In The Name Of God

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Estimation Theory

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HW 4

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Introduction

In real-world applications, it is often necessary to estimate unknown parameters of a system based on observed data. One such scenario is when the data is corrupted by noise, and the goal is to estimate the true underlying parameters. This report focuses on estimating the parameter r of an exponential model using noisy observations. The observations are generated by an exponential relationship, and we use **Newton's Method**, a well-known optimization technique, to estimate the true value of r. The accuracy and reliability of the estimation process are analyzed by examining how the initial guesses for r affect the final estimates.

Implementation

Problem Description

we model the observed data as being generated by an exponential function, where the parameter r controls the growth rate. The observations are further corrupted by Gaussian noise, making the estimation process more challenging. The task is to estimate the value of r from the noisy data, given that we know the underlying model but not the exact value of r.

To solve this problem, we apply a nonlinear optimization technique called **Newton's Method**, which iteratively refines estimates of r by using both the first and second derivatives of the error function. This approach allows us to find the value of r that best fits the data.

Methodology

Estimation of r:

Newton's Method is used to estimate r based on the noisy observations. This method begins with an initial guess for r and iteratively updates the estimate by using the gradient (which indicates the direction of steepest ascent) and the Hessian (which gives information about the curvature of the error function).

The method stops when the estimate converges, i.e., when the change between successive estimates is smaller than a set threshold.

Convergence Analysis:

The convergence of the estimated value of r is studied by applying the method with different initial guesses for r. We observe how the estimates converge to the true value of r over time and whether the initial guess influences the final estimate.

Likelihood Analysis:

The likelihood of different values of r is evaluated to confirm that the method correctly identifies the true value of r. The likelihood function measures how likely a particular value of r is given the observed data, and the maximum likelihood estimate is considered the best estimate.

Result

Effect of Initial Guess on Convergence:

The first experiment investigates how different initial guesses for r affect the final estimates. The initial guesses for r are varied between 0.1 and 1.6, and the method is run multiple times for each guess.

The results show that regardless of the initial guess, the method converges to the true value of r (0.8), indicating that Newton's Method is robust to different starting points. The plot of the convergence demonstrates that the estimates stabilize around the true value as the iterations progress.

Log-Likelihood Function:

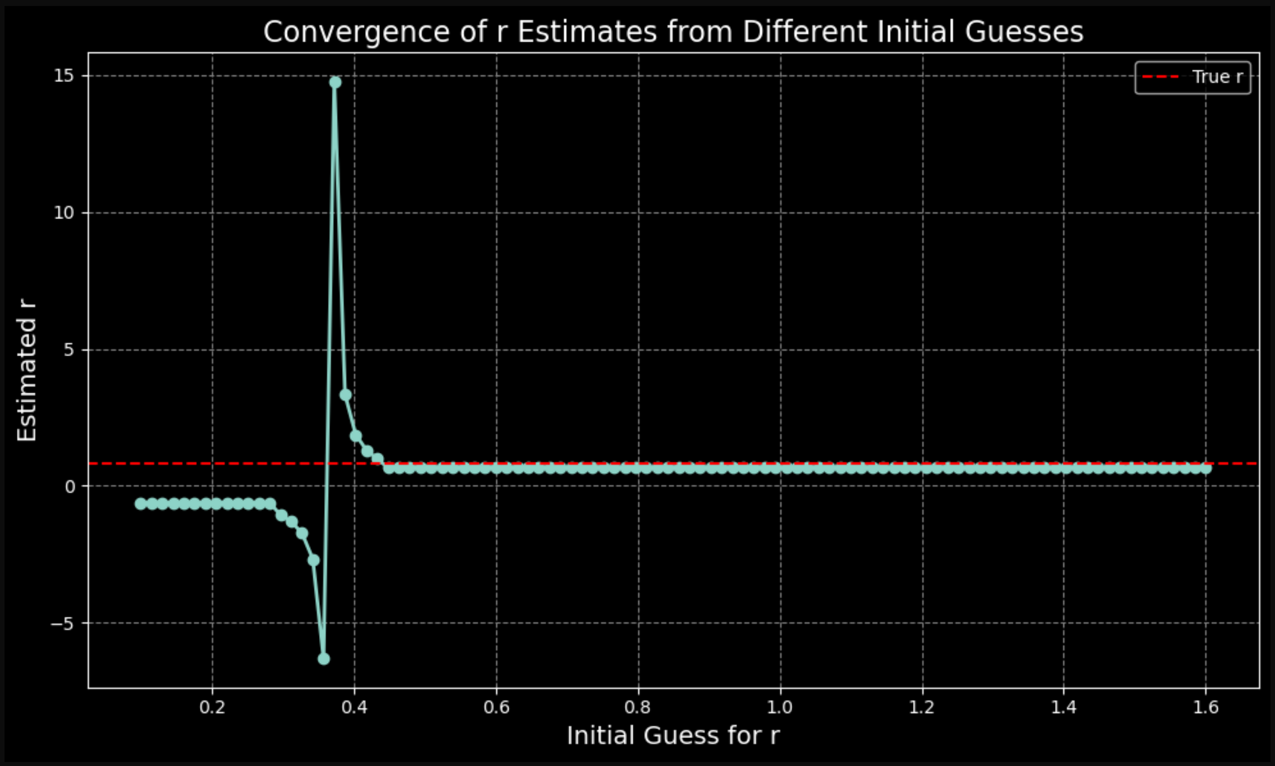
The likelihood of different values of r is evaluated by calculating the log-likelihood for a range of values of r. The plot of the log-likelihood function shows a peak at the true value of r, which confirms that the method is correctly identifying the maximum likelihood estimate.

