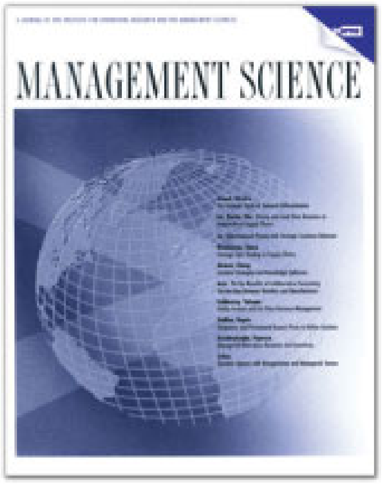

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The Allocation of Aircraft to Routes—An Example of Linear Programming Under Uncertain Demand

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THE ALLOCATION OF AIRCRAFT TO ROUTES—AN EXAMPLE OF LINEAR PROGRAMMING UNDER UNCERTAIN DEMAND

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Summary

The purpose of this paper is to illustrate an application of linear programming to the problem of allocation of aircraft to routes in order to maximize expected profits when there is uncertain customer demand. The approach is intuitive; the theoretical basis of this work is found in an earlier study. The allocations are compared with those obtained under the usual procedure of assuming a fixed demand equal to the expected value. The computational procedure is similar to the fixed demand case, with only slightly more computational effort required.

This paper is intended both for readers interested in routing (and analogous resource allocation) problems and for those interested in studying an example of an application of linear programming under uncertainty.

1. Introduction

There are many business, economic, and military problems that have the following characteristics in common: a limited quantity of capital equipment or final product must be allocated among a number of final use activities, where the level of demand for each of these activities, and hence the payoff, is uncertain; and further, once the allocation is made, it is not economically feasible to reallocate because of geographical separation of the activities, because of differences in form of the final products, or because of a minimum lead time between the decision and its implementation. Examples of such problems are (1) scheduling transport vehicles over a number of routes to meet a demand in some future period and (2) allocating quantities of a commodity at discrete time intervals among several storage or distribution points while the future demand for the commodity is unknown. It is assumed, however, that demand can be forecast or estimated as a distribution of values, each with a specified probability of being the actual value.

The general area where the techniques of this paper apply may be schematized broadly as problems where

(a) Alternative sets of activity levels can be chosen consistent with given resources;
(b) Each set of chosen activity levels provides the facilities or stocks to meet an unknown demand whose distribution is assumed known;
(c) Profits depend on the costs of the facilities, on stocks, and on the revenues from the demand;

and where the general objective is to determine that set of activity levels that maximize profits.

The paper entitled "Linear Programming Under Uncertainty" [2] forms the theoretical basis for the present paper. Our purpose is to illustrate the procedural steps on an example which, in fact, originally inspired the referenced theoretical work in this area. Thus, little in the way of rigorous theory will be attempted, although each step will be justified intuitively.

The method is explained by the use of a model for routing aircraft. Several types of aircraft are allocated over a number of routes; the monthly demand for service over each route is assumed to be known only as a distribution of probable values. The aircraft are so allocated as to minimize the sum of cost of performing the transportation, plus the expected value of the revenue lost through the failure to serve all the traffic that actually developed.

For purposes of month-to-month scheduling, an air-transport operator would, presumably, feel better about having to make an estimate of the range and general distribution of future travel (or shipment) over his routes than about having to commit himself to a single expected value. Indeed, he might feel that the optimal assignment should be insensitive to a wide range of demand distribution, and that an assignment based on expected values (as if these were known fixed demands) would be misleading. It is suggested that the reader make sensitivity tests by modifying the demand distributions given in the illustrative example.

Passenger demand, of course, occurs on a day-by-day, in fact, on a flight-by-flight basis. The assumed number of passengers per type aircraft per given type flight may be thought of as an ideal number which can be increased slightly by decreasing the amount of air freight and by "smoothing" the demand by encouraging the customers to take open reservations on alternative flights as opposed to less certain reservations on the desired flight. In spite of these possible adjustments, traveler preferences and the inevitable last-minute cancellations do cause loss of seat carrying capacity. However, the best way to reflect these effects of the daily variations in demand are beyond the scope of this paper. For our purpose here, either the aircraft passenger capability or the demand may be thought of as adjusted downward to reflect the loss due to daily variations of demand.

The method employed is simple, and the example used can be solved by hand in an hour or two. Larger problems can be solved with computing machines.

In a previously published paper, [1] the method was applied to the same example, assuming the demand on each route to be known;[1] this paper continues the analysis to show how to handle a frequency distribution of demand over each route. A different allocation is found to be optimal in this case.

This paper will describe the problem, briefly indicate the nature of the solution based on expected values, show the method of solving the problem using stochastic values for demand and, finally, compare the two solutions.

[1] This was equivalent to using the expected value of demand, rather than taking account of the whole frequency distribution, as in this paper.

2. Review of Fixed Demand Example

The fixed demand example, used to illustrate the method, takes a fixed fleet of four types of aircraft, as shown in (1). These aircraft have differences in speeds,

(1)

Assumed Aircraft Fleet

Type	Description	Number Available
A	Post-War 4-Engine	10
B	Post-War 2-Engine	19
C	Pre-War 2 Engine	25
D	Pre-War 4-Engine	15

ranges, payload capacities, and cost characteristics. The assumed routes and expected traffic loads (the distribution of demand will be discussed later) are shown in (2).

(2)

Traffic Load by Route

Route	Route Miles[3]	Expected Number of Passengers[2]	Price 1-Way Ticket
1. N.Y.-L.A. (1 stop)	2,475	25,000	$130
2. N.Y.-L.A. (2 stop)	2,475	12,000	130
3. N.Y.-Dallas (0 stop)	1,381	18,000	70
4. N.Y.-Dallas (1 stop)	1,439	9,000	70
5. N.Y.-Boston (0 stop)	185	60,000	10

Since this paper proposes to illustrate the applicability of a method in solving problems in which several realistic elements are considered, it is assumed that not all aircraft can carry their full loads on all routes and that the obtainable utilization varies from route to route. Specifically, Type B is assumed to be able to operate at only 75 per cent payload on Route 3, and Type D at 80 per cent on Route 1; whereas Type C cannot fly either Route 1 or Route 3, and Type B cannot fly Route 1. Utilization is defined as the average number of hours of useful work performed per month by each aircraft assigned to a particular route. Utilization of 300 hours per month is assumed on Routes 1 and 2; 285 on Routes 3 and 4; and 240 on Route 5.

The assumed dollar costs per 100 passenger-miles are shown in (3). These costs do not include any capital costs such as the cost of the aircraft and ground facilities. They represent variable costs such as the cost of gasoline, salaries of the crew, and servicing the aircraft.

A second source of "costs" is due to the loss of revenues when not enough air-

[2] This is the expected number of full one-way trips per month to be carried on each route. If a passenger gets off en route and is replaced by another passenger, it is counted as one full trip.

