حل تمارین فصل 2

تمرین: تبدیل لاپلاس توابع زیر بدست آورید:

$$x(t) = \sin t \times u(t-1) = \sin(t-1+1)\,u(t-1)$$

$$x(t) = \sin(t-1)\underbrace{\cos(1)}\,u(t-1) + \underbrace{\sin 1}\cos(t-1)\,u(t-1)$$

مقدار ثابت

$$X(s) = \cos(1) \times \left(\frac{1}{s^2+1}\right) \times e^{-s} + \sin(1) \times \left(\frac{s}{s^2+1}\right) \times e^{-s}$$

$$y(t) = \sin(t-1)\,u(t) = \sin t \times \cos 1 \times u(t) - \sin 1 \cos t \times u(t)$$

$$Y(s) = \cos 1 \times \left(\frac{1}{s^2+1}\right) - \sin(1) \times \left(\frac{s}{s^2+1}\right)$$

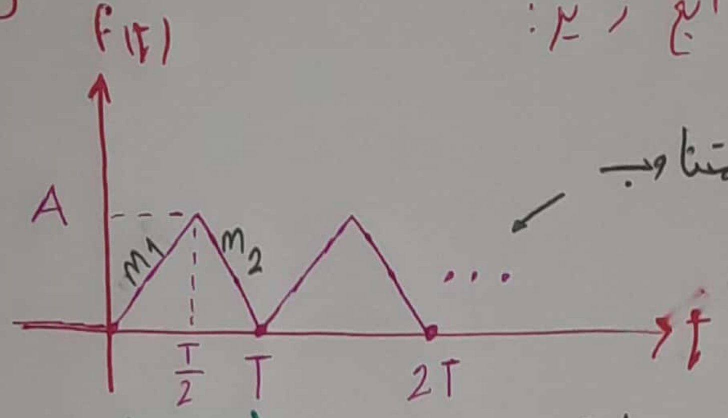
$$z(t) = \sin(t-1)\,u(t-1)$$

$\sin t\, u(t)$ اول هم شیفتو

$$Z(s) = \left(\frac{1}{s^2+1}\right) \times e^{-s}$$

ص 2

تمرین: تبدیل لاپلاس توابع زیر:



$$m_1 = \frac{A}{\frac{T}{2}} = \frac{2A}{T} \qquad m_2 = \frac{0-A}{\frac{T}{2}} = \frac{-2A}{T}$$

$$\hat{f}(t) = \frac{2A}{T} r(t) - \frac{4A}{T} r\left(t - \frac{T}{2}\right) + \frac{2A}{T} r(t-T)$$

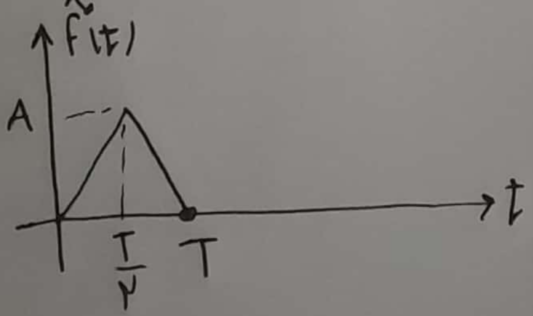
$$\hat{f}(t) = \frac{2A}{T}\left(t\,u(t) - 2\left(t - \frac{T}{2}\right)u\left(t - \frac{T}{2}\right) + (t-T)\,u(t-T)\right)$$

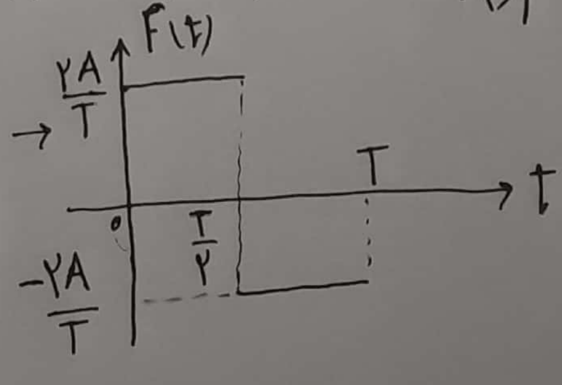
$$\hat{F}(s) = \frac{2A}{T}\left(\frac{1}{s^2} - 2e^{-\frac{T}{2}s} \times \frac{1}{s^2} + e^{-Ts}\frac{1}{s^2}\right)$$

$$F(s) = \frac{\hat{F}(s)}{1-e^{-Ts}} \xrightarrow{T=T} \cancel{F(s) =}$$

روش دوم:

$$f(t) \xleftrightarrow{s} F(s) \qquad f'(t) \xleftrightarrow{s} F_1(s) \qquad F(s) = \frac{F_1(s)}{s}$$





$$\hat{f}'(t) = \frac{2A}{T}u(t) - \frac{4A}{T}u\left(t - \frac{T}{2}\right) + \frac{2A}{T}u(t-T)$$

