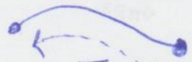
انتگرال: خط ← $\bar{dl}$ عنصر طول

سطح ← $\bar{ds}$ عنصر سطح

حجم ← dv عنصر حجم

$A(u_1, u_2, u_3)$



$\bar{dl}$ $\quad B(u_1+du_1, u_2+du_2+u_3+du_3)$

$\bar{dl}_1 : \begin{cases} u_1 \to u_1 + du_1 \\ u_2 = \text{ثابت} \\ u_3 = \text{ثابت} \end{cases} \qquad \bar{dl}_2 : \begin{cases} u_2 \to u_2 + du_2 \\ u_1 = \text{ثابت} \\ u_3 = \text{ثابت} \end{cases}$

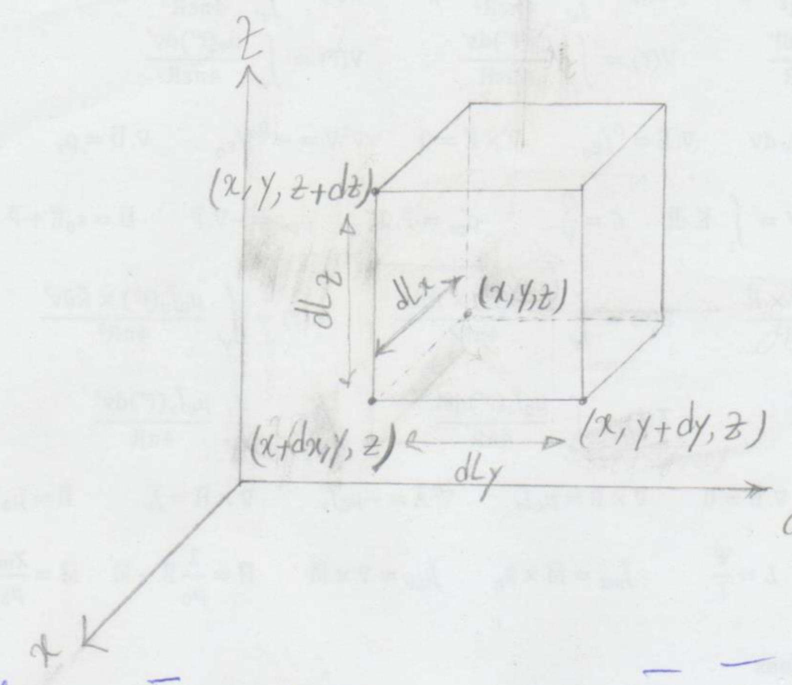
$\bar{dl}_3 : \begin{cases} u_3 \to u_3 + du_3 \\ u_1 = \text{ثابت} \\ u_2 = \text{ثابت} \end{cases}$

$\bar{dl}_i$ با du_i $(i=1,2,3)$ برابر نیست .

$$\bar{dl}_i = h_i \, du_i \, \hat{a}_{l_i} \, , \quad i = 1,2,3$$

$$h_i(u_1, u_2, u_3)$$

دستگاه مختصات کارتزین:

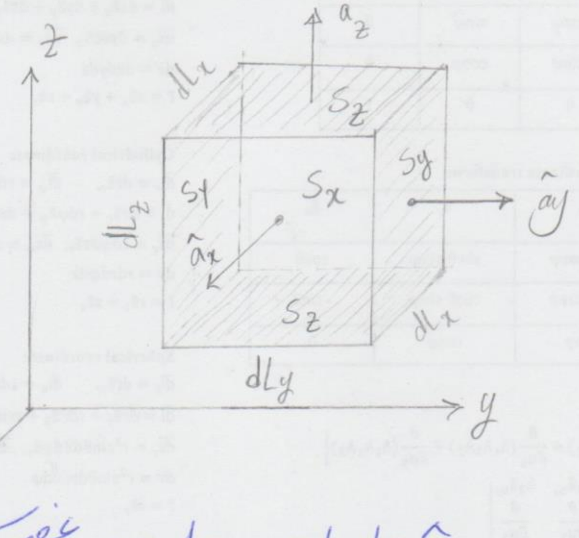


بردار جزء طول

جزء طول:

$$\overline{dL_x} = dx\,\hat{a}_x, \quad h_x = 1$$
$$\overline{dL_y} = dy\,\hat{a}_y, \quad h_y = 1$$
$$\overline{dL_z} = dz\,\hat{a}_z, \quad h_z = 1$$

$$\overline{dL} = \overline{dL_x} + \overline{dL_y} + \overline{dL_z} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$$



