Panel Regression with Endogenous Regime Switching

Yoosoon Chang[*], Alireza Marahel[†], Joon Y. Park[‡]

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Abstract

This paper investigates the time variation in the Captial Asset Pricing Model (CAPM) betas by introducing a new approach that models panel regressions with endogenous regime-switching using a latent autoregressive factor. For our estimation, we model the CAPM using portfolio returns sorted on book-to-market ratio, where the factor loadings, the pricing errors, and the volatility of the error terms can vary across high and low volatility states of the market. We find that the behavior of this asset pricing model significantly differs across different volatility regimes and its performance improves significantly, especially when it is evaluated during the times where the market is in the low volatility regime.

1 Introduction

The Capital Asset Pricing Model (CAPM) introduced by Sharpe (1964) and Lintner (1965) is among the first and most important benchmark models in the asset pricing literature where it considers the market return as the sole factor to explain the variations observed in the stock excess returns. However, numerous papers, including Fama and French (1992), have evaluated the performance of the CAPM with constant factor loadings and found that the estimated betas do not explain the variation observed in the average returns across different portfolios. After this point, many researchers have tried to propose an asset-pricing model that employs multiple factors to explain the excess stock returns, given in the form of $R^e_{i,t} = \alpha_i + \sum_{n=1}^{n=N} \beta_{n,i} f_{n,i} + \varepsilon_{i,t}$, where $R^e_{i,t}$ is the excess return and $f_{n,i}$'s are the risk factors. The number of different factors proposed is quite overwhelming. Harvey, Liu, and Zhu (2015) documented 316 significant factors pricing the cross-section of stock returns

[*]Department of Economics, Indiana University.
[†]Department of Economics, Indiana University: amarahel@iu.edu.
[‡]Department of Economics, Indiana University.

identified by the literature, with the majority being in the last 15 years. The main objective of introducing new factors is to construct a model that could explain the observed cross-sectional variations in the stock returns more than what is already established. However, the abnormal cross-sectional returns were still found to be persistent since the pricing errors seem to remain significant in all the models proposed so far. A possible explanation for the failure of the original CAPM and other asset-pricing models is the dramatic intertemporal variation in the stock prices, which we believe is the main reason that these traditional models cannot perform well. The main issue in these models is one of their principal assumptions which states the volatility level in the market and betas are both constant over time. This inspires the idea that there is more than a single regime existing in the market. Given this hypothesis, there is literature that follows this logic and studies the models with time-varying betas. Broadly speaking, there are two types of approaches to implement the time variation of factor loadings to the model specification. One way to do so is to consider continuous changes in the betas. For instance, Jagannathan and Wang (1996) evaluate the performance of conditional CAPM where it assumes that the CAPM holds in a conditional sense and the betas and the market risk premium can vary over time. Among this type of papers, some use instrumental variables to proxy time-variation observed in the factor loading and market risk premium and to identify the covariance between them (e.g. Lettau and Ludvigson (2002); Petkova and Zhang (2005)). In another group of articles, there has been an effort to apply the mentioned hypothesis to expand the multi-factor models with multiple regimes, each of which is associated with a different distribution of asset returns (e.g. Tu (2010); Abdymomunov and Morley (2011); Chen and Kawaguchi (2018)). To put it another way, the regime-switching models proposed by these articles create a setting that assumes that the stock excess returns are drawn from different distributions, with a well-defined stochastic process determining the likelihood that each return is drawn from a given distribution. However, they simply apply an exogenous Markov-switching model, which was introduced by Hamilton (1989), where the process of determining the regimes is completely independent of all other features of the model.

In another approach, to test the hypothesis that the time variation in the betas is discrete (having multiple regimes in the model), this paper proposes a new approach to model panel regressions with endogenous regime-switching using an autoregressive latent factor that was first introduced by Chang, Choi, and Park (2017). Under our specification, the state of the market–high volatility or low volatility state–is determined by whether a latent regime factor, that is extracted from the observed time series, takes a value above or below a threshold level. The innovation of the latent factor is assumed to be correlated with the previous stock return shock and as a result, the shock to the stock returns will affect the stock market regime-switching in the following period. There are a couple of advantages of using this regime switching model. First, the ability of our model to extract the latent factor enables us to efficiently get more information on the regime-switching from the observed stock return data and look for the key determinant of the state of the market. Second, our model implies that the future state of the market is determined by not only the current state but also the realized values of the stock return, which is what one may normally expect.

Our empirical findings, consistent with the discussion made in section 4, demonstrates that the time-varying betas can help explaining the portfolio returns much better than the original CAPM, especially when market volatility level is relatively low. The results reported by previous articles, obtain from applying the regime-switching specification to an asset pricing model, commonly provided contrary evidence to the theoretical positive relationship between risk and return. To justify the anomalies observed to the contrary of this theory, these articles have discussed that even though the investors may react to the information about the true volatility regimes, it is more reasonable to assume that there is a time delay in the process of digesting the news and information about the current volatility level of the market. However, before we could make such a judgment, we think there should be a distinction between the aggregate uncertainty level in the market and the relative uncertainty observed in each portfolio with respect to the market which will be discussed in section 4. It is demonstrated that the positive relationship between risk and return still holds with respect to the relative risk observed in the portfolios, but only when the market is in the low volatility regime.

The model introduced in this paper can be applied to any multi-factor asset pricing model given in the form of $R^e_{i,t} = \alpha_i + \sum_{n=1}^{n=N} \beta_{n,i} f_{n,i} + \varepsilon_{i,t}$. To show how our model works, we consider the CAPM that measures the systematic risk of a security relative to the overall market. The overall market excess return, which is among the most promising factors expressed in the literature, is the only risk factor used in this model. We expect the model to correctly identify the price of risk when the market is in the low volatility regime. Our model specification may simply be extended by adding an additional latent factor to consider the possibility that the pricing errors can follow a separate state process. Additionally, this endogenous regime-switching specification can be further extended to a version that considers more than two volatility regimes to evaluate the performance of any asset pricing model.