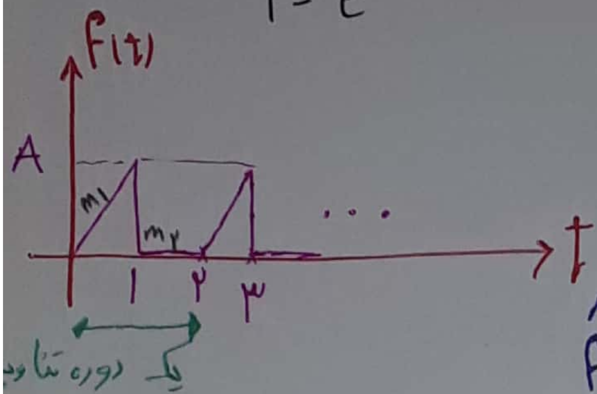
ادامه حل در ص 3

$$\hat{F}_1(s) = \frac{2A}{T}\left(\frac{1}{s} - 2e^{-\frac{T}{2}s}\times\frac{1}{s} + e^{-Ts}\times\frac{1}{s}\right)$$

$$\hat{F}(s) = \frac{\hat{F}_1(s)}{s} \longrightarrow \hat{F}(s) = \frac{2A}{T}\left(\frac{1}{s^2} - \frac{2}{s^2}e^{-\frac{T}{2}s} + \frac{1}{s^2}\times e^{-Ts}\right)$$

$$F(s) = \frac{\hat{F}(s)}{1-e^{-Ts}}$$

تمرین:



$m_1 = A$

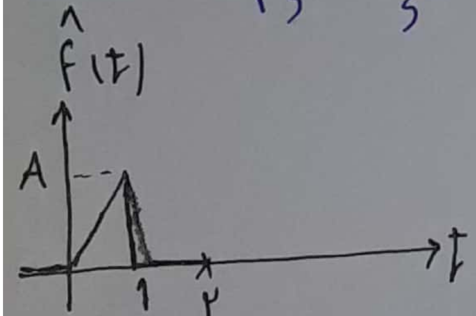
$m_2 = 0$

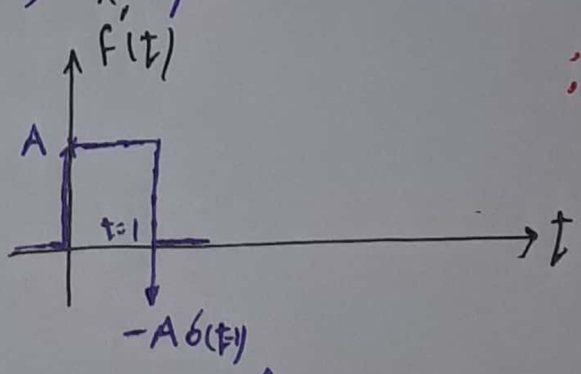
$$\hat{f}(t) = Ar(t) - Ar(t-1) - Au(t-1)$$

$$\hat{f}(t) = A\left(t\,u(t) - (t-1)u(t-1) - u(t-1)\right)$$

$$\hat{F}(s) = A\left(\frac{1}{s^2} - \frac{1}{s^2}\times e^{-s} - \frac{1}{s}\times e^{-s}\right) \rightarrow F(s) = \frac{\hat{F}(s)}{1-e^{-2s}}$$

روش دوم:





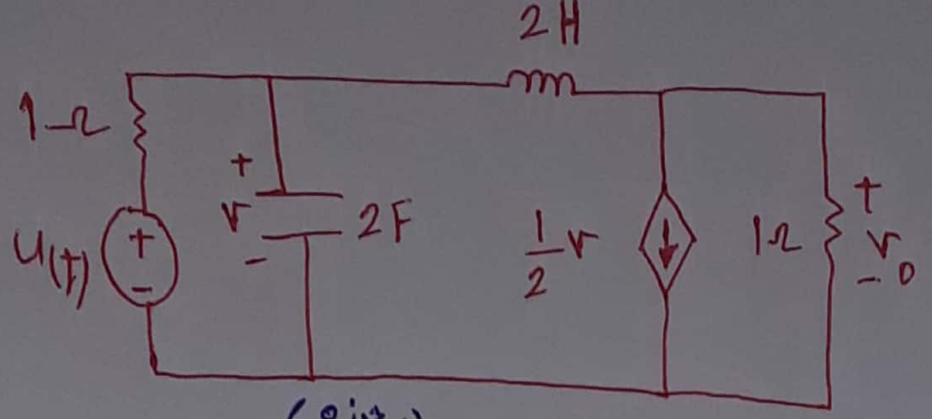
$$\hat{f}'(t) = Au(t) - Au(t-1) - A\delta(t-1) \rightarrow \hat{F}_1(s) = A\left(\frac{1}{s} - \frac{1}{s}\times e^{-s} - e^{-s}\right)$$

$$\hat{F}(s) = \frac{\hat{F}_1(s)}{s} \rightarrow \hat{F}(s) = A\left(\frac{1}{s^2} - \frac{1}{s^2}\times e^{-s} - \frac{1}{s}e^{-s}\right)$$