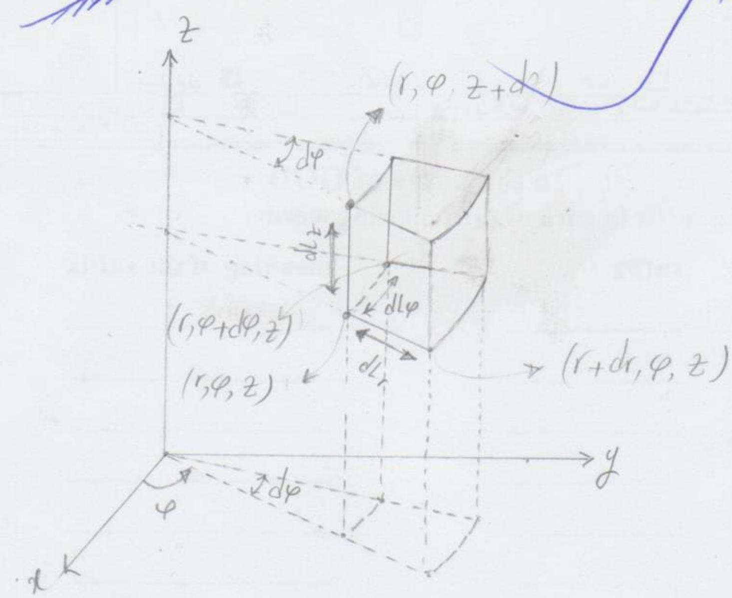
جزء سطح:

$$\begin{cases} ds_x = \pm dL_y\, dL_z\, \hat{a}_x \\ ds_y = \pm dL_x\, dL_z\, \hat{a}_y \\ ds_z = \pm dL_x\, dL_y\, \hat{a}_z \end{cases} \longrightarrow \text{جزء سطح:} \begin{cases} ds_x = \pm dy\, dz\, \hat{a}_x \\ ds_y = \pm dx\, dz\, \hat{a}_y \\ ds_z = \pm dx \cdot dy\, \hat{a}_z \end{cases}$$

جزء حجم:

$$dv = dL_x \cdot dL_y\, dL_z = dx \cdot dy \cdot dz$$

دستگاه مختصات استوانه‌ای :



برای عنصر طول :

عنصر طول :

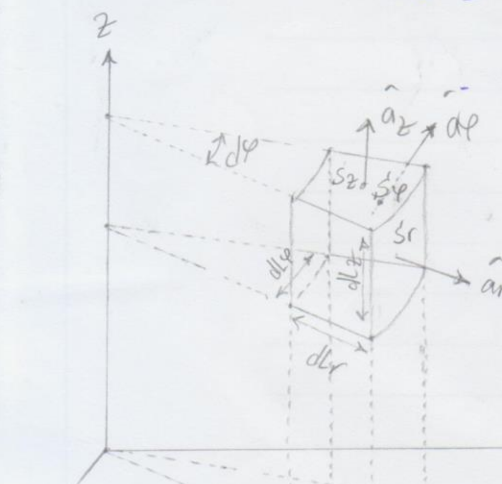
$$\overline{dL_r} = dr\,\hat{a}_r\,,\quad h_r = 1$$

$$\overline{dL_\varphi} = r\,d\varphi\,\hat{a}_\varphi\,,\quad h_\varphi = 1$$

$$\overline{dL_z} = dz\,\hat{a}_z\,,\quad h_z = 1$$

$$\overline{dL} = \overline{dL_r} + \overline{dL_\varphi} + \overline{dL_z}$$

$$= dr\,\hat{a}_r + r\,d\varphi\,\hat{a}_\varphi + dz\,\hat{a}_z$$



عنصر سطح :

$$\overline{ds_r} = \pm\, dL_\varphi\, dL_z\, \hat{a}_r = \pm\, r\,d\varphi\, dz\, \hat{a}_r$$