[3] *Official Airline Guide*, July, 1954, p. 276. The N. Y.-Los Angeles routes are via Chicago and via Chicago Denver; the stop en route between New York and Dallas is at Memphis.

craft are assigned to the route to meet the passenger demand. In this case, the loss of revenue is the same as the price of a one-way ticket shown in the E row of (3).

Dollar costs per 100 passenger-miles

	Type Aircraft	Route				
		N.Y. to L.A. 1-Stop	N.Y. to L.A. 2-Stops	N.Y. to Dallas 0-Stop	N.Y. to Dallas 1-Stop	N.Y. to Boston 0-Stop
	A	\$0.45	\$0.57	\$0.45	\$0.47	\$0.64
(3)	B	—	.64	.83	.63	.88
	C	—	.92	—	.93	1.13
	D	.74	.61	.59	.62	.81
		Dollar costs per passenger turned away[4]				
	E	130 (13)	130 (13)	70 (7)	70 (7)	10 (1)

Based on the speeds, ranges, payload capacities and turnaround times, passenger capabilities were determined. The resultant potential number of passengers (in hundreds) p_{ij} that can be hauled per month per aircraft type i on route j is shown in (4); see staggered right figure in each box. By multiplying these numbers by the corresponding costs per 100 passenger-miles given in (3) and the number of miles given in (2), the monthly cost per aircraft can also be obtained. This is given in the lower left figure in each box which is the cost c_{ij} in thousands of dollars per month per aircraft type i assigned to the route j. The revenue losses c_{5j} in thousands of dollars per 100 passengers not carried are given in the E row of (4); finally we define $p_{5j} = 1$.[5] The staggered layout of (4) was chosen so as to identify the corresponding data found in Table 1, the latter being the *work sheet* upon which the entire problem is solved.

The basic problem consists of determining the number of aircraft of each type to assign to each route consistent with aircraft availabilities (1) and in determining how much revenue will be lost due to failure of allocated aircraft to meet passenger demand on various routes (2) and (3). Since many alternative allocations are possible, our specific objective will be to find that allocation that minimizes total costs where costs are defined as operating costs plus lost revenues based on the cost factors given in (3).

This may be formulated mathematically as a linear programming problem. Let x_{ij} denote the unknown quantity of the i^{th} type aircraft assigned to j^{th} route where $i = 1, 2, \cdots, m - 1$ and $j = 1, 2, \cdots, n - 1$. If x_{in} denotes the surplus or unallocated aircraft, then (5) states that the sum of allocated and unallocated aircraft of each type accounts for the total available aircraft a_i. If x_{mj} denotes the number of passengers in hundreds turned away, then equation (6) states that

[4] Figures shown in parentheses are 1000's of dollars lost per 100 passengers turned away. (Throughout this paper, passengers are measured in units of hundreds.)

[5] This will make it easier to form the passenger balance or "column" equations (7).

Passenger capabilities and costs per aircraft per month

(4)

Type Aircraft	Route				
	N.Y. to L.A. 1-Stop	N.Y. to L.A. 2-Stop	N.Y. to Dallas 0-Stop	N.Y. to Dallas 1-Stop	N.Y. to Boston 0-Stop
A Passengers (00) Costs ($000)	$p_{11} = 16$ $c_{11} = 18$	$p_{12} = 15$ $c_{12} = 21$	$p_{13} = 28$ $c_{13} = 18$	$p_{14} = 23$ $c_{14} = 16$	$p_{15} = 81$ $c_{15} = 10$
B Passengers (00) Costs ($000)	*	$p_{22} = 10$ $c_{22} = 15$	$p_{23} = 14$ $c_{23} = 16$	$p_{24} = 15$ $c_{24} = 14$	$p_{25} = 57$ $c_{25} = 9$
C Passengers (00) Costs ($000)	*	$p_{32} = 5$ $c_{32} = 10$	*	$p_{34} = 7$ $c_{34} = 9$	$p_{35} = 29$ $c_{35} = 6$
D Passengers (00) Costs ($000)	$p_{41} = 9$ $c_{41} = 17$	$p_{42} = 11$ $c_{42} = 16$	$p_{43} = 22$ $c_{43} = 17$	$p_{44} = 17$ $c_{44} = 15$	$p_{45} = 55$ $c_{45} = 10$
E (Deficit)	Losses per 100 passengers not hauled				
Passengers (00) Costs ($000)	$p_{51} = 1$ $c_{51} = 13$	$p_{52} = 1$ $c_{52} = 13$	$p_{53} = 1$ $c_{53} = 7$	$p_{54} = 1$ $c_{54} = 7$	$p_{55} = 1$ $c_{55} = 1$

the sum of passenger carrying capability of each type aircraft allocated to the j^{th} route, $p_{ij}\, x_{ij}$, plus the unsatisfied demand accounts for the total demand, d_j. Relation (7) states that all unknown quantities x_{ij} must be either positive or zero. Finally, if z is total costs, it is the sum of all the individual operating costs of each allocation, $c_{ij}\, x_{ij}$, plus the revenues lost by unsatisfied demands $c_{mj}\, x_{mj}$, see equation (8).

Fixed demand model

Find numbers x_{ij}, and the minimum value of z such that for $i = 1, 2, \cdots, m$; and $j = 1, 2, \cdots, n$

(5) Row Sums: $x_{i1} + x_{i2} + \cdots + x_{in} = a_i, \quad (i \neq m)$

(6) Col. Sums: $p_{1j}x_{1j} + p_{2j}x_{2j} + \cdots + p_{mj}x_{mj} = d_j$

(7) $x_{ij} \geqq 0$

(8) $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij} = z$

Any set of x_{ij} satisfying (5), (6), (7) is termed a *feasible solution*, and a feasible choice which minimizes the total cost z of the assignment given by (8) is called an *optimal (feasible) solution.*

The optimal assignment of aircraft to routes based on fixed demand as developed in the earlier study is shown in Table 1. The values assigned to the unknowns x_{ij} appear underlined in the upper left of each box unless $x_{ij} = 0$ in which case it is omitted; the entire layout takes the form:

x_{ij}		
		p_{ij}
	c_{ij}	

The sums by *rows* of x_{ij} entries in Table 1 equated to availabilities yield equations (5). The sums by *columns* of x_{ij} weighted by corresponding values of p_{ij} equated to demands yield equations (6); the x_{ij} weighted by corresponding c_{ij} and summed over the entire table yields (8). As noted earlier, Table 1 is actually the work sheet upon which the entire problem is solved. Later on we shall discuss a revision of this work sheet for solving problems with variable demand. All figures in the table, except for the upper left entries, x_{ij} and values of the so called "implicit prices" u_i and v_j shown in the margins, are constants which do not change during the course of computation. The values of the variables x_{ij}, u_i and v_j, however, will change during the course of successive iterations of the simplex method as adapted for this problem. For this reason it is customary to cover the work sheet with clear acetate and to enter the variable information with a grease pencil which can be easily erased; alternatively, a blackboard or semi-transparent tissue paper overlays can be used. The detailed rules for obtaining the optimal solution shown are given in [1] and will not be repeated here because a more general set of rules for the uncertain demand case will be given which, of course, could be used for the expected demand case.