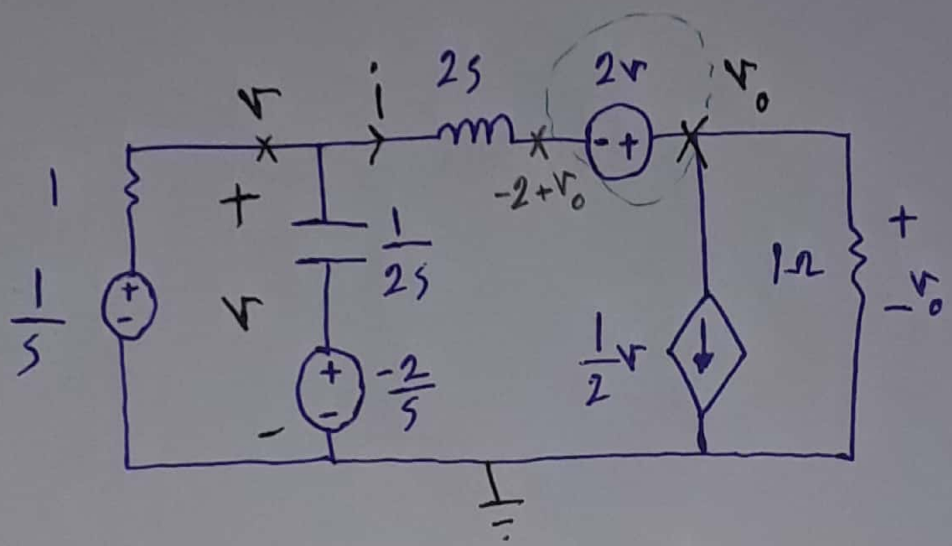
$$F(s) = \frac{\hat{F}(s)}{1-e^{-Ts}} = \frac{\hat{F}(s)}{1-e^{-2s}}$$

تمرین 1) $v_C(0^-)=-2V$, $i_L(0^-)=1A$

$v_o(t)=?$



در حوزه لاپلاس



$$KCL @ v: \frac{v-\frac{1}{s}}{1}+\frac{v+\frac{2}{s}}{\frac{1}{2s}}+\frac{v-v_o+2}{2s}=0 \xrightarrow{\times 2s}$$

$$2sv-2+4s^2(v)+8s+v-v_o+2=0 \longrightarrow$$

$$-v_o+(4s^2+2s+1)v=-8s \quad (I)$$

$$KCL @ v_o: \frac{v_o-2-v}{2s}+\frac{1}{2}v+\frac{v_o-0}{1}=0 \xrightarrow{\times 2s}$$

$$v_o-2-v+sv+2sv_o=0 \rightarrow (2s+1)v_o+(s-1)v=2 \quad (II)$$

ادامه حل در ص 2

ادامه حل تمرین 1-

$$\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \underset{\text{ضرب}}{*} \begin{bmatrix} -1 & 4s^2+2s+1 \\ 2s+1 & s-1 \end{bmatrix} = \begin{bmatrix} -8s \\ 2 \end{bmatrix}$$

$$v_0 = \frac{\begin{vmatrix} -8s & 4s^2+2s+1 \\ 2 & s-1 \end{vmatrix}}{\begin{vmatrix} -1 & 4s^2+2s+1 \\ 2s+1 & s-1 \end{vmatrix}} = \frac{-8s^2+8s-8s^2-4s-2}{-s+1-8s^2-4s^2-2s-4s^2-2s-1} =$$

$$v_0(s) = \frac{2(8s^2-2s+1)}{s(8s^2+8s+5)}$$

* نکته: ۰ محاسبه لاپلاس معکوس ۰ زمانی که ریشه مختلط داریم (قطب)

$s=0 \qquad 8s^2+8s+5=0 \to \Delta<0 \to$ قطب مختلط

$$F(s) = \frac{X(s)}{(s+a)^2+b^2} = \frac{k_1 s + k_2}{(s+a)^2+b^2} = \frac{k_1(s+a)}{(s+a)^2+b^2} + \frac{k_2-k_1 a}{(s+a)^2+b^2}$$

$$f(t) = F^{-1}\{F(s)\} = k_1 e^{-at}\cos bt + \frac{k_2-k_1 a}{b} e^{-at}\sin bt$$

$$v_0(s) = \frac{\frac{1}{4}(8s^2-2s+1)}{s\left(s^2+s+\frac{5}{8}\right)} = \frac{A}{s} + \frac{k_1\left(s+\frac{1}{2}\right)+k_2-\frac{1}{2}k_1}{\underbrace{\left(s+\frac{1}{2}\right)^2+\frac{3}{8}}_{s^2+s+\frac{5}{8}}}$$

محاسبه A, k_1, k_2

ادامه حل در ص 3

ص ۶

ادامه حل تمرین ۱:

$$\frac{\frac{1}{4}(8s^2-2s+1)}{s\left(s^2+s+\frac{5}{8}\right)}=\frac{A}{s}+\frac{K_1\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\frac{3}{8}}+\frac{K_2-\frac{1}{2}K_1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{8}} \ast$$

$$\xrightarrow{\times s}\frac{\frac{1}{4}(8s^2-2s+1)}{\left(s^2+s+\frac{5}{8}\right)}=A+\frac{K_1 s\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\frac{3}{8}}+\frac{\left(K_2-\frac{1}{2}K_1\right)s}{\left(s+\frac{1}{2}\right)^2+\frac{3}{8}}$$

$$s=0:\ \frac{\frac{1}{4}}{\frac{5}{8}}=A\longrightarrow A=\frac{2}{5}$$

$$s=\infty:\ \frac{8}{4}=A+K_1\longrightarrow K_1=2-A=\frac{8}{5}$$

$$s=-1:\ \frac{\frac{1}{4}(8+2+1)}{(1+1)\left(1-1+\frac{5}{8}\right)}=+\frac{2}{5}-\frac{\frac{8}{5}\left(-1+\frac{1}{2}\right)}{\frac{1}{4}+\frac{3}{8}}+\frac{\left(K_2-\frac{4}{5}\right)(-1)}{\frac{1}{4}+\frac{3}{8}}$$

