به نام خدا

پاسخ تمرین سری ۹ ریاضی مهندسی (سری فوریه)

۱- انتگرال فوریه

$$\begin{cases} g(x)=\frac{\pi}{2}e^{-x}\cos x & x>0 \\ g(x)=-g(-x) & x<0 \end{cases}$$

تابع فرد $\leftarrow$ $B(\omega)\checkmark$

$A(\omega)=0$

$$f(x)=\frac{1}{\pi}\int_0^\infty \left[\cancel{A(\omega)\cos\omega x}+B(\omega)\sin\omega x\right]d\omega$$

$$A(\omega)=\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx = 2\int_0^\infty f(x)\cos\omega x\,dx = 0$$

$$B(\omega)=\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx = 2\int_0^\infty \frac{\pi}{2}e^{-x}\cos x\sin\omega x\,dx = \pi\int_0^\infty e^{-x}\cos x\sin\omega x\,dx$$

$$=\pi\int_0^\infty e^{-x}\left[\frac{1}{2}\left[\sin(1+\omega)x+\sin(\omega-1)x\right]\right]dx$$

$$=\frac{\pi}{2}\Big[\underbrace{\int_0^\infty e^{-x}\sin(\omega+1)x\,dx}_{1}+\underbrace{\int_0^\infty e^{-x}\sin(\omega-1)x\,dx}_{2}\Big]$$

مشتق	انتگرال
e^{-x}	$\sin(\omega+1)x$
$-e^{-x}$	$\frac{-1}{\omega+1}\cos(\omega+1)x$
e^{-x}	$\frac{-1}{(\omega+1)^2}\sin(\omega+1)x$

1 $\Rightarrow$
$$\underbrace{\int_0^\infty e^{-x}\sin(\omega+1)x\,dx}_{I_1}=\frac{-e^{-x}}{\omega+1}\cos(\omega+1)x\Big|_0^\infty-\cancel{\frac{e^{-x}}{(\omega+1)^2}\sin(\omega+1)x}\Big|_0^\infty-\underbrace{\frac{1}{(\omega+1)^2}\int e^{-x}\sin(\omega+1)x\,dx}_{I_1/(\omega+1)^2}$$

$$\Rightarrow\left(1+\frac{1}{(\omega+1)^2}\right)I_1=\frac{1}{\omega+1}\longrightarrow \boxed{I_1=\frac{\omega+1}{(\omega+1)^2+1}}$$

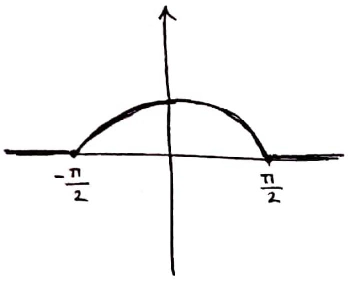
مشتق	انتگرال
e^{-x}	$\sin(\omega-1)x$
$-e^{-x}$	$\frac{-1}{\omega-1}\cos(\omega-1)x$
e^{-x}	$\frac{-1}{(\omega-1)^2}\sin(\omega-1)$

2 $\Rightarrow$
$$\underbrace{\int_0^\infty e^{-x}\sin(\omega-1)x\,dx}_{I_2}=\frac{-e^{-x}}{\omega-1}\cos(\omega-1)x-\cancel{\frac{e^{-x}}{(\omega-1)^2}\sin(\omega-1)x}\Big|_0^\infty-\underbrace{\frac{1}{(\omega-1)^2}\int e^{-x}\sin(\omega-1)x}_{I_2/(\omega-1)^2}$$

$$\Rightarrow\left(1+\frac{1}{(\omega-1)^2}\right)I_2=\frac{1}{\omega-1}\longrightarrow \boxed{I_2=\frac{(\omega-1)}{(\omega-1)^2+1}}$$

$$B(\omega)=\frac{\pi}{2}\left[\frac{\omega+1}{(\omega+1)^2+1}+\frac{\omega-1}{(\omega-1)^2+1}\right]=\boxed{\frac{\pi\omega^3}{\omega^4+4}}\longrightarrow f(x)=\int_0^\infty \frac{\omega^3}{\omega^4+4}\sin\omega x\,d\omega \checkmark$$

$$\int_0^{\infty} \frac{\cos\frac{\pi w}{2}\cos wx}{1-w^2}\,dw = \begin{cases} \frac{\pi}{2}\cos x & |x|<\frac{\pi}{2} \quad -\frac{\pi}{2}<x<\frac{\pi}{2} \\ 0 & |x|>\pi/2 \quad x>\pi/2,\ x<-\frac{\pi}{2} \end{cases}$$ -۲



تابع زوج

$A(w)$ ✓

$B(w)=0$

$$f(x) = \frac{1}{\pi}\int_0^{\infty} A(w)\cos wx\,dw$$

$$A(w) = \int_{-\infty}^{\infty} f(x)\cos wx\,dx = 2\int_0^{\infty}\frac{\pi}{2}\cos x\cos wx\,dx = \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\left[\cos(1+w)x+\cos(1-w)x\right]dx$$

$$= \frac{\pi}{2}\left[\frac{1}{1+w}\sin(1+w)x + \frac{1}{1-w}\sin(1-w)x\right]_0^{\pi/2} = \frac{\pi}{2}\left[\frac{\sin(1+w)\pi/2}{1+w} + \frac{\sin(1-w)\pi/2}{1-w}\right]$$

$$* \quad \sin(1\mp w)\pi/2 = \sin\left(\pi/2 \mp \frac{w\pi}{2}\right) = \underbrace{\sin\pi/2}_{1}\cos\frac{w\pi}{2} \mp \sin\frac{w\pi}{2}\underbrace{\cos\pi/2}_{0} = \cos\frac{w\pi}{2}$$

$$\Rightarrow A(w) = \frac{\pi}{2}\left[\frac{2\cos\frac{w\pi}{2}}{1-w^2}\right] = \boxed{\frac{\pi\cos\frac{w\pi}{2}}{1-w^2}}$$

$$f(x) = \int_0^{\infty}\frac{\cos\frac{w\pi}{2}}{1-w^2}\cos wx\,dw$$ ✓

$$* \quad \sin(a\pm b)x = \sin ax\cos bx \pm \sin bx\cos ax$$