QI and QC Homework 1

15 Farvardin - 21 Farvardin 1404

Full Name : **Student's ID :**

1 Question 1: Not Operation Representation

Let $\{|0\rangle, |1\rangle\}$ be an orthonormal basis in the Hilbert space $\mathbf{C}^2$. The NOT operation (unitary operator) is defined as

$$|0\rangle \to |1\rangle, \quad |1\rangle \to |0\rangle$$

1.1 (a)

Find the unitary operator U_{NOT} which implements the NOT operation with respect to the basis $\{|0\rangle, |1\rangle\}$.

1.2 (b)

Let

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Find the matrix representation of U_{NOT} for this basis.

1.3 (c)

Let

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Find the matrix representation of U_{NOT} for this basis.

2 Question 2: The Operator-Schmidt decomposition

The operator-Schmidt decomposition of a linear operator Q acting in the product Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of two finite-dimensional Hilbert spaces ($\dim \mathcal{H}_1 = m, \dim \mathcal{H}_2 = n$)

with $\mathcal{H}_1 = \mathbf{C}^m$ and $\mathcal{H}_2 = \mathbf{C}^n$ can be constructed as follows. Let X, Y be $d \times d$ matrices over C. Then we can define a scalar product $\langle X, Y \rangle := \operatorname{tr}\left(XY^\dagger\right)$. Using this inner product we can define an orthonormal set of $d \times d$ matrices $\left\{X_j : j = 1, 2, \ldots, d^2\right\}$ which satisfies the condition

$$\langle X_j, X_k \rangle = \operatorname{tr}\left(X_j X_k^\dagger\right) = \delta_{jk}$$

Thus we can write the matrix Q as

$$Q = \sum_{j=1}^{m^2} \sum_{k=1}^{n^2} c_{jk} A_j \otimes B_k$$

where $\left\{A_j : j = 1, 2, \ldots, m^2\right\}$ and $\left\{B_k : k = 1, 2, \ldots, n^2\right\}$ are fixed orthonormal bases of $m \times m$ and $n \times n$ matrices in the Hilbert spaces $\mathbf{C}^m$ and $\mathbf{C}^n$ respectively, and c_{jk} are complex coefficients. Thus $C = (c_{jk})$, with $j = 1, 2, \ldots, m^2$ and $k = 1, 2, \ldots, n^2$ is an $m^2 \times n^2$ matrix. The singular value decomposition theorem states that the matrix C can be written as

$$C = U \Sigma V^\dagger$$

where U is an $m^2 \times m^2$ unitary matrix, V is an $n^2 \times n^2$ unitary matrix and Σ is an $m^2 \times n^2$ diagonal matrix. The matrix Σ is of the form

$$\Sigma = \begin{pmatrix} s_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & s_{n^2} \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}$$

It is assumed that C, U and V are calculated in orthonormal bases, for example the standard basis. Thus we obtain

$$Q = \sum_{j=1} \sum_{k=1} \sum_{\ell=1} U_{j\ell} s_\ell V_{\ell k} A_j \otimes B_k$$

where s_ℓ is the ℓ-th diagonal entry of the $m^2 \times n^2$ diagonal matrix Σ. Defining

$$H_\ell = \sum_{j=1}^{m^2} U_{j\ell} A_j$$

$$K_\ell = \sum_{k=1}^{n^2} V_{\ell k} B_k$$

where $\ell = 1, 2, \ldots, n^2$ we find the operator-Schmidt decomposition

$$Q = \sum_{\ell=1}^{n^2} s_\ell H_\ell \otimes K_\ell$$

2.1 (a)

Consider the CNOT gate

$$U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Find the operator-Schmidt decomposition of U_{CNOT}.

2.2 (b)

Consider the SWAP operator

$$U_{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Find the operator-Schmidt decomposition of U_{SWAP}.

2.3 (c)*

Let

$$Z = \left(\sqrt{1-p}I_2 \otimes I_2 + i\sqrt{p}\sigma_x \otimes \sigma_x\right)\left(\sqrt{1-p}I_2 \otimes I_2 + i\sqrt{p}\sigma_z \otimes \sigma_z\right)$$

where σ_x, σ_y and σ_z are the Pauli spin matrices. Find the operator-Schmidt decomposition of Z.

3 Question 3 : Properties Of Density Matrix

Consider the operator (4×4 matrix) in the Hilbert space C^4

$$\rho = \frac{1}{4}(1-\epsilon)I_4 + \epsilon(|0\rangle \otimes |0\rangle)(\langle 0| \otimes \langle 0|)$$

where ϵ is a real parameter with $\epsilon \in [0, 1]$ and

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Does ρ define a density matrix?

4 Question 4 : Partial Trace

Consider the 4×4 matrix (density matrix)

$$|\mathbf{u}\rangle\langle\mathbf{u}| = \begin{pmatrix} u_1\bar{u}_1 & u_1\bar{u}_2 & u_1\bar{u}_3 & u_1\bar{u}_4 \\ u_2\bar{u}_1 & u_2\bar{u}_2 & u_2\bar{u}_3 & u_2\bar{u}_4 \\ u_3\bar{u}_1 & u_3\bar{u}_2 & u_3\bar{u}_3 & u_3\bar{u}_4 \\ u_4\bar{u}_1 & u_4\bar{u}_2 & u_4\bar{u}_3 & u_4\bar{u}_4 \end{pmatrix}$$

in the product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \equiv \mathbf{C}^4$, where $\mathcal{H}_A = \mathcal{H}_B = \mathbf{C}$.

4.1 (a)

Calculate

$$\mathrm{tr}_A(|\mathbf{u}\rangle(\mathbf{u}\ |)$$

where the basis is given by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes I_2, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes I_2$$

and I_2 denotes the 2×2 unit matrix.

4.2 (b)

Find

$$\mathrm{tr}_B(|\mathbf{u}\rangle\langle\mathbf{u}|)$$

where the basis is given by

$$I_2 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad I_2 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

**If you need any help contact me via whatsapp or email : sama.ahanyazar@ut.ac.ir*
**Send homework in Elearn or via email*