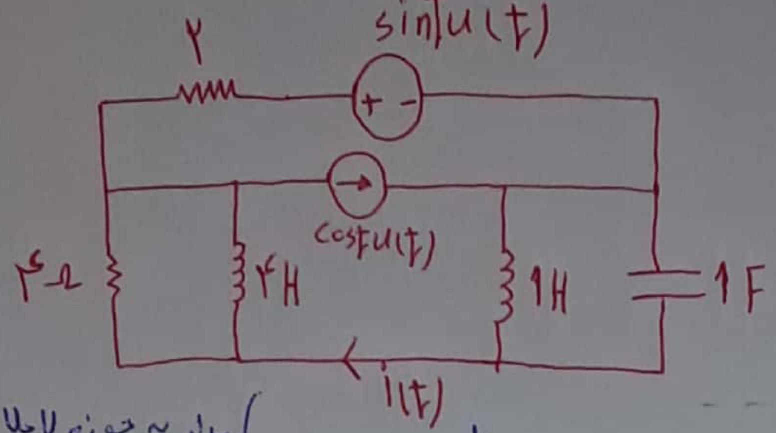
$$\longrightarrow K_2=-0.9,\ K_1=1.6,\ A=0.4$$

$$r_0(t)=e^{-\frac{t}{2}}\left(\frac{8}{5}\cos(0.612t)-\frac{13.73}{5}\sin(0.612t)\right)+\frac{2}{5}u(t)$$

matlab test: $\frac{8}{5}e^{-\frac{t}{2}}\left[\cos\frac{\sqrt{2}\times\sqrt{3}}{4}t-\frac{17}{24}\sqrt{2}\times\sqrt{3}\sin\frac{\sqrt{2}\sqrt{3}}{4}t\right]+\frac{2}{5}$

تمرین 2: $i(t) = ?\ ;\ t > 0$

شرایط اولیه صفر است.

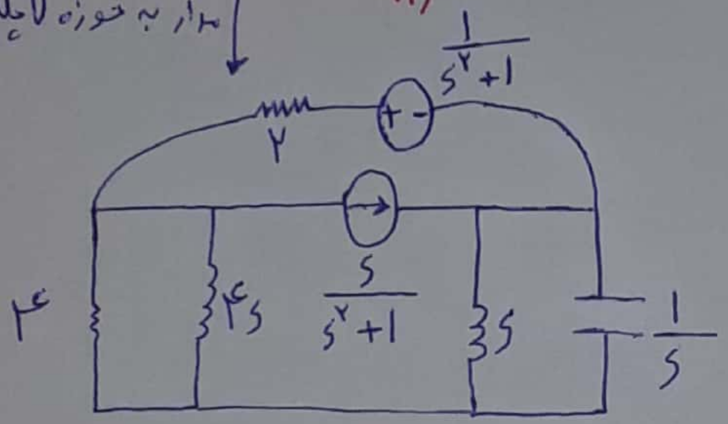


$i_{L_1}(t=0^-) = i_{L_2}(t=0^-) = 0$

$v_C(0^-) = v_C(0^+) = 0$

$i_{L1}(t=0^+) = i_{L2}(t=0^+) = 0$

مدار به حوزه لاپلاس



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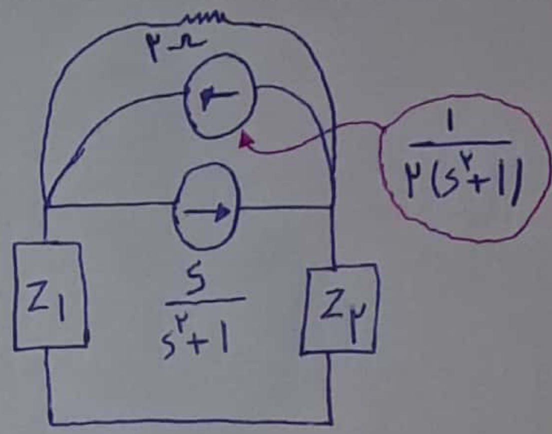
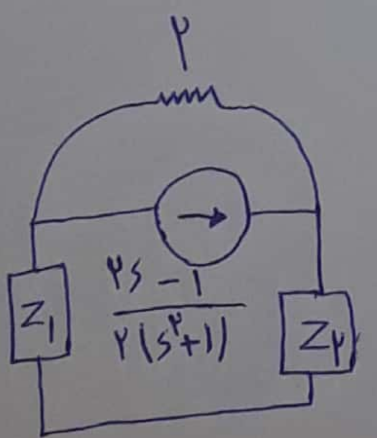


$Z_{1(s)} = 4 \,||\, 4s = \frac{4s}{s+1}$

$Z_2(s) = s \,||\, \frac{1}{s} = \frac{s}{s^2+1}$



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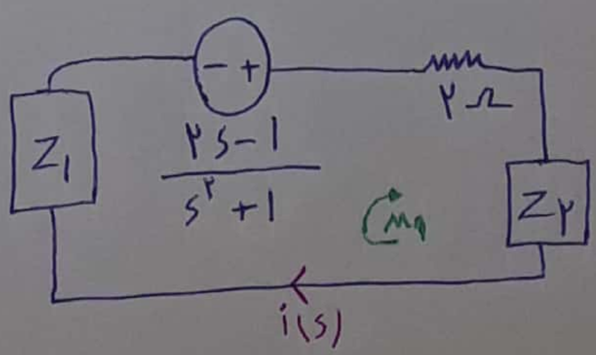


