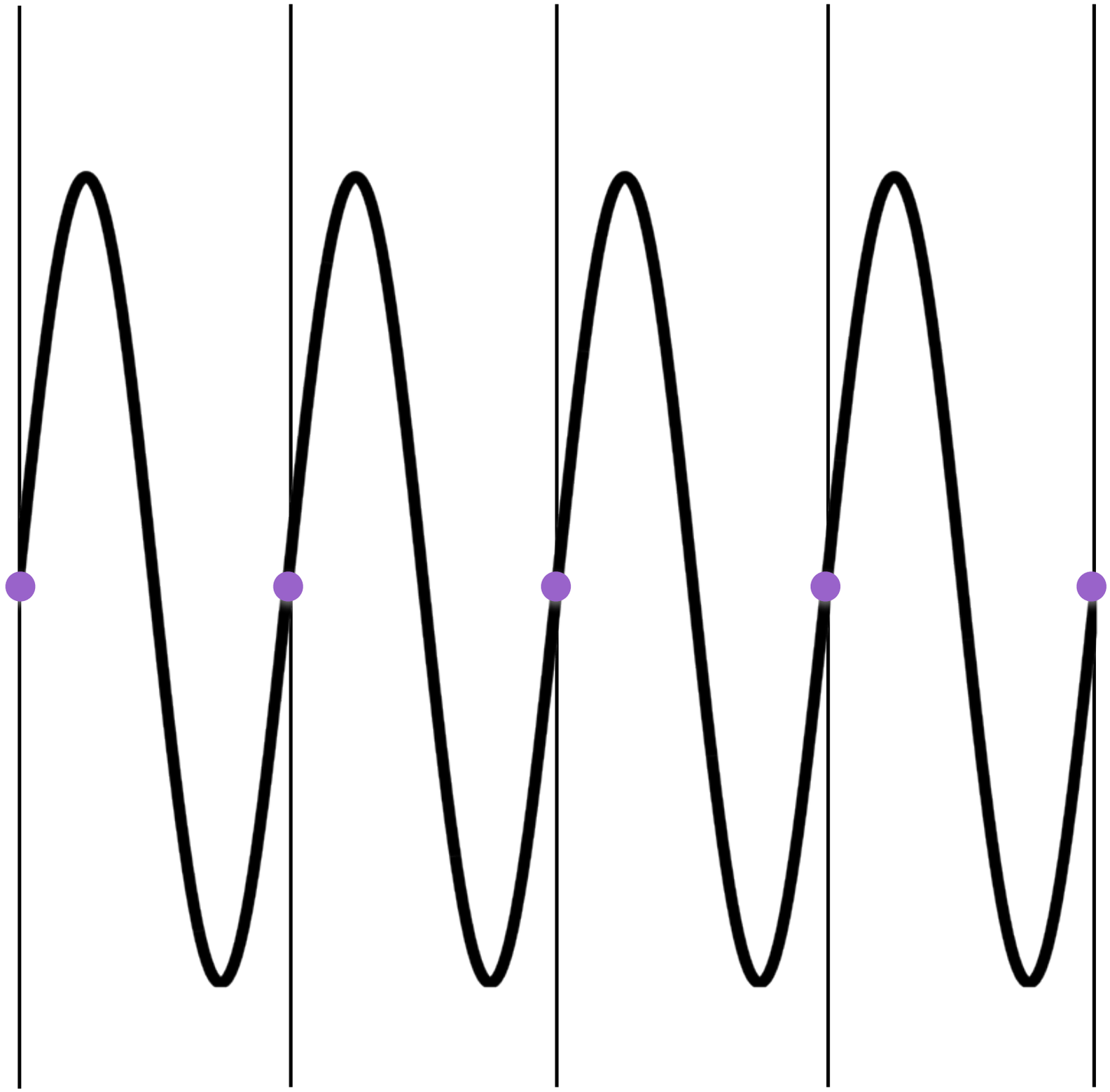


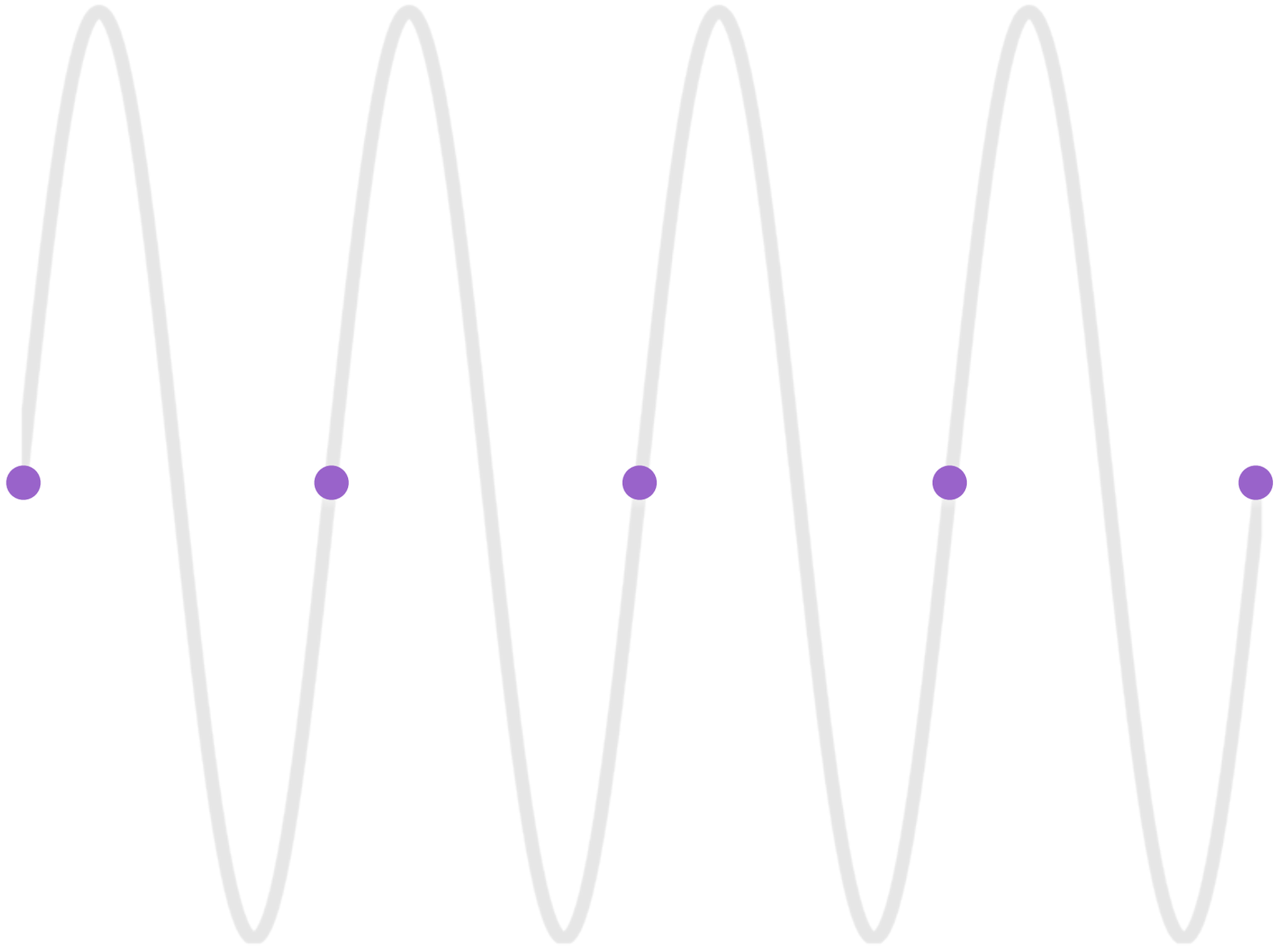
Why does differentiation amplify noise?

$$\mathcal{F}\left\{\frac{d^n f(x)}{dx^n}\right\} = (i2\pi\omega)^n F(\omega)$$

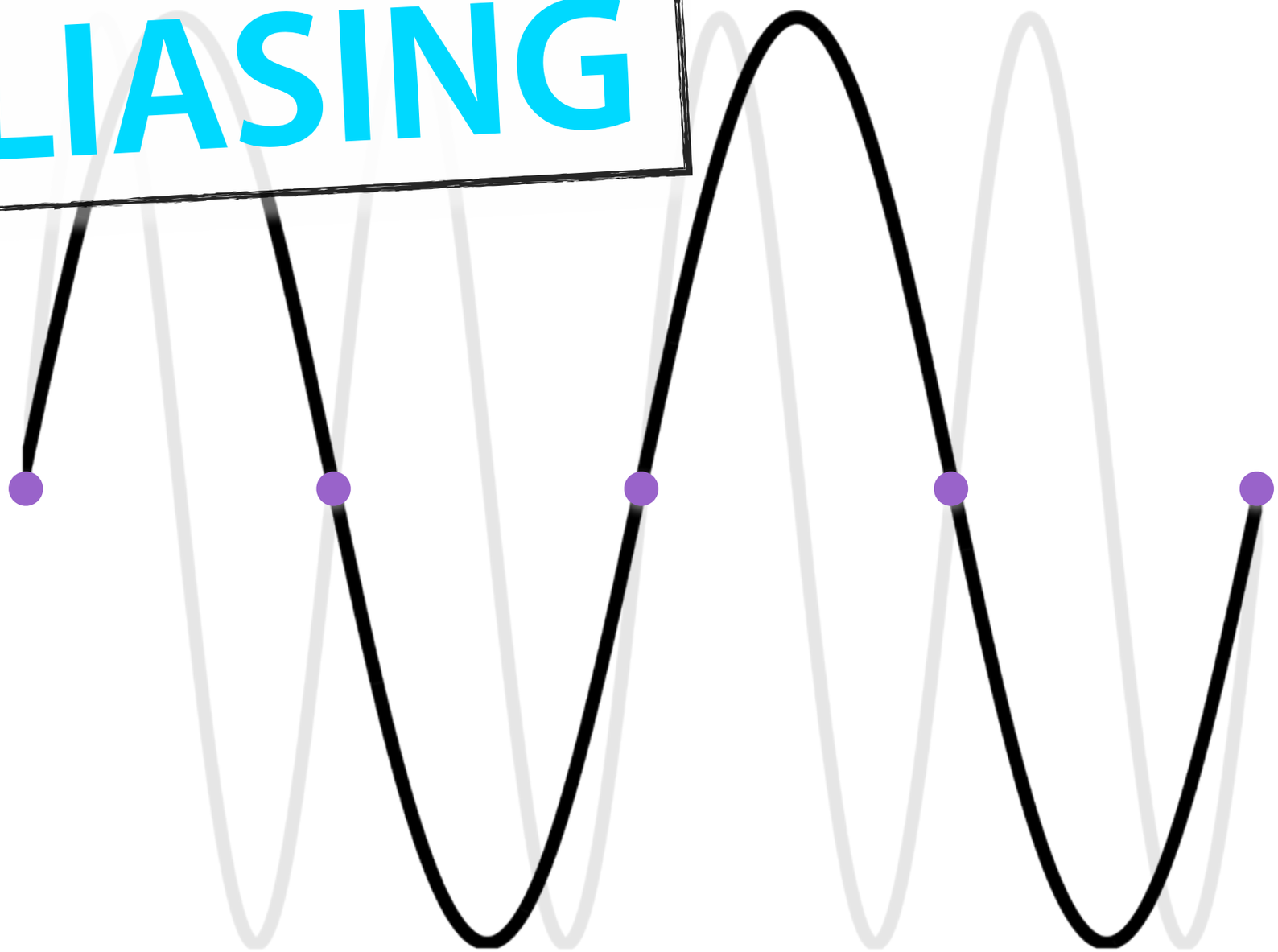


What is the source of this artifact?