2 Model

In this section, we introduce a panel regression model with endogenous regime switching and describe how it can simplify to a model based on the conventional regime switching.

The model for a panel (y_{it}) is specified as

$$y_{it} = \alpha_i(s_t) + \beta_i(s_t)' x_t + \varepsilon_{it}(s_t) \tag{1}$$

for $t = 1, ..., T$ and $i = 1, ..., N$, where (x_t) is a vector of covariates and $\varepsilon_{it}(s_t)$ represents the regime dependent error term, which is further specified as

$$\varepsilon_{it}(s_t) = \pi_i(s_t) u_t + \sigma_i e_{it}, \tag{2}$$

where in turn (u_t) and (e_{it}) are normal random variables with zero mean and unit variance, independent from each other, and also both serially and cross-sectionally. The specification in (2) implies that the error term in our panel regression a factor structure with a single common factor u_t whose loadings $\pi_i(s_t)$ are regime dependent. Only the common factor component has regime dependence, and all the idiosyncratic components are set to be independent of regimes. The coefficients $\alpha_i(s_t)$ and $\beta_i(s_t)$ in (1) are also set to be dependent upon regimes determined by the common state variable (s_t). In our model, the state variable (s_t) is defined by

$$s_t = 1\{w_t \geq \tau\} \tag{3}$$

where τ is an unknown threshold, $1\{\cdot\}$ is the indicator function, and (w_t) is a latent autoregressive factor generated as

$$w_t = \lambda w_{t-1} + v_t \tag{4}$$

with $|\lambda| < 1$ and i.i.d. standard normal innovations (v_t). Therefore, we have two regimes, denoted as 1 and 0 respectively according to the value of (s_t), depending upon whether the latent factor (w_t) takes a value above or below the threshold τ. Chang, Choi, and Park (2017) show that the transition given by (3) and (4) have a one-to-one correspondence with those of the general two-state Markov transitions: we can find a pair of λ and τ so that (3) and (4) yield the same transition for any two-state Markov transition, as well as the choice of a pair of λ and τ in (3) and (4) uniquely determine a two-state Markov transition.

The reformulation of a two-sate Markov transition as in (3) and (4) has some clear advantages. First, by introducing a latent factor, we may extract information on the *strength* of regimes, as well as regimes themselves. Secondly, and more importantly, our formulation makes it possible to allow for endogeneity in the regime switching. In fact, we introduce correlation between the common factor of the error term (u_t) and (v_t) in (2) and (4), and in particular let

$$\rho = \mathbb{E}(u_t v_{t+1}) \tag{5}$$

and allow $\rho \neq 0$. For nonzero ρ, the regime determined by the value of the latent factor w_{t+1} at time $t+1$ is affected by the realization of $\varepsilon_{it}(s_t)$ at time t (which itself depend on the realization of the regime at time t), implying the presence of a feedback effect of the common factor of the error term (u_t) on the regime. As in the conventional regime switching model, we assume that (u_t) and (v_t) are all jointly normal, and that they have zero mean and, for identification, unit variance. With this specification, if $\rho < 0$, the lagged common factor of the innovation u_t of the time series y_t at time t becomes negatively correlated with the innovation v_{t+1} of the latent autoregressive factor w_{t+1} at time $t+1$. This implies that a negative shock to y_t in the current period will cause an increase to the volatility level in the next period. The opposite is true when $\rho > 0$.

Under our specification in (1), (2) and (5), we can decompose (u_t) as

$$u_t = \rho v_{t+1} + \sqrt{1-\rho^2}\eta_t \tag{6}$$

with $|\rho| \leq 1$, and (η_t) is an i.i.d. standard normal random variable being independent of (v_t) at all leads and lags. With (1) and (6) taken together, our model may be rewritten as

$$y_{it} = \alpha_i(s_t) + \beta_i(s_t)' x_t + \rho\pi_i(s_t)v_{t+1} + \sqrt{1-\rho^2}\pi_i(s_t)\eta_t + \sigma_i e_{it}, \tag{7}$$

which is a general panel regression with regime switching, where we allow for both fixed and random effects, as well as heterogeneity. clearly, our model specified by (1), (2), (3), (4), (5) and (6) may also be given by (7) with (3) and (4).

To simplify our notation, let us label the states as 1 (low volatility) or 0 (high volatility) when state variable takes a value of 1 or 0, respectively. If we denote ξ_i as a generic notation for the state dependent parameters of the model, e.g. $\alpha_i(s_t)$, we may write

$$\xi_i(s_t) = \xi_{i,0}(1-s_t) + \xi_{i,1}s_t,$$

where $\xi_{i,0}$ and $\xi_{i,1}$ are the values the state dependent parameter can get, depending on whether we have $w_t < \tau$ or $w_t \geq \tau$. For the identification of the parameters of our model, we characterize the state of the market by its uncertainty, similar to what investors tend to do. We assume that $\pi_{i,0} > \pi_{i,1}$ for all i, which simply means that the level of uncertainty is relatively higher in the high volatility state than the low volatility state (note that $Var(\varepsilon_{it}(s_t)) = \pi_i^2(s_t) + \sigma_i^2$). If $\lambda = 1$, the latent factor (w_t) becomes a random walk and we further have to face the issue of joint identification for the initial value w_0 of (w_t) and the threshold level τ. In this case, the latent autoregressive process becomes $w_t = w_{t-1} + \sum_{t=1}^{T} v_t$ for all t. We set $w_0 = 0$ since any transformation of the form w_0 to $w_0 + c$ for any constant

c will result in the transformation of w_t to $w_t + c$ and τ to $\tau + c$, and will not affect state process (s_t) defined in (3). However, in the case of $|\lambda| < 1$, the identification problem of the initial value w_0 of (w_t) does not arise and if we let

$$w_0 =_d \mathbb{N}\left(0, \frac{1}{1-\lambda^2}\right),$$

the latent factor (w_t) becomes a strictly stationary process. Therefore, one may easily see that the autoregressive parameter λ determines the level of persistency observed in the regime changes. In particular, if the regime changes in the market is highly persistent in a specific time period, the autoregressive parameter will be close to 1 for that period.