In (9) we have a convenient summary serving to identify and define the numerical data entered in Table 1 and to give the test for optimality.

	Constants:	a_i = number of aircraft available of type i
		d_j = expected passenger demand in 100's per month on route j
		p_{ij} = passenger carrying capability in 100's per month per aircraft type i assigned to route j ($p_{mj} = 1$ by definition)
(9)		c_{ij} = costs in 1000's of dollars per month per aircraft type i assigned to route j (c_{mj} is per 100 passengers turned away)
	x_{ij} Entries:	x_{ij} = number of aircraft type i assigned to j^{th} route (x_{mj} is 100's of passengers turned away)
	Omitted x_{ij} Entries:	$x_{ij} = 0$ if upper left entry in box is missing
	Implicit Prices:	u_i and v_j are determined such that $u_i + p_{ij}v_j = c_{ij}$, ($u_m = v_n = 0$) for (i, j) boxes corresponding to $x_{ij} > 0$, i.e., non-omitted x_{ij} entries
	Test For Optimality:	Solution is optimal if for *all* (i, j) boxes $u_i + p_{ij}v_j \leqq c_{ij}$

3. Extension of Examples to Uncertain Demand

Up to this point the problem is identical to that described and solved in our previous paper. Now, to introduce the element of uncertain demand, we assume not a known (expected) demand on each route but a known *frequency distribution* of demand. The assumed frequency distributions are shown in (10). Thus on Route 1 (N.Y. to L.A.—1-Stop) it is assumed that either 20, 22, 25, 27, or 30 thousand passengers will want transportation during the month. On the other hand, for the N.Y. to L.A.—2-Stop either 5,000 or 15,000 passengers will want transportation with probabilities 30% or 70% respectively, etc. The assumed traffic distributions are, of course, hypothetical to illustrate our method. The demand distributions on the five routes varied over wide ranges and have different characteristics; Route 1 is flat, Route 2 is U-shaped, Routes 3, 4, 5 are unimodular but have differing degrees of concentration about the mode. Route 4 has a distribution with a very long tail that may reflect a realistic traffic situation.

Assumed Distribution of Passenger Demand

(λ_{hj} = Probability of Demand d_{hj})

(10)

Route	Hundreds of Passengers	Approx. Mean (00)	Probability of Passenger Demand	Probability of Equaling or Exceeding Demand
(1)	$200 = d_{11}$		$.2 = \lambda_{11}$	$1.0 = \gamma_{11}$
	$220 = d_{21}$		$.05 = \lambda_{21}$	$.8 = \gamma_{21}$
	$250 = d_{31}$	250	$.35 = \lambda_{31}$	$.75 = \gamma_{31}$
	$270 = d_{41}$		$.2 = \lambda_{41}$	$.4 = \gamma_{41}$
	$300 = d_{51}$		$2 = \lambda_{51}$	$.2 = \gamma_{51}$
(2)	$50 = d_{12}$	120	$.3 = \lambda_{12}$	$1.0 = \gamma_{12}$
	$150 = d_{22}$		$.7 = \lambda_{22}$	$.7 = \gamma_{22}$
(3)	$140 = d_{13}$		$.1 = \lambda_{13}$	$1.0 = \gamma_{13}$
	$160 = d_{23}$		$.2 = \lambda_{23}$	$.9 = \gamma_{23}$
	$180 = d_{33}$	180	$.4 = \lambda_{33}$	$.7 = \gamma_{33}$
	$200 = d_{43}$		$.2 = \lambda_{43}$	$.3 = \gamma_{43}$
	$220 = d_{53}$		$.1 = \lambda_{53}$	$.1 = \gamma_{53}$
(4)	$10 = d_{14}$		$.2 = \lambda_{14}$	$1.0 = \gamma_{14}$
	$50 = d_{24}$		$.2 = \lambda_{24}$	$.8 = \gamma_{24}$
	$80 = d_{34}$	90	$.3 = \lambda_{34}$	$.6 = \gamma_{34}$
	$100 = d_{44}$		$.2 = \lambda_{44}$	$.3 = \gamma_{44}$
	$340 = d_{54}$		$.1 = \lambda_{54}$	$.1 = \gamma_{54}$
(5)	$580 = d_{15}$		$.1 = \lambda_{15}$	$1.0 = \gamma_{15}$
	$600 = d_{25}$	600	$.8 = \lambda_{25}$	$.9 = \gamma_{25}$
	$620 = d_{35}$		$.1 = \lambda_{35}$	$.1 = \gamma_{35}$

To illustrate the essential character of the linear programming problem for the case of uncertain demand let us focus our attention on a single route—say, Route 1—with probability distribution of demand as given in (10). Let us suppose that aircraft assigned to Route 1 are capable of hauling 100 Y_1 passengers. The first 200(00) units of this capability are certain to be used and revenues from this source (negative costs) will be 13(000) $= k_1$ per unit. The next 20(00) units of this capability will be used with probability $\gamma_{21} = .8$. Indeed, 80% of the time the demand will be 220(00) or greater, while 20% of the time it will be 200(00); hence, the expected revenues per unit from this increment is $.8 \times 13 = 10.4$ or $10.4 = k_1\gamma_{21}$. On the third increment of 30(00) units (22,001 to 25,000 seats) the expected revenue is $.75 \times 13 = 9.8 = k_1\gamma_{31}$ per unit since there is a 25% chance that none of these units will be used and 75% that all will be used. For the fourth increment of 20(00) units (25,001 to 27,000) the expected revenue is $.4 \times 13 = 5.2 = k_1\gamma_{41}$ per unit. For the fifth increment of 3,000 units (27,001 to 30,000) it is $.2 \times 13 = 2.6 = k_1\gamma_{51}$ per unit. For the sixth increment, which is the number of units assigned above the 30,000 mark, the expected revenue per unit is $.0 \times 13 = 0$ per unit since it is certain that none of these units can be used. It is clear that no assignments above the 30,000 are worthwhile and hence the last increment can be omitted. The index $h = 1, 2, 3, 4, 5$, will be used to denote the 1st, 2nd, $\cdots$, 5th increment of demand.