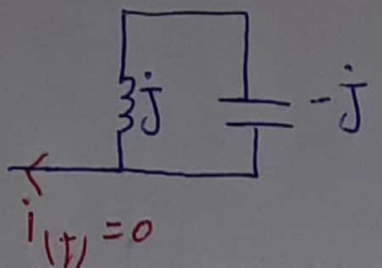
$$\text{KVL @ } M_1:\quad i_{(s)} = \frac{\dfrac{2s-1}{s^2+1}}{Z_1 + Z_2 + 2}$$

$$Z_1 + Z_2 + 2 = \frac{6s^3 + 3s^2 + 7s + 2}{(s^2+1)(s+1)}$$

ادامه حل ص بعد

حل ۵

با کمی دقت متوجه می شویم در $\omega=1$ سلف $1H$ با خازن $1F$ تشدید دارد.
پس پاسخ حالت دائمی $i(t)=0$



پس $i(t)$ تنها پاسخ حالت گذرا دارد که همان قطب های $i(s)$ می باشد.

$$i(s) = \frac{(2s-1)(s+1)}{4s^3+3s^2+7s+2}$$

هدف معادله قطب های $i(s)$

$$4s^3+3s^2+7s+2=0 \rightarrow s \simeq -0.3$$ ریشه معادله (با سعی و خطا)

$s+0.3=0 \rightarrow$ ریشه معادله $\rightarrow$

$$\begin{array}{l|l}
4s^3+3s^2+7s+2 & s+0.3 \\ \hline
-4s^3-1.8s^2 & 4s^2+1.2s+6.64 \\ \cline{1-1}
1.2s^2+7s & \\
-1.2s^2-0.36s & \\ \cline{1-1}
6.64s+2 & \\
-6.64s-1.99 & \\ \cline{1-1}
\simeq 0 &
\end{array}$$

$$(s+0.3)\underbrace{(4s^2+1.2s+6.64)}_{\Delta<0}=0$$

$$4s^2+1.2s+6.64=0 \rightarrow s^2+0.2s+1.1=0 \quad \begin{cases} 2\alpha=1 \rightarrow \alpha=0.5 \\ \omega_0=\sqrt{1.1}\simeq 1 \end{cases}$$

$\omega_0 > \alpha \rightarrow$ میرای ضعیف $\quad e^{-\alpha t}(A\cos\omega_d t + B\sin\omega_d t) \quad \omega_d=\sqrt{\omega_0^2-\alpha^2}=0.86$

$$i(t) = 0 + Ae^{-0.3t} + e^{-0.5t}(B\cos\omega_d t + C\sin\omega_d t)$$

ادامه حل ص بعد

ص ۶

$$i_{(t)} = \underset{?}{A}\, e^{-0.3t} + e^{-0.5t}\left(\underset{?}{B}\cos 0.84t + \underset{?}{C}\sin 0.84t\right) + 0$$

$$I_{(s)} = \frac{2s^2+s-1}{6s^3+3s^2+7s+2}$$

اگر $F_{(s)}$ تبدیل لاپلاس $f_{(t)}$ باشد: می دانیم

$$f_{(0^+)} = \lim_{s\to\infty} sF_{(s)} \quad , \quad \frac{df}{dt}(0^+) = \lim_{s\to\infty}\left[s^2F_{(s)} - s f_{(0^+)}\right]$$

$$\frac{d^2f}{dt^2}(0^+) = \lim\left[s^3F_{(s)} - s^2 f_{(0^+)} - s f'_{(0^+)}\right]$$

$$\underbrace{i_{(t=0)}}_{i_{(0^+)}} = \lim_{s\to\infty}\frac{2s^3+s^2-1}{6s^3+3s^2+7s+2} = \frac{1}{3} = i\,(t=0^+)$$

$$\frac{di}{dt}(0^+) = \lim\left[\frac{2s^4+s^3-s^2}{6s^3+3s^2+7s+2} - \frac{s}{3} = \frac{6s^4-6s^4+3s^3-3s^3-3s^2-7s^2}{6s^3+3s^2+7s}\right] = \frac{-10}{3}$$

$$\frac{d^2i}{dt^2}(0^+) = \lim_{s\to\infty}\left[\frac{2s^5+s^4-s^3}{6s^3+3s^2+7s+2} - \frac{s^2}{3} + \frac{10s}{3}\right] = \lim_{s\to\infty}\left[\frac{3s^4}{77s^4}\right] = \frac{3}{77}$$

$$i_{(0)} = A + B = \frac{1}{3}\ (\mathrm{I}) \quad , \quad \frac{di}{dt}(0) = -0.3A - 0.5B + 0.84C = -\frac{10}{3}\ (\mathrm{II})$$

$$\frac{di}{dt} = -0.3A\, e^{-0.3t} - 0.5\, e^{-0.5t}(B\cos 0.84t + \sin \ldots) + e^{-0.5t}(0.84B\sin 0.84t + 0.84 \cos 0.84t)$$