If we let $\rho = 0$, the state process defined in (3) reduces to conventional Markov switching process where the innovation $\varepsilon_t(s_t)$ of the time series y_t becomes independent of the innovation v_{t+1} of the latent autoregressive factor w_{t+1}. To see how this works, we assume $\rho = 0$ for the rest of this section. It follows that the transition probabilities will depend on the latent factor autoregressive coefficient λ and the threshold level τ. We may easily see that

$$\mathbb{P}\{s_t = 0 | w_{t-1}\} = \mathbb{P}\{w_t < \tau | w_{t-1}\} = \Phi\left(\tau - \lambda w_{t-1}\right) \tag{8}$$

$$\mathbb{P}\{s_t = 1 | w_{t-1}\} = \mathbb{P}\{w_t \geq \tau | w_{t-1}\} = 1 - \Phi\left(\tau - \lambda w_{t-1}\right). \tag{9}$$

If we let $|\lambda| < 1$, it follows that the transition probabilities of the state process (s_t) from the low volatility state to the low volatility state and from the high volatility state to the high volatility state is given by

$$\mathbb{P}\{s_t = 0 | s_{t-1} = 0\} = \frac{\int_{-\infty}^{\tau\sqrt{1-\lambda^2}} \Phi\left(\tau - \frac{\lambda x}{\sqrt{1-\lambda^2}}\right) \varphi(x) dx}{\Phi(\tau\sqrt{1-\lambda^2})} \tag{10}$$

$$\mathbb{P}\{s_t = 1 | s_{t-1} = 1\} = 1 - \frac{\int_{\tau\sqrt{1-\lambda^2}}^{\infty} \Phi\left(\tau - \frac{\lambda x}{\sqrt{1-\lambda^2}}\right) \varphi(x) dx}{1 - \Phi(\tau\sqrt{1-\lambda^2})}. \tag{11}$$

Let us define the conditional transition density $p(s_t | s_{t-1})$ as

$$p(s_t | s_{t-1}) = (1 - s_t)\omega + s_t(1 - \omega) \tag{12}$$

where $\omega = \omega(s_{t-1})$ is the transition probability of (s_t) to the low volatility state conditional on the previous state and the past values of the observed times series and is given by

$$\omega(s_{t-1}) = \frac{\left[(1-s_{t-1})\int_{-\infty}^{\tau\sqrt{1-\lambda^2}} + s_{t-1}\int_{\tau\sqrt{1-\lambda^2}}^{\infty}\right]\Phi\left(\tau - \frac{\lambda x}{\sqrt{1-\lambda^2}}\right)\varphi(x)dx}{(1-s_{t-1})\Phi(\tau\sqrt{1-\lambda^2}) + s_{t-1}\left[1-\Phi(\tau\sqrt{1-\lambda^2})\right]}.$$

On the other hand, if we let $\lambda = 1$, the state process (s_t) defined in (3) becomes nonstationary and its transition evolves with time t. For $t = 1$, the transitions are given by $\mathbb{P}\{s_1 = 0|s_0 = 0\} = \Phi(\tau)$ where $\mathbb{P}\{s_{t-1} = 0\} = 1$ if $\tau > 0$, and $\mathbb{P}\{s_1 = 1|s_0 = 1\} = 1 - \Phi(\tau)$ where $\mathbb{P}\{s_{t-1} = 1\} = 1$ if $\tau \leq 0$. For $t \geq 2$, we define the transition probabilities explicitly as functions of time as

$$\mathbb{P}\{s_t = 0|s_{t-1} = 0\} = \frac{\int_{-\infty}^{\tau/\sqrt{t-1}} \Phi\left(\tau - x\sqrt{t-1}\right)\varphi(x)dx}{\Phi\left(\tau/\sqrt{t-1}\right)} \tag{13}$$

$$\mathbb{P}\{s_t = 1|s_{t-1} = 1\} = 1 - \frac{\int_{\tau/\sqrt{t-1}}^{\infty} \Phi\left(\tau - x\sqrt{t-1}\right)\varphi(x)dx}{1-\Phi\left(\tau/\sqrt{t-1}\right)}. \tag{14}$$

3 Estimation

Our model can be estimated by the maximum likelihood method. For the maximum likelihood estimation of our model based on the sample (y_{it}) for $i = 1, ..., N$ and $t = 1, ..., T$, we let $y_t = (y_{1t}, ..., y_{Nt})'$ and

$$\mathcal{F}_t = \sigma\Big((y_s)_{s=1}^t\Big)$$

which is the information given by $y_1, ..., y_t$ for $t = 1, ..., T$. The log-likelihood function is then given by

$$\ell(y_1, ..., y_T) = \log p(y_1) + \sum_{t=2}^{T} \log p(y_t|\mathcal{F}_{t-1})$$

where $p(\cdot)$ and $p(\cdot|\cdot)$ denote the density and conditional density functions, respectively. The objective is to maximize the log-likelihood function over a matrix of unknown parameters $\theta \in \Theta$, which includes, for example, the coefficients of the model such as α_i, the latent autoregressive factor coefficient λ, etc. Then, the maximum likelihood estimator $\hat{\theta}$ of θ is given by

$$\hat{\theta} = \arg\max_{\theta \in \Theta} \ell(y_1, ..., y_T)$$

where θ consists of the set of state dependent coefficients $(\alpha_{i0}, \alpha_{i1})$ and (β_{i1}, β_{i1}), the volatility parameters (π_{i0}, π_{i1}) and σ_i, as well as the correlation coefficient ρ, the autoregressive coefficient of the latent factor λ, and the threshold level τ.

As in the conventional regime switching model, the log-likelihood function can be obtained in two stages: prediction and updating steps. In what follows, we let $\varepsilon_t(s_t)$ denote $(\varepsilon_{1t}(s_t), ..., \varepsilon_{Nt}(s_t))'$, analogous to the definition of (y_t).

