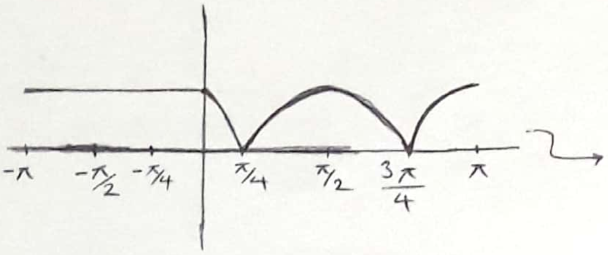
بنام خدا

تمرینات سری سوم ریاضی مهندسی (سری فوریه)

۱- $f(x)=\begin{cases} 1 & -\pi<x<0 \\ \cos^2 2x & 0<x<\pi \end{cases}$



تابع نه زوج نه فرد

$$\cos^2 x=\frac{1+\cos 2x}{2}$$

$$\cos^2 2x=\frac{1+\cos 4x}{2}$$

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n\cos\frac{2n\pi}{T}x+b_n\sin\frac{2n\pi}{T}x \qquad T=2\pi$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{1}{2\pi}\left(\int_{-\pi}^{0}1\,dx+\int_{0}^{\pi}\cos^2 2x\,dx\right)=\frac{1}{2\pi}\left(x\Big]_{-\pi}^{0}+\int_0^{\pi}\frac{1+\cos 4x}{2}\,dx\right)$$

$$=\frac{1}{2\pi}\left(\pi+\frac{1}{2}\left(x+\cancel{\frac{1}{4}\sin 4x}^{\,0}\right)_0^{\pi}\right)=\frac{1}{2\cancel{\pi}}\left(\cancel{\pi}+\frac{1}{2}(\cancel{\pi})\right)=\boxed{\frac{3}{4}}$$

$$a_n=\frac{2}{T}\int_{-\pi}^{\pi} f(x)\cos\frac{2n\pi}{T}x\,dx=\frac{2}{2\pi}\left(\int_{-\pi}^{0}1\cos nx\,dx+\int_{0}^{\pi}\underbrace{\cos^2 2x}_{\frac{1+\cos 4x}{2}}\cos nx\,dx\right)$$

$$=\frac{1}{\pi}\left(\cancel{\frac{1}{n}\sin nx}^{\,0}\Big]_{-\pi}^{0}+\frac{1}{2}\left(\int_0^{\pi}\cos nx\,dx+\int_0^{\pi}\cos 4x\cos nx\,dx\right)\right)$$

$$=\frac{1}{2\pi}\left(\cancel{\frac{1}{n}\sin nx}^{\,0}\Big]_0^{\pi}+\int_0^{\pi}\frac{1}{2}\left[\cos(4+n)x+\cos(4-n)x\right]dx\right)$$

$$=\frac{1}{4\pi}\left(\cancel{\frac{1}{4+n}\sin(4+n)x}^{\,0}+\cancel{\frac{1}{4-n}\sin(4-n)x}^{\,0}\right]_0^{\pi}=\boxed{0} \qquad n\neq 4$$

دقت کنید $a_n=0$ است اما در مخرج جمله $4+n$ داریم پس باید a_n را برای $\underline{\underline{n=4}}$ بصورت جداگانه حساب کنیم.

$$a_4=\frac{2}{2\pi}\left(\int_{-\pi}^{0}1\cos 4x\,dx+\int_0^{\pi}\cos^2 2x\cos 4x\,dx\right)=\frac{1}{4\pi}\int_0^{\pi}\left(\cos(4+4)x+\cancel{\cos(4-4)x}^{\,1}\right)dx$$

$$=\frac{1}{4\pi}\left(\cancel{\frac{1}{8}\sin 8x}^{\,0}+x\right]_0^{\pi}=\frac{1}{4\pi}(\pi)=\boxed{\frac{1}{4}} \qquad n=4$$

$$\cos ax\cos bx=\frac{1}{2}\left[\cos(a+b)x+\cos(a-b)x\right]$$

$$\sin ax\cos bx=\frac{1}{2}\left[\sin(a+b)x+\sin(a-b)x\right]$$

$$b_n = \frac{2}{T}\int_{T_0} f(x)\sin\frac{2n\pi}{T}x\,dx = \frac{2}{2\pi}\left(\int_{-\pi}^{0} 1\sin nx\,dx + \int_{0}^{\pi} \underbrace{\cos^2 2x}_{\frac{1+\cos 4x}{2}}\sin nx\,dx\right)$$

$$= \frac{1}{\pi}\left\{\left(\frac{-1}{n}\cos nx\right)_{-\pi}^{0} + \frac{1}{2}\left(\int_0^\pi \sin nx\,dx + \int_0^\pi \frac{1}{2}\left[\sin(n+4)x + \sin(n-4)x\right]dx\right)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-1}{n} + \frac{(-1)^n}{n} + \frac{1}{2}\left(\left[\frac{-1}{n}\cos nx\right]_0^\pi + \frac{1}{2}\left[\frac{-1}{n+4}\cos(n+4)x - \frac{1}{n-4}\cos(n-4)x\right]_0^\pi\right)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{(-1)^n-1}{n} + \frac{1}{2n}\left(1-(-1)^n\right) + \frac{1}{4}\left[\frac{-\cos(n+4)\pi}{n+4} - \frac{\cos(n-4)\pi}{n-4} + \frac{1}{n+4} + \frac{1}{n-4}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{(-1)^n-1}{2n} + \frac{1}{4}\left[\frac{1-\cos(n+4)\pi}{n+4} + \frac{1-\cos(n-4)\pi}{n-4}\right]\right\} \quad n \neq 4$$

