ص ۱ جلسه پنجم

✓ فرم مختلط سری فوریه

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$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi}{T}x + b_n \sin\frac{2n\pi}{T}x$$

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \cos\theta = \frac{1}{2}\left[e^{(i\theta)} + e^{(-i\theta)}\right]$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \qquad \sin\theta = \frac{1}{2j}\left[e^{(i\theta)} - e^{(-i\theta)}\right]$$

اثبات برای $T=2\pi$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

اثبات می‌کنیم که

$$\rightarrow f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{j\frac{2\pi}{T}nx}$$

$$a_0 = \frac{1}{2\pi}\int_{T} f(x)\,dx \rightarrow a_0 = \frac{1}{T}\int_{\langle T\rangle} f(x)\,dx$$

$$\rightarrow C_n = \frac{1}{T}\int_{T} f(x) e^{-j\frac{2\pi}{T}xn}\,dx$$

$$a_n = \frac{1}{\pi}\int_{2\pi} f(x)\cos nx\,dx \rightarrow a_n = \frac{2}{T}\int_{\langle T\rangle} f(x)\cos\frac{2n\pi}{T}x\,dx$$

$$b_n = \frac{1}{\pi}\int_{2\pi} f(x)\sin nx\,dx \rightarrow b_n = \frac{2}{T}\int_{\langle T\rangle} f(x)\sin\frac{2n\pi}{T}x\,dx$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$\Rightarrow f(x) = a_0 + \sum_{n=1}^{\infty}\left\{\frac{a_n}{2}\left[e^{jnx} + e^{-jnx}\right] + \frac{b_n}{2j}\left[e^{jnx} - e^{-jnx}\right]\right\}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} e^{jnx}\underbrace{\left[\frac{a_n}{2} + \frac{b_n}{2j}\right]}_{C_n} + e^{-jnx}\underbrace{\left[\frac{a_n}{2} - \frac{b_n}{2j}\right]}_{K_n}$$

$$C_n = \frac{a_n}{2} + \frac{b_n}{2j} = \frac{1}{2}(a_n - jb_n) = \frac{1}{2}\left[\frac{1}{\pi}\int f(x)\cos nx\,dx + (-j)\times\frac{1}{\pi}\int f(x)\sin nx\,dx\right]$$

$$\frac{1}{2\pi}\left[\int f(x)(\cos nx - j\sin nx)\,dx\right] = \frac{1}{2\pi}\left[\int_{\langle 2\pi\rangle} f(x) e^{-jnx}\,dx\right]$$

$$K_n = \frac{a_n}{2} - \frac{b_n}{2j} = \frac{1}{2}(a_n + jb_n) = \frac{1}{2}\left[\frac{1}{\pi}\int f(x)\cos nx\,dx + j\times\frac{1}{\pi}\int f(x)\sin nx\,dx\right]$$

$$\frac{1}{2\pi}\left[\int f(x)(\cos nx + j\sin nx)\,dx\right] = \frac{1}{2\pi}\int f(x) e^{jnx}\,dx = C_{-n}$$

$a_0 = C_0$

* ختم:

$$f(x) = C_0 + \sum_{n=1}^{\infty} C_n e^{jnx} + \sum_{n=1}^{\infty} C_{-n} e^{-jnx} = \sum_{n=-\infty}^{+\infty} C_n e^{jnx}$$

s.a.m

مثال ۱: $-\pi<x<\pi$ ، $f(x)=e^x$ و $f(x)=f(x+2\pi)$. الف) ضرایب فرم حقیقی سری فوریه (a_0 و a_n و b_n) را محاسبه کنید. ب) ضرایب فرم مختلط سری فوریه c_n را محاسبه کنید. ج) رابطه $c_n=\frac{a_n}{2}-j\frac{b_n}{2}$ را تحقیق کنید.