Prediction The prediction step is defined as

$$p(y_t|\mathcal{F}_{t-1}) = \sum_{s_t} p(y_t|s_t, \mathcal{F}_{t-1}) p(s_t|\mathcal{F}_{t-1}).$$

We may easily deduce that $p(y_t|s_t, \mathcal{F}_{t-1}) = p(y_t|s_t)$, and that

$$p(y_t|s_t) = \left(\frac{1}{2\pi}\right)^{N/2} \left(\sqrt{\det \Omega(s_t)}\right)^{-1} \exp\left(-\frac{1}{2}\varepsilon_t'(s_t)\Omega^{-1}(s_t)\varepsilon_t(s_t)\right) \tag{15}$$

with $\varepsilon_{it}(s_t)$ specified by

$$\varepsilon_{it}(s_t) = y_{it} - \alpha_i(s_t) - \beta_i'(s_t)x_t$$

for $i = 1, .., N$, where

$$\Omega(s_t) = \begin{pmatrix} \pi_1^2(s_t) + \sigma_1^2 & \pi_1(s_t)\pi_2(s_t) & \cdots & \pi_1(s_t)\pi_N(s_t) \\ \pi_2(s_t)\pi_1(s_t) & \pi_2^2(s_t) + \sigma_2^2 & \cdots & \pi_2(s_t)\pi_N(s_t) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_N(s_t)\pi_1(s_t) & \pi_N(s_t)\pi_2(s_t) & \cdots & \pi_N^2(s_t) + \sigma_N^2 \end{pmatrix} \tag{16}$$

is the covariance matrix of $\varepsilon_t(s_t)$. Note that $p(\varepsilon_t(s_t)) =_d \mathbb{N}(0, \Omega(s_t))$.

Moreover, we have

$$p(s_t|\mathcal{F}_{t-1}) = \sum_{s_{t-1}} p(s_t|s_{t-1}, \mathcal{F}_{t-1}) p(s_{t-1}|\mathcal{F}_{t-1}) \tag{17}$$

As in the conventional Markov switching filter, $p(s_{t-1}|\mathcal{F}_{t-1})$ is obtained from the previous updating step, which will be given below. Therefore, it suffices to get $p(s_t|s_{t-1}, \mathcal{F}_{t-1})$. Note that

$$p(w_t|w_{t-1},\mathcal{F}_{t-1}) = p(w_t|w_{t-1},y_{t-1}) = p(w_t|w_{t-1},\varepsilon_{t-1}(s_{t-1}))$$

where w_{t-1} is independent of $\varepsilon_{t-1}(s_{t-1})$.

Let $|\rho| < 1$ and $|\lambda| < 1$. It follows that

$$p(v_{t+1}|s_t,\varepsilon_t(s_t)) =_d \mathbb{N}\left(\rho\pi'(s_t)\Omega^{-1}(s_t)\varepsilon_t(s_t), 1-\rho^2\pi'(s_t)\Omega^{-1}(s_t)\pi(s_t)\right). \tag{18}$$

where $\pi = (\pi_1,...,\pi_N)'$. Appendix A provides some useful insights to find the determinant and inverse of the covariance matrix $\Omega(s_t)$, analytically. For the sake of simplicity of our notation, we drop (s_t) from the state dependent parameters and variables.

It follows that

$$\mathbb{P}\Big\{s_t = 0|s_{t-1} = 0,\mathcal{F}_{t-1}\Big\} = \frac{\int_{-\infty}^{\tau\sqrt{1-\lambda^2}} \Phi\left(\frac{\tau-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}} - \frac{\lambda x}{\sqrt{(1-\lambda^2)\left(1-\rho^2\pi'\Omega^{-1}\pi\right)}}\right)\varphi(x)dx}{\Phi\left(\tau\sqrt{1-\lambda^2}\right)}$$

and

$$\mathbb{P}\Big\{s_t = 1|s_{t-1} = 1,\mathcal{F}_{t-1}\Big\} = 1 - \frac{\int_{\tau\sqrt{1-\lambda^2}}^{\infty} \Phi\left(\frac{\tau-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}} - \frac{\lambda x}{\sqrt{(1-\lambda^2)\left(1-\rho^2\pi'\Omega^{-1}\pi\right)}}\right)\varphi(x)dx}{1-\Phi\left(\tau\sqrt{1-\lambda^2}\right)}$$

with $\varepsilon_{i,t-1}$ given by

$$\varepsilon_{i,t-1} = y_{i,t-1} - \alpha_i - \beta_i' x_{t-1}$$

for $i = 1,..,N$. Similar to Chang, Choi, and Park (2017), let us define the conditional transition density $p(s_t|s_{t-1},\mathcal{F}_{t-1})$ as

$$p(s_t|s_{t-1},\mathcal{F}_{t-1}) = (1-s_t)\omega_\rho + s_t(1-\omega_\rho) \tag{19}$$

where $\omega_\rho = \omega_\rho(s_{t-1},\mathcal{F}_{t-1})$ is the transition probability of (s_t) to the low volatility state conditional on the previous state and the past values of the observed times series. We can easily see that

$$\omega_\rho = \frac{\left[(1-s_{t-1})\int_{-\infty}^{\tau\sqrt{1-\lambda^2}} + s_{t-1}\int_{\tau\sqrt{1-\lambda^2}}^{\infty}\right]\Phi\left(\frac{\tau-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}} - \frac{\lambda x}{\sqrt{(1-\lambda^2)(1-\rho^2\pi'\Omega^{-1}\pi)}}\right)\varphi(x)dx}{(1-s_{t-1})\Phi\left(\tau\sqrt{1-\lambda^2}\right) + s_{t-1}\left[1-\Phi\left(\tau\sqrt{1-\lambda^2}\right)\right]} \tag{20}$$