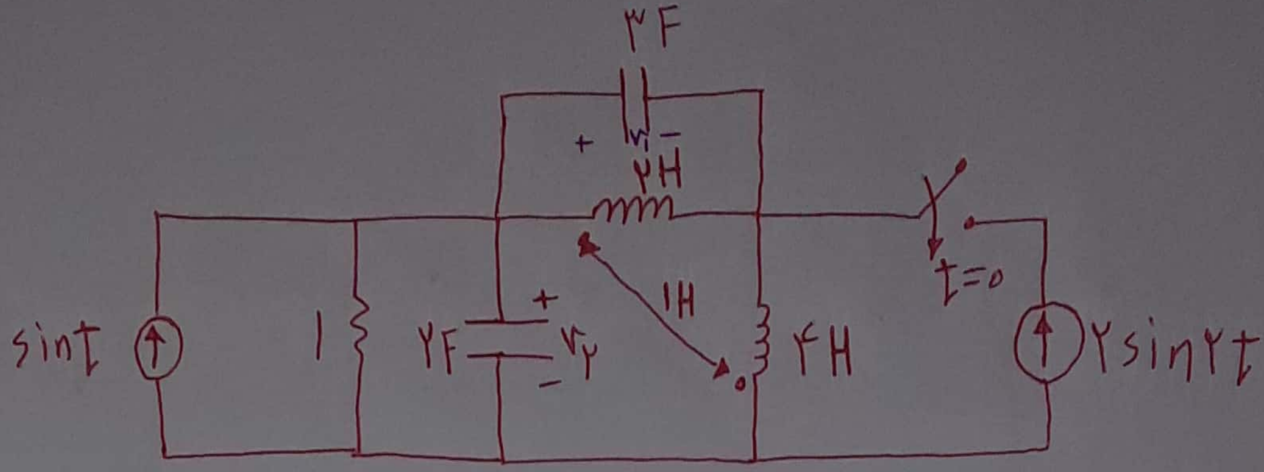
$$\frac{d^2i}{dt^2}(0) = 0.09A + 0.25B - 0.5\times 0.84C + 0.84C \times 0.84 - 0.5\times 0.84B = \frac{3}{77}$$

$$0.09A - 0.18B + 0.3C = \frac{3}{77}\ (\mathrm{III})$$

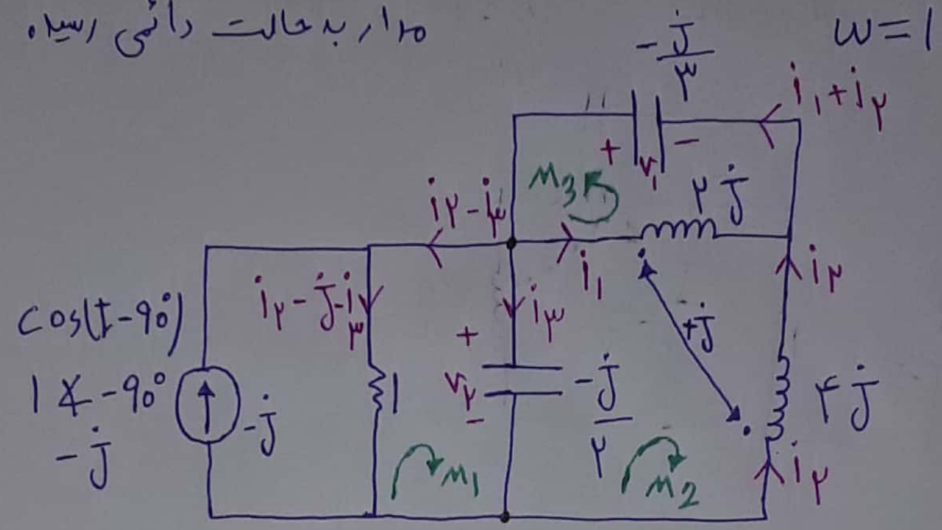
$$(\mathrm{I}), (\mathrm{II}), (\mathrm{III}) \rightarrow \begin{cases} A = 6 \\ B = -5.5 \\ C = -5 \end{cases}$$

$$i_{(t)} = 6e^{-0.3t} + e^{-0.5t}\left(-5.5\cos 0.84t - 5\sin 0.84t\right) + 0$$

تمرین ۳:



$t<0$: مدار به حالت دائمی رسیده $\qquad \omega = 1$



$$KVL@m_1: -i_2 + j + i_3 - \frac{j}{2} i_3 = 0 \rightarrow -i_2 + \left(1-\frac{j}{2}\right) i_3 = -j \quad (I)$$

$$KVL@m_2: +\frac{j}{2} i_3 + 2j i_1 + j i_2 - \left(4j(i_2) + j i_1\right) = 0$$

$$\rightarrow j i_1 - 3j i_2 + \frac{j}{2} i_3 = 0 \rightarrow i_1 - 3 i_2 + \frac{i_3}{2} = 0 \quad (II)$$

$$KVL@m_3: 2j i_1 + j i_2 - \frac{j}{3}(i_1+i_2) = 0 \rightarrow \frac{5}{3} j i_1 + \frac{2}{3} j i_2 = 0$$

$$\rightarrow 5 i_1 = -2 i_2 \rightarrow +2 i_2 + 5 i_1 = 0 \quad (III)$$

$(I),(II),(III) \rightarrow$

$$i_1 = -0.03 + 0.05j = 0.06 \angle 120° \qquad i_1 + i_2 = 0.04 - 0.08j$$

$$v_1 = \frac{j}{3}(i_1+i_2) = 0.026 + 0.01j$$

$$i_2 = 0.07 - 0.13j = 0.15 \angle -61°$$

$$i_3 = 0.51 - 0.87j \rightarrow v_2 = -\frac{j}{2} i_3 = -0.43 - 0.25j = 0.5 \angle 210°$$

