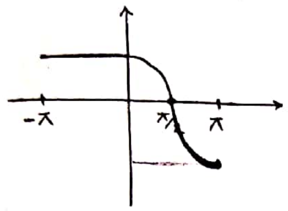
" به نام خدا "

پاسخ تمرین سری 7 ریاضی مهندسی (سری فوریه)

صورت مختلط سری فوریه تابع $f(x)=\begin{cases}1 & -\pi<x<0\\ \cos x & 0<x<\pi\end{cases}$ را بیابید.



$\rightarrow T=2\pi$

$$\cos x=\frac{e^{ix}+e^{-ix}}{2}$$
$$e^{i\theta}=\cos\theta+i\sin\theta$$

$$f(x)=\sum_{n=-\infty}^{\infty}C_n e^{inx}\quad,\quad C_n=\frac{1}{2\pi}\int f(x)e^{-inx}dx$$

$$C_n=\frac{1}{2\pi}\left(\int_{-\pi}^{0}1e^{-inx}dx+\int_{0}^{\pi}\cos x\,e^{-inx}dx\right)$$

$$=\frac{1}{2\pi}\left(\frac{1}{-in}e^{-inx}\Big]_{-\pi}^{0}+\int_0^{\pi}\left(\frac{e^{ix}+e^{-ix}}{2}\right)e^{-inx}dx\right)=\frac{1}{2\pi}\left(\frac{i}{n}-\frac{i}{n}e^{in\pi}+\frac{1}{2}\left[\int_0^{\pi}e^{ix}e^{-inx}dx+\int_0^{\pi}e^{-ix}e^{-inx}dx\right]\right)$$

$$=\frac{i}{2n\pi}\left(1-e^{in\pi}\right)+\frac{1}{4\pi}\left[\frac{e^{(1-n)ix}}{(1-n)i}+\frac{e^{-(1+n)ix}}{-(1+n)i}\right]_0^{\pi}$$

$$=\frac{i}{2n\pi}\left(1+(-1)^{n+1}\right)+\frac{1}{4\pi}\left[\frac{e^{(1-n)i\pi}}{(1-n)i}+\frac{e^{-(1+n)i\pi}}{-(1+n)i}-\frac{1}{(1-n)i}-\frac{1}{-(1+n)i}\right]$$

$$=\frac{i}{2n\pi}\left(1+(-1)^{n+1}\right)+\frac{i}{4\pi}\left[\frac{e^{(1-n)i\pi}-1}{(n-1)}+\frac{e^{-i\pi(1+n)}-1}{(n+1)}\right]$$

$$e^{i\pi}=\cos\pi+i\sin\pi=-1$$
$$e^{-ni\pi}=\cos n\pi-i\sin n\pi=(-1)^n$$
$$e^{-2i\pi}=\cos 2\pi-i\sin 2\pi=1$$

$$=\frac{i}{2n\pi}\left(1+(-1)^{n+1}\right)+\frac{in}{2\pi}\left(\frac{(-1)^{n+1}-1}{n^2-1}\right)\qquad \underline{n\neq 0,1}$$

$$C_0=\frac{1}{2\pi}\left(\int_{-\pi}^{0}dx+\int_0^{\pi}\cos x\,dx\right)=\frac{1}{2\pi}\left(x\Big]_{-\pi}^{0}+\cancel{\sin x}\Big]_0^{\pi}\right)=\boxed{\frac{1}{2}}$$

$$C_1=\frac{1}{2\pi}\left(\int_{-\pi}^{0}e^{-ix}dx+\int_0^{\pi}\cos x\,e^{-ix}dx\right)=\frac{1}{2\pi}\left(\frac{1}{-i}e^{-ix}\Big]_{-\pi}^{0}+\frac{1}{2}\int_0^{\pi}\underbrace{e^{ix}e^{-ix}}_{1}dx+\frac{1}{2}\int_0^{\pi}\underbrace{e^{-ix}e^{-ix}}_{e^{-2ix}}dx\right)$$

$$=\frac{1}{2\pi}\left[i-ie^{i\pi}+\frac{\pi}{2}+\frac{i}{4}e^{-2i\pi}-\frac{i}{4}\right]=\boxed{\frac{i}{\pi}+\frac{1}{4}}$$

$$\rightarrow f(x)=\frac{1}{2}+\left(\frac{i}{\pi}+\frac{1}{4}\right)e^{ix}+\sum_{\substack{n=-\infty\\ n\neq 0,1}}^{\infty}\left[\frac{i}{2n\pi}\left(1+(-1)^{n+1}\right)+\frac{in}{2\pi}\left(\frac{(-1)^{n+1}-1}{n^2-1}\right)\right]e^{inx}\ \checkmark$$