Now, if we let $\lambda = 1$, the progression of latent autoregressive factor process defined in (4) becomes a random walk, which makes the state process defined in (3) nonstationary and its transition evolves with time t. For $t = 1$, $\omega_\rho(s_0) = \Phi(\tau)$ with $\mathbb{P}\{s_0 = 0\} = 1$ and $\mathbb{P}\{s_0 = 1\} = 1$ when $\tau > 0$ and $\tau \leq 0$, respectively. For $t \geq 2$

$$\omega_\rho = \frac{\left[(1-s_{t-1})\int_{-\infty}^{\tau/\sqrt{t-1}} + s_{t-1}\int_{\tau/\sqrt{t-1}}^{\infty}\right]\Phi\left(\frac{\tau-\rho\pi'\Omega^{-1}\varepsilon_{t-1}-x\sqrt{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}}\right)\varphi(x)dx}{(1-s_{t-1})\Phi\left(\tau/\sqrt{t-1}\right) + s_{t-1}\left[1-\Phi\left(\tau/\sqrt{t-1}\right)\right]}. \tag{21}$$

The conditional transition of y_t is fully specified by (20) or (21) in case of $|\rho| < 1$. Appendix B provides the steps required to obtain the above expressions.

If $|\rho| = 1$, we will have perfect endogeneity and the conditional transition of the state process (s_t) in (20) and (21) is not valid anymore. Under this condition, the current shock ε_t of the model will fully describes the realization of latent factor w_{t+1} in the next period. We modify the transition probability ω_ρ of (s_t) to the low volatility state, conditional on the previous state and past values of the observed time series for different values of the autoregressive coefficient λ, as follows:

(i) $\lambda = 0$

$$\omega_\rho = 1\{\rho\pi'\Omega^{-1}\varepsilon_{t-1} < \tau\}$$

(ii) $0 < \lambda < 1$

$$\begin{aligned}\omega_\rho = (1-s_{t-1})\min&\left(1, \frac{\Phi\left(\frac{\sqrt{1-\lambda^2}}{\lambda}(\tau-\rho\pi'\Omega^{-1}\varepsilon_{t-1})\right)}{\Phi\left(\tau\sqrt{1-\lambda^2}\right)}\right)\\ +s_{t-1}\max&\left(0, \frac{\Phi\left(\frac{\sqrt{1-\lambda^2}}{\lambda}(\tau-\rho\pi'\Omega^{-1}\varepsilon_{t-1})\right)-\Phi\left(\tau\sqrt{1-\lambda^2}\right)}{1-\Phi\left(\tau\sqrt{1-\lambda^2}\right)}\right)\end{aligned}$$

(iii) $-1 < \lambda < 0$

$$\omega_\rho = (1-s_{t-1}) \max\left(0, \frac{\Phi\left(\tau\sqrt{1-\lambda^2}\right) - \Phi\left(\frac{\sqrt{1-\lambda^2}}{\lambda}(\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1})\right)}{\Phi\left(\tau\sqrt{1-\lambda^2}\right)}\right)$$
$$+ s_{t-1} \min\left(1, \frac{1 - \Phi\left(\frac{\sqrt{1-\lambda^2}}{\lambda}(\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1})\right)}{1 - \Phi\left(\tau\sqrt{1-\lambda^2}\right)}\right)$$

(iv) $\lambda = 1$

$$\omega_\rho = (1-s_{t-1}) \min\left(1, \frac{\Phi\left(\frac{1}{\sqrt{t-1}}(\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1})\right)}{\Phi\left(\tau/\sqrt{t-1}\right)}\right)$$
$$+ s_{t-1} \max\left(0, \frac{\Phi\left(\frac{1}{\sqrt{t-1}}(\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1})\right) - \Phi\left(\tau/\sqrt{t-1}\right)}{1 - \Phi\left(\tau/\sqrt{t-1}\right)}\right)$$

Clearly, the transition density of the state process (s_t) depends on the lagged values of the observed times series and consequently, is not a Markov process. However, if we let $\rho = 0$, the model simplifies to the standard 1st order Markov process, independent of (y_t) similar to the conventional Markov switching model.

Updating The updating step is exactly the same as that of the conventional Markov switching model, and given by

$$p(s_t|\mathcal{F}_t) = p(s_t|y_t, \mathcal{F}_{t-1}) = \frac{p(y_t|s_t, \mathcal{F}_{t-1})p(s_t|\mathcal{F}_{t-1})}{p(y_t|\mathcal{F}_{t-1})} \tag{22}$$

where $p(y_t|s_t, \mathcal{F}_{t-1})$ is given by (15). Therefore, we can easily obtain $p(s_t|\mathcal{F}_t)$ from $p(s_t|\mathcal{F}_{t-1})$ and $p(y_t|\mathcal{F}_{t-1})$ computed from the prediction step.

Latent Factor As mentioned before, the way we defined our regime switching filter for the state process (s_t) enables us to easily extract the latent autoregressive factor (w_t) through the prediction and updating steps defined in (17) and (22). In the prediction step for the latent factor, we may write

$$p(w_t, s_{t-1}|\mathcal{F}_{t-1}) = p(w_t|s_{t-1}, \mathcal{F}_{t-1})p(s_{t-1}|\mathcal{F}_{t-1}) \tag{23}$$

where $p(s_{t-1}|\mathcal{F}_{t-1})$ is obtained from the previous updating step for the state process (s_t).

