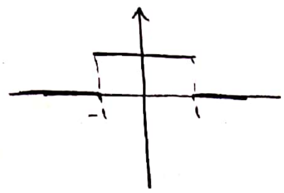
«به نام خدا»

پاسخ تمرین سری سوم ریاضی مهندسی (سری فوریه)

۱- انتگرال فوریه :

الف) $f(x) = \begin{cases} 1 & |x|<1 \quad (-1<x<1) \\ 0 & |x|>1 \quad (x>1,\ x<-1) \end{cases}$



$\rightarrow$ تابع زوج

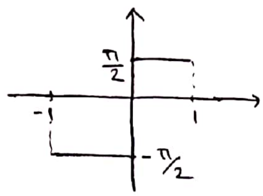
$B(\omega)=0$

$A(\omega)=\checkmark$

$$A(\omega)=\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx = 2\int_{0}^{\infty} f(x)\cos\omega x\,dx = 2\int_{0}^{1} 1\cos\omega x\,dx = \frac{2}{\omega}\sin\omega x\Big]_0^1 = \boxed{\frac{2\sin\omega}{\omega}}$$

$$\Rightarrow f(x)=\frac{1}{\pi}\int_0^{\infty}\left[A(\omega)\cos\omega x+\cancel{B(\omega)\sin\omega x}\right]d\omega = \boxed{\frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega}{\omega}\cos\omega x\,d\omega} \checkmark$$

ب) $f(x) = \begin{cases} \pi/2 & 0<x<1 \\ -\pi/2 & -1<x<0 \end{cases}$



$\rightarrow$ تابع فرد

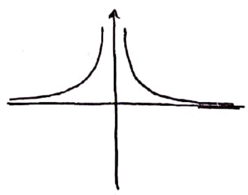
$B(\omega)=\checkmark$

$A(\omega)=0$

$$B(\omega)=\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx = 2\int_{0}^{\infty} f(x)\sin\omega x\,dx = 2\int_{0}^{1}\frac{\pi}{2}\sin\omega x\,dx = \frac{-\pi}{\omega}\cos\omega x\Big]_0^1 = \boxed{\frac{-\pi}{\omega}(\cos\omega-1)}$$

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\left[\cancel{A(\omega)\cos\omega x}+B(\omega)\sin\omega x\right]d\omega = \boxed{\int_0^{\infty}\frac{1-\cos\omega}{\omega}\sin\omega x\,d\omega} \checkmark$$

ج) $f(x)=e^{-|x|} = \begin{cases} e^{-x} & x>0 \\ e^{x} & x<0 \end{cases}$



$\rightarrow$ تابع زوج

$A(\omega)=\checkmark$

$B(\omega)=0$

$$A(\omega)=\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx = 2\int_{0}^{\infty} f(x)\cos\omega x\,dx = 2\int_{0}^{\infty} e^{-x}\cos\omega x\,dx$$

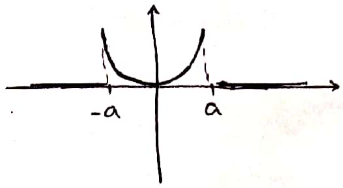
مشتق	انتگرال
e^{-x} (+)	$\cos\omega x$
$-e^{-x}$ (−)	$\frac{1}{\omega}\sin\omega x$
e^{-x} (+)	$\frac{-1}{\omega^2}\cos\omega x$

$$\Rightarrow \underbrace{\int_0^{\infty} e^{-x}\cos\omega x\,dx}_{I} = \cancel{\frac{e^{-x}}{\omega}\sin\omega x\Big]_0^{\infty}} - \frac{e^{-x}}{\omega^2}\cos\omega x\Big]_0^{\infty} - \underbrace{\int_0^{\infty}\frac{e^{-x}}{\omega^2}\cos\omega x\,dx}_{I/\omega^2}$$

$$\Rightarrow \left(1+\frac{1}{\omega^2}\right)I = \frac{1}{\omega^2} \longrightarrow \boxed{I=\frac{1}{1+\omega^2}} \xrightarrow{A(\omega)=2I} \boxed{A(\omega)=\frac{2}{1+\omega^2}}$$

$$f(x)=\frac{1}{\pi}\int_0^{\infty}A(\omega)\cos\omega x\,d\omega = \frac{2}{\pi}\int_0^{\infty}\frac{1}{1+\omega^2}\cos\omega x\,d\omega \checkmark$$

$$ج)\quad f(x)=\begin{cases} x^2 & |x|<a \rightarrow -a<x<a \\ 0 & |x|>a \rightarrow \begin{matrix} x>a \\ x<-a \end{matrix} \end{cases}$$