The number of assigned units in each increment, however, can be viewed as an unknown that depends on the *total* (passenger hauling) capability assigned to Route $j = 1$. Thus if the total assigned is $Y_1 = 210(00)$ then the part of this total belonging to the first increment, denoted by y_{11}, is $y_{11} = 200(00)$ and the part belonging to the second increment, denoted by y_{21}, is $y_{21} = 10(00)$. The amounts in the higher increments are $y_{hi} = 0$ for $i = 3, 4, 5$. To review, the passenger-carrying capacity Y_j is determined by the number of aircraft assigned to route j so that

$$Y_j = p_{1j}\,x_{1j} + p_{2j}\,x_{2j} + p_{3j}\,x_{3j} + p_{4j}\,x_{4j}\,. \tag{11}$$

On the other hand, Y_j itself breaks down into five increments

$$Y_j = y_{1j} + y_{2j} + y_{3j} + y_{4j} + y_{5j} \tag{12}$$

for routes $j = 1, 3, 4$, and correspondingly fewer for $j = 2, 5$. Regardless of the total Y_j the amount y_{hj} belonging to each increment is bounded by the total size of each increment which we denote by b_{hj}; the latter, however, is simply the change in demand level so that

$$\begin{aligned}
0 &\leqq y_{1j} \leqq d_{1j} &&= b_{1j}\\
0 &\leqq y_{2j} \leqq d_{2j} - d_{1j} &&= b_{2j}\\
0 &\leqq y_{3j} \leqq d_{3j} - d_{2j} &&= b_{3j}\\
0 &\leqq y_{4j} \leqq d_{4j} - d_{3j} &&= b_{4j}\\
0 &\leqq y_{5j} \leqq d_{5j} - d_{4j} &&= b_{5j}\,.
\end{aligned} \tag{13}$$

The total expected revenue from route j is, therefore,

$$k_j(\gamma_{1j}\,y_{1j} + \gamma_{2j}\,y_{2j} + \cdots + \gamma_{5j}\,y_{5j}) \tag{14}$$

where k_j is revenue (in thousands) per 100 passengers carried on route j and as seen in (10) the probability γ_{ij} of exceeding or equaling demand d_{ij} is related to λ_{ij} the probability of demand by

$$
\begin{aligned}
1 = \gamma_{1j} &= \lambda_{1j} + \lambda_{2j} + \lambda_{3j} + \lambda_{4j} + \lambda_{5j} \\
\gamma_{2j} &= \lambda_{2j} + \lambda_{3j} + \lambda_{4j} + \lambda_{5j} \\
\gamma_{3j} &= \lambda_{3j} + \lambda_{4j} + \lambda_{5j} \\
\gamma_{4j} &= \lambda_{4j} + \lambda_{5j} \\
\gamma_{5j} &= \lambda_{5j}
\end{aligned}
\tag{15}
$$

and the values of λ_{hj} are given in (10). For example, the total expected revenues for Route 1 are

$$13(1.0y_{11} + .8y_{12} + .75y_{13} + .4y_{14} + .2y_{15}). \tag{16}$$

The most important fact to note about the linear form (16) is the decrease in the successive values of the coefficients, γ_{hj}. Moreover, this will always be the case whatever the distribution of demand since the probability of equaling or exceeding a given demand level d_{hj} decreases with increasing values of demand.

Suppose now y_{11}, y_{21}, $\cdots$, are treated as unknown variables in a linear programming problem subject only to (12) and (13) where the objective is to maximize revenues. Let us suppose further that Y_1 is fixed. It is clear, since the coefficient of y_{11} is largest in the maximizing form (14), y_{11} will be chosen as large as possible consistent with (12) and (13); for the chosen value y_{11}, the next increment y_{21} will be chosen as large as possible consistent with (12) and (13), etc. Thus, we need only specify y_{h1} by restrictions (12) and (13), because *when the maximum is reached* the values of the variables y_{11}, y_{21}, $\cdots$ are precisely the *incremental values* associated with Y_1, which we discussed earlier, (12). Even if passenger capability Y_1 is not fixed, as in the case about to be considered, it should be noted that whatever be the value of Y_1 the values of y_{11}, y_{21} $\cdots$ which minimize an over-all cost form such as in (20) below must maximize (14) for $j = 1$ and hence the incremental values of Y_1 will be generated by y_{11}, y_{21}, $\cdots$.

The linear programming problem in the case of uncertain demand is shown (17), (18), (19), (20). Thus expected costs are defined as total outlays (first term) plus the expected loss of revenue due to shortage of seats (last two terms),

Uncertain Demand Model

Find numbers x_{ij} and y_{hj} and the minimum value of z such that for $i = 1, 2, \cdots, m; j = 1, 2, \cdots, n; h = 1, 2, \cdots, r$.

(17) Row Sums: $x_{i1} + x_{i2} + \cdots + x_{in} = a_i$

(18) Column Sums: $p_{1j}x_{1j} + p_{2j}x_{2j} + \cdots + p_{nj}x_{nj} = y_{1j} + y_{2j} + \cdots + y_{rj}$

(19) $x_{ij} \geq 0, \quad 0 \leq y_{hj} \leq b_{hj}$

(20) Expected costs: $z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \left[R_0 - \sum_{j=1}^{n} k_j \sum_{h=1}^{r} \gamma_{hj} y_{1j} \right]$

where R_0 is the expected revenue if sufficient seats were supplied for all customers.

For the problem at hand the bounds, b_{hj}, and the expected revenues, $k_j\gamma_{hj}$, per unit for the "incremental variables" y_{hj} can be computed from probability distributions (10) via (13) and (15).

The numerical values of the constants for the stochastic case are tabulated in (21).

Incremental Bounds, b_{hj}, and Expected Revenues $k_j\gamma_{hj}$ per Unit of Passenger Carrying Capacity Assigned

(21)

	Route 1		Route 2		Route 3		Route 4		Route 5	
Increment h	b_{h1}	$k_1\gamma_{h1}$	b_{h2}	$k_2\gamma_{h2}$	b_{h3}	$k_3\gamma_{h3}$	b_{h4}	$k_4\gamma_{h4}$	b_{h5}	$k_5\gamma_{h5}$
1	200	$k_1 = 13$	50	$k_2 = 13$	140	$k_3 = 7$	10	$k_4 = 7$	580	$k_5 = 1$
2	20	$.8k_1 = 10.4$	100	$.7k_2 = 9.1$	20	$.9k_3 = 6.3$	40	$.8k_4 = 5.6$	20	$.9k_5 = .9$
3	30	$.75k_1 = 9.8$	**		20	$.7k_3 = 4.9$	30	$.6k_4 = 4.2$	20	$.1k_5 = .1$
4	20	$.4k_1 = 5.2$	**		20	$.3k_3 = 2.1$	20	$.3k_4 = 2.1$	**	
5	30	$.2k_1 = 2.6$	**		20	$.1k_3 = .7$	240	$.1k_4 = .7$	**	

** Only two increments for route 2 and three increments for route 5 are needed to describe distribution of demand.

4. Rules for Computation

The work sheet for determining the optimal assignment under uncertain demand is shown in Table 2. The entries in the "x_{ij}" boxes and "y_{hj}" boxes take the form

$$\boxed{\begin{array}{ll} x_{ij} & \\ & p_{ij} \\ & c_{ij} \end{array}} \qquad \boxed{\begin{array}{ll} y_{hj} & \\ & -1 \\ & -k_j\gamma_{hj} \end{array}}$$

To form the new row equations (17), the x_{ij} entries are summed to yield the a_i values given in the aircraft available column. To form the column equations (18), the x_{ij} entries are multiplied by p_{ij}, the y_{hj} by -1, and summed down to yield zero.

Step 1. To initiate the computation any set of nonnegative values may be assigned to the unknowns x_{ij} and y_{hj} provided they satisfy the equations and thereby constitute a feasible solution.