$$\overline{ds_\varphi} = \pm\, dL_r\, dL_z = \pm\, dr\, dz\, \hat{a}_\varphi$$

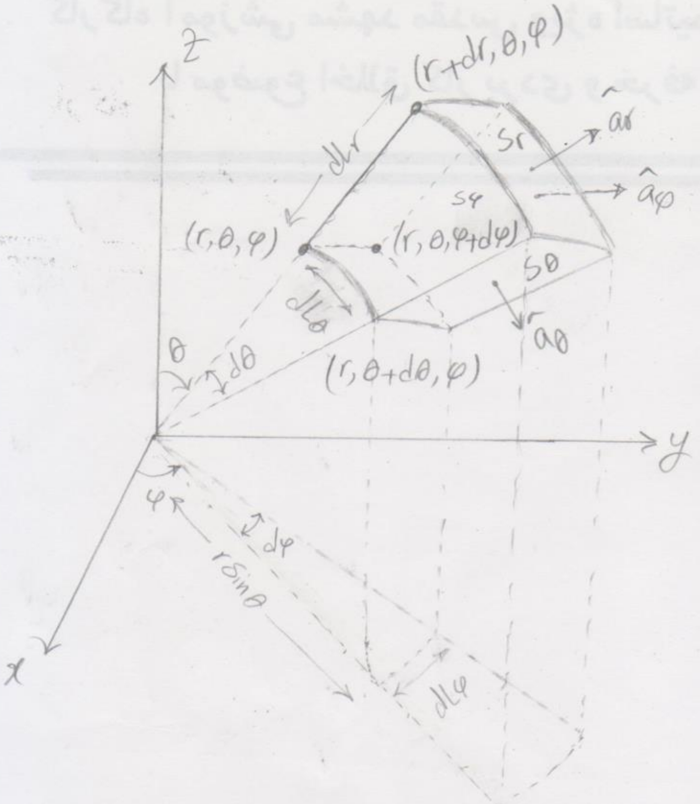
$$ds_z = \pm\, dL_r\, dL_\varphi = \pm\, r\,dr\,d\varphi\,\hat{a}_z$$

عنصر حجم :

$$dv = dL_r \cdot dL_\varphi \cdot dL_z$$

$$= r\,dr\,d\varphi\,dz$$

دستگاه مختصات کروی :



بردار عنصر طول

$$\underline{\overline{dL}} = \overline{dL_r} + \overline{dL_\theta} + \overline{dL_\varphi}$$
$$= dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\varphi\,\hat{a}_\varphi$$

عنصر طول :

$$\overline{dL_r} = dr\,\hat{a}_r \,, \quad h_r = 1$$
$$\overline{dL_\theta} = r\,d\theta\,\hat{a}_\theta \,, \quad h_\theta = r$$
$$dL_\varphi = r\sin\theta\,d\varphi\,\hat{a}_\varphi \,, \quad h_\varphi = r\sin\theta$$

عنصر سطح :

$$\overline{ds_r} = \pm\, dL_\theta\, dL_\varphi\, \hat{a}_r = \pm\, r^2 \sin\theta\, d\theta\, d\varphi\, \hat{a}_r$$
$$\overline{ds_\theta} = \pm\, dL_r . dL_\varphi\, \hat{a}_\theta = \pm\, r\sin\theta\, dr\, d\varphi\, \hat{a}_\theta$$
$$ds_\varphi = \pm\, dL_r .\, dL_\theta\, \hat{a}_\varphi = \pm\, r\, dr\, d\theta\, \hat{a}_\varphi$$

عنصر حجم :

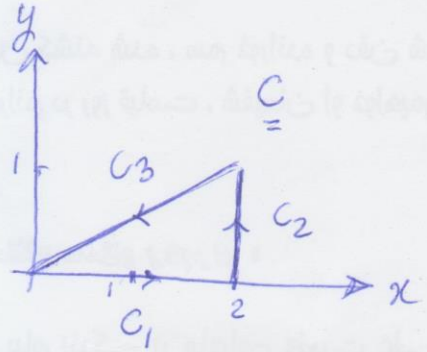
$$dv = dL_r .\, dL_\theta .\, dL_\varphi = r^2 \sin\theta\, dr . d\theta\, d\varphi$$

انتگرال خط

$$\int_C \bar{A}\cdot d\bar{l}\ ,\quad \int_C \bar{A}\times d\bar{l}\ ,\quad \int_C \bar{A}\, dl\ ,\quad \int_C A\, d\bar{l}$$