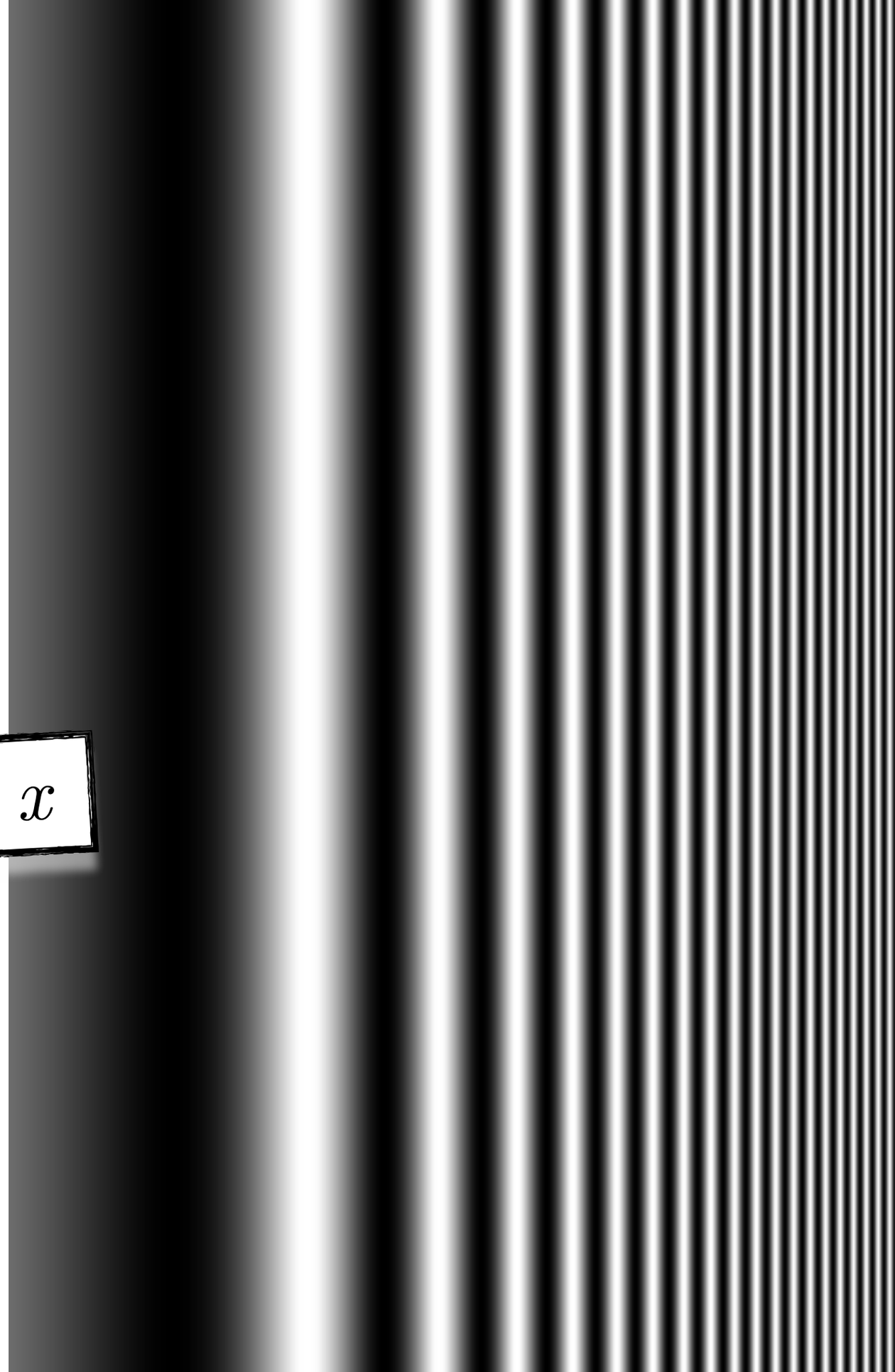


ALIASING



$$f(x, y) = \sin(2^x x)$$

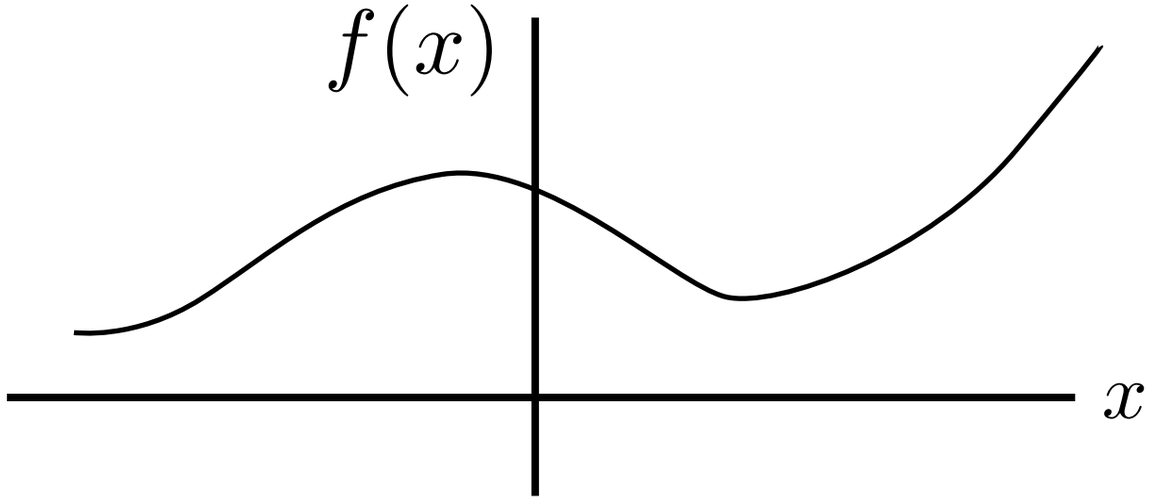
frequency increase with x



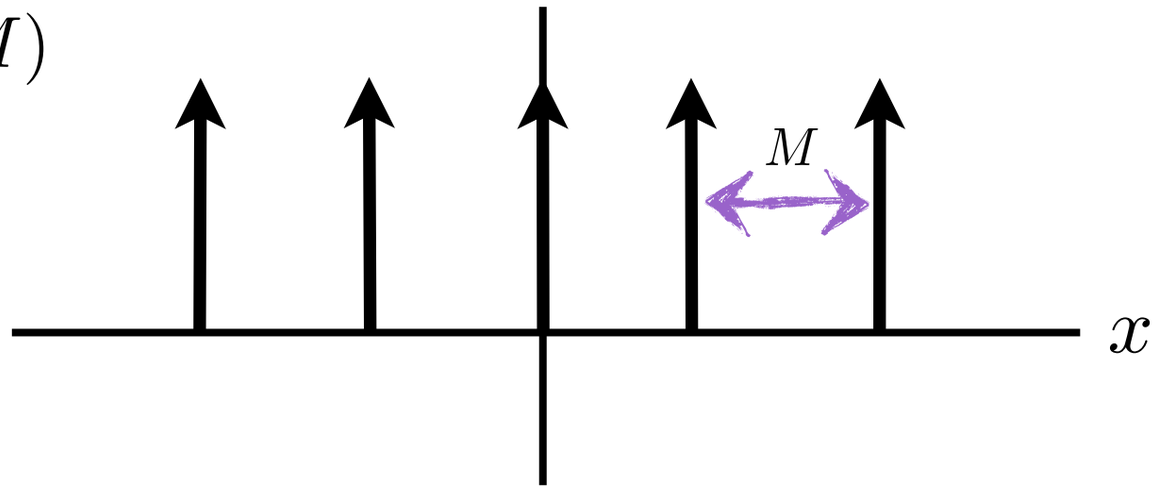

ALIASING
not enough samples

Aliasing as seen in the frequency domain

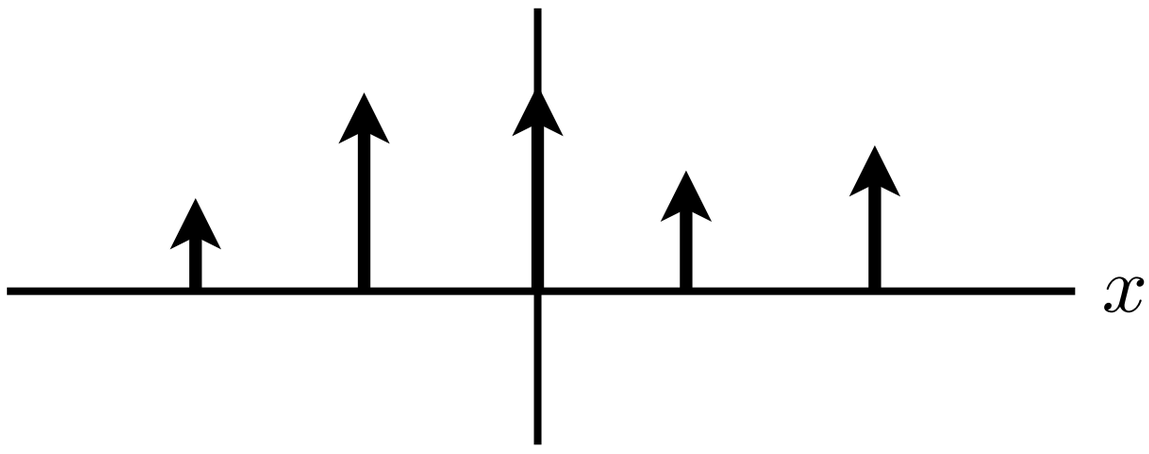
spatial domain



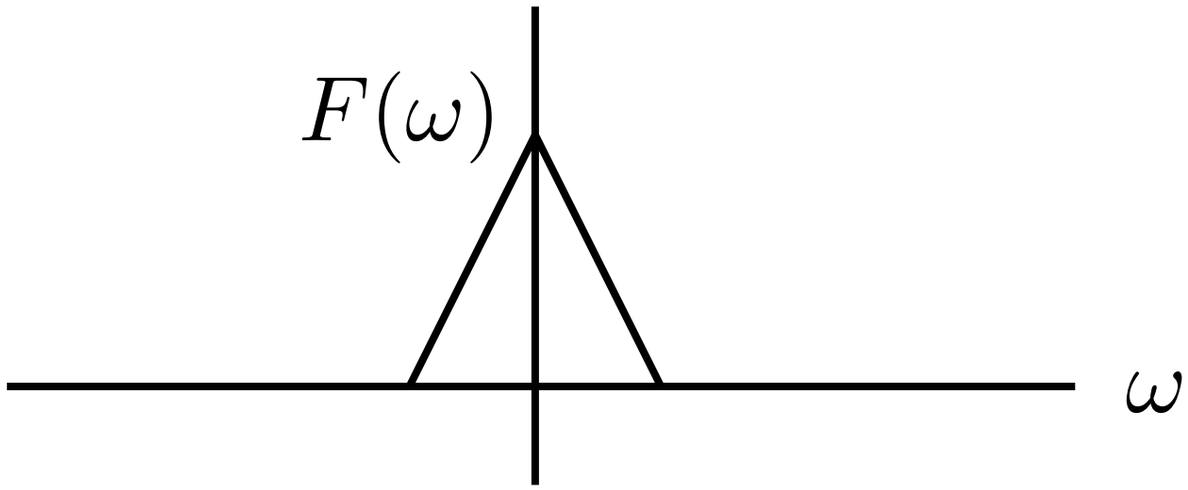
$$\mathrm{comb}(x) = \sum_{k=-\infty}^{\infty} \delta(x - kM)$$



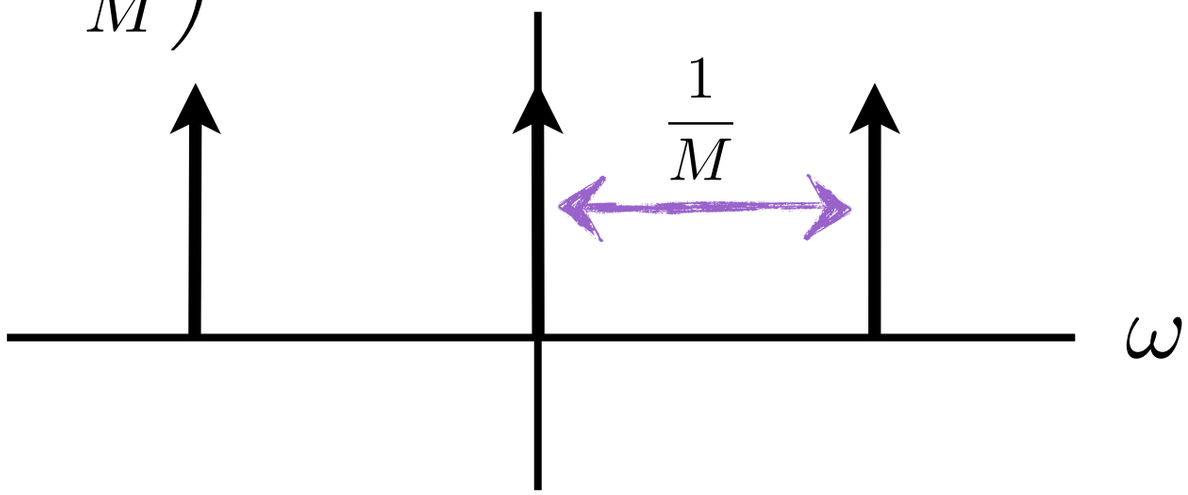
multiply



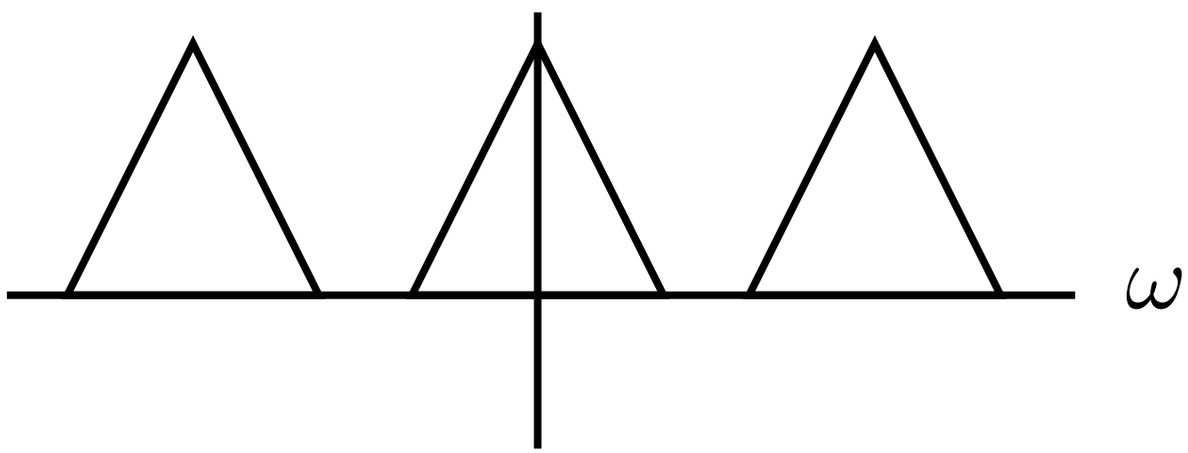
frequency domain

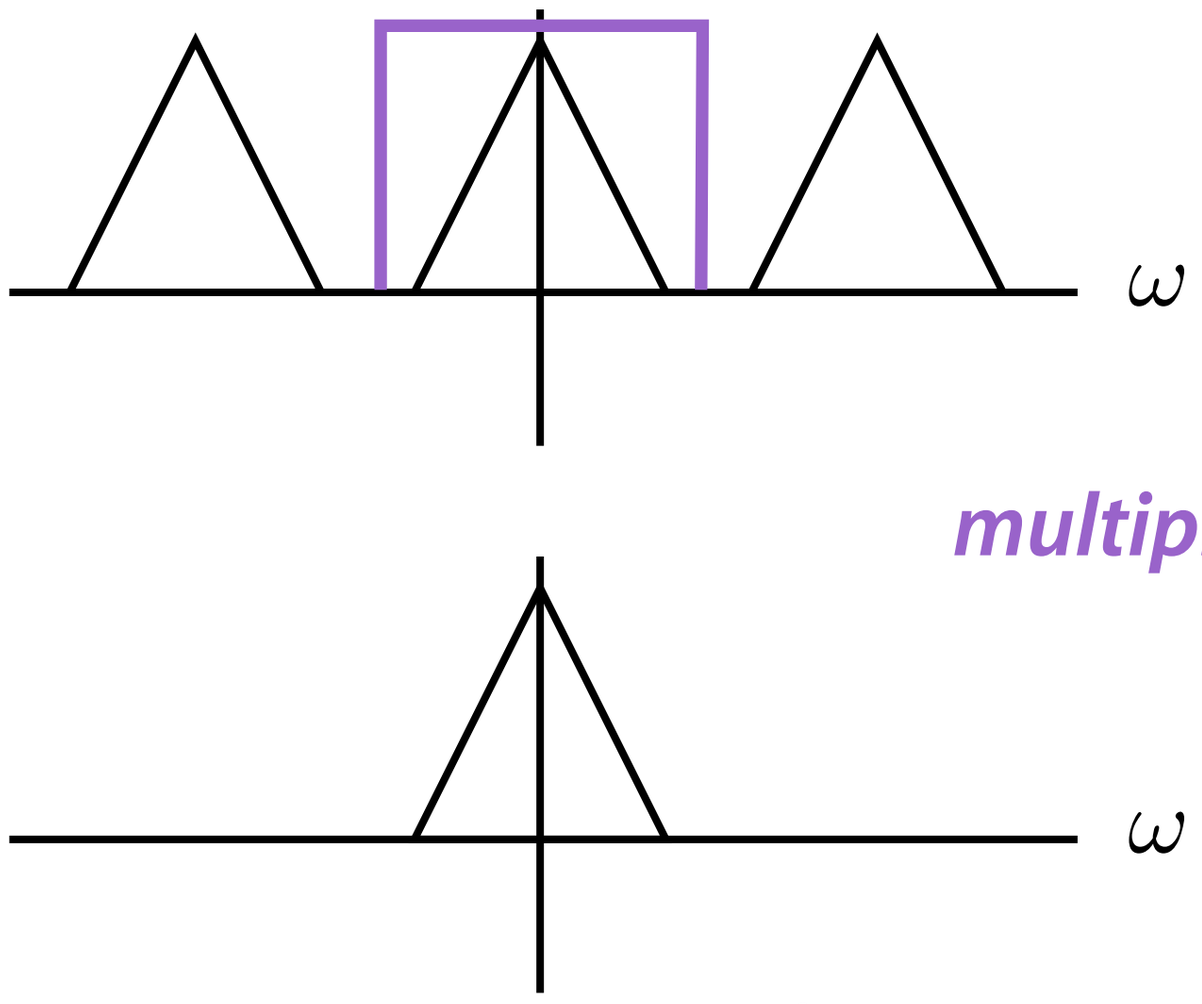


$$\mathrm{comb}(\omega) = \frac{1}{M} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{1}{M}\right)$$