ادامه حل ص بعد

ص

ادامه حل تمرین ۳:

$$i_1(t) = 0.06 \angle 120^\circ = 0.06\cos(t+120^\circ) \to i_1(0^-) = -0.03$$

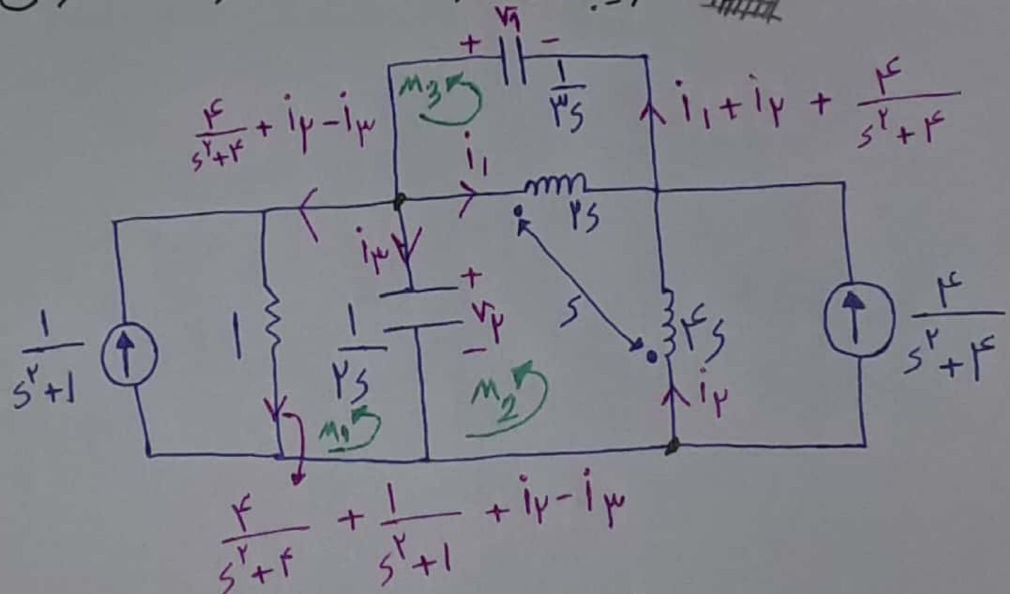
$$i_2(t) = 0.15 \angle -61^\circ = 0.15\cos(t-61^\circ) \to i_2(0^-) = 0.07$$

$$v_1(t) = 0.026 + 0.01j = 0.03 \angle 21^\circ \to v_1(0^-) = 0.03\cos(21^\circ) = 0.03$$

$$v_2(t) = 0.5 \angle 210^\circ = 0.5\cos(t+210^\circ) \Rightarrow v_2(0^-) = -0.43$$

با توجه به مقادیر ~~اولیه~~ به دست آمده تقریبا از شرایط اولیه صرف نظر می کنیم.

$t \geq 0:$



$$KVL@M_1: -\frac{1}{2s} i_3 + \frac{4}{s^2+4} + \frac{1}{s^2+1} + i_2 - i_3 = 0$$

$$\to i_2 - \left(1 + \frac{1}{2s}\right) i_3 = \frac{-4}{s^2+4} - \frac{1}{s^2+1} \quad (I)$$

$$KVL@M_2: 4s i_2 + s i_1 - (2s i_1 + s i_2) + \frac{i_3}{2s} = 0$$

$$\to -s i_1 + 3s i_2 + \frac{1}{2s} i_3 = 0 \quad (II)$$

$$KVL@M_3: 2s i_1 + s i_2 + \frac{1}{3s}\left(i_1 + i_2 + \frac{4}{s^2+4}\right) = 0$$

$$\to \left(2s + \frac{1}{3s}\right) i_1 + \left(s + \frac{1}{3s}\right) i_2 = \frac{-4}{3s(s^2+4)} \quad (III)$$

ص ۹

مثلا با حل ۳ معادله مجهول ص قبل :

$$i_3 = \frac{2s^2(21s^2+4)\left[\left(\frac{1}{s^2+1}\right)+\frac{4}{s^2+4}\right]}{21s^5+42s^4+4s^3+14s^2+1} - \frac{8s^2}{A}$$