C: یک منحنی تکه‌ای دلخواه

مثال:



$$\int \bar{A}\cdot d\bar{l}$$

$$\bar{A} = (2x+y^2)\,\hat{a}_x + (3y-4x)\,\hat{a}_y$$

حل: با محدوده بودن (مثال ثابت)

$$\oint_C \bar{A}\cdot d\bar{l} = \int_{C_1} \bar{A}\cdot d\bar{l} + \int_{C_2} \bar{A}\cdot d\bar{l} + \int_{C_3} \bar{A}\cdot d\bar{l}$$

$$C_1: \begin{cases} y=0, & \text{ثابت} \\ z=0, & \text{ثابت} \\ 0\leqslant x\leqslant 2 \end{cases} \longrightarrow d\bar{L} = d\bar{L}_x = dx\,\hat{a}_x$$

$$\bar{A} = 2x\,\hat{a}_x - 4x\,\hat{a}_y\ ,\quad \bar{A}\cdot d\bar{l} = (2x\hat{a}_x - 4x\,\hat{a}_y)\cdot dx\,\hat{a}_n = 2x\,dx$$

$$\int_{C_1} \bar{A}\cdot d\bar{l} = \int_{x=0}^{x=2} 2x\,dx = x^2\Big|_{x=0}^{2} = \underline{\underline{4}}$$

$$C_2: \begin{cases} x = 2 \quad \text{ثابت} \\ z = 0 \quad \text{ثابت} \\ 0 \leq y \leq 1 \end{cases} \rightarrow \overline{dl} = \overline{dl}_y = dy\, a_y$$

$$\bar{A} = (4+y^2)\,\hat{a}_x + (3y-8)\,\hat{a}_y$$

$$\bar{A}\cdot\overline{dl} = \left[(4+y^2)\hat{a}_x + (3y-8)\hat{a}_y\right]\cdot dy\,\hat{a}_y = (3y-8)\,dy$$

$$\int_{C_2} \bar{A}\cdot\overline{dl} = \int_{y=0}^{y=1} (3y-8)\,dy = \frac{3}{2}y^2 - 8y \Big|_{y=0}^{1} = \frac{3}{2} - 8 = \frac{-13}{2}$$

$$C_3: \begin{cases} z = 0, \quad \text{ثابت} \\ 2 \leq x \leq 0 \\ 1 \leq y \leq 0 \end{cases} \qquad \overline{dl} = \overline{dl}_x + \overline{dl}_y = dx\,\hat{a}_x + dy\,\hat{a}_y$$

$$\begin{cases} y = \frac{1}{2}x \\ dy = \frac{1}{2}dx \end{cases} \quad ; C_3 \text{ معادله خط}$$

$$\bar{A} = (2x+y^2)\,\hat{a}_x + (3y-4x)\,\hat{a}_y \Big|_{y=\frac{1}{2}x} = (2x + \frac{1}{4}x^2)\,\hat{a}_x + (\frac{3}{2}x - 4x)\,\hat{a}_y$$

$$= (2x + \frac{x^2}{4})\,\hat{a}_x - \frac{5}{2}x\,\hat{a}_y$$

$$\bar{A}\cdot\overline{dl} = \left[(2x + \frac{x^2}{4})\hat{a}_x - \frac{5}{2}x\,\hat{a}_y\right]\cdot(dx\,\hat{a}_x + \frac{1}{2}dx\,\hat{a}_y)$$

$$= (2x + \frac{x^2}{4})\,dx - \frac{5}{2}x \cdot \frac{1}{2}dx = (\frac{3}{4}x + \frac{x^2}{4})\,dx$$

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$$\int_{C_3} \bar{A}\cdot d\bar{l} = \int_{x=2}^{0} \left(\frac{3}{4}x + \frac{x^2}{4}\right)dx = \frac{3}{8}x^2 + \frac{x^3}{12}\Big|_{x=2}^{0} = -\frac{13}{6}$$

پس $\oint_C \bar{A}\cdot d\bar{l} = +4 - \frac{13}{2} - \frac{13}{6} = -\frac{14}{3}$

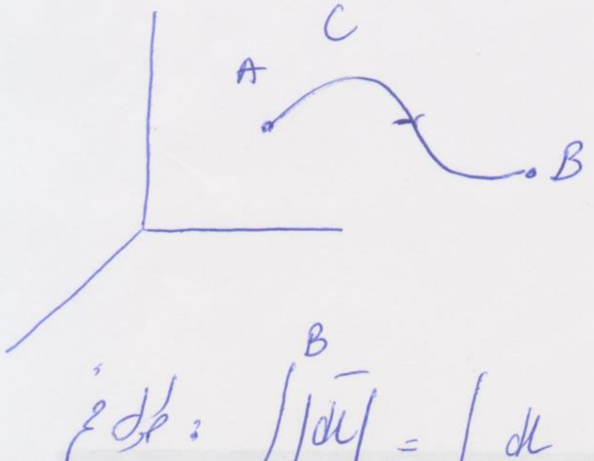
راه حل دوم :

$$\oint_C \bar{A}\cdot d\bar{l} = \oint_C (A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z)\cdot(dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z)$$

$$= \oint_C (A_x\,dx + A_y\,dy + A_z\,dz)$$

با محدوده خودتان (حل در کتاب)