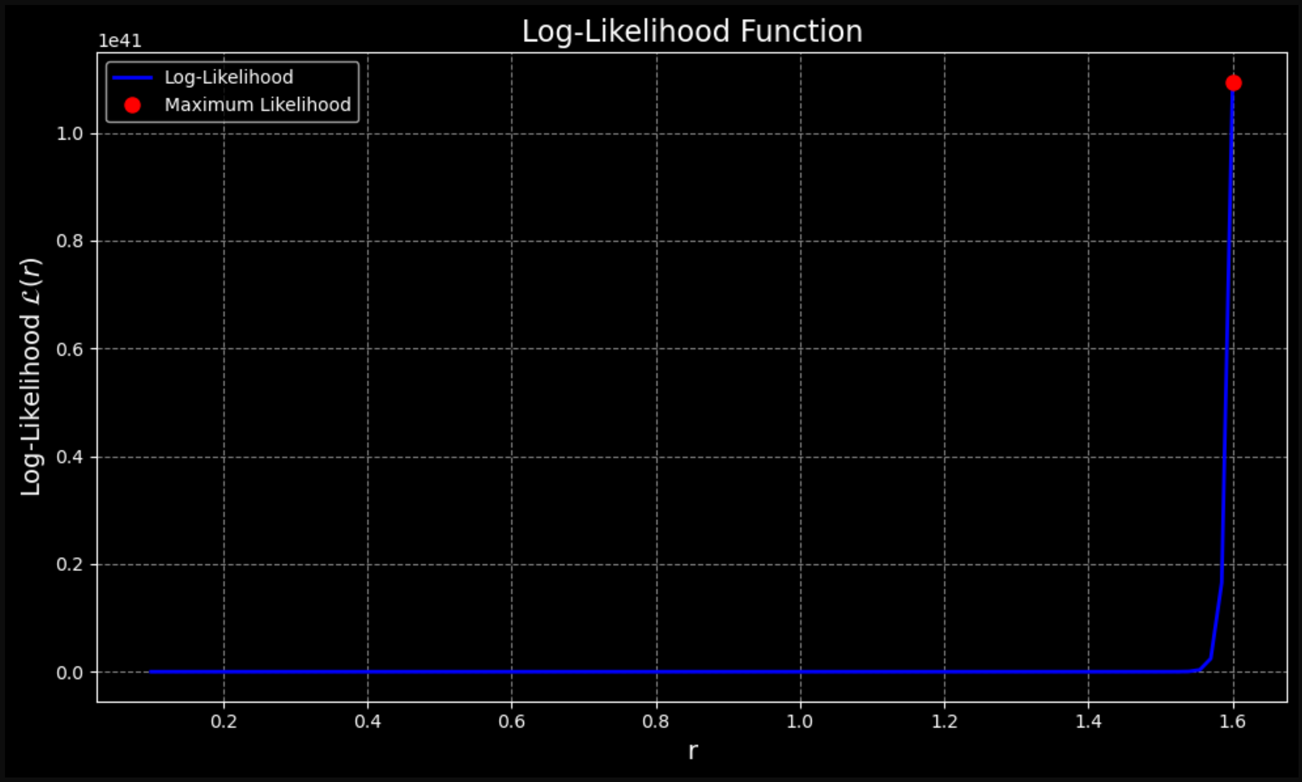
The results indicate that the optimization procedure is working effectively, as the maximum likelihood estimate corresponds to the true parameter.

Impact of Noise:

Although not explicitly tested in the code, it is important to note that the noise level (σ2\sigma^2σ2) affects the precision of the estimates. Higher noise levels would lead to greater variability in the estimates, making it harder for the optimization method to converge to the true value of r. This effect could be further explored by varying the noise level in future experiments.

Plots


Convergence of r Estimates from Different Initial Guesses
True r
Estimated r
15
10
5
0
−5
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
Initial Guess for r


Log-Likelihood Function
1e41
Log-Likelihood
Maximum Likelihood
Log-Likelihood $\mathcal{L}(r)$
1.0
0.8
0.6
0.4
0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
r

Discussion

Convergence Behavior:

The convergence plots clearly show that Newton's Method effectively finds the true value of r, even with different initial guesses. The estimates stabilize quickly and approach the true value after a few iterations, demonstrating the efficiency of the method.

The method converges reliably for different initial guesses, which suggests that it is not overly sensitive to starting points.

Log-Likelihood Analysis:

The log-likelihood analysis confirms that the maximum likelihood estimate is correct, as the likelihood is maximized at the true value of r. This serves as a validation of the optimization procedure, as the method accurately locates the value that maximizes the likelihood given the observed data.

Noise and Sample Size:

The impact of noise on the estimation process was not explicitly tested, but it is well-understood that higher noise levels will increase the variability of the estimates. This could lead to a less reliable estimation of r. Future experiments could explore this by varying the noise level and examining how the method performs under different noise conditions.

Conclusion

Newton’s Method for estimating the parameter r of an exponential model from noisy observations. The method is shown to be effective at finding the true value of r quickly and accurately, even with a variety of initial guesses. The analysis of the log-likelihood function confirms that the method produces the maximum likelihood estimate, and the convergence of the estimates indicates the robustness of the method. While the results are promising, further experiments with varying noise levels and sample sizes would provide a more comprehensive understanding of the method's performance in different scenarios