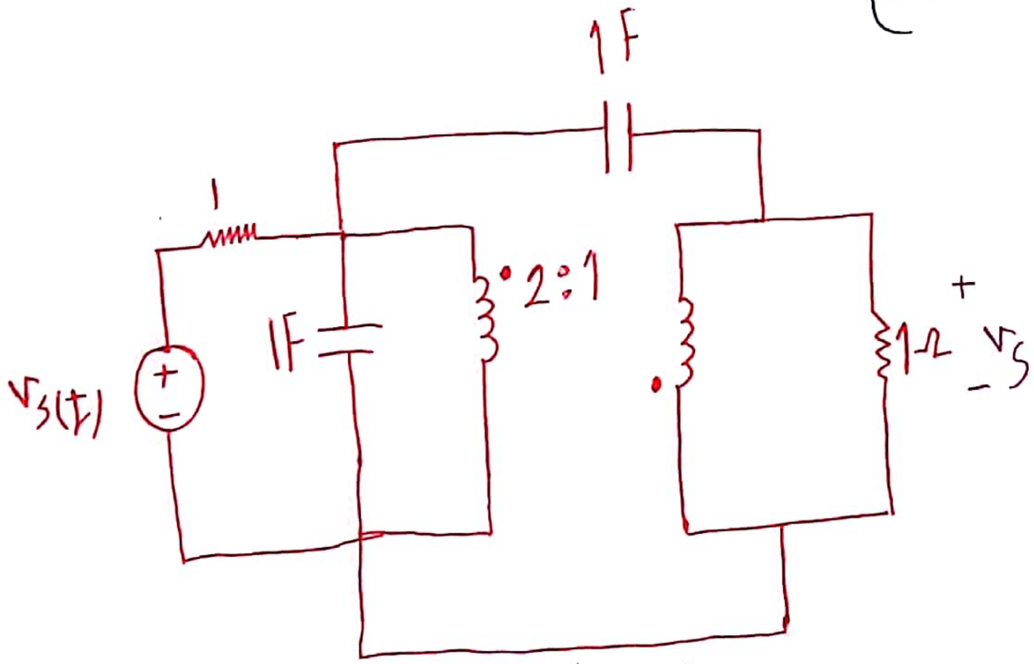
$$A = (s^2+4)(21s^5+42s^4+4s^3+14s^2+1)$$

$$v_2 = \frac{i_3}{2s} \longrightarrow$$ با محاسبه لاپلاس معکوس v_2 بدست می آید

$$v_1 = \frac{-1}{3s}\left(i_1+i_2+\frac{4}{s^2+4}\right)$$

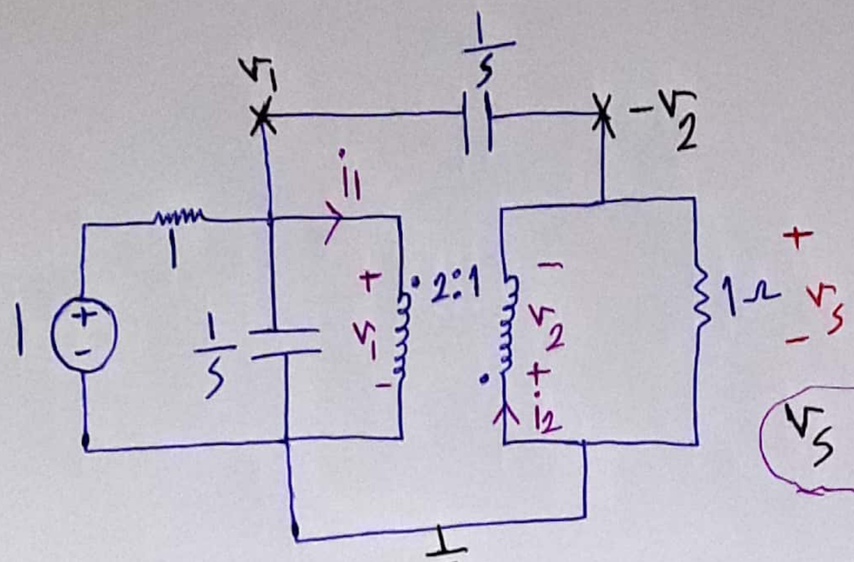
تمرین: پاسخ ضربه مدار شکل زیر با فرض شرایط اولیه صفر

پاسخ ضربه ← $v_s(t)=\delta(t)$ ورودی



مدار به حوزه لاپلاس ببرید

ص



$v_s = -v_2$

$\frac{v_1}{v_2} = \frac{2}{1} \rightarrow v_1 = 2v_2 \; (\mathrm{I})$ $\quad 2i_1 + i_2 = 0 \rightarrow i_2 = -2i_1 \; (\mathrm{II})$

KCL @ v_1: $v_1 - 1 + sv_1 + i_1 + (v_1 + v_2)s = 0$

$\xrightarrow{(\mathrm{I})}$ ~~$v_1 - 1 + sv_1 + i_1 + sv_1$~~ $(1+2s)v_1 + sv_2 = 1 - i_1 \quad (*)$

$\xrightarrow{\mathrm{I}} 2(2s+1)v_2 + sv_2 = 1 - i_1 \rightarrow (5s+2)v_2 + i_1 = 1 \quad (**)$

KCL @ v_2: $-v_2 - i_2 - (v_2 + v_1)s = 0 \rightarrow -sv_1 - (s+1)v_2 - i_2 = 0$

$\xrightarrow{\mathrm{I},\,\mathrm{II}} -2sv_2 - (s+1)v_2 + 2i_1 = 0 \rightarrow -(3s+1)v_2 + 2i_1 = 0$

$\rightarrow i_1 = \frac{(3s+1)v_2}{2} \quad \xrightarrow{(**)} (5s+2)v_2 + \frac{3s+1}{2}v_2 = 1 \rightarrow$

$v_{2(s)} = \frac{2}{13s+5} \rightarrow v_{2(t)} = \frac{2}{13}\left(\frac{1}{s+\frac{5}{13}}\right) \rightarrow v_{2(t)} = \frac{2}{13} e^{-\frac{5}{13}t} u(t)$

$\rightarrow v_{s(t)} = -\frac{2}{13} e^{-\frac{5}{13}t} u(t)$