In order to compute $p(w_t, s_{t-1}|\mathcal{F}_{t-1})$, we need to find the transition density of the latent factor conditional on the previous state and the information based the lagged values of the observed time series. The expression for this transition density is derived for different values the autoregressive coefficient λ of the latent factor process and the endogeneity parameter ρ, as follows:

(i) $|\lambda| < 1$ and $|\rho| < 1$

$$p(w_t|s_{t-1} = 1, \mathcal{F}_{t-1}) = \frac{1 - \Phi\left(\sqrt{\frac{1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi}{1-\rho^2\pi'\Omega^{-1}\pi}}\left(\tau - \frac{\lambda(w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1})}{1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi}\right)\right)}{1 - \Phi\left(\tau\sqrt{1-\lambda^2}\right)} \times \mathbb{N}\left(\rho\pi'\Omega^{-1}\varepsilon_{t-1}, \frac{1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi}{1-\lambda^2}\right)$$

$$p(w_t|s_{t-1} = 0, \mathcal{F}_{t-1}) = \frac{\Phi\left(\sqrt{\frac{1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi}{1-\rho^2\pi'\Omega^{-1}\pi}}\left(\tau - \frac{\lambda(w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1})}{1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi}\right)\right)}{\Phi\left(\tau\sqrt{1-\lambda^2}\right)} \times \mathbb{N}\left(\rho\pi'\Omega^{-1}\varepsilon_{t-1}, \frac{1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi}{1-\lambda^2}\right)$$

(ii) $|\lambda| < 1$ and $|\rho| = 1$

- $0 < \lambda < 1$

$$p(w_t|s_{t-1} = 1, \mathcal{F}_{t-1}) = \frac{\frac{\sqrt{1-\lambda^2}}{\lambda}\varphi\left(\frac{w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\lambda}\sqrt{1-\lambda^2}\right)}{1-\Phi\left(\tau\sqrt{1-\lambda^2}\right)} 1\{w_t \geq \lambda\tau + \rho\pi'\Omega^{-1}\varepsilon_{t-1}\}$$

$$p(w_t|s_{t-1} = 0, \mathcal{F}_{t-1}) = \frac{\frac{\sqrt{1-\lambda^2}}{\lambda}\varphi\left(\frac{w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\lambda}\sqrt{1-\lambda^2}\right)}{\Phi\left(\tau\sqrt{1-\lambda^2}\right)} 1\{w_t < \lambda\tau + \rho\pi'\Omega^{-1}\varepsilon_{t-1}\}$$

- $-1 < \lambda < 0$

$$p(w_t|s_{t-1} = 1, \mathcal{F}_{t-1}) = \frac{\frac{\sqrt{1-\lambda^2}}{\lambda}\varphi\left(\frac{w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\lambda}\sqrt{1-\lambda^2}\right)}{1-\Phi\left(\tau\sqrt{1-\lambda^2}\right)} 1\{w_t \leq \lambda\tau + \rho\pi'\Omega^{-1}\varepsilon_{t-1}\}$$

$$p(w_t|s_{t-1} = 0, \mathcal{F}_{t-1}) = \frac{\frac{\sqrt{1-\lambda^2}}{\lambda}\varphi\left(\frac{w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\lambda}\sqrt{1-\lambda^2}\right)}{\Phi\left(\tau\sqrt{1-\lambda^2}\right)} 1\{w_t > \lambda\tau + \rho\pi'\Omega^{-1}\varepsilon_{t-1}\}$$

(iii) $\lambda = 1$ and $|\rho| < 1$

$$p(w_t|s_{t-1} = 1, \mathcal{F}_{t-1}) = \frac{1 - \Phi\left(\sqrt{\frac{t-\rho^2\pi'\Omega^{-1}\pi}{(t-1)(1-\rho^2\pi'\Omega^{-1}\pi)}}\left(\tau - \frac{(t-1)(w_t - \rho\pi'\Omega^{-1}\varepsilon_{t-1})}{t-\rho^2\pi'\Omega^{-1}\pi}\right)\right)}{1 - \Phi\left(\tau/\sqrt{t-1}\right)} \times \mathbb{N}\left(\rho\pi'\Omega^{-1}\varepsilon_{t-1}, t - \rho^2\pi'\Omega^{-1}\pi\right)$$

$$p(w_t|s_{t-1} = 0, \mathcal{F}_{t-1}) = \frac{\Phi\left(\sqrt{\frac{t-\rho^2\pi'\Omega^{-1}\pi}{(t-1)(1-\rho^2\pi'\Omega^{-1}\pi)}}\left(\tau - \frac{(t-1)(w_t - \rho\pi'\Omega^{-1}\varepsilon_{t-1})}{t-\rho^2\pi'\Omega^{-1}\pi}\right)\right)}{\Phi\left(\tau/\sqrt{t-1}\right)} \times \mathbb{N}\left(\rho\pi'\Omega^{-1}\varepsilon_{t-1}, t - \rho^2\pi'\Omega^{-1}\pi\right)$$

(iv) $\lambda = 1$ and $|\rho| = 1$

$$p(w_t|s_{t-1} = 1, \mathcal{F}_{t-1}) = \frac{\frac{1}{\sqrt{t-1}}\varphi\left(\frac{w_t - \rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{t-1}}\right)}{1 - \Phi\left(\tau/\sqrt{t-1}\right)} 1\{w_t \geq \tau + \rho\pi'\Omega^{-1}\varepsilon_{t-1}\}$$

$$p(w_t|s_{t-1} = 0, \mathcal{F}_{t-1}) = \frac{\frac{1}{\sqrt{t-1}}\varphi\left(\frac{w_t - \rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{t-1}}\right)}{\Phi\left(\tau/\sqrt{t-1}\right)} 1\{w_t < \tau + \rho\pi'\Omega^{-1}\varepsilon_{t-1}\}.$$

Similar to the state process (s_t), the updating step for the latent autoregressive factor is given by

$$p(w_t, s_{t-1}|\mathcal{F}_t) = \frac{p(y_t|w_t, s_{t-1}, \mathcal{F}_{t-1})p(w_t, s_{t-1}|\mathcal{F}_{t-1})}{p(y_t|\mathcal{F}_{t-1})}. \tag{24}$$

It follows that

$$p(w_t|\mathcal{F}_t) = \sum_{s_{t-1}} p(w_t, s_{t-1}|\mathcal{F}_t),$$

which enables us to extract the inferred factor,

$$\mathbb{E}(w_t|\mathcal{F}_t)) = \int w_t p(w_t|\mathcal{F}_t) dw_t$$

for $t = 1, ..., T$, when the parameters that maximize the likelihood function are found.