مثال: طول خم منحنی :



طول خم : $\int_A^B |d\bar{l}| = \int_C dl \quad , \quad d\bar{l} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$

$$|d\bar{l}| = (dx^2 + dy^2 + dz^2)^{1/2}$$

$$= dx\left(1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right)^{1/2}$$

انتگرال سطح : $\int_S \bar{A}\cdot\bar{ds}$ و $\bar{ds} = ds\,\hat{a}_n$

$\hat{a}_n$: بردار عمود بر سطح

- در حالت کلی برای $\bar{ds}$ دو جهت مخالف یکدیگر وجود دارد
- اگر سطح بسته باشد، جهت $\bar{ds}$ عمود بر سطح و به سمت خارج سطح است.

نحوه محاسبه بردار عمود بر سطح :

$f(u_1, u_2, u_3) = K$ معادله کلی سطح S

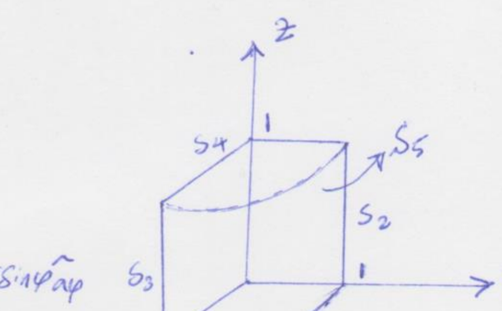
$$\bar{ds} = ds\,\hat{a}_n \quad , \quad \hat{a}_n = \frac{\bar{N}}{|N|}$$

$$\bar{N} = \nabla f = \frac{\partial f}{\partial x}\hat{a}_x + \frac{\partial f}{\partial y}\hat{a}_y + \frac{\partial f}{\partial z}\hat{a}_z$$

مثال :

$\oint_S \bar{A}\cdot\bar{ds}$

$\bar{A} = r\cos\varphi\,\hat{a}_r - r\sin\varphi\,\hat{a}_\varphi$



حل :

$$\oint_S \bar{A}\cdot\bar{ds} = \int_{S_1}\bar{A}\cdot\bar{ds} + \int_{S_2}\bar{A}\cdot\bar{ds} + \int_{S_3}\bar{A}\cdot\bar{ds} + \int_{S_4}\bar{A}\cdot\bar{ds} + \int_{S_5}\bar{A}\cdot\bar{ds}$$

دانشگاه محقق اردبیلی

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$\underline{S_1}$: $\begin{cases} z=0 \\ 0 \leq \varphi \leq \pi/2 \\ 0 \leq r \leq 1 \end{cases} \rightarrow$ $\overline{ds} = ds_z = \pm r\,dr\,d\varphi\,\hat{a}_z$

$\overline{ds} = -r\,dr\,d\varphi\,\hat{a}_z$

$\overline{A}\cdot\overline{ds} = (r\cos\varphi\,\hat{a}_r - r\sin\varphi\,\hat{a}_\varphi)\cdot(-r\,dr\,d\varphi\,\hat{a}_z) = 0 \rightarrow \int_{S_1} \overline{A}\cdot\overline{ds} = 0$

$\underline{S_2}$: $\begin{cases} \varphi = \pi/2 \\ 0 \leq r \leq 1 \\ 0 \leq z \leq 1 \end{cases} \rightarrow$ $\overline{ds} = \overline{ds}_\varphi = \pm dr\,dz\,\hat{a}_\varphi$

$\overline{ds} = +dr\,dz\,\hat{a}_\varphi$

$\overline{A}\cdot\overline{ds} = (r\cos\varphi\,\hat{a}_r - r\sin\varphi\,\hat{a}_\varphi)\cdot(dr\,dz\,\hat{a}_\varphi) = -r\sin\varphi\,dr\,dz$