↳ تابع زوج $A(\omega)\checkmark$, $B(\omega)=0$

$$A(\omega)=\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx = 2\int_0^{\infty} f(x)\cos\omega x\,dx = 2\int_0^{a} x^2\cos\omega x\,dx$$

مشتق	انتگرال
x^2 (+)	$\cos\omega x$
$2x$ (−)	$\frac{1}{\omega}\sin\omega x$
2 (+)	$\frac{-1}{\omega^2}\cos\omega x$
0	$\frac{-1}{\omega^3}\sin\omega x$

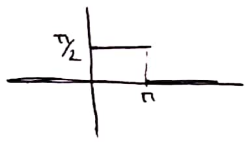
$$=2\left[\frac{x^2}{\omega}\sin\omega x+\frac{2x}{\omega^2}\cos\omega x-\frac{2}{\omega^3}\sin\omega x\right]_0^a$$

$$=2\left[\frac{a^2}{\omega}\sin a\omega+\frac{2a}{\omega^2}\cos a\omega-\frac{2}{\omega^3}\sin a\omega\right]=\frac{2}{\omega}\left[\left(a^2-\frac{2}{\omega^2}\right)\sin a\omega+\frac{2a}{\omega}\cos a\omega\right]$$

$$\rightarrow f(x)=\frac{1}{\pi}\int_0^{\infty} A(\omega)\cos\omega x\,d\omega=\frac{2}{\pi}\int_0^{\infty}\left[\left(a^2-\frac{2}{\omega^2}\right)\sin a\omega+\frac{2a}{\omega}\cos a\omega\right]\frac{\cos\omega x}{\omega}\,d\omega \quad \checkmark$$

۲- نشان دهید:

$$الف)\quad \int_0^{\infty}\frac{1-\cos\pi\omega}{\omega}\sin\omega x\,d\omega=\begin{cases} \pi/2 & 0<x<\pi \\ 0 & x>\pi \end{cases}$$



بسط فرد می‌دهیم

تابع فرد

با توجه به این عبارت فقط کافی است انتگرال سینوسی ($B(\omega)$) را بدست آوردم.

$$B(\omega)=\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx=2\int_0^{\infty} f(x)\sin\omega x\,dx=2\int_0^{\pi}\frac{\pi}{2}\sin\omega x\,dx=-\frac{\pi}{\omega}\cos\omega x\Big]_0^{\pi}=\boxed{-\frac{\pi}{\omega}\left[\cos\pi\omega-1\right]}$$

$$f(x)=\frac{1}{\pi}\int_0^{\infty}B(\omega)\sin\omega x\,d\omega=\int_0^{\infty}\frac{1-\cos\pi\omega}{\omega}\sin\omega x\,d\omega=\begin{cases} \pi/2 & 0<x<\pi \\ 0 & x>\pi \end{cases} \quad \checkmark$$

$$ب)\quad \int_0^{\infty}\frac{\cos\omega x}{1+\omega^2}\,d\omega=\frac{\pi}{2}e^{-x} \qquad x>0$$

$\underbrace{\frac{\cos\omega x}{1+\omega^2}}_{A(\omega)\checkmark \rightarrow \text{تابع زوج}}$

$$A(\omega)=\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx=2\int_0^{\infty}\frac{\pi}{2}e^{-x}\cos\omega x\,dx$$

در سوال اول تمرین ج انتگرال معادل محاسبه شده است.

$$\Rightarrow \boxed{A(\omega)=\frac{\pi}{1+\omega^2}}$$

$$f(x)=\frac{1}{\pi}\int_0^{\infty}A(\omega)\cos\omega x\,d\omega=\int_0^{\infty}\frac{1}{1+\omega^2}\cos\omega x\,d\omega=\frac{\pi}{2}e^{-x} \qquad x>0 \quad \checkmark$$

ج) $$\int_0^\infty \frac{\cos\omega x + \omega \sin \omega x}{1+\omega^2}\, d\omega = \begin{cases} 0 & x<0 \\ \pi/2 & x=0 \\ \pi e^{-x} & x>0 \end{cases} \longrightarrow \text{نقطه ناپیوستگی} : f(x) = \frac{f(0^+)+f(0^-)}{2} = \frac{\pi e^{-x} + 0}{2} = \frac{\pi + 0}{2} = \boxed{\pi/2}$$ ✓