4 Empirical Results

In this section, we examine the performance of the Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Lintner (1965), which remains a benchmark asset pricing model in the academic literature, under the model specification described in section 2. Under our specification, the timing of changes in beta, which corresponds to changes in the market volatility levels, is determined directly by the return data through our endogenous regime switching specification rather than being imposed exogenously through a Markov-switching process.

To estimate the regime-dependent CAPM, we consider the monthly data for excess stock returns on value-weighted tertile and decile portfolios of all stocks listed on the New York Stock Exchange (NYSE), AMEX and NASDAQ sorted separately by BE/ME ratios (BE/ME portfolios) for July 1963–June 2022 (708 months). At the end of each June, stocks are allocated to three and ten BE/ME groups (Low to High) using NYSE breakpoints. In the sort for June of year t, B is book equity at the end of the fiscal year ending in year $t-1$ and M is the market cap at the end of December of year $t-1$, adjusted for changes in shares outstanding between the measurement of B and the end of December. This data is readily available and are downloaded from Kenneth R. French's official website.

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Table I and Table II report the estimation results for the model using 3 and 10 portfolios sorted on BE/ME, respectively. The parameters estimated for the low volatility level of the market are consistent with the positive correlation between risk and returns. The portfolios that have higher book-to-market ratios carry a higher level of risk relative to the market and consequently, they should have betas bigger than 1. On the other hand, the portfolios with lower book-to-market ratios are expected to have betas less than 1 as they bear lower levels of risk compared to the market. As a general rule, for a model to price the relative risk of portfolios correctly, it should provide betas that are increasing in book-to-market ratios. For instance, the average returns of the three portfolios sorted on the book-to-market ratio during the period 1964 - 2021 are given by 0.9339, 0.9551, and 1.1553 as we move from the portfolio with the lowest book-to-market ratio to the portfolio with the highest book-to-market ratios. This shows that portfolios with higher book-to-market ratio had higher average returns because of the higher level of risk they bear and therefore, the model should consider higher betas for them. The betas for the low volatility regime reported in Table I follow this rule perfectly. However, we do not observe this relationship in the high volatility regime as one may expect. This shows that the poor performance of the CAPM with constant coefficients, where the estimated betas do not explain the variation observed

Table I: Estimated parameters of the regime-dependent CAPM (3 Portfolios Sorted on BE/ME)

		Low	2	High
High Volatility	α	0.3103	0.7465	1.6142
		(0.5126)	(0.5930)	(1.1154)
	β	1.1731***	0.6788***	0.5939***
		(0.8179)	(0.2340)	(0.1484)
	π	0***	2.109***	0.8296***
		(0.4560)	(0.2077)	(0.6332)
Low Volatility	α	0.3701*	0.3579***	0.3177***
		(0.6727)	(0.8469)	(0.9399)
	β	0.9583***	1.0073***	1.1435***
		(0.6647)	(0.5079)	(0.6218)
	π	0***	0.9426***	0.8296***
		(0.4560)	(0.2373)	(0.3648)
	σ	0.9558***	0.0001***	1.8496***
		(0.0666)	(0.3500)	(0.5996)
	ρ		-0.1549	
			(0.4721)	
	λ		0.8328	
			(0.4554)	
	τ		-1.381***	
			(1.2613)	

Notes: The standard errors are calculated using bootstrap method and reported in parenthesis. Significance is computed using bootstrap confidence intervals. *p<0.1; **p<0.05; ***p<0.01

in the average returns across different portfolios, is caused by unpredictable and unexplained relationship between the portfolio returns and market factor in the high volatility regime. This evidence is a possible explanation for the failure of the original CAPM which assumes that the volatility level in the market and beta are both constant over time. Furthermore, the reported results run contrary to the theoretical positive relationship between the risk and the return. To justify the anomalies observed to the contrary of this theory, there has been a debate in the literature that claims even though the investors react to information about the true volatility regimes, it is more reasonable to assume that there is a time delay in the process of digesting the news and information about the current volatility level of the market. However, before we could make that kind of judgment, we think there should be a distinction between the aggregate uncertainty level in the market and the relative uncertainty observed in each portfolio (or each stock specifically) with respect to the market uncertainly level. Under our specification, the positive relationship between the risk and return still holds with respect to the relative risk observed in the portfolios, but only when the market is in the low volatility regime. By investigating the data, we can easily observe that the periods of high volatility experience big negative returns with a lot of variations. As one may expect, the goal during these times won't be carrying higher levels of risk to have higher returns since we don't know what rules govern the behavior of different portfolios, taking into account that every investor also may behave differently based on their speculation of how the price of different portfolios will change. Therefore, we should not expect a portfolio to have a higher return when the market is in the high volatility regime compared to the time that market is in the low volatility regime. What we have found using our model specification is consistent with some of the previous studies. For instance, Lakonishok et al. (1994) report that the betas for portfolios with higher book-to-market ratios are higher than betas for portfolios with lower book-to-market ratios in good times (low volatility state).

To further demonstrate the importance of the distinction between the volatility levels of the market, we have estimated a version of the model in which both states are identical. The results are reported in Table III and Table IV. For instance, if we consider the case of the 3 portfolio setup, we can no longer observe the increasing pattern over the betas similar to what we've discussed before. The same holds for the case of the 5 portfolio setup. Additionally, the pricing errors in the state-independent CAPM are bigger in magnitudes compared to the low volatility regime in the state-dependent version, except for the first portfolio for both cases. This further indicates the importance of the differentiation between different states of the market.