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n\cos\frac{2n\pi}{T}x+b_n\sin\frac{2n\pi}{T}x$$

$$a_0=\frac{1}{T}\int_{\langle T\rangle} f(x)\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^x\,dx=\frac{1}{2\pi}\left[e^{\pi}-e^{-\pi}\right]=(e^{\pi}-e^{-\pi})/2\pi=\frac{1}{\pi}\sinh\pi$$

$$a_n=\frac{2}{T}\int_{\langle T\rangle} f(x)\cos\frac{2n\pi}{T}x\,dx=\frac{2}{2\pi}\int_{-\pi}^{\pi} e^x\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos nx\,dx$$

انتگرال	مشتق
e^x	$\cos nx$
e^x	$\frac{1}{n}\sin nx$
e^x	$-\frac{1}{n^2}\cos nx$

$$\int e^x\cos nx\,dx=\frac{e^x}{n}\sin nx+\frac{e^x}{n^2}\cos nx+\int e^x\left(-\frac{1}{n^2}\cos nx\right)dx$$

$$I\left(1+\frac{1}{n^2}\right)=\frac{e^x}{n}\sin nx+\frac{e^x}{n^2}\cos nx$$

$$I\left(\frac{n^2+1}{n^2}\right)=\frac{e^x}{n}\sin nx+\frac{e^x}{n^2}\cos nx$$

$$I=\left(\frac{n^2}{n^2+1}\right)\left[\frac{e^x}{n}\sin nx+\frac{e^x}{n^2}\cos nx\right]_{-\pi}^{\pi}=\left(\frac{n^2}{1+n^2}\right)\left[\frac{e^{\pi}}{n}\sin n\pi+\frac{e^{\pi}}{n^2}\cos n\pi-\left(\frac{e^{-\pi}}{n}\sin(-n\pi)+\frac{e^{-\pi}}{n^2}\cos(-n\pi)\right)\right]$$

$$\left(\frac{\cancel{n^2}}{1+n^2}\right)\left(\frac{e^{\pi}}{\cancel{n^2}}\cos n\pi-\frac{e^{-\pi}}{\cancel{n^2}}\cos n\pi\right)=\frac{\cos n\pi}{(1+n^2)}(e^{\pi}-e^{-\pi})$$

$$a_n=\frac{1}{\pi}\frac{\cos n\pi}{(1+n^2)}(e^{\pi}-e^{-\pi})=\frac{\cos n\pi}{(1+n^2)}\times\frac{e^{\pi}-e^{-\pi}}{\pi}=\frac{2\times(-1)^n}{\pi(1+n^2)}\sinh\pi \checkmark$$

$$b_n=\frac{2}{T}\int_0^{T} f(x)\sin\frac{2n\pi}{T}x\,dx=\frac{2}{2\pi}\int_{-\pi}^{\pi} e^x\sin nx\,dx$$

انتگرال	مشتق
e^x	$\sin nx$
e^x	$-\frac{1}{n}\cos nx$
e^x	$-\frac{1}{n^2}\sin nx$

$$\int_{-\pi}^{\pi} e^x\sin nx\,dx=-\frac{e^x}{n}\cos nx+\frac{e^x}{n^2}\sin nx+\int -\frac{e^x}{n^2}\sin nx\,dx$$

$$I\left(1+\frac{1}{n^2}\right)=\frac{-e^x}{n}\cos nx+\frac{e^x}{n^2}\sin nx$$

$$I=\frac{n^2}{n^2+1}\left[-\frac{e^x}{n}\cos nx+\frac{e^x}{n^2}\sin nx\right]_{-\pi}^{\pi}=\frac{n^2}{n^2+1}\left[-\frac{e^{\pi}}{n}\cos n\pi+\frac{e^{\pi}}{n^2}\sin n\pi-\left(-\frac{e^{-\pi}}{n}\cos n\pi+\frac{e^{-\pi}}{n^2}\sin n\pi\right)\right]$$

$$\frac{n^2}{1+n^2}\left[-\frac{e^{\pi}}{n}\cos n\pi+\frac{e^{-\pi}}{n}\cos n\pi\right]=\frac{n\cos n\pi}{1+n^2}\left[e^{-\pi}-e^{\pi}\right]=\frac{1}{\pi}\times\frac{-2n\cos n\pi}{1+n^2}\left[\frac{e^{\pi}-e^{-\pi}}{2}\right]=\frac{-2n\cos n\pi}{1+n^2}\sinh\pi$$

$$C_n=\frac{1}{T}\int_{\langle T\rangle} f(x)\,e^{-j\frac{2n\pi}{T}x}\,dx=\frac{1}{2\pi}\int e^{x}e^{-jnx}\,dx=\frac{1}{2\pi}\int e^{(1-jn)x}\,dx=$$