Step 2. Circle* *any* $(m + n)$ of x_{ij} and y_{hj} entries where $m + n$ is the number of row plus column equations. These circles can be arbitrarily selected except that they must have the property that if the fixed values assigned to the other

* Entries referred to as "arbitrarily selected and circled" entries appear in bold face figures in the tables.

noncircled variables and the constant terms were *arbitrarily changed to other values* then the circled variables would be determined uniquely in terms of the latter. Such a circled set of variables is called a *basic set* of variables; the array of coefficients associated with this set in the equations (17) and (18) is referred to as the *basis* in the theory of the simplex method [4].

Note: One simple way of selecting a basic set is shown in Table 3. One x_{ij} entry is arbitrarily selected and circled* in each row corresponding to a row equation, and one y_{hj} is arbitrarily selected and circled in each column corresponding to a column equation. In general, it is suggested that entries be circled* that appear to have a chance of having a positive value in an optimum solution; in case of y_{hj} values the last entry in the column that appears likely to be positive in an optimum solution should be circled and other y_{hj} above it in the column be set equal to b_{hj}.

Step 3. Compute for (i, j) and (h, j) combinations corresponding to circled entries, implicit prices u_i and v_j associated with equations by determining values of u_i and v_j satisfying the equations

$$u_i + p_{ij}v_j = c_{ij} \qquad (x_{ij} \text{ circled}) \tag{22}$$

$$0 + (-1)v_j = -k_j\gamma_{hj} \qquad (y_{hj} \text{ circled}). \tag{23}$$

There are always $(m + n)$ equations (22) and (23) in $(m + n)$ unknowns u_i and v_j which can be shown always to have a unique solution [4]. They can be solved by inspection, for it can be shown that the system is either completely triangular or at worst contains subsystems—some triangular and some triangular if one unknown is specified.[6]

Step 4. Compute for each box corresponding to x_{ij} or y_{hj}

$$\delta_{ij} = (u_i + p_{ij}v_j) - c_{ij} \qquad (\text{for } x_{ij} \text{ box}) \tag{24}$$

$$\delta'_{hj} = (0 - v_j) - (-k_j\gamma_{hj}) \qquad (\text{for } y_{hj} \text{ box}) \tag{25}$$

In practice, one of the δ_{ij} or δ'_{hj} is recorded; the others are computed and compared with it and the largest in absolute value is used. It can be shown [4] that if the x_{ij} or y_{hj} value associated with a non-circled entry is changed to

$$x_{ij} \pm \theta \quad \text{or} \quad y_{hj} \pm \theta \qquad \theta \geqq 0 \tag{26}$$

the other non-circled variables remaining invariant, and the circled variables adjusted, then the expected costs z will change to z' where

$$z' = z \mp \theta\delta_{ij} \quad \text{or} \quad z' = z \mp \theta\delta'_{hj} \tag{27}$$

Thus it pays to *increase* x_{ij} or y_{hj} if δ_{ij} or $\delta'_{hj} > 0$, unless $y_{hj} = b_{hj}$, its upper bound, in which case no increase in y_{hj} is allowed; also it pays to *decrease* x_{ij} or y_{hj} if δ_{ij} or $\delta'_{hj} < 0$ unless $x_{ij} = 0$ or $y_{hj} = 0$, in which case no decrease is allowed.

[6] This is the analogue for the "generalized" transportation problem (6), (7), (8), (9) of the well known theorem for the standard transportation problem that all bases are triangular. Its proof is similar.

Test for Optimality: According to the theory of the simplex method [3] if the *non-circled* variables satisfy the following conditions:

(1) they are all at either their upper or lower bounds,

(2) their corresponding δ_{ij} and $\delta'_{kj} \leqq 0$, if they are at their lower bound, and

(3) their corresponding δ_{ij} and $\delta'_{kj} \geqq 0$ if they are at their upper bound,

then the solution is optimal and the algorithm terminates. Otherwise there are δ_{ij} or δ'_{ij} for which a decrease or increase (depending on whether the sign is negative or positive) in the corresponding variable is allowed; let the largest among them in absolute value be denoted by δ_{rs} or δ'_{rs}.

Step 5. Leaving all noncircled entries fixed except for the value of the variable corresponding to the (r, s) box determined in Step 4, modify the value of x_{rs} (or y_{rs}) to

$$x_{rs} + \theta \text{ (or } y_{rs} + \theta) \text{ if } \delta_{rs} > 0 \text{ (or } \delta'_{rs} > 0) \qquad \text{or to} \tag{28}$$

$$x_{rs} - \theta \text{ (or } y_{rs} - \theta) \text{ if } \delta_{rs} < 0 \text{ (or } \delta'_{rs} < 0) \tag{29}$$

where $\theta \geqq 0$ is unknown and recompute the values of circled variables as linear functions of θ. Choose the value of $\theta = \theta^*$ at the largest value possible consistent with keeping all basic (circled) variables [whose values now depend on θ] between their upper and lower bounds; in the next cycle correct the values of the circled variables on the assumption $\theta = \theta^*$.

Also, if at the value $\theta = \theta^*$ one (or more) of the circled variables attains its upper or lower bound, in the next cycle drop *any one* of these variables (never drop more than one) from the basic set and circle the variable x_{rs} instead. Should it happen that it is x_{rs} that attains its upper or lower bound at $\theta = \theta^*$, the set of circled variables is the same as before; their values, however, are changed to allow x_{rs} to be fixed at its new bound.

Start the next cycle of the iterative procedure by returning to Step 3.

5. Numerical Solution of the Routing Problem

For our starting solution we used for values of x_{ij} the best solution of the earlier study, assuming fixed demands equal to the expected values of the distribution.[7] These are shown in Table 3. These x_{ij} will meet the expected demands so that $Y_j = b_j$ except for route 5 where there is a deficit of 100 and $Y_5 = 500$, see (11). These Y_j are broken down into the successive incremental values shown below the double line, see (12).

Next, one of the variables in each row is circled. The selected variables are x_{11}, x_{22}, x_{35}, x_{43}; each appears likely to be in an optimal solution, however, x_{43} has been circled rather than x_{41}, which may be a better choice. Next, the last positive entry in each column is circled, i.e., the variables y_{31}, y_{22}, y_{33}, y_{44}, y_{15}.

[7] In the humorous parady by Paul Gunther entitled "Use of Linear Programming in Capital Budgeting," *Journal of the Operations Research Society of America*, May, 1955, it will be recalled that Mrs. Efficiency wondered why Mr. O. R. did not start out with a good guess. In this paper you will note that we followed Mrs. Efficiency's suggestion and have started with a guess at the final solution rather than going through the customary use of artificial variables and a phase one of the simplex process.