مقدار b_n برابر با مقدار روبرو است اما باید دقت کنیم مخرج کسرها به ازای $n=4$ صفر خواهد شد در نتیجه b_4 را به صورت جداگانه حساب میکنیم

$$b_4 = \frac{2}{2\pi}\left(\int_{-\pi}^{0} 1\sin 4x\,dx + \int_0^\pi \cos^2 2x\sin 4x\,dx\right)$$

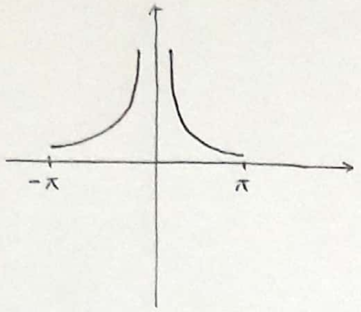
$$= \frac{1}{\pi}\left\{\left[\cancelto{0}{\frac{-1}{4}\cos 4x}\right]_{-\pi}^{0} + \frac{1}{2}\int_0^\pi \sin 4x\,dx + \frac{1}{4}\int_0^\pi\left(\sin(4+4)x + \cancelto{0}{\sin(4-4)x}\right)dx\right\}$$

$$= \frac{1}{2\pi}\left\{\left[\cancelto{0}{\frac{-1}{4}\cos 4x}\right]_0^\pi + \frac{1}{4}\left[\cancelto{0}{\frac{-1}{8}\cos 8x}\right]_0^\pi\right\} = \boxed{0} \quad \underline{n=4}$$

$$\longrightarrow f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{2n\pi}{T}x + b_n\sin\frac{2n\pi}{T}x$$

$$f(x) = \frac{3}{4} + \frac{1}{4}\cos 4x + \sum_{n=1}^{\infty}\left\{\frac{(-1)^n-1}{2n\pi} + \frac{1}{4\pi}\left[\frac{1-\cos(n+4)\pi}{n+4} + \frac{1-\cos(n-4)\pi}{n-4}\right]\right\}\sin nx$$

۲-



$$f(x)=e^{-|x|} \qquad -\pi<x<\pi$$

$$\begin{cases} e^{x} & -\pi<x<0 \\ e^{-x} & 0<x<\pi \end{cases}$$

تابع زوج $\begin{cases} a_0 \checkmark \\ a_n \checkmark \\ b_n=0 \end{cases}$

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n \cos\frac{2n\pi}{T}x + b_n \sin\frac{2n\pi}{T}x \qquad T=2\pi$$

$$a_0=\frac{1}{T}\int_{(T)} f(x)\,dx=\frac{1\times 2}{2\pi}\int_0^{\pi} e^{-x}\,dx=\frac{1}{\pi}\left(-e^{-x}\right)_0^{\pi}=\frac{1}{\pi}\left(-e^{-\pi}+1\right)=\boxed{\frac{1}{\pi}\left(1-e^{-\pi}\right)}$$

$$a_n=\frac{2}{T}\int_{(T)} f(x)\cos\frac{2n\pi}{T}x\,dx=\frac{2\times 2}{2\pi}\int_0^{\pi} e^{-x}\cos nx\,dx$$

مشتق	انتگرال
e^{-x} $+$	$\cos nx$
$-e^{-x}$ $-$	$\frac{1}{n}\sin nx$
e^{-x} $+\int$	$\frac{-1}{n^2}\cos nx$

$$=\frac{2}{\pi}\left(\cancel{\frac{e^{-x}}{n}\sin nx}-\frac{e^{-x}}{n^2}\cos nx\Big]_0^{\pi}-\frac{1}{n^2}\int e^{-x}\cos nx\,dx\right)$$

$$\Rightarrow \underbrace{\int_0^{\pi} e^{-x}\cos nx\,dx}_{I}=\left(\frac{-e^{-\pi}}{n^2}(-1)^n+\frac{1}{n^2}\right)-\underbrace{\frac{1}{n^2}\int e^{-x}\cos nx\,dx}_{\frac{-I}{n^2}}$$

$$\Rightarrow \left(1+\frac{1}{n^2}\right)I=\left(-\frac{e^{-\pi}}{n^2}(-1)^n+\frac{1}{n^2}\right)\Rightarrow I=\frac{1-e^{-\pi}(-1)^n}{n^2+1}$$

$$a_n=\frac{2}{\pi}I$$

$$\Rightarrow \boxed{a_n=\frac{2\left(1-e^{-\pi}(-1)^n\right)}{\pi(n^2+1)}}$$

$$f(x)=\frac{1}{\pi}\left(1-e^{-\pi}\right)+\sum_{n=1}^{\infty}\frac{2}{\pi(1+n^2)}\left(1-e^{-\pi}(-1)^n\right)\cos nx \quad \checkmark$$