$$\frac{1}{2\pi(1-jn)}e^{(1-jn)x}\Big|_{-\pi}^{\pi}=\frac{1}{2\pi(1-jn)}\left[e^{(1-jn)\pi}-e^{(1-jn)(-\pi)}\right]=\frac{1}{2\pi(1-jn)}\left[e^{\pi}\times e^{-jn\pi}-e^{-\pi}e^{+jn\pi}\right]$$

$$\frac{1}{2\pi(1-jn)}\left[e^{\pi}(\cos(-n\pi)+i\sin(-n\pi))-e^{-\pi}(\cos n\pi+i\sin(n\pi))\right]=\frac{1\times\cos n\pi}{2\pi(1-jn)}\left[e^{\pi}-e^{-\pi}\right]=\frac{(-1)^n\sinh\pi}{\pi(1-jn)}$$

$$\frac{(-1)^n\sinh\pi}{\pi(1-jn)}\times\frac{(1+jn)}{(1+jn)}=\frac{(-1)^n\sinh\pi(1+jn)}{\pi(1+n^2)}=\underbrace{\frac{(-1)^n\sinh\pi\times1}{\pi(1+n^2)}}_{a_n/2}+j\,\frac{(-1)^n\sinh\pi\times n}{\pi(1+n^2)}=$$

$$\frac{a_n}{2}-j\frac{b_n}{2}$$

مثال ۲: ضریب C_0 در بسط سری فوریه مختلط تابع $-1<x<1$ و $f(x)=x$ را بدست آورید

$$C_n=\frac{1}{T}\int_{\langle T\rangle} f(x)\,e^{-j\frac{2n\pi}{T}x}\,dx\Rightarrow C_0=\frac{1}{T}\int_{\langle T\rangle} f(x)\,e^{-j\frac{2n\pi}{T}x0}\,dx$$ (قضیه صفحه ۷۷)

$$C_0=\frac{1}{2}\int_{-1}^{1}x\,dx=\frac{1}{2}\times\frac{x^2}{2}\Big|_{-1}^{1}=\frac{1}{4}(1^2-(-1)^2)=0$$

مثال ۳: با بسط فوریه تابع متناوب $-1<x<1$ و $f(x)=\cosh x$ ، حاصل سری مقابل:

$$C_n=\frac{1}{2}\left(\frac{a_n}{2}+\frac{b_n}{2j}\right)$$

$S=\sum_{n=-\infty}^{n=\infty}\frac{(-1)^n}{1+n^2\pi^2}$ را بدست آورید. (آزمایشی کارشناسی ارشد ۸۰)

$f(x)=\cosh x:$ تابع زوج است بنابراین a_n مقدار دارد و b_n برابر صفر خواهد بود.

$$b_n=0\ ,\quad a_n=\frac{2}{T}\int_{\langle T\rangle}f(x)\cos\frac{2n\pi}{T}x\,dx=\frac{2}{2}\int_{-1}^{1}\cosh x\cos n\pi x\,dx=\int_{-1}^{1}\cosh x\cos n\pi x\,dx$$

مشتق	انتگرال
$\cosh x$ +	$\cos n\pi x$
$\sinh x$ −	$\frac{1}{n\pi}\sin n\pi x$
$\cosh x$	$-\frac{1}{n^2\pi^2}\cos n\pi x$

$$\int_{-1}^{1}\cosh x\cos n\pi x\,dx=\frac{1}{n}\sin n\pi x\cosh x-\sinh x\left(-\frac{1}{n^2\pi^2}\cos n\pi x\right)+\int\cosh x\left(-\frac{1}{n^2\pi^2}\right)\cos n\pi x\,dx$$

$$I\left(1+\frac{1}{n^2\pi^2}\right)=\frac{1}{n\pi}\sin n\pi x+\frac{1}{n^2\pi^2}\sinh x\cos n\pi x\Big|_{-1}^{1}$$

$$I\left(1+\frac{1}{n^2\pi^2}\right)=\left(\frac{1}{n^2\pi^2}\sinh(1)\cos n\pi\right)-\left(\frac{1}{n^2\pi^2}\sinh(-1)\cos n\pi\right)$$

$$I\left(1+\frac{1}{n^2\pi^2}\right)=\frac{\cos n\pi}{n^2\pi^2}\times2\sinh(1)\Rightarrow I=\frac{1\times\cos n\pi}{1+n^2\pi^2}\times2\sinh(1)=\frac{2(-1)^n\sinh(1)}{1+n^2\pi^2}$$