In all there are $m + n = 9$ circled variables. The implicit values, u_i and v_j, are determined by solving the nine equations:

(30.1) $$u_1 + p_{11}v_1 = c_{11} \qquad (p_{11} = 16, c_{11} = 18)$$

(30.2) $$u_2 + p_{22}v_2 = c_{22} \qquad (p_{22} = 10, c_{22} = 15)$$

(30.3) $$u_3 + p_{35}v_5 = c_{35} \qquad (p_{35} = 29, c_{35} = 6)$$

(30.4) $$u_4 + p_{43}v_3 = c_{43} \qquad (p_{43} = 22, c_{43} = 17)$$

(30.5) $$0 + (-1)v_1 = -k_1\gamma_{31} \qquad (k_1\gamma_{31} = 9.8)$$

(30.6) $$0 + (-1)v_2 = -k_2\gamma_{22} \qquad (k_2\gamma_{22} = 9.1)$$

(30.7) $$0 + (-1)v_3 = -k_3\gamma_{33} \qquad (k_3\gamma_{33} = 4.9)$$

(30.8) $$0 + (-1)v_4 = -k_4\gamma_{44} \qquad (k_4\gamma_{44} = 2.1)$$

(30.9) $$0 + (-1)v_5 = -k_5\gamma_{15} \qquad (k_5\gamma_{15} = 1.0)$$

This permits the computation of δ_{ij} and δ'_{hj}, see (24) and (25). As a check $\delta_{ij} = 0$ and $\delta'_{hj} = 0$ for (i, j) and (h, j) corresponding to circled variables. The largest value of δ_{ij} or δ'_{hj} in absolute value is

$$\delta_{24} = [-76 + 15\ (2.1)] - 14 = -58.5,$$

hence a decrease in the variable x_{24} with adjustments of the circled variables will result in a decrease in the expected costs of 58.5 units per unit decrease in x_{24}. If $x_{24} = 6$ is changed to $x_{24} = 6 - \theta$, then in order to satisfy column 4 equations the circled variable $y_{44} = 10$ must be modified to $y_{44} = 10 - 15\theta$ (all other variables in column 4 are fixed). Also to satisfy row equation 2, $x_{22} = 8$ must be modified to $x_{22} = 8 + \theta$, and this in turn causes $y_{22} = 70$ to be changed to $y_{22} = 70 + 10\theta$ in order to satisfy column equation 2. The largest value of θ is $\theta^* = 10/15$ at which value $y_{44} = 0$.

The numerical values of the variables appearing in Table 4 are obtained from those of Table 3 by setting $\theta = \theta^* = 10/15$. The variable x_{24} becomes a new circled variable in place of y_{44} which hit its lower bound, zero; the other variables to be circled remain the same as in Table 3. Computing the new set of implicit prices the largest δ_{ij} in absolute value which can increase or decrease according to sign of δ_{ij} is $\delta_{23} = 23.4$. Changing x_{23} to $5 - \theta$ requires that the variables x_{22}, y_{22}, y_{33} be modified as shown, Table 4. The maximum value of θ is $\theta = \theta^* = 20/14$ at which value $y_{33} = 0$. The new solution in which x_{23} replaces y_{33} as a circled variable is given in Table 5. In Table 5 the decrease in non-circled variable x_{41} causes changes in the variables x_{43}, x_{22}, x_{23}, y_{31}, y_{22}. The largest value of $\theta = 9/16$ at which value y_{22} hits its *upper bound* $b_{22} = 100$.

In the passage from Table 6 to Table 7 we have become a little fancy and have taken a "double" step. The maximum increase is $\theta = 80/29$ at which point y_{15} hits its upper bound $b_{15} = 580$. It is easy to see that if next the incremental variable y_{25} is increased that δ_{32} associated with x_{32} should be changed to $\delta_{32} + 29(\gamma_{15} - \gamma_{25})k_5 = -4.5 + 29(1.0 - .9) = -1.6$; therefore, it is economical to increase y_{25} as well as y_{15}. However, it can be shown that signs of δ_{32} would become positive if the next increment, y_{35}, were considered. The maximum value of $\theta = \theta^* = 100/29$.

In the passage from Table 7 to 8, it will be noted that the variable y_{33} is again brought into solution having been dropped earlier. The maximum value of θ is 22/20 at which value y_{33} reaches its upper bound, so that the new solution, Table 8, has the same set of circled variables and hence the same implicit values as Table 7. Moreover, the solution is *optimal* since all non-circled variables are either at their upper or lower bounds—those at upper bounds have corresponding $\delta_{ij} \geqq 0$ and those at lower bounds have $\delta_{ij} \leqq 0$.

In comparing this solution (Table 8) with the optimal solution for the fixed demand case (Table 1), it is interesting to note that the chief difference appears to be a general tendency to shift the total seats made available on a route to a *mode* of the distribution rather than to the *mean* of the distribution for those distributions with sharp peaks. The total seats made available to routes with flat distributions of demand appear to be at highest level attainable with the residual passenger-carrying potential.

To compute the expected costs of the various solutions the first step, see (20), is to determine what the expected revenues R_0 would be if sufficient seating capacity were furnished at all times to supply all passengers that show. From (2) or (16) it is easy to see that

$$R_0 = 13(250) + 13(120) + 7(180) + 7(90) + 1(600) = 7300$$

or $7,300,000.

Comparative Costs of Various Solutions

(31)

Table	Expected Revenues for Seats Supplied (1)	Expected Lost Revenues* (2)	Operating Costs (3)	Net Expected Cost (Thousands) (2) + (3)
3	−6534	766	900	1,666
4	−6574	726	901	1,627
5	−6607	693	901	1,594
6	−6638	662	899	1,561
7	−6641	659	883	1,542
8	−6659	641	883	1,524

* Data in column (2) are obtained by subtracting the expected revenues for seats supplied, column (1), from $R_0 = 7300$, the expected revenues if unlimited number of seats were supplied.

It is seen that the solution presented in the earlier paper [1] assuming demands to be exactly equal to the expected values of demand has a net expected cost of $1,666,000. [It is interesting to note that if the demands were fixed and equal to expected demands, the costs would only be $1,000,000, see Table 1. The 67% increase in net cost for the variable demand case is due to 13,400 additional passengers (on the average) being turned away because of the distributions of demand assumed in (10)]. The successive improvements in the solution, Tables 3 to 8, reduced the net expected costs from \$1,666,000 to \$1,524,000 for the optimal solution.

In the illustration the best solution obtained by pretending that demands are fixed at these expected values has a 9% higher expected cost than that for the best solution obtained by using the assumed distributions of demand. It is also seen that very little additional computational effort was required to take account of this uncertainty of demand.

References

[1] FERGUSON, ALLEN R., AND GEORGE B. DANTZIG, "The Problem of Routing Aircraft," *Aeronautical Engineering Review*, April, 1955, Vol. 14, No. 4.

[2] DANTZIG, GEORGE B., "Linear Programming Under Uncertainty," *Management Sciences*, April–July, 1955.

[3] DANTZIG, GEORGE B., "Upper Bounds, Secondary Constraints, and Block Triangularity in Linear Programming," *Econometrica*, April, 1955.