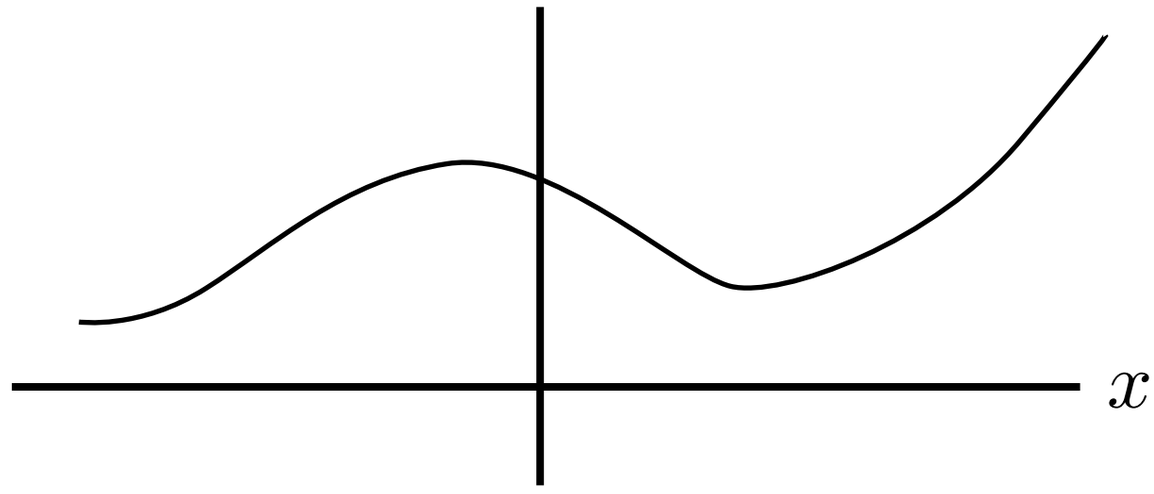
convolve





multiply by box filter

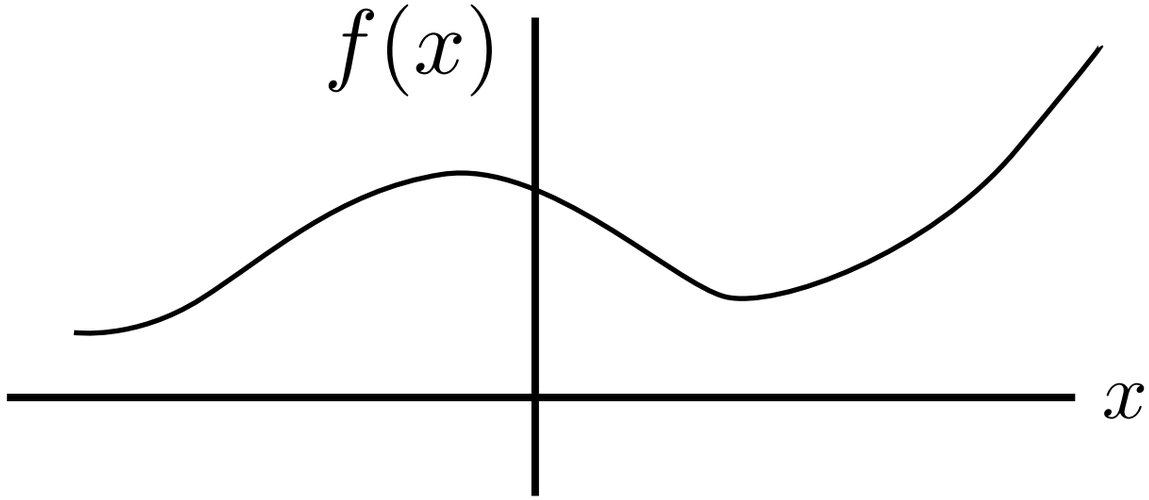
inverse Fourier transform



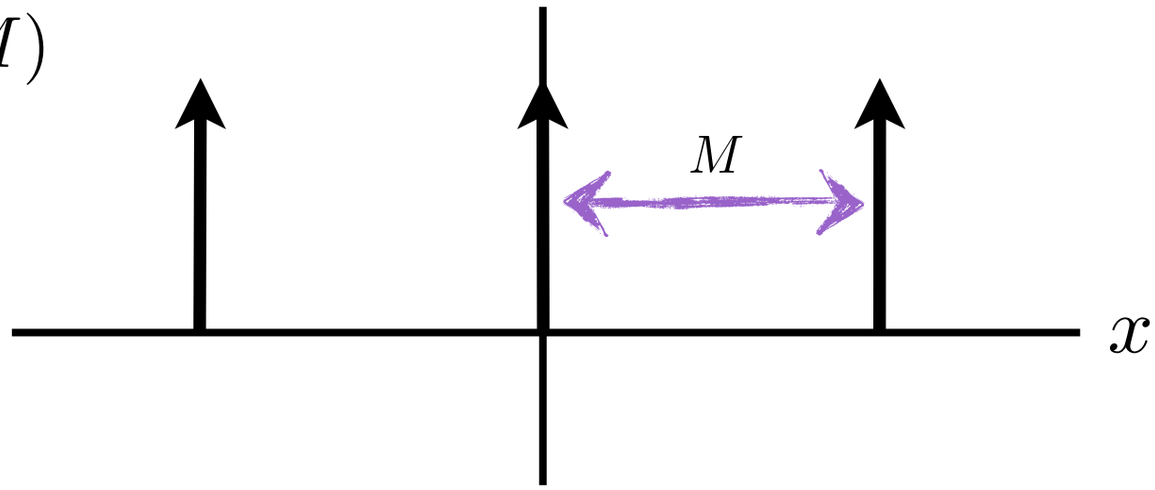
$$f(x) = \left(\sum_{k=-\infty}^{\infty} f(kM)\delta(x - kM) \right) * \operatorname{sinc}\left(\frac{\pi x}{M}\right)$$

$$= \sum_{k=-\infty}^{\infty} f(kM)\operatorname{sinc}\left(\frac{\pi}{M}(x - kM)\right)$$

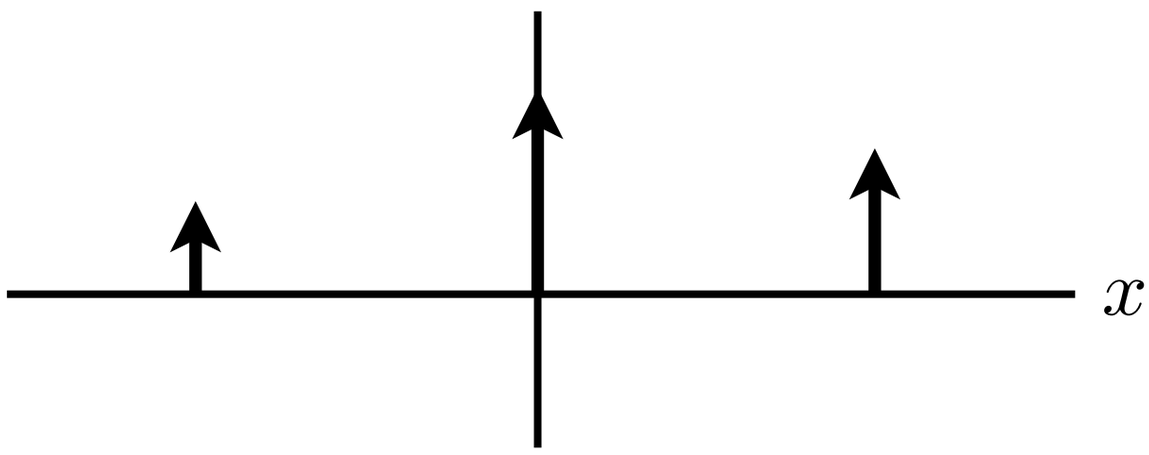
spatial domain



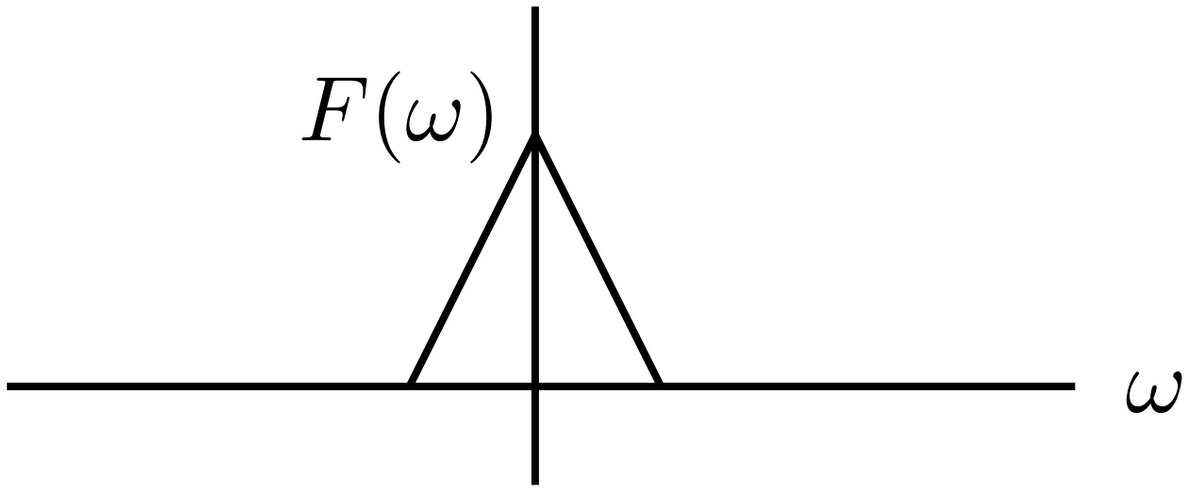
$$\mathrm{comb}(x) = \sum_{k=-\infty}^{\infty} \delta(x - kM)$$



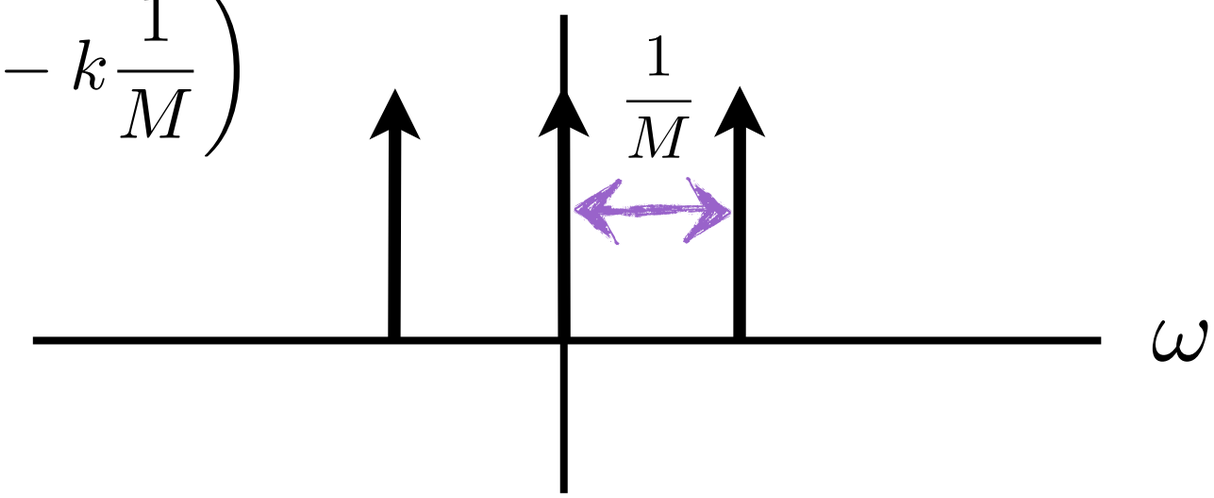
multiply



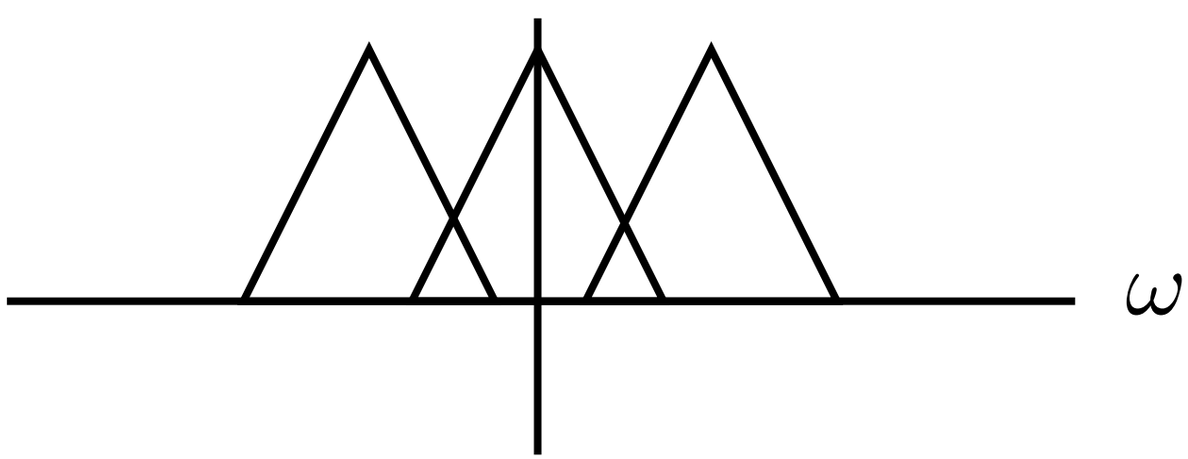
frequency domain

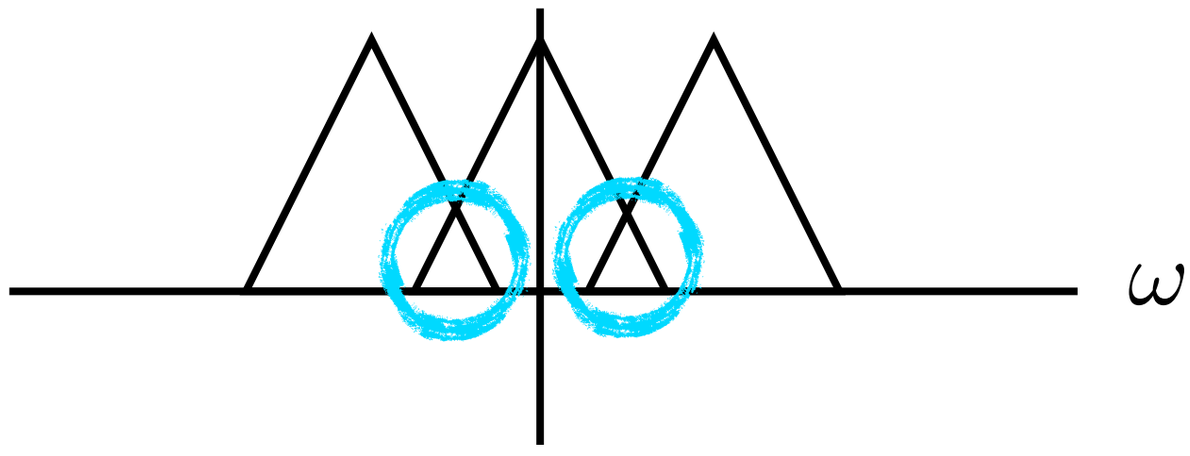


$$\mathrm{comb}(\omega) = \frac{1}{M} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{1}{M}\right)$$



