Linear algebra

Alireza Qanbari

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Vector spaces

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- The second operation, scalar multiplication, takes any scalar a and any vector v and outputs a new vector av.

Vector spaces

Axiom	Signification
Associativity of addition	$u + (v + w) = (u + v) + w$
Commutativity of addition	$u + v = v + u$
Identity element of addition	There exists an element 0 in V, called the zero vector (or simply zero), such that v + 0 = v for all v in V.

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$$T(u+v) = T(u) + T(v), \quad T(av) = aT(v)$$

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$$T(au + bv) = T(au) + T(bv) = aT(u) + bT(v)$$

Subspaces, span, and basis

The study of those subsets of vector spaces that are in themselves vector spaces under the induced operations is fundamental, similarly as for many mathematical structures. These subsets are called linear subspaces. More precisely, a linear subspace of a vector space V over a field F is a subset W of V such that $u + v$ and au are in W, for every u, v in W, and every a in F.

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Examples

given a linear map $T : V \to W$, the image $T(V)$ of V, and the inverse image $T^{-1}(0)$ of 0 (called kernel or null space), are linear subspaces of W and V, respectively.

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where $v_1 + v_2 + \cdots + v_k$ are in S, and $a_1 + a_2 + \cdots + a_k$, ak are in F form a linear subspace called the span of S. The span of S is also the intersection of all linear subspaces containing S. In other words, it is the (smallest for the inclusion relation) linear subspace containing S.

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If any basis of V (and therefore every basis) has a finite number of elements, V is a finite-dimensional vector space. If U is a subspace of V, then $dimU \leq dimV$. In the case where V is finite-dimensional, the equality of the dimensions implies $U = V$.

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If U_1 and U_2 are subspaces of V, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2),$$

where $U_1 + U_2$ denotes the span of $U_1 \cup U_2$.[4]

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