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TABLE 1

Optimal Assignment for Fixed Demand

Operating Costs and Lost Revenues = \$1,000,000

Type Aircraft	Route (1) N. Y. to L. A. 1-Stop	(2) N. Y. to L.A. 2-Stop	(3) N. Y. to Dallas 0-Stop	(4) N. Y. to Dallas 1-Stop	(5) N. Y. to Boston 0-Stop	(6) Surplus Aircraft	Aircraft Available	Implicit Prices u_i
(1)A	*10* 16 18	15 21	28 18	23 16	81 10	0 0	10 = a_1	−171
(2)B	**	*8* 10 15	*5* 14 16	*6* 15 14	57 9	0 0	19 = a_2	−51
(3)C	**	*7.8* 5 10	**	7 9	*17.2* 29 6	0 0	25 = a_3	−23
(4)D	*10* 9 17	11 16	*5* 22 17	17 15	55 10	0 0	15 = a_4	−89
(5)E Deficit	1 13	1 13	1 7	1 7	*100* 1 1	0 0	**	0
Demand d_j	250	120	180	90	600	**		
Implicit Prices v_j	11.8	6.6	4.8	4.33	1	0		

TABLE 2

*Work Sheet for Determining Optimal Assignment under Uncertain Demand**

Type Aircraft	Route						Aircraft Available	Implicit Prices u_i
	(1) N.Y. to L.A. 1-Stop	(2) N.Y. to L.A. 2-Stop	(3) N.Y. to Dallas 0-Stop	(4) N.Y. to Dallas 1-Stop	(5) N.Y. to Boston 0-Stop	(6) Surplus Aircraft		
(1)A	x_{11} $p_{11} = 16$ $c_{11} = 18$	x_{12} 15 21	x_{13} 28 18	x_{14} 23 16	x_{15} 81 10	x_{16} 0 0	10	u_1
(2)B	***	x_{22} 10 15	x_{23} 14 16	x_{24} 15 14	x_{25} 57 9	x_{26} 0 0	19	u_2
(3)C	***	x_{32} 5 10	***	x_{34} 7 9	x_{35} 29 6	x_{36} 0 0	25	u_3
(4)D	x_{41} 9 17	x_{42} 11 16	x_{43} 22 17	x_{44} 17 15	x_{45} 55 10	x_{46} 0 0	15	u_4
Increment (1)	$y_{11} \leqq 200$ −1 −13	$y_{12} \leqq 50$ −1 −13	$y_{13} \leqq 140$ −1 −7	$y_{14} \leqq 10$ −1 −7	$y_{15} \leqq 580$ −1 −1	***	**	0
(2)	$y_{21} \leqq 20$ −1 −10.4	$y_{22} \leqq 100$ −1 −9.1	$y_{23} \leqq 20$ −1 −6.3	$y_{24} \leqq 40$ −1 −5.6	$y_{25} \leqq 20$ −1 −.9	***	**	0
(3)	$y_{31} \leqq 30$ −1 −9.8	***	$y_{33} \leqq 20$ −1 −4.9	$y_{34} \leqq 30$ −1 −4.2	$y_{35} \leqq 20$ −1 −.1	***	**	0

(4)	$y_{41} \leqq 20$ −1 −5.2	***	$y_{43} \leqq 20$ −1 −2.1	$y_{44} \leqq 20$ −1 −2.1	***	***	**	0
(5)	$y_{51} \leqq 30$ −1 −2.6	***	$y_{53} \leqq 20$ −1 −.7	$y_{54} \leqq 240$ −1 −.7	***	***	**	0
Net	0	0	0	0	0			
Implicit Prices v_j	v_1	v_2	v_3	v_4	v_5	0		

** Corresponding row or column has no equation.

*** Box not used because aircraft type cannot fly required range or fewer increments are needed to describe the distribution of demand on the route.

TABLE 3—Cycle 0

Work Sheet for Determining Optimal Assignment under Uncertain Demand

$\delta_{24} = 58.4$, $\theta = 10/15$, Exp. Cost = \$1,666,000

Type Aircraft	Route						Aircraft Available	Implicit Prices π_i
	(1) N.Y. to L.A. 1-Stop	(2) N.Y. to L.A. 2-Stop	(3) N.Y. to Dallas 0-Stop	(4) N.Y. to Dallas 1-Stop	(5) N.Y. to Boston 0-Stop	(6) Surplus Aircraft		
(1)A	10 16 18	15 21	28 18	23 16	81 10	0 0	10	−139
(2)B	***	$8+\theta$ 10 15	5 14 16	$6-\theta$ 15 14	57 9	0 0	19	−76
(3)C	***	7.8 5 10	***	7 9	17.2 29 6	0 0	25	−23
(4)D	10 9 17	11 16	5 22 17	17 15	55 10	0 0	15	−91
Increment (1)	200 −1 −13	50 −1 −13	140 −1 −7	10 −1 −7	500 −1 −1	***	**	0
(2)	20 −1 −10.4	$70+10\theta$ −1 −9.1	20 −1 −6.3	40 −1 −5.6	−1 −.9	***	**	0
(3)	30 −1 −9.8	***	20 −1 −4.9	30 −1 −4.2	−1 −.1	***	**	0

(4)	−1 −5.2	***	−1 −2.1	10 − 15θ −1 −2.1	***	***	**	0
(5)	−1 −2.6	***	−1 −.7	−1 −.7	***	***	**	0
Net	0	0	0	0	0			
Implicit Prices v_i	9.8	9.1	4.9	2.1	1	0		

NOTE: Bold face figures are referred to as circled entries in the text.

TABLE 4—Cycle 1

Work Sheet for Determining Optimal Assignment under Uncertain Demand

$\delta_{22} = 23.4,\ \theta = 20/14$, Exp. Cost = \$1,627,000

Type Aircraft	Route (1) N.Y. to L.A. 1-Stop	(2) N.Y. to L.A. 2-Stop	(3) N.Y. to Dallas 0-Stop	(4) N.Y. to Dallas 1-Stop	(5) N.Y. to Boston 0-Stop	(6) Surplus Aircraft	Aircraft Available	Implicit Prices u_i
(1)A	10 16 18	15 21	28 18	23 16	81 10	0 0	10	−139
(2)B	***	8.7 + θ 10 15	5 − θ 14 16	5.3 15 14	57 9	0 0	19	−76
(3)C	***	5 10	***	7 9	17.2 29 6	0 0	25	−23
(4)D	10 9 17	11 16	5 22 17	17 15	55 10	0 0	15	−91
Increment (1)	200 −1 −13	50 −1 −13	140 −1 −7	10 −1 −7	500 −1 −1	***	**	0
(2)	20 −1 −10.4	77 + 10θ −1 −9.1	20 −1 −6.3	40 −1 −5.6	−1 −.9	***	**	0
(3)	30 −1 −9.8	***	20 − 14θ −1 −4.9	30 −1 −4.2	−1 −.1	***	**	0

(4)	−1 −5.2	***	−1 −2.1	−1 −2.1	***	***	**	0
(5)	−1 −2.6	***	−1 −.7	−1 −.7	***	***	**	0
Net	0	0	0	0	0			
Implicit Prices v_j	9.8	9.1	4.9	6	1	0		

NOTE: Bold face figures are referred to as circled entries in the text.