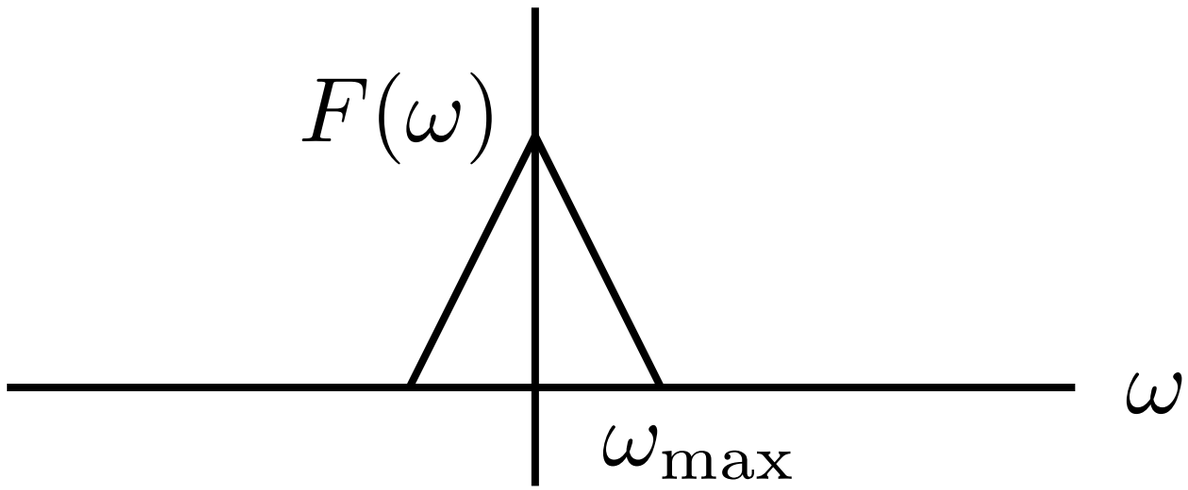
convolve



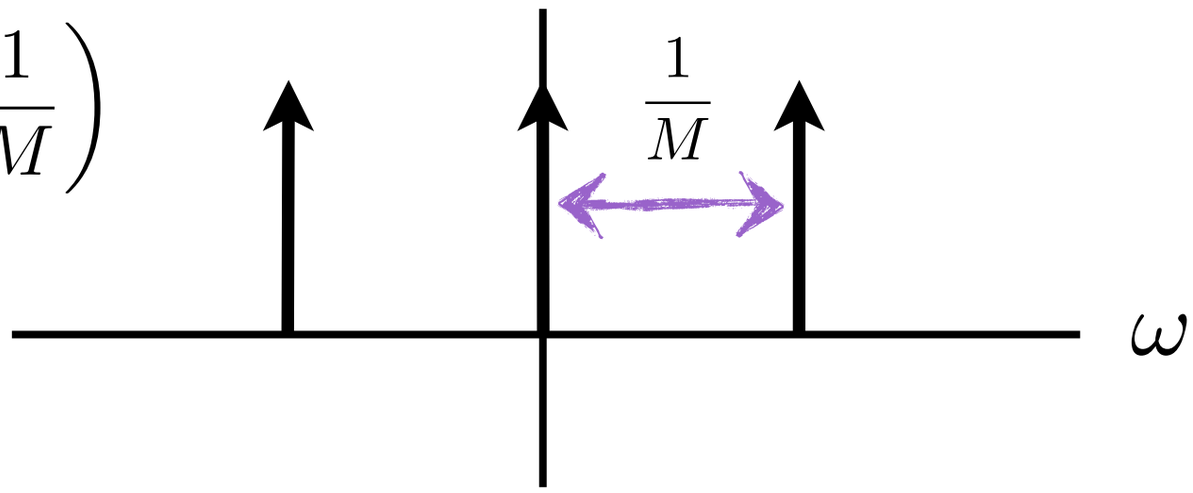


Signal is **ALIASED**

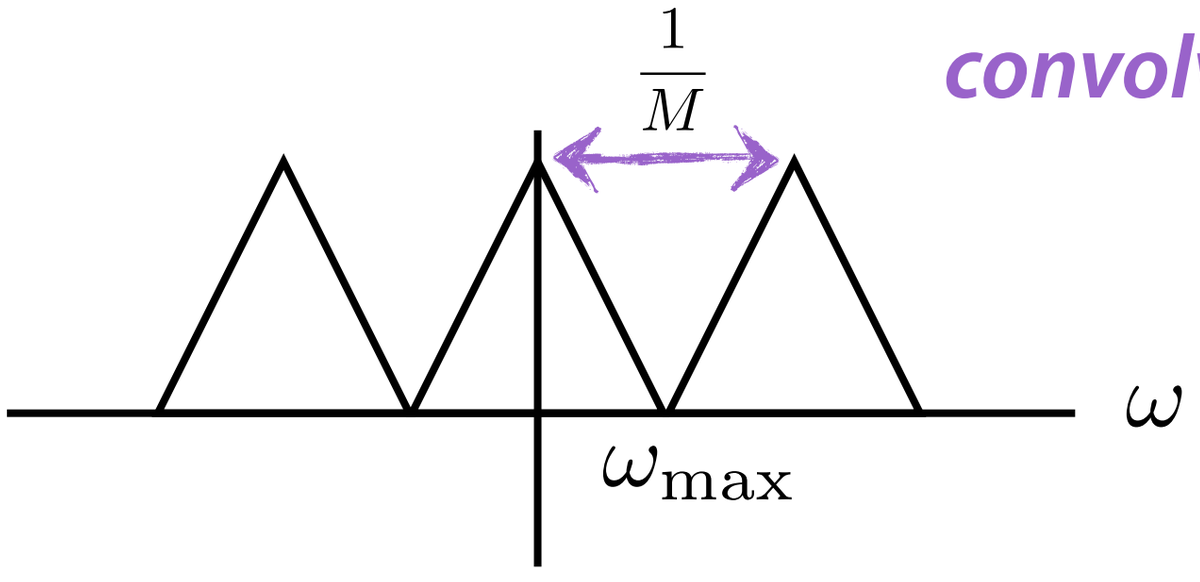
Nyquist Sampling Theorem

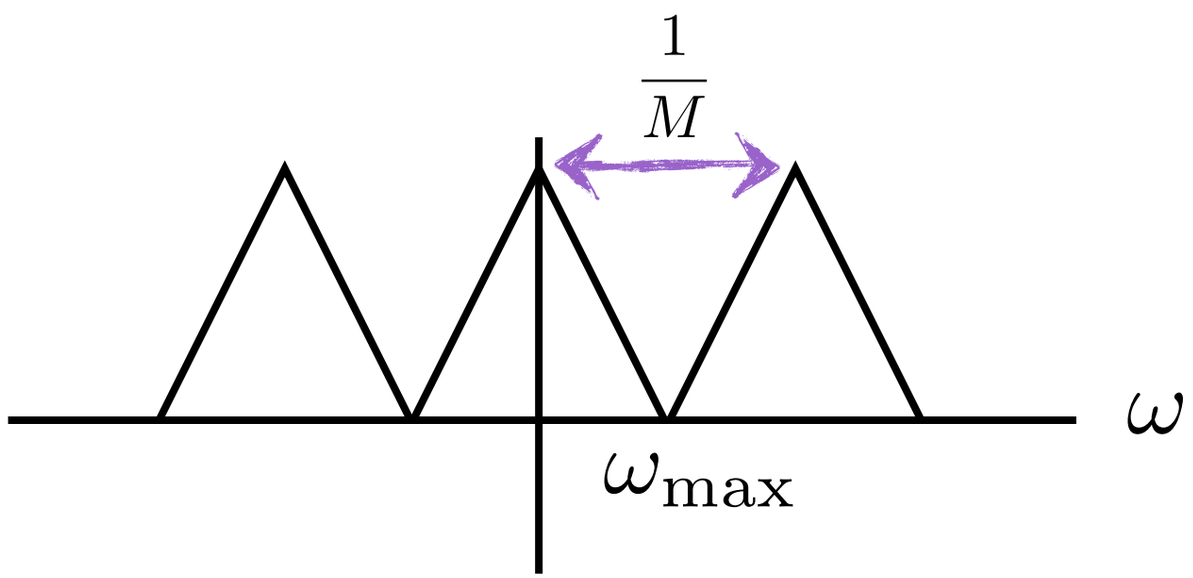


$$\mathrm{comb}(\omega) = \frac{1}{M} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{1}{M}\right)$$

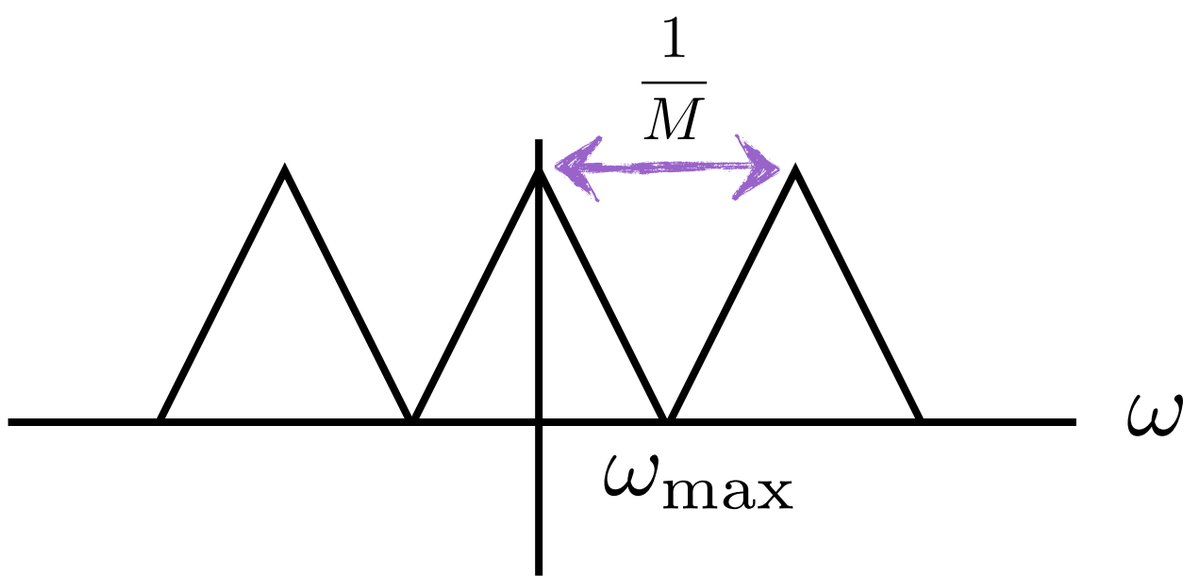


convolve



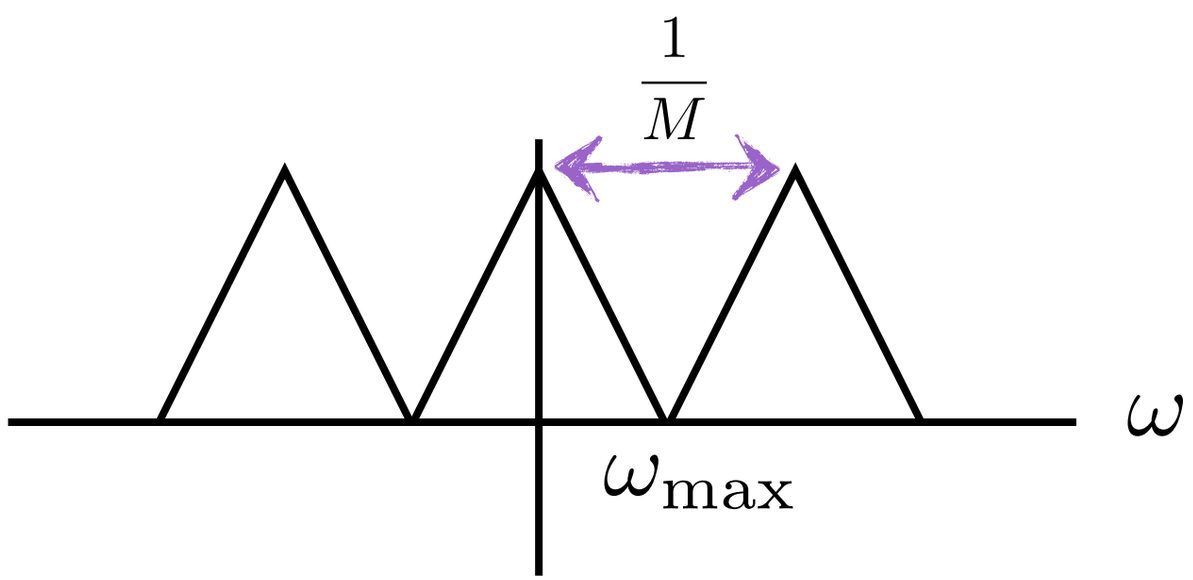


When does aliasing not occur?



Nyquist Sampling Theorem

$$\frac{1}{M} > 2\omega_{\max}$$



Nyquist Sampling Theorem

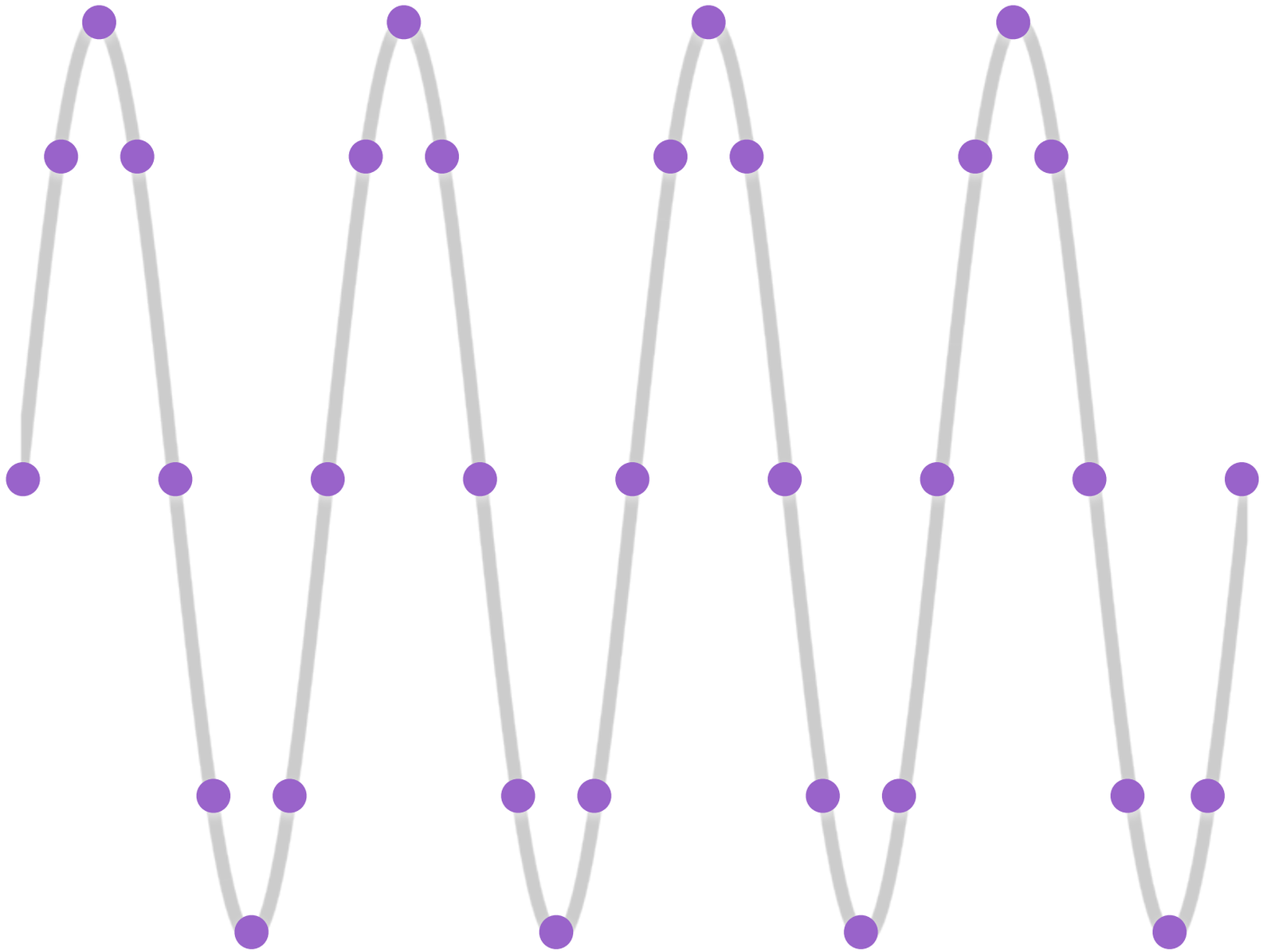
$$\frac{1}{M} > 2\omega_{\max}$$

Sampling frequency must be greater than twice the signal's highest frequency

How can we avoid aliasing?