$\int_{S_2} \overline{A}\cdot\overline{ds} = \int_{r=0}^{1}\int_{z=0}^{1} -r\sin\varphi\,dr\,dz \Big|_{\varphi=\pi/2} = -\frac{r^2}{2}\Big|_{r=0}^{1}\, z\Big|_{z=0}^{1} = -\frac{1}{2}$

$\underline{S_3}$: $\begin{cases} \varphi = 0 \\ 0 \leq r \leq 1 \\ 0 \leq z \leq 1 \end{cases} \rightarrow$ $\overline{ds} = \overline{ds}_\varphi = \pm dr\,dz\,\hat{a}_\varphi$

$\overline{ds} = -dr\,dz\,\hat{a}_\varphi$

$\overline{A}\cdot\overline{ds} = (r\cos\varphi\,\hat{a}_r - r\sin\varphi\,\hat{a}_\varphi)\cdot(-dr\,dz\,\hat{a}_\varphi) = +r\sin\varphi\,dr\,dz \Big|_{\varphi=0} = 0$

$\int_{S_3} \overline{A}\cdot\overline{ds} = 0$

$$S_4: \begin{cases} z=1 \\ 0\leq r\leq 1 \\ 0\leq \varphi \leq \pi/2 \end{cases} \rightarrow \bar{ds} = ds_z = \pm\, r\,dr\,d\varphi\,\hat{a}_z$$

$$\bar{ds} = +\,r\,dr\,d\varphi\,\hat{a}_z$$

$$\bar{A}\cdot\bar{ds} = (r\cos\varphi\,\hat{a}_r - r\sin\varphi\,\hat{a}_\varphi)\cdot(r\,dr\,d\varphi\,\hat{a}_z) = $$

$$\int_{S_4}\bar{A}\cdot\bar{ds} = 0$$

$$S_5: \begin{cases} r=1 \\ 0\leq \varphi \leq \pi/2 \\ 0\leq z\leq 1 \end{cases} \rightarrow \bar{ds} = \bar{ds_r} = \pm\, r\,d\varphi\,dz\,\hat{a}_r$$

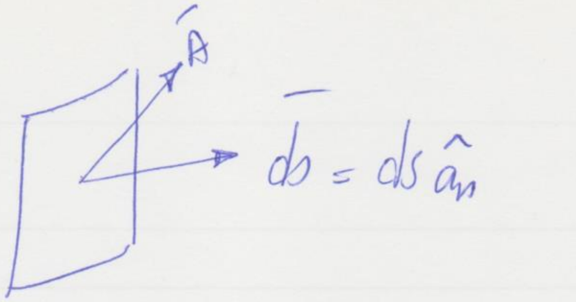
$$\bar{ds} = +\,r\,d\varphi\,dz\,\hat{a}_r$$

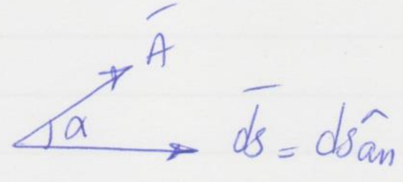
$$\bar{A}\cdot\bar{ds} = (r\cos\varphi\,\hat{a}_r - r\sin\varphi\,\hat{a}_\varphi)\cdot(r\,d\varphi\,dz\,\hat{a}_r) = r^2\cos\varphi\,d\varphi\,dz\Big|_{r=1} = \cos\varphi\,d\varphi\,dz$$

$$\int_{S_5}\bar{A}\cdot\bar{ds} = \int_{z=0}^{1}\int_{\varphi=0}^{\pi/2}\cos\varphi\,d\varphi\,dz = \sin\varphi\,.\,z\Big|_{\varphi=0}^{\pi/2}\Big|_{z=0}^{1} = 1$$

$$\therefore \oint_S \bar{A}\cdot\bar{ds} = 0 - \frac{1}{2} + 0 + 0 + 1 = \frac{1}{2}$$

$\bar{A}\cdot\bar{ds}$:





$$\bar{A}\cdot\bar{ds} = |\bar{A}|\, ds \cdot \cos\alpha$$

$$= \underbrace{|\bar{A}| \cos\alpha}\cdot ds$$

چه میزان $\bar{A}$، در راستای عمود بر سطح مولفه دارد

چه میزان $\bar{A}$ از سطح خارج میشود: شار، فوران از عنصر سطح $\bar{ds}$

$\int_s \bar{A}\cdot\bar{ds}$: میزان شار، فوران از سطح S

انتگرال حجم:

$$\int_v |\bar{A}|\, dv$$