TABLE 5—Cycle 2

Work Sheet for Determining Optimal Assignment under Uncertain Demand

$\delta_{41} = 56.8$, $\theta = 9/16$, Exp. Cost = \$1,594,000

Type Aircraft	Route						Aircraft Available	Implicit Prices u_i
	(1) N.Y. to L.A. 1-Stop	(2) N.Y. to L.A. 2-Stop	(3) N.Y. to Dallas 0-Stop	(4) N.Y. to Dallas 1-Stop	(5) N.Y. to Boston 0-Stop	(6) Surplus Aircraft		
(1)A	10 16 18	15 21	28 18	23 16	81 10	0 0	10	−139
(2)B	***	$10.1 + 1.6\theta$ 10 15	$3.6 - 1.6\theta$ 14 16	5.3 15 14	57 9	0 0	19	−76
(3)C	***	7.8 5 10	***	7 9	17.2 29 6	0 0	25	−23
(4)D	$10 - \theta$ 9 17	11 16	$5 + \theta$ 22 17	17 15	55 10	0 0	15	−128
Increment (1)	200 −1 −13	50 −1 −13	140 −1 −7	10 −1 −7	500 −1 −1	***	**	0
(2)	20 −1 −10.4	$91 + 16\theta$ −1 −9.1	20 −1 −6.3	40 −1 −5.6	−1 −.9	***	**	0
(3)	$30 - 9\theta$ −1 −9.8	***	−1 −4.9	30 −1 −4.2	−1 −.1	***	**	0

(4)	−1 −5.2	***	−1 −2.1	−1 −2.1	***	***	**	0
(5)	−1 −2.6	***	−1 −.7	−1 −.7	***	***	**	0
Net	0	0	0	0	0			
Implicit Prices v_j	9.8	9.1	6.6	6	1	0		

NOTE: Bold face figures are referred to as circled entries in the text.

TABLE 6—Cycle 3

Work Sheet for Determining Optimal Assignment under Uncertain Demand

$\delta_{32} = 5.5,\ \theta = 100/29 = 3.45$, Exp. Cost = \$1,561,000

Type Aircraft	Route: (1) N.Y. to L.A. 1-Stop	(2) N.Y. to L.A. 2-Stop	(3) N.Y. to Dallas 0-Stop	(4) N.Y. to Dallas 1-Stop	(5) N.Y. to Boston 0-Stop	(6) Surplus Aircraft	Aircraft Available	Implicit Prices u_i
(1) A	10 16 18	15 21	28 18	23 16	81 10	0 0	10	−139
(2) B	***	$11 + .5\theta$ 10 15	$2.7 - .5\theta$ 14 16	5.3 15 14	57 9	0 0	19	−40
(3) C	***	$7.8 - \theta$ 5 10	***	7 9	$17.2 + \theta$ 29 6	0 0	25	−23
(4) D	$9.4 - .3\theta$ 9 17	11 16	$5.6 + .3\theta$ 22 17	17 15	55 10	0 0	15	−71
Increment (1)	200 −1 −13	50 −1 −13	140 −1 −7	10 −1 −7	$500 + 29\theta$ −1 −1	***	**	0
(2)	20 −1 −10.4	100 −1 −9.1	20 −1 −6.3	40 −1 −5.6	−1 −.9	***	**	0
(3)	$25 - 2.7\theta$ −1 −9.8	***	−1 −4.9	30 −1 −4.2	−1 −.1	***	**	0

(4)	−1 −5.2	***	−1 −2.1	−1 −2.1	***	***	**	0
(5)	−1 −2.6	***	−1 −.7	−1 −.7	***	***	**	0
Net	0	0	0	0	0			
Implicit Prices v_j	9.8	5.5	4	3.6	1	0		

NOTE: Bold face figures are referred to as circled entries in the text.

TABLE 7—Cycle 4

Work Sheet for Determining Optimal Assignment under Uncertain Demand

$\delta_{33} = -.9$, $\theta = 20/22 = .9$, Exp. Cost = \$1,542,000

Type Aircraft	Route: (1) N.Y. to L.A. 1-Stop	(2) N.Y. to L.A. 2-Stop	(3) N.Y. to Dallas 0-Stop	(4) N.Y. to Dallas 1-Stop	(5) N.Y. to Boston 0-Stop	(6) Surplus Aircraft	Aircraft Available	Implicit Prices u_i
(1)A	10 16 18	15 21	28 18	23 16	81 10	0 0	10	−139
(2)B	***	12.8 10 15	.9 14 16	5.3 15 14	57 9	0 0	19	−40
(3)C	***	4.3 5 10	***	7 9	20.7 29 6	0 0	25	−18
(4)D	$8.3 - \theta$ 9 17	11 16	$6.7 + \theta$ 22 17	17 15	55 10	0 0	15	−71
Increment (1)	200 −1 −13	50 −1 −13	140 −1 −7	10 −1 −7	580 −1 −1	***	**	0
(2)	20 −1 −10.4	100 −1 −9.1	20 −1 −6.3	40 −1 −5.6	20 −1 −.9	***	**	0
(3)	$15 - 9\theta$ −1 −9.8	***	$+22\theta$ −1 −4.9	30 −1 −4.2	−1 −.1	***	**	0

(4)	−1 −5.2	***	−1 −2.1	−1 −2.1	***	***	**	0
(5)	−1 −2.6	***	−1 −.7	−1 −.7	***	***	**	0
Net	0	0	0	0	0			
Implicit Prices v_i	9.8	5.5	4	3.6	.8	0		

NOTE: Bold face figures are referred to as circled entries in the text.

TABLE 8—Cycle 5 (Optimal)

Work Sheet for Determining Optimal Assignment under Uncertain Demand

Minimum Expected Cost $1,524,000

Type Aircraft	Route						Aircraft Available	Implicit Prices u_i
	(1) N.Y. to L.A. 1-Stop	(2) N.Y. to L.A. 2-Stop	(3) N.Y. to Dallas 0-Stop	(4) N.Y. to Dallas 1-Stop	(5) N.Y. to Boston 0-Stop	(6) Surplus Aircraft		
(1)A	10 16 18	15 21	28 18	23 16	81 10	0 0	10	−139
(2)B	***	12.8 10 15	.9 14 16	5.3 15 14	57 9	0 0	19	−40
(3)C	***	4.3 5 10	***	7 9	20.7 29 6	0 0	25	−18
(4)D	7.4 9 17	11 16	7.6 22 17	17 15	55 10	0 0	15	−71
Increment (1)	200 −1 −13	50 −1 −13	140 −1 −7	10 −1 −7	580 −1 −1	***	**	0
(2)	20 −1 −10.4	100 −1 −9.1	20 −1 −6.3	40 −1 −5.6	20 −1 −.9	***	**	0
(3)	7 −1 −9.8	***	20 −1 −4.9	30 −1 −4.2	−1 −.1	***	**	0

(4)	−1 −5.2	***	−1 −2.1	−1 −2.1	***	***	**	0
(5)	−1 2.6	***	−1 −.7	−1 −.7	***	***	**	0
Net	0	0	0	0	0			
Implicit Prices v_j	9.8	5.5	4	3.6	.8	0		

NOTE: Bold face figures are referred to as circled entries in the text.