1

Adjust sampling rate



Sample signal more often

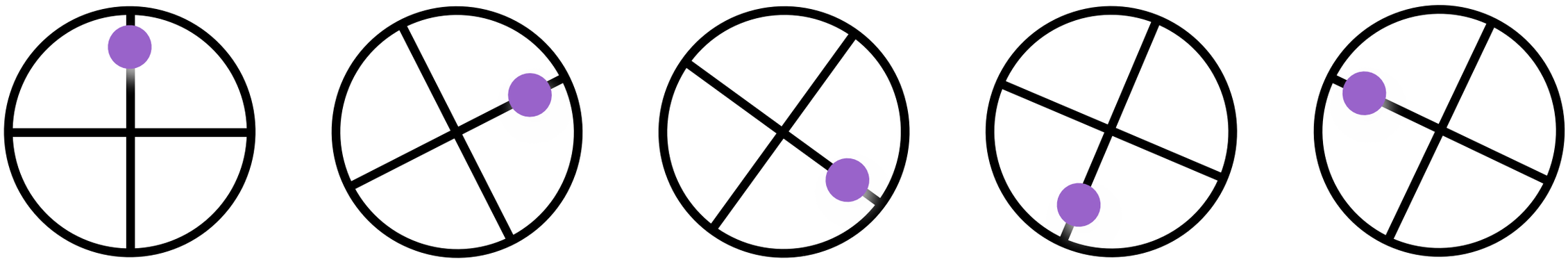
2

Prefilter image

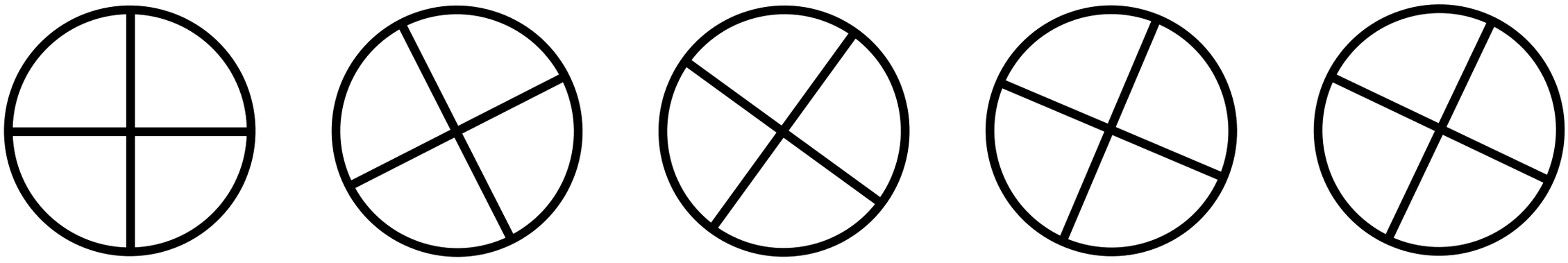
Remove some HIGH FREQUENCIES

smooth prior to sampling

Lens acts as a low-pass filter


frame 0
frame 1
frame 2
frame 3
frame 4

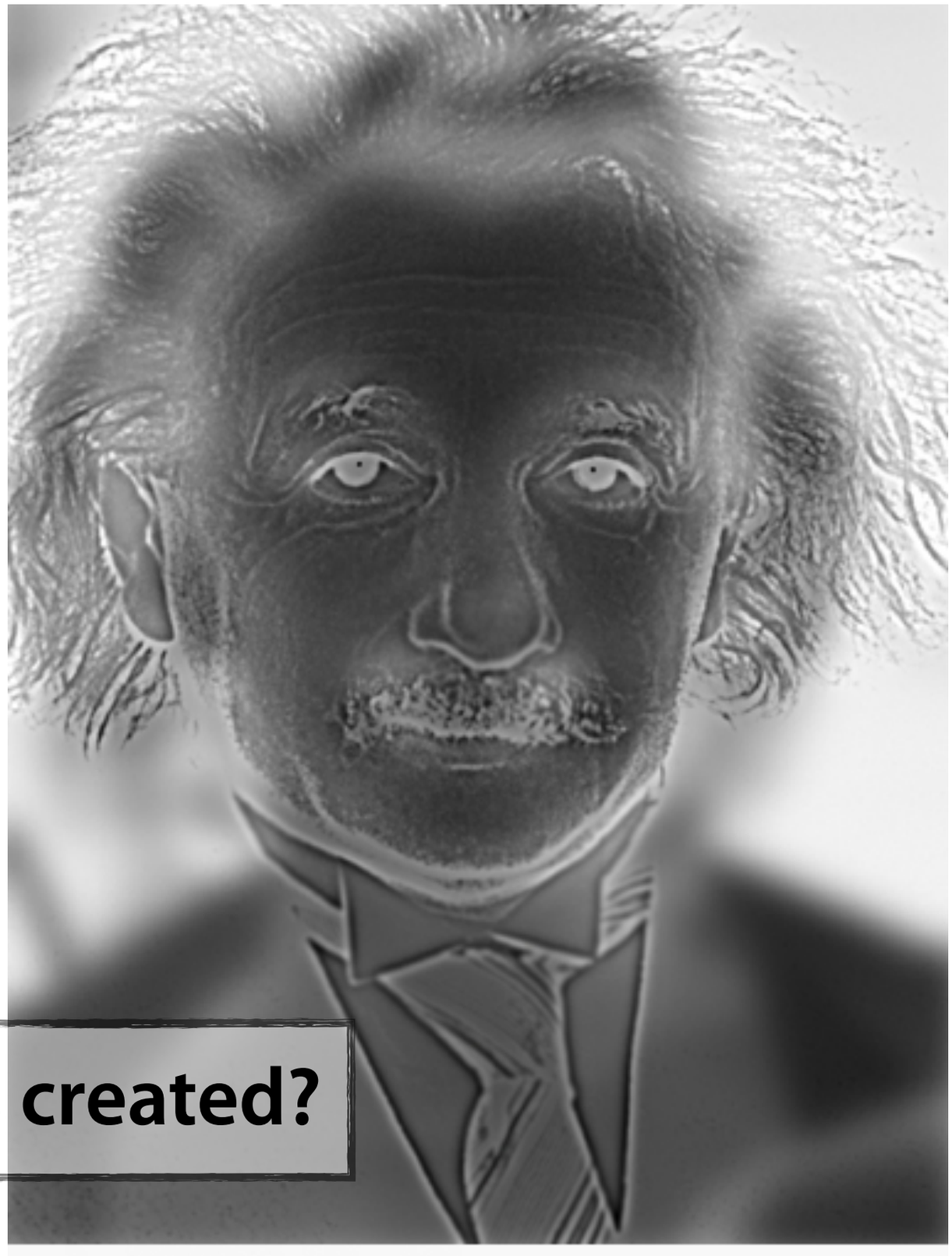

time



frame 0 frame 1 frame 2 frame 3 frame 4

time

Without the dot wheel appears to slowly rotate backwards!



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How was this created?