$\underbrace{\qquad\qquad}_{B(\omega)\checkmark,\ A(\omega)\checkmark}$

(تابع نه زوج نه فرد)

$$A(\omega) = \int_{-\infty}^{\infty} f(x)\cos\omega x\, dx = \cancel{\int_{-\infty}^{0} 0} + \int_0^\infty \pi e^{-x} \cos\omega x\, dx = \boxed{\frac{\pi}{1+\omega^2}}$$

انتگرال در قسمت ج سوال ۱ حل شده است.

$$B(\omega) = \int_{-\infty}^{\infty} f(x) \sin\omega x\, dx = \cancel{\int_{-\infty}^{0} 0} + \int_0^\infty \pi e^{-x} \sin \omega x\, dx$$

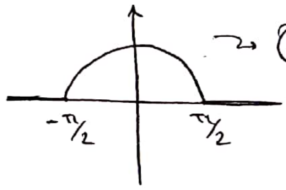
مشتق	انتگرال
e^{-x} (+)	$\sin\omega x$
$-e^{-x}$ (−)	$\frac{-1}{\omega}\cos\omega x$
e^{-x} (+∫)	$\frac{-1}{\omega^2}\sin\omega x$

$$\Rightarrow \underbrace{\int_0^\infty e^{-x} \sin\omega x\, dx}_{I} = -\frac{e^{-x}}{\omega}\cos\omega x - \cancel{\frac{e^{-x}}{\omega^2}\sin\omega x}\Big]_0^\infty - \frac{1}{\omega^2}\underbrace{\int_0^\infty e^{-x}\sin\omega x\, dx}_{I/\omega^2}$$

$$\Rightarrow \left(1+\frac{1}{\omega^2}\right) I = \frac{1}{\omega} \longrightarrow \boxed{I = \frac{\omega}{1+\omega^2}} \xRightarrow{B(\omega)=\pi I} \boxed{B(\omega) = \frac{\omega\pi}{1+\omega^2}}$$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^\infty \left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right] d\omega = \int_0^\infty \left[\left(\frac{1}{1+\omega^2}\right)\cos\omega x + \frac{\omega}{1+\omega^2}\sin\omega x\right] d\omega = \begin{cases} 0 & x<0 \\ \pi e^{-x} & x>0 \end{cases}$$ ✓

د) $$\int_0^\infty \frac{\cos\frac{\pi\omega}{2}\cos\omega x}{1-\omega^2}\, d\omega = \begin{cases} \frac{\pi}{2}\cos x & |x|<\pi/2 \rightarrow -\frac{\pi}{2}<x<\pi/2 \\ 0 & |x|>\pi/2 \rightarrow x>\pi/2,\ x<-\pi/2 \end{cases}$$



→ تابع زوج

$A(\omega)\checkmark,\ B(\omega)=0$

$$A(\omega) = \int_{-\infty}^{\infty} f(x)\cos\omega x\, dx = 2\int_0^{\pi/2} \frac{\pi}{2}\cos x \cos\omega x\, dx = \frac{\pi}{2}\int_0^{\pi/2}\left[\cos(1+\omega)x + \cos(1-\omega)x\right] dx$$

$$= \frac{\pi}{2}\left[\frac{1}{1+\omega}\sin(1+\omega)x + \frac{1}{1-\omega}\sin(1-\omega)x\right]_0^{\pi/2} = \frac{\pi}{2}\left[\frac{\overset{\cos\frac{\pi\omega}{2}}{\cancel{\sin(1+\omega)\pi/2}}}{1+\omega} + \frac{\overset{\cos\frac{\omega\pi}{2}}{\cancel{\sin(1-\omega)\pi/2}}}{1-\omega}\right] \star$$

$$\longrightarrow A(\omega) = \frac{\pi}{2}\left[\frac{2\cos\frac{\omega\pi}{2}}{1-\omega^2}\right] = \boxed{\frac{\pi\cos\omega\pi/2}{1-\omega^2}} \Rightarrow f(x) = \int_0^\infty \frac{\cos\omega\pi/2}{1-\omega^2}\cos\omega x\, d\omega$$ ✓

$\star$ $\sin(a\pm b)x = \sin ax \cos bx \pm \sin bx \cos ax$

$$\sin(1\pm\omega)\pi/2 = \overset{1}{\cancel{\sin\pi/2}}\cos\frac{\omega\pi}{2} \pm \sin\frac{\omega\pi}{2}\overset{0}{\cancel{\cos\pi/2}} = \cos\omega\pi/2$$