دیورژانس = ؟

الف – $f_1(R) = R^n a_R$

$$\nabla \cdot f_1 = \frac{1}{R^2 \sin\theta}\left[\frac{\partial}{\partial R}\left(R^2 \sin\theta A_R\right) + \frac{\partial}{\partial \theta}\left(\underbrace{R\sin\theta A_\theta}_{0}\right) + \frac{\partial}{\partial \varphi}\left(\underbrace{R A_\varphi}_{0}\right)\right]$$

$$= \frac{1}{R^2 \sin\theta}\left[\frac{\partial}{\partial R}\left(\underbrace{R^2 \sin\theta \cdot R^n}_{R^{(2+n)}\sin\theta}\right)\right] = \frac{1}{R^2 \cancel{\sin\theta}} \times (2+n)\cancel{\sin\theta} \times R^{(n+1)} = (n+2)R^{(n-1)}$$

ب – $f_2(R) = {}^{k}/_{R^2}\, a_R$

$$\nabla \cdot f_2 = \frac{1}{R^2 \cancel{\sin\theta}}\left[\frac{\partial}{\partial R}\left(R^2 \cancel{\sin\theta} \times {}^{k}/_{R^2}\right)\right] = 0$$

f → تابع عددی

A → تابع برداری

اثبات → $\nabla\cdot(fA) = f\nabla\cdot A + A\cdot\nabla f$

$$A = A_x a_x + A_y a_y + A_z a_z \rightarrow \nabla\cdot(fA_x a_x + fA_y a_y + fA_z a_z) = \frac{\partial}{\partial x}(fA_x)+\frac{\partial}{\partial y}(fA_y)+\frac{\partial}{\partial z}(fA_z)$$

$$f\frac{\partial A_x}{\partial x} + f\frac{\partial A_y}{\partial y} + f\frac{\partial A_z}{\partial z} + A_x\frac{\partial f}{\partial x} + A_y\frac{\partial f}{\partial y} + A_z\frac{\partial f}{\partial z}$$

$$= f\Big(\underbrace{\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}}_{\nabla\cdot A}\Big) + \underbrace{A_x\frac{\partial f}{\partial x} + A_y\frac{\partial f}{\partial y} + A_z\frac{\partial f}{\partial z}}_{(\underbrace{A_x a_x + A_y a_y + A_z a_z}_{A})\cdot(\underbrace{\frac{\partial f}{\partial x}a_x + \frac{\partial f}{\partial y}a_y + \frac{\partial f}{\partial z}a_z}_{\nabla f})}$$

$$= f\nabla\cdot A + A\cdot\nabla f$$

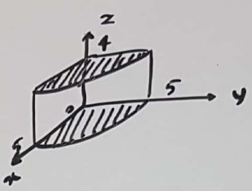
29- 2

$A = r^2 a_r + 2Z a_z$

$r=5$ ← نواحی محدود →

$z=0$

$z=4$



اثبات قضیه دیورژانس = ?

$$\int_v \nabla \cdot A \, dv = \oint_s A \cdot ds$$

$$\nabla \cdot A = \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_r) + \frac{\partial}{\partial \varphi}(A_\varphi) + \frac{\partial}{\partial z}(r A_z)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(\overbrace{r \cdot r^2}^{r^3}) + \frac{\partial}{\partial \varphi}(0) + \frac{\partial}{\partial z}(r \cdot 2z)\right]$$

$$= \frac{1}{r}\left[3r^2 + 2r\right] = 3r + 2$$

$$\int \nabla \cdot A \, dv = \int_0^4\int_0^{2\pi}\int_0^5 (3r+2)\, r\,dr\,d\varphi\,dz = \iiint (3r^2+2r)\,dr\,d\varphi\,dz = (r^3+r^2)\Big]_0^5 \,(\varphi)\Big]_0^{2\pi}\, z\Big]_0^4 = (125+25)(2\pi)(4)$$

$$= \boxed{1200\pi}$$

$$\oint A \cdot ds = \underbrace{\int_{S_1} A \cdot ds_1}_{\text{سطح بالا}} + \underbrace{\int_{S_2} A \cdot ds_2}_{\text{سطح پایین}} + \underbrace{\int_{S_3} A \cdot ds_3}_{\text{سطح جانبی}}$$