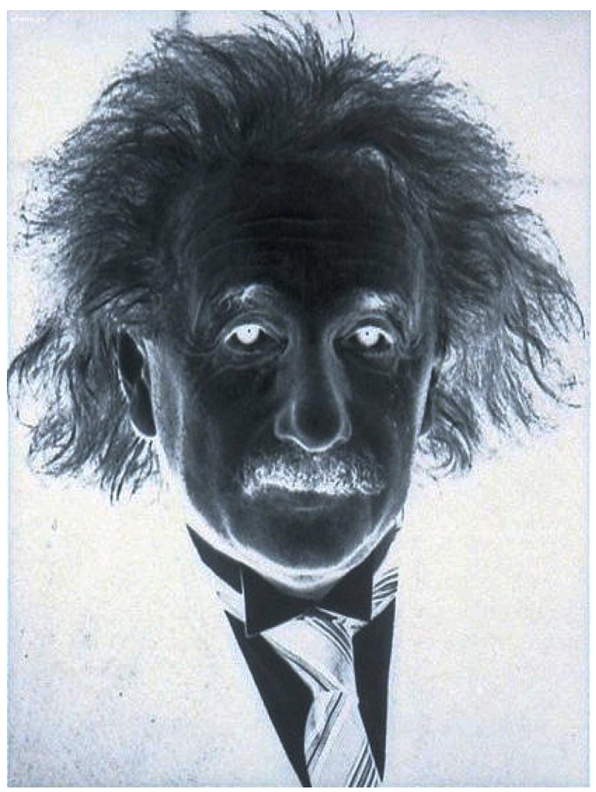
$$E[x,y] * (F[x,y] + G[x,y])$$

$$= (E[x,y] * F[x,y]) + (E[x,y] * G[x,y])$$

$*$ $g_1[x, y]$

lowpass filter

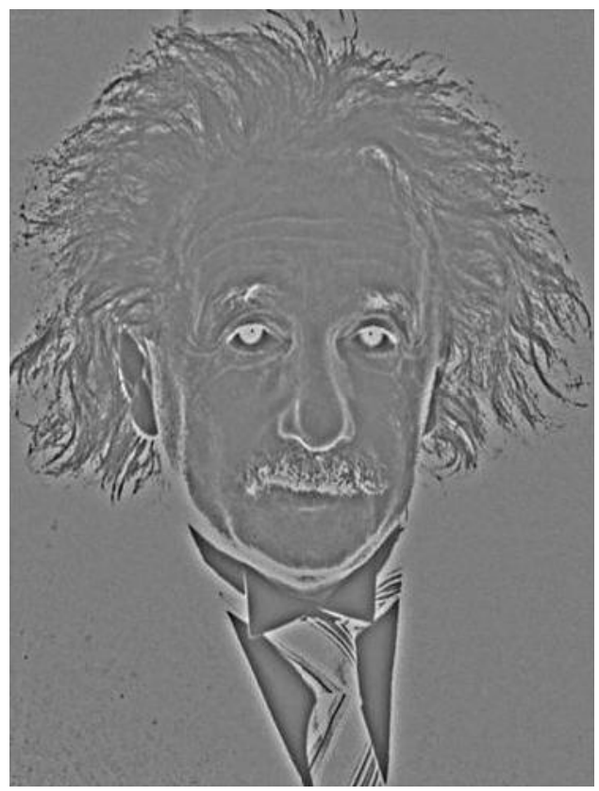
$=$



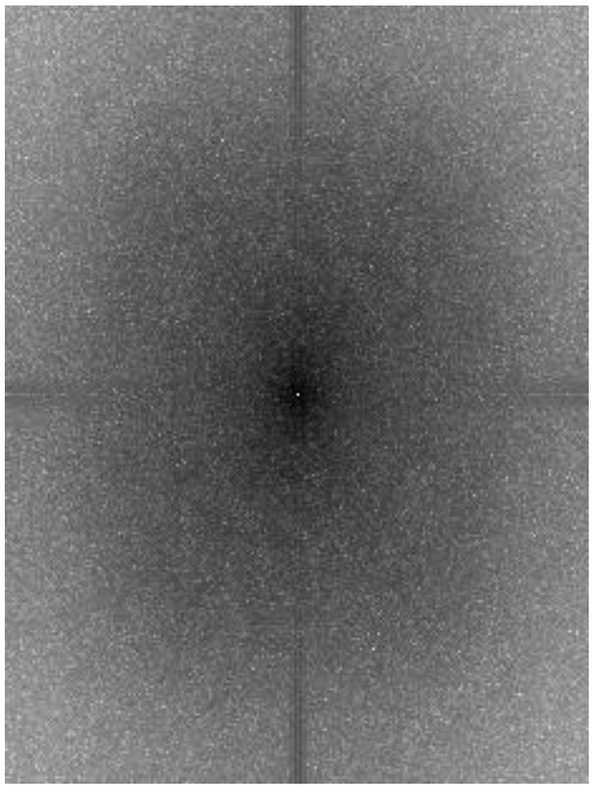
$*$ $(\delta[x, y] - g_2[x, y])$ $=$

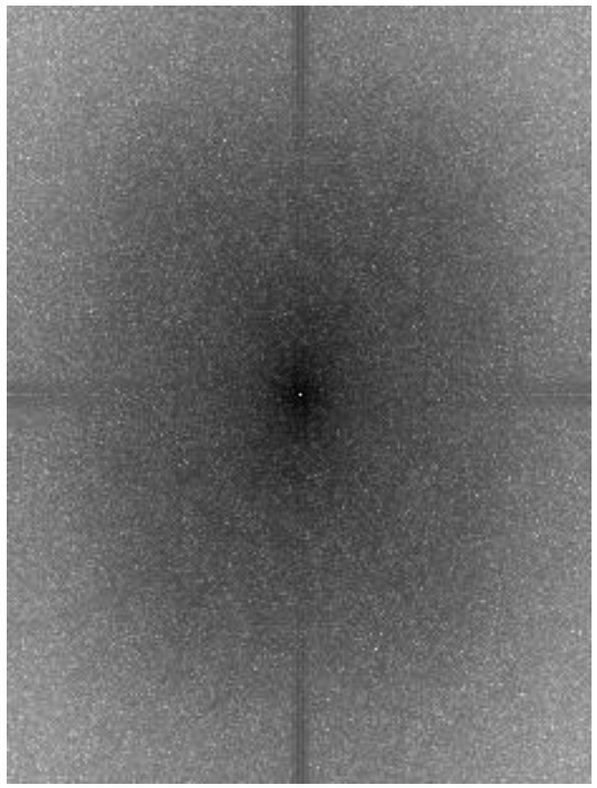
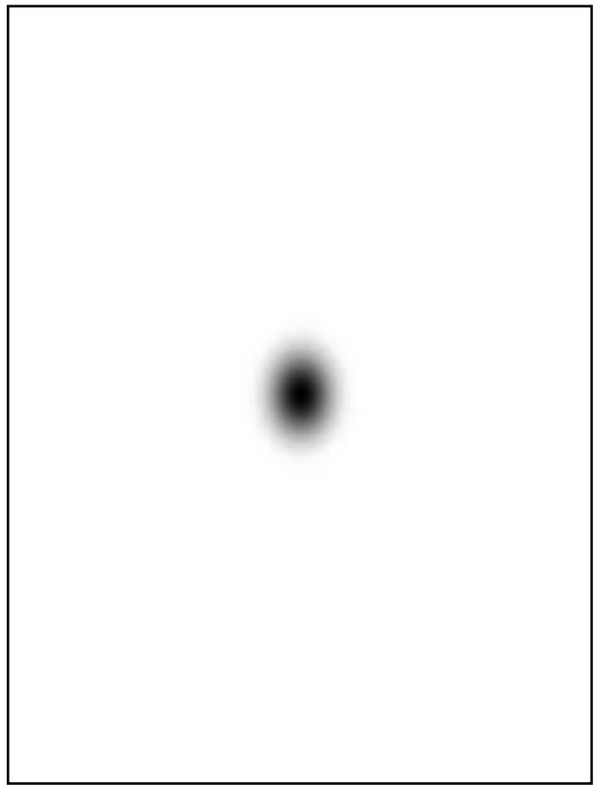
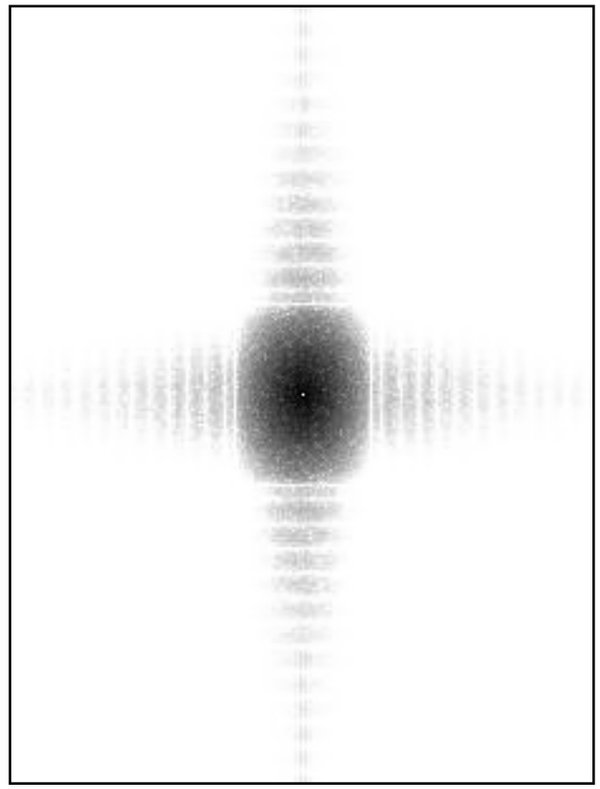
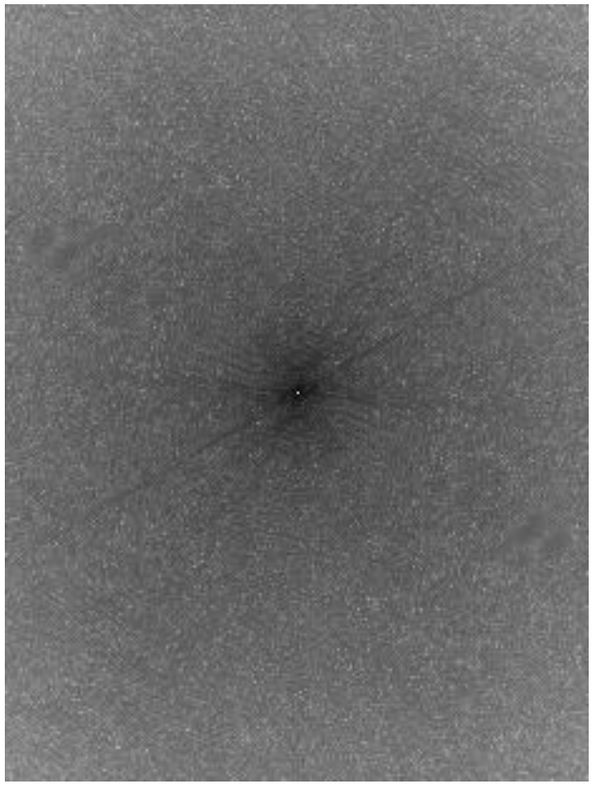
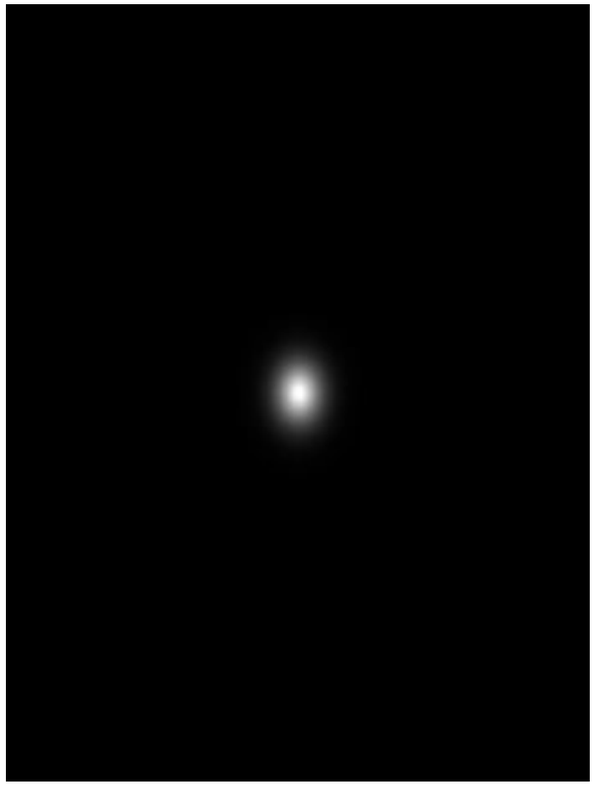
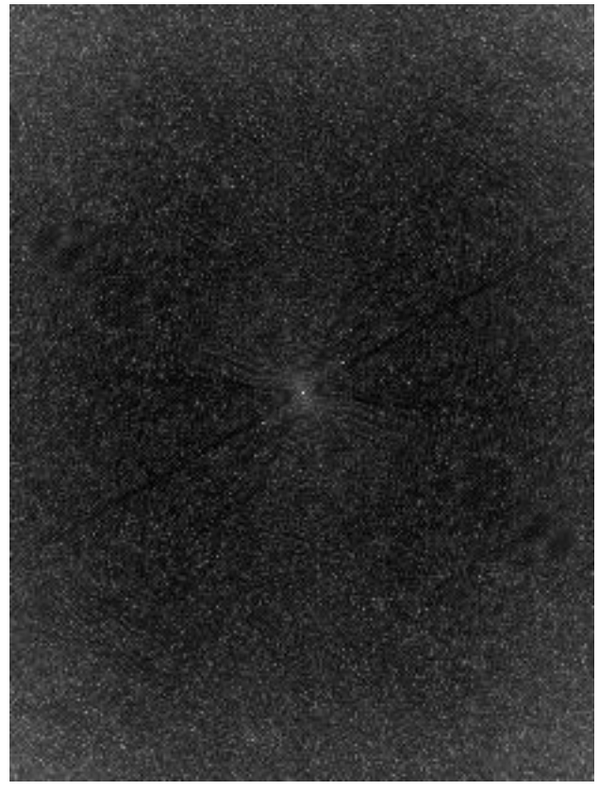
highpass filter

$+$

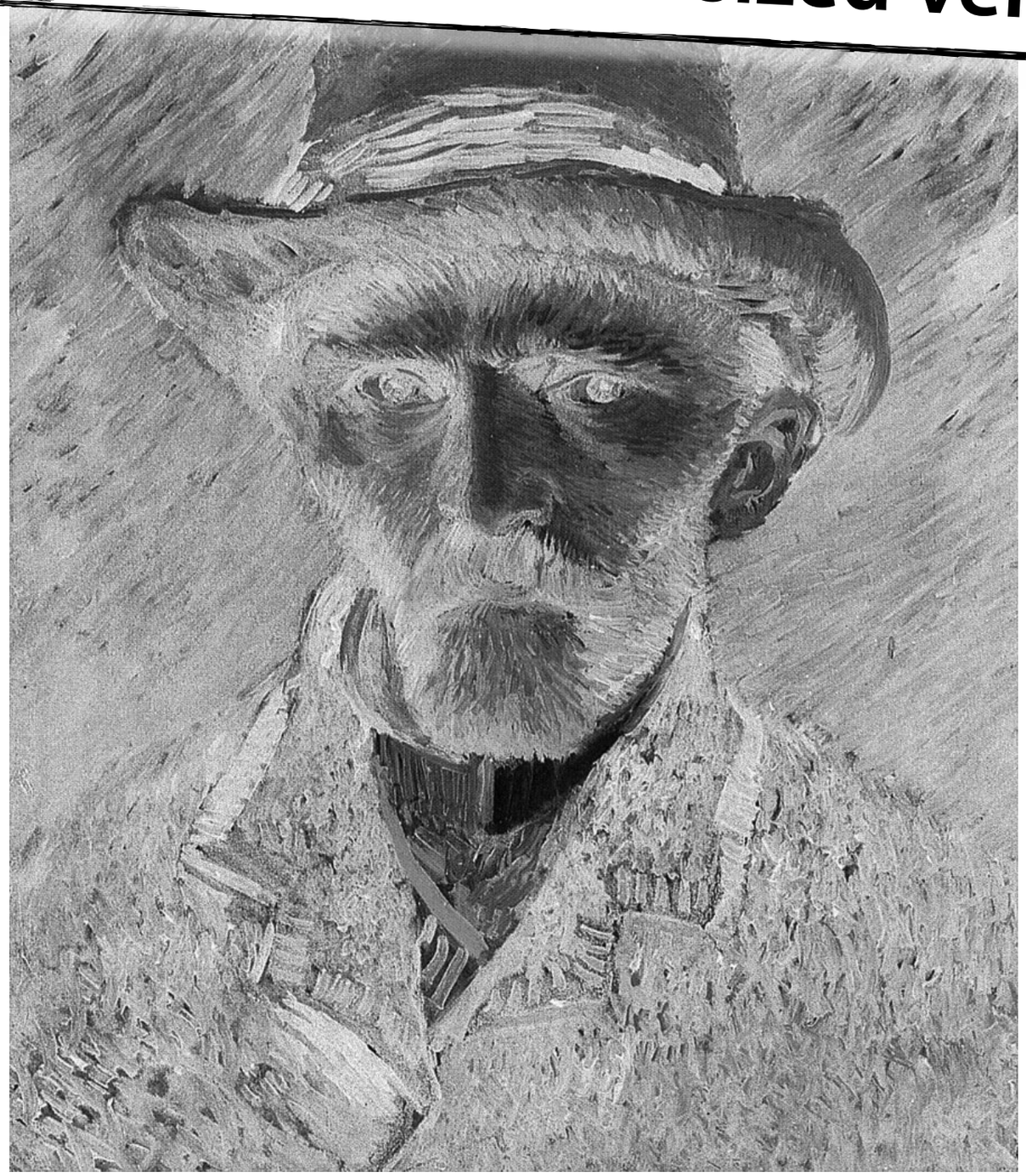


Frequency Domain



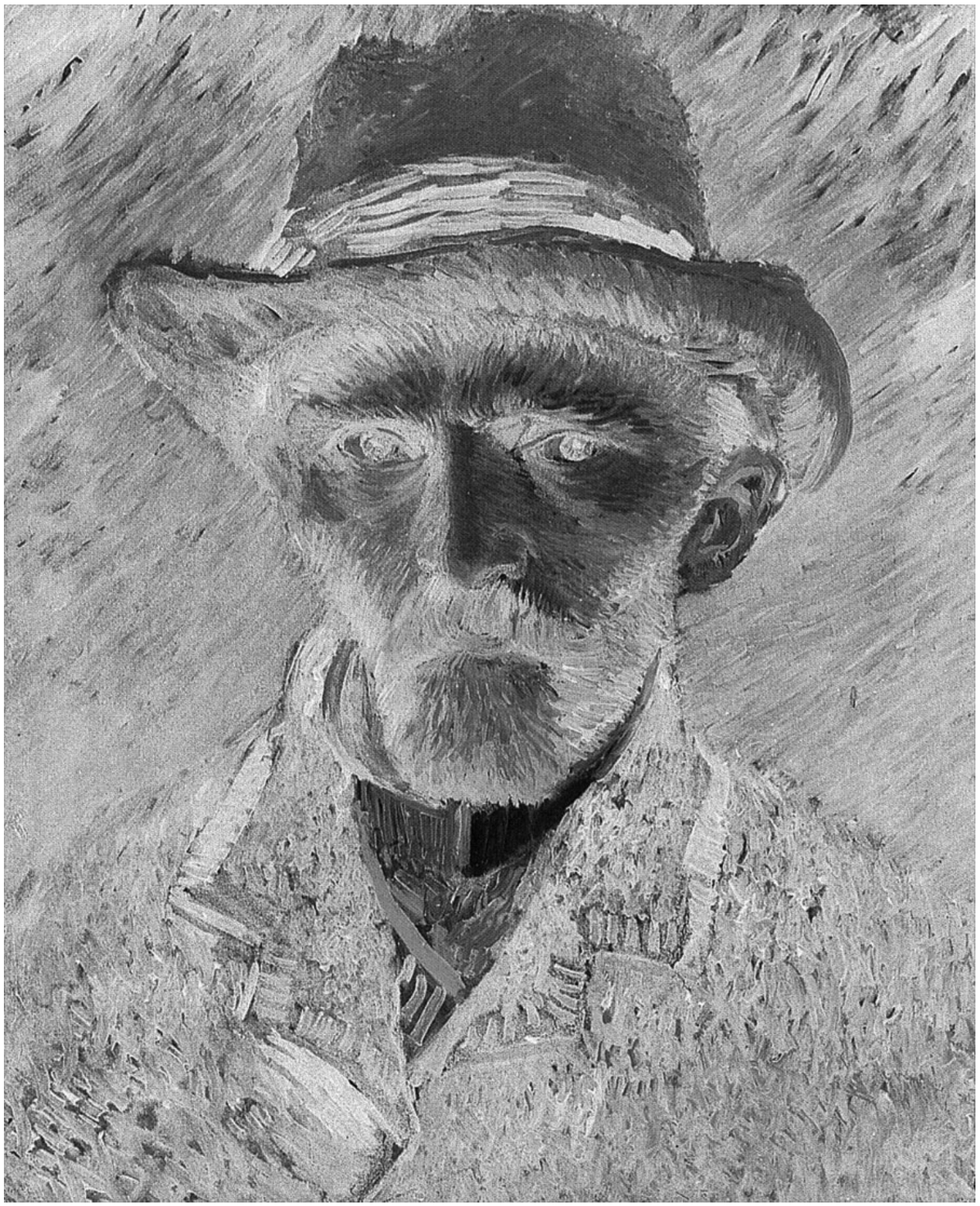







How to generate a half-sized version?



1/4 size (4x zoom)

Smooth then subsample

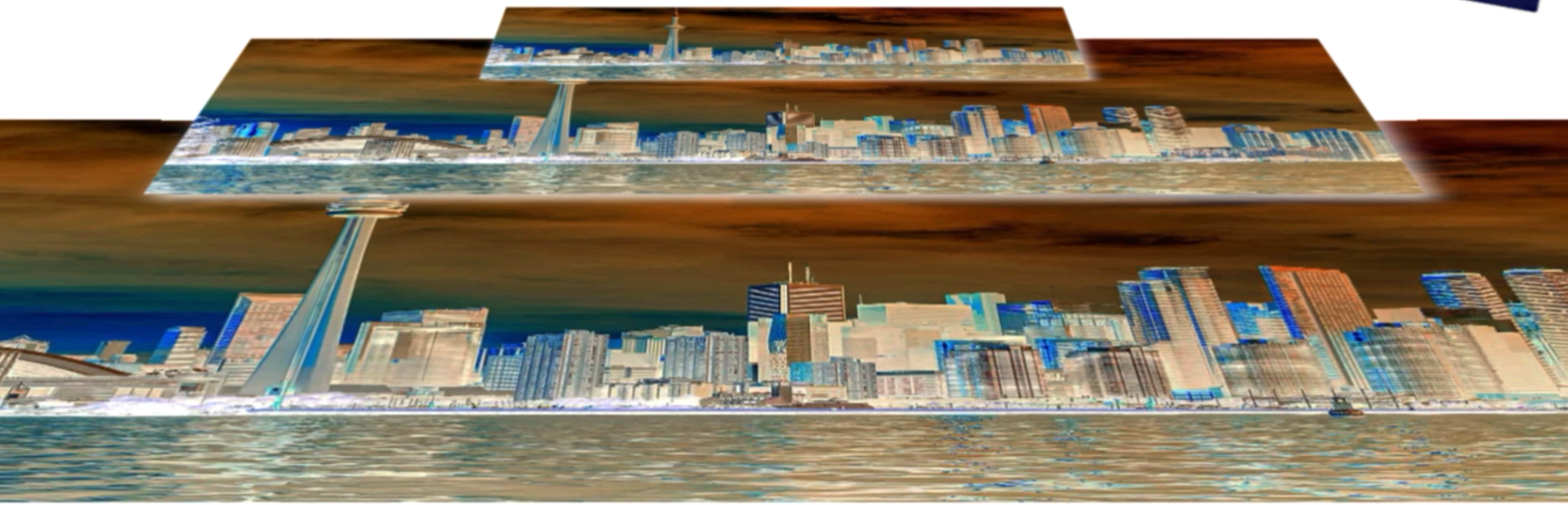


original

binomial

repeated smoothing and subsampling

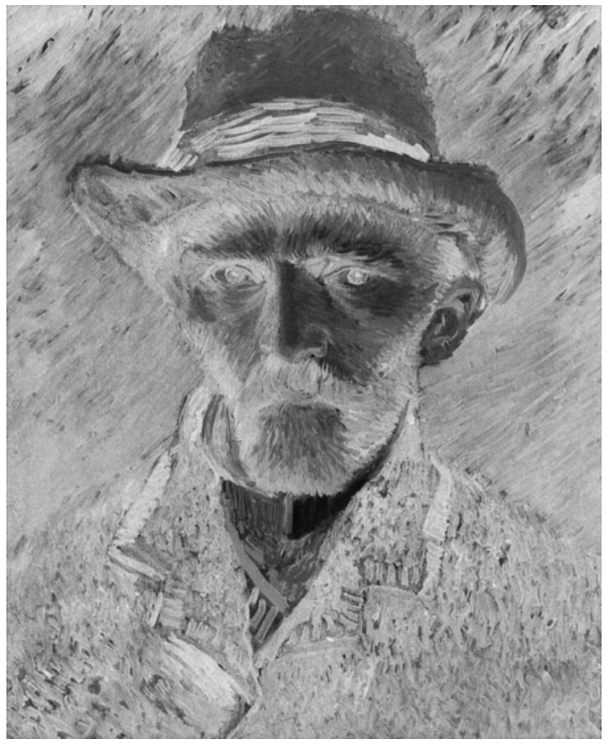
Gaussian Pyramid



binomial

repeated smoothing and subsampling

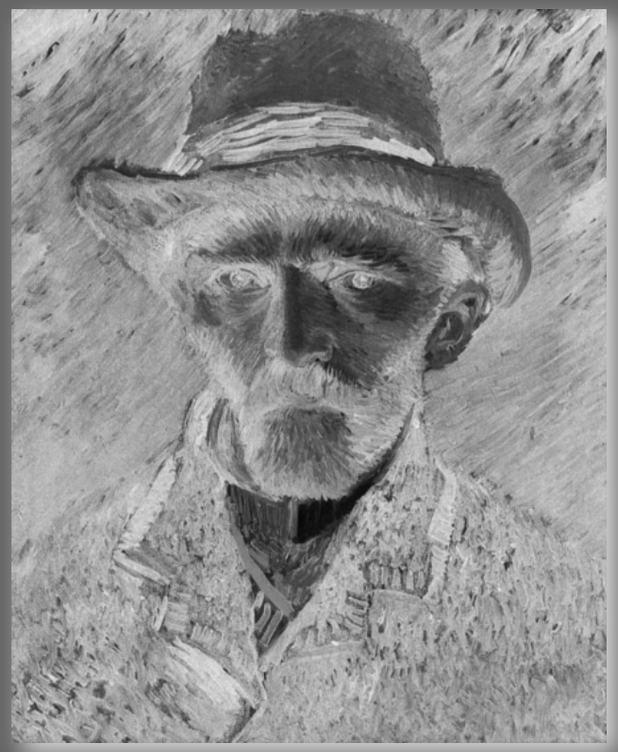
Gaussian Pyramid



1/2 size

```
>> f = (1/16)*[1 4 6 4 1]; % Binomial filter
>> tmp = imfilter(im, f,  'symmetric');
>> im_smooth = imfilter(tmp,  f', 'symmetric');
>> im_smooth = im_smooth(1:2:end, 1:2:end);
>> imshow(im_smooth, [])
```

Gaussian Pyramid



1/2 size

```
>> f = (1/16)*[1  4  6  4  1]; % Binomial filter
>> tmp = imfilter(im, f,  'symmetric');
>> im_smooth = imfilter(tmp,  f', 'symmetric');
>> im_smooth = im_smooth(1:2:end, 1:2:end);
>> imshow(im_smooth, [])
```

Pascal's Triangle

1

1 1

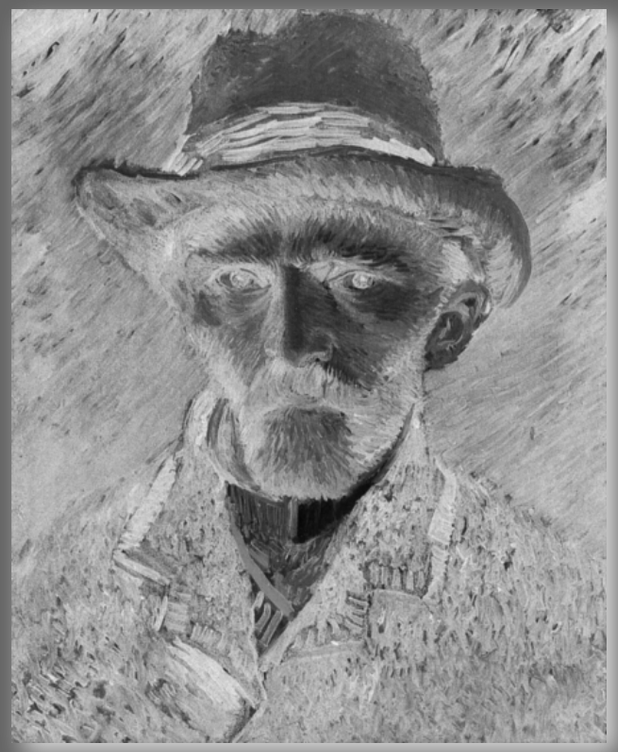
1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Gaussian Pyramid



1/2 size

```
>> f = (1/16)*[1 4 6 4 1]; % Binomial filter
>> tmp = imfilter(im, f,  'symmetric');
>> im_smooth = imfilter(tmp,  f', 'symmetric');
>> im_smooth = im_smooth(1:2:en
>> imshow(im_smooth, [])
```