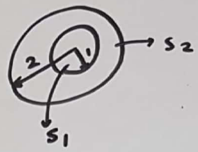
$$\begin{cases} ds_1 = r\,dr\,d\varphi\, a_z \\ ds_2 = -r\,dr\,d\varphi\, a_z \\ ds_3 = r\,d\varphi\,dz\, a_r \end{cases}$$

$$= \int_0^{2\pi}\int_0^5 2\underbrace{z}_{z=4}\, r\,dr\,d\varphi + \int_0^{2\pi}\int_0^5 -2\underbrace{z}_{z=0}\, r\,dr\,d\varphi + \int_0^4\int_0^{2\pi} \underbrace{r^2}_{r=5} \cdot r\,d\varphi\,dz = \underbrace{2(4) \times \frac{r^2}{2}\Big]_0^5 \times 2\pi}_{\frac{16\pi}{2} \times 25 = 200\pi} + 0 + \underbrace{125 \times 2\pi \times 4}_{1000\pi} = \boxed{1200\pi}$$

بدین ترتیب قضیه دیورژانس اثبات می‌شود.

$D = \cos^2\varphi / R^3 \; a_R$

$\begin{cases} R=1 \\ R=2 \end{cases} \xrightarrow{\text{در ناحیه‌ی بین}}$ ۲ پوسته‌ی کروی



الف) $\oint D\cdot ds = ? \longrightarrow \vec{ds} = R^2 \sin\theta \, d\theta d\varphi \, a_R$

$\oint D\cdot ds = \oint \frac{R^2 \sin\theta \cdot \cos^2\varphi}{R^3} d\theta d\varphi = \int_0^{2\pi}\int_0^{\pi} \frac{\sin\theta \cos^2\varphi}{R} d\theta d\varphi$

$\int \sin\theta \, d\theta = -\cos\theta \Big]_0^{\pi} = -(\cos\pi - \cos 0) = \boxed{2}$

$\int \cos^2\varphi \, d\varphi = \int \frac{1+\cos 2\varphi}{2} d\varphi = \frac{1}{2}\varphi + \frac{1}{2}\sin 2\varphi \times \frac{1}{2} \Big]_0^{2\pi}$

$= \frac{1}{2}(2\pi) + \frac{1}{4}(\underbrace{\sin 4\pi - \sin 0}_{0}) = \boxed{\pi}$

$\oint D\cdot ds = 2\pi / R$
- $\to$ روی سطح S_1 $(R_1=1) \Rightarrow -2\pi \to$ منفی بودن به این خاطر است که عمود بر سطح S_1 در جهت $-a_R$ می‌باشد.
- $\to$ روی سطح S_2 $(R_2=2) \Rightarrow \pi$ $\downarrow$ عمود بر سطح S_2 در جهت a_R است.

$\Rightarrow \oint_{\text{کل}} D\cdot ds = \oint_{S_1} D\cdot ds + \oint_{S_2} D\cdot ds = -2\pi + \pi = \boxed{-\pi}$

32-2

ب) $\int \nabla\cdot D \, dv = ?$

$\nabla\cdot D = \frac{1}{R^2 \sin\theta} \frac{\partial}{\partial R}\left(R^2 \sin\theta \times \frac{\cos^2\varphi}{R^3}\right)$

$= \frac{\cos^2\varphi}{R^2 \cancel{\sin\theta}} \times \cancel{\sin\theta} \frac{\partial}{\partial R}\left(\frac{1}{R}\right)^{-1/R^2}$

$= -\frac{\cos^2\varphi}{R^4}$

$\int_0^{2\pi}\int_0^{\pi}\int_1^{2} -\frac{\cos^2\varphi}{\cancel{R^4}_{R^2}} \cancel{R^2} \sin\theta \, dR d\theta d\varphi$

$\int -\frac{1}{R^2} dR = \frac{1}{R}\Big]_1^2 = \frac{1}{2} - 1 = \boxed{-\frac{1}{2}}$

$\int \sin\theta \, d\theta = -\cos\theta \Big]_0^{\pi} = \boxed{2}$

$\int \underbrace{\cos^2\varphi}_{\frac{1+\cos 2\varphi}{2}} d\varphi = \frac{1}{2}\varphi + \frac{1}{2}\times\frac{1}{2}\sin 2\varphi \Big]_0^{2\pi} = \boxed{\pi}$

$= (-\frac{1}{2})(2)(\pi) = \boxed{-\pi}$