separable filtering

BLOCKY ARTIFACTS due to compression

5

major steps

Step 1

Convert colour space

Convert RGB image to YCrCb


VICTORIA ST.
339
GOULD ST.
32
RYERSON UNIVERSITY

Y


RYERSON UNIVERSITY

Cr

RYERSON UNIVERSITY

Cb

Step 2

Resample colour channels

Y

Cr

Cb

Subsample colour channels by factor of two


Y
VICTORIA ST.
339
GOULD ST.
32
RYERSON UNIVERSITY

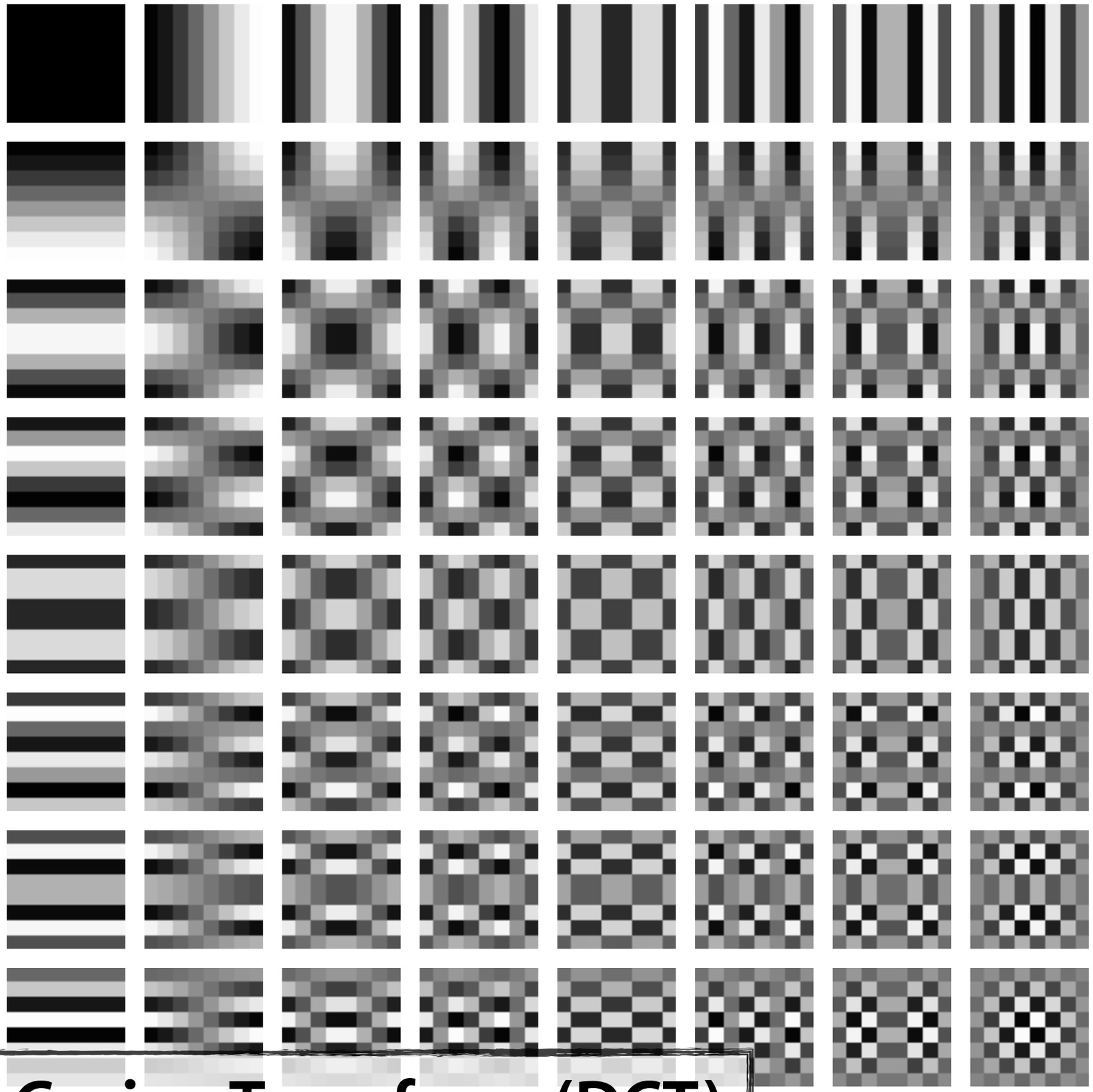

Cr


Cb

Step 3

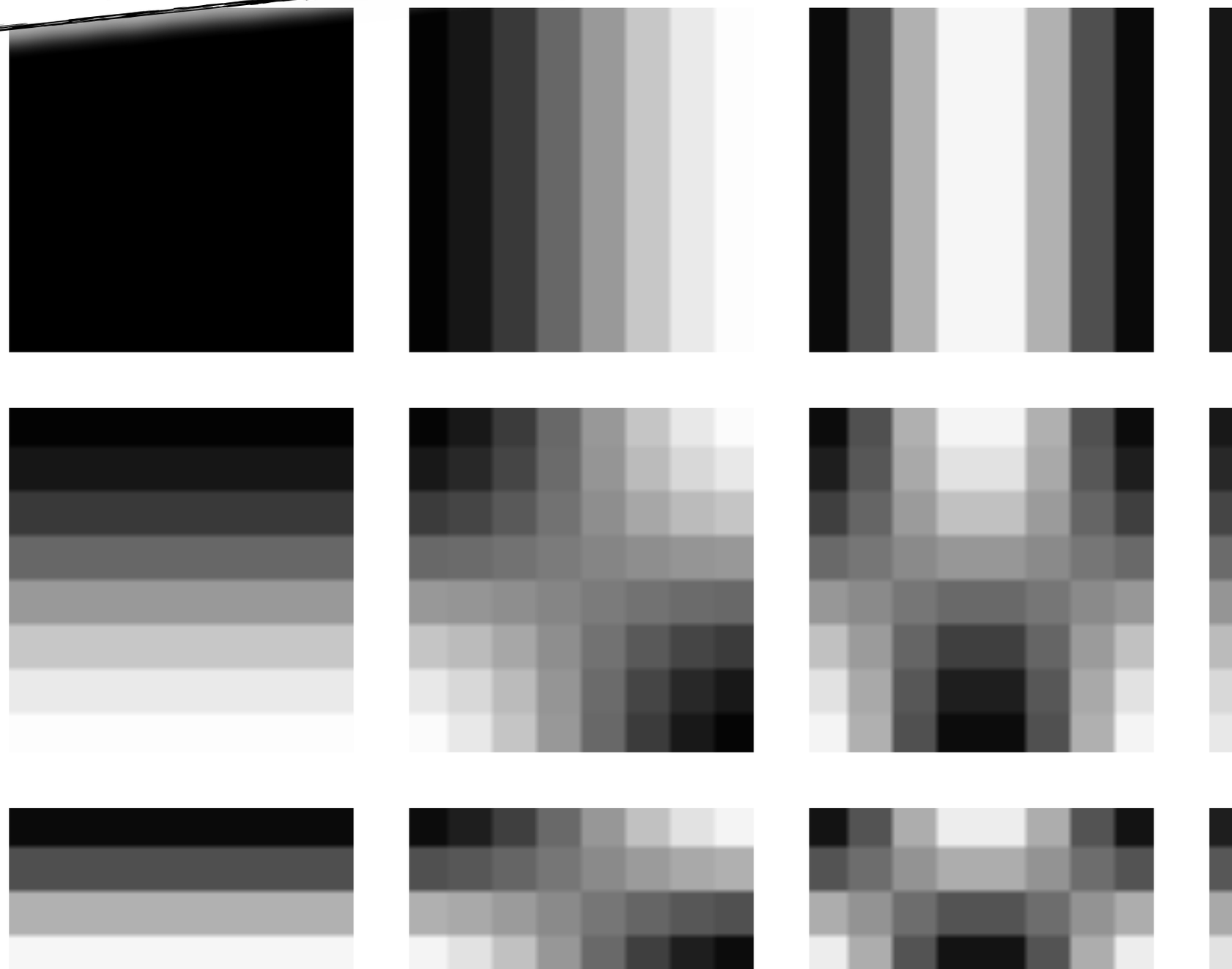
Compute DCT for each 8x8 image patch

map image patches to a set of real numbers

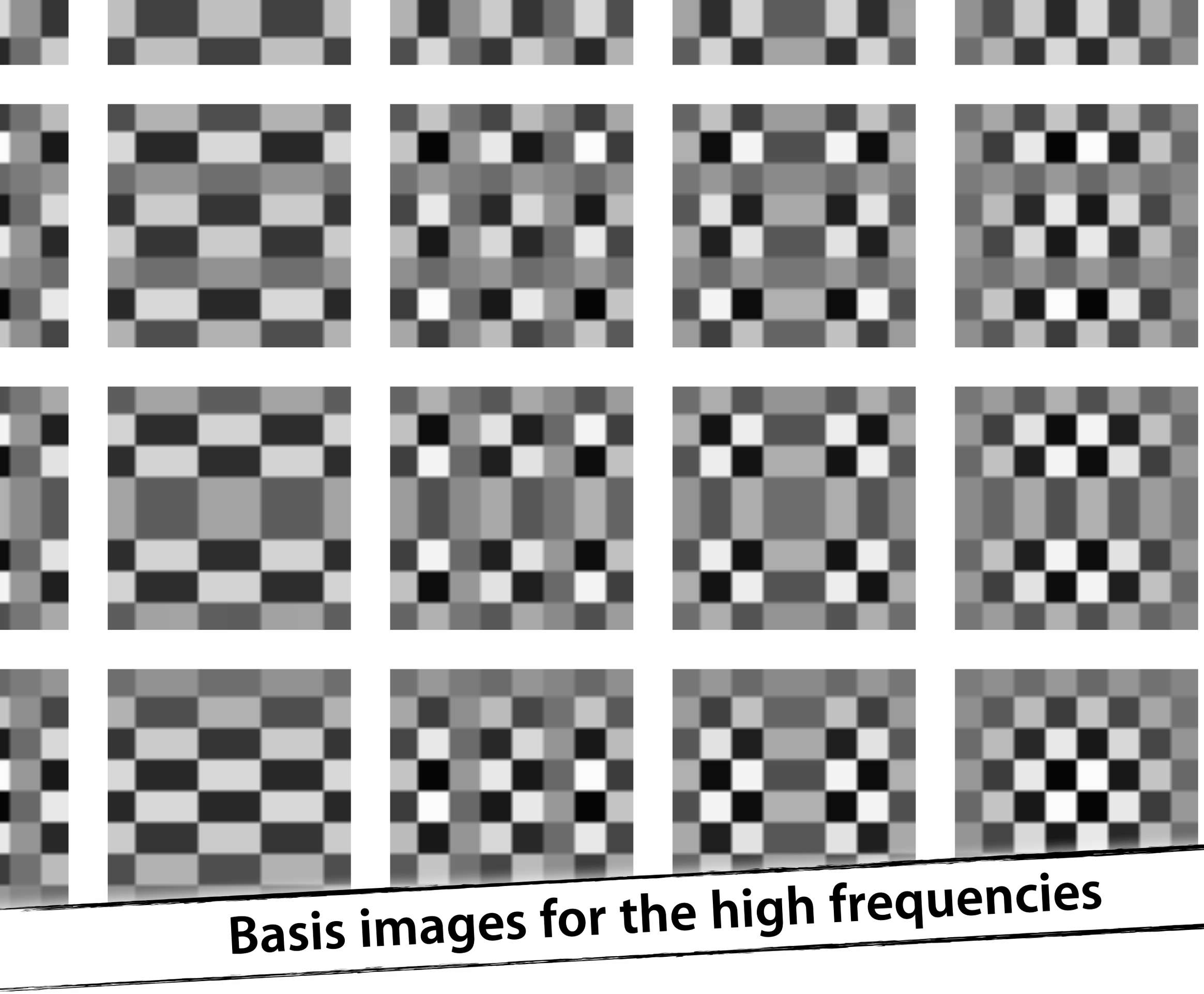


Discrete Cosine Transform (DCT)

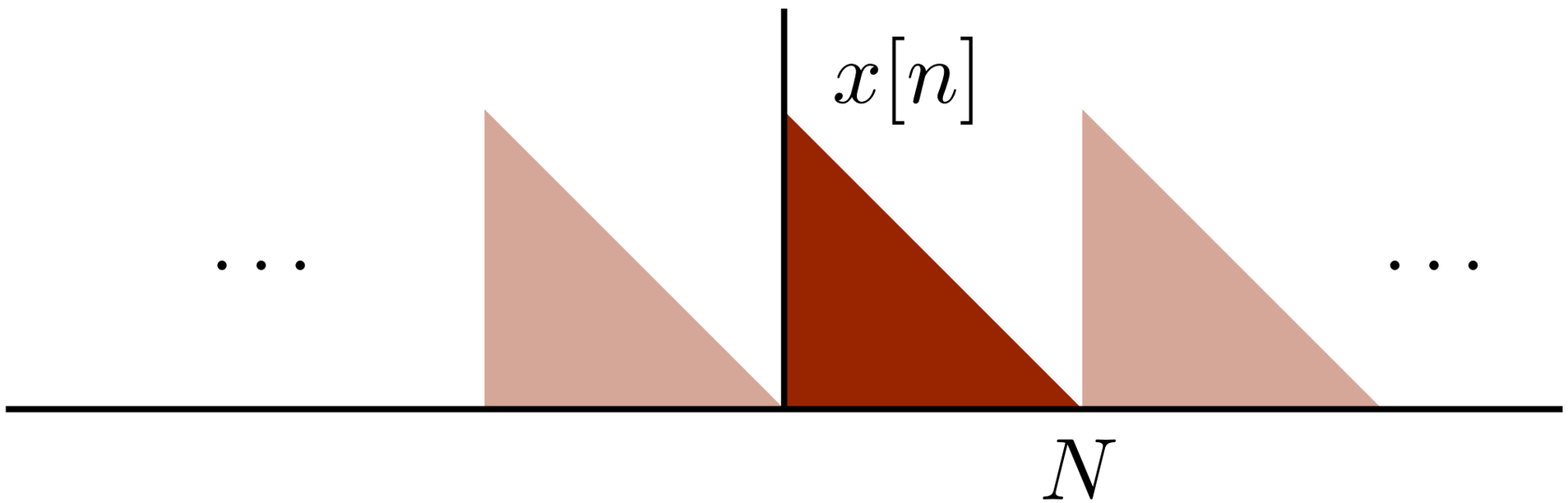
DC component capturing the average intensity



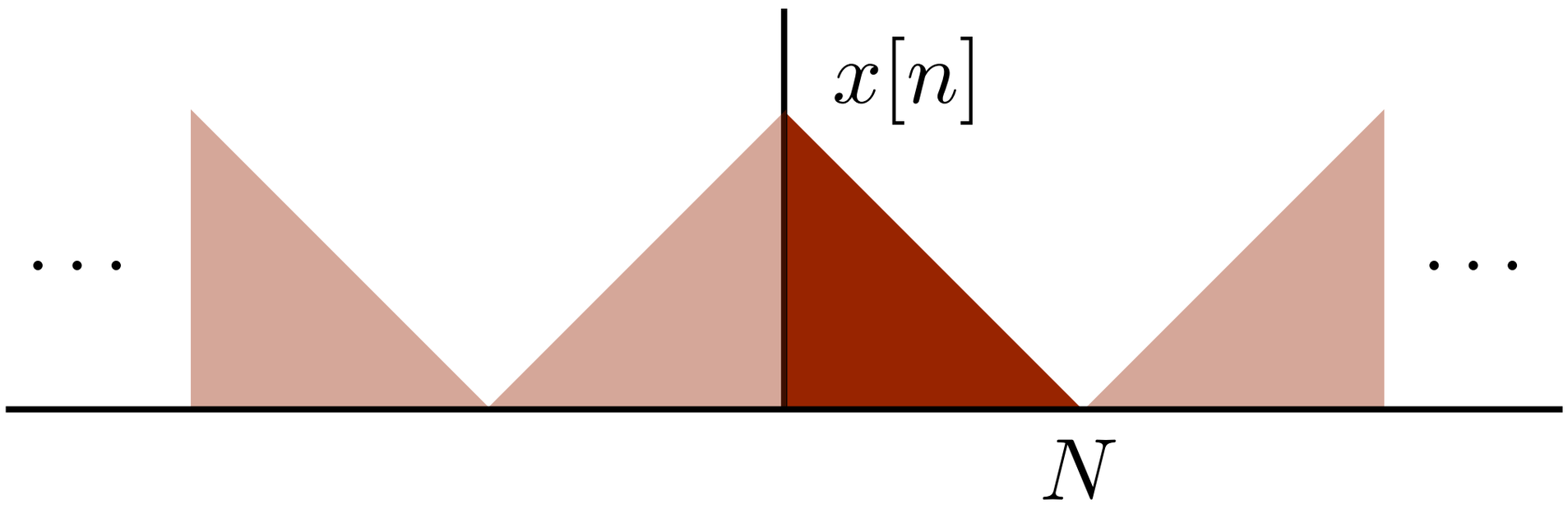
Basis images for the low frequencies



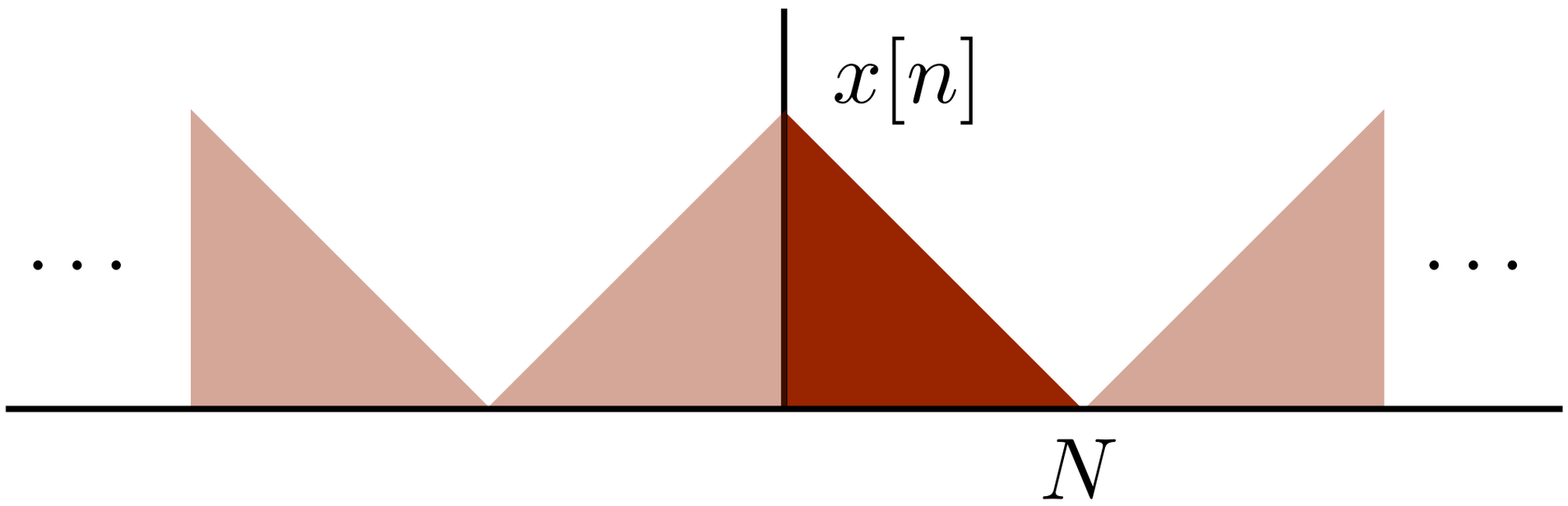
Basis images for the high frequencies



DFT assumes periodic extension beyond boundaries



DCT adds mirrored version and then periodic extension



Performing DFT without the complex numbers

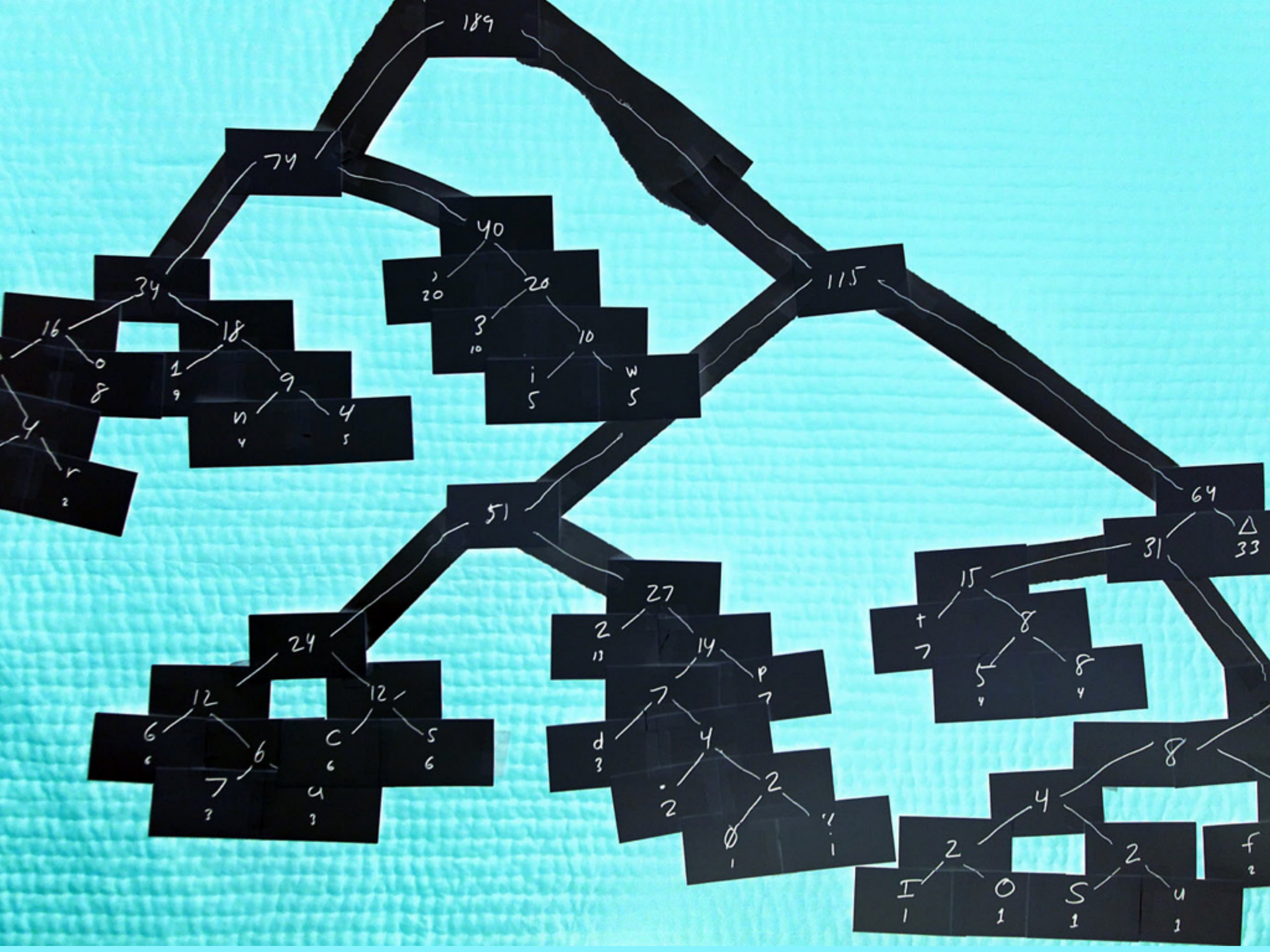
Step 4

Quantize DCT coefficients

quantization level depends on frequency

Step 5

Lossless compression using Huffman coding


189
74
40
115
34
20
20
3
10
10
i
5
w
5
16
18
0
8
1
9
9
n
4
4
5
51
64
24
27
31
33
12
12
6
C
6
5
6
7
4
2
14
7
d
3
4
2
2
0
1
1
15
7
8
5
4
8
4
8
4
2
I
1
O
1
S
1
u
1
f
2