The results reported in both Table I and Table II are inline with what was described before. The only anomaly is the third portfolio in the 5 portfolio setup reported in Table II

Table II: Regime-Dependent CAPM (10 Portfolios Sorted on BE/ME, 1963 - 2022)

		Growth	2	3	4	5	6	7	8	9	Value
Low Volatility	α	0.256***	0.3131***	0.349***	0.3604***	0.4262***	0.5402***	0.461***	0.5043***	0.6707***	0.5451***
		(0.8739)	(0.8102)	(0.7647)	(0.7159)	(0.6438)	(0.5196)	(0.6028)	(0.5672)	(0.4139)	(0.5646)
	β	0.9718***	0.9922***	0.9922***	1.0455***	0.9684***	1.0063***	0.9946***	1.0336***	1.0968***	1.2474***
		(0.1586)	(0.1347)	(0.1296)	(0.0877)	(0.1462)	(0.11)	(0.1176)	(0.0943)	(0.0824)	(0.1974)
	π	0***	0***	0***	0.013***	0.2956***	0.7313***	1.1836***	1.6909***	1.9429***	2.5532***
		(1.0998)	(1.0998)	(1.0998)	(1.0709)	(0.8636)	(0.5078)	(0.1732)	(0.5491)	(0.8241)	(1.4943)
High Volatility	α	0.3714	1.1813**	0.8452*	0.6872	0.59	0.7681	0.4425	1.19*	0.9583*	1.2918*
		(1.0013)	(0.4111)	(0.4585)	(0.6922)	(0.81)	(0.793)	(0.9638)	(0.5483)	(0.5971)	(0.7209)
	β	1.3648***	1.0022***	0.8987***	0.6843***	0.6324***	0.5269***	0.5815***	0.5486***	0.5871***	0.6761***
		(0.2758)	(0.1276)	(0.1948)	(0.3724)	(0.4258)	(0.5214)	(0.4822)	(0.5321)	(0.4752)	(0.4134)
	π	0***	0.7565***	2.0395***	2.3022***	2.3338***	3.2171***	3.3854***	1.8696***	2.0142***	2.5532***
		(1.0998)	(0.7676)	(1.1837)	(1.8611)	(2.044)	(3.2091)	(3.2328)	(1.7085)	(1.3986)	(1.5384)
	σ	1.7021***	1.3273***	1.2584***	1.3577***	1.4489***	1.3971***	1.4566***	1.4758***	1.5132***	2.2371***
		(0.6455)	(0.254)	(0.1756)	(0.2524)	(0.3148)	(0.2415)	(0.3177)	(0.3827)	(0.3879)	(1.0845)
	ρ					0.2495					
						(1.0597)					
	λ					0.6137***					
						(0.5566)					
	τ					1.3666***					
						(0.5553)					

Notes: The standard errors are calculated using wild residual bootstrap method and reported in parenthesis. Significance is computed using wild residual bootstrap confidence intervals. *p<0.1; **p<0.05; ***p<0.01

Table III: Estimated parameters of regime-independent CAPM (3 Portfolios Sorted on BE/ME)

	Low	2	High
α	0.361	0.4382	0.5948
β	1.0172	0.9173	0.9951
π	0	0.9476	1.7965
σ	1.0483	1.0929	1.4927
ρ		-0.1477	
λ		0.9939	
τ		1.6436	

Table IV: Estimated parameters of regime-independent CAPM (5 Portfolios Sorted on BE/ME)

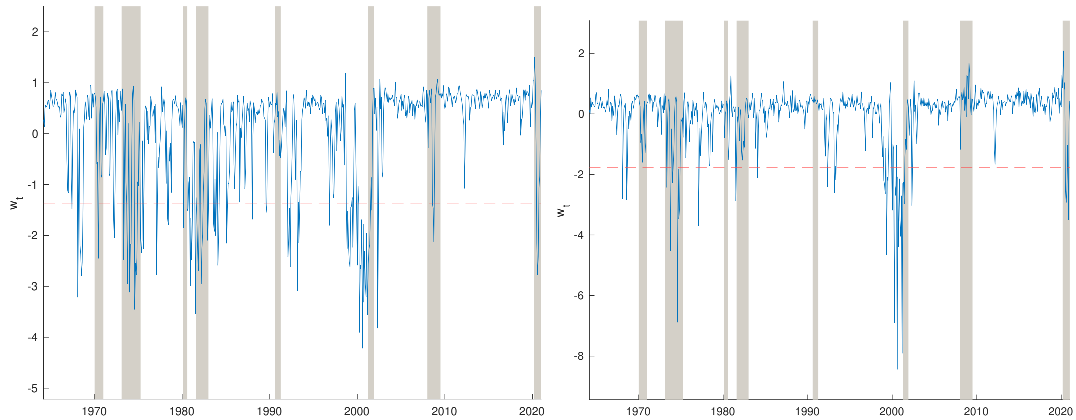
	Low	2	3	4	High
α	0.3606	0.3953	0.4963	0.4881	0.6232
β	1.0281	0.9703	0.8983	0.9236	1.0433
π	0	0.3857	1.0018	1.9605	1.8378
σ	1.355	1.1934	1.2832	0.6809	1.9764
ρ			-0.2627		
λ			0.1402		
τ			-9.9465		

where the beta is slightly decreased compared to the second portfolio. But the overall trend for betas from low to high book-to-market ratio is increasing. Generally, the pricing errors are smaller in the low volatility regime than their corresponding value in the other regime, except for the portfolios with the lowest BE/ME which the trend is reversed both for 3 and 5 portfolio setups. The estimated value for ρ is nonzero which provides evidence for the presence of endogeneity in the regime-switching of the market volatility. Furthermore, the negative values of ρ in both cases mean that a positive shock to the time series will transfer as a negative shock to the latent factor which increases the probability of entering the high volatility state.

The left- and right-hand side graphs in Figure I present the extracted latent factors for 3 and 5 portfolios sorted on BE/ME ratio with the recession periods announced by the National Bureau of Economic Research (NBER), respectively. Note that when the value of the latent factor is below the threshold τ (red dashed line), the market is in the high volatility regimes and it is expected to be synchronous with the periods of financial crises or macroeconomic recession periods. When the latent factor is above the threshold τ the market is in the low volatility regime which is defined to be as the behavior of the market in normal times. The recession periods stated by the NBER are all recognized by the extracted latent factor as periods of high stock market volatility. These results suggest that there is a link between market volatility and the state of the economy. The only drawback of our result is the unrecognized financial crisis in 2008 by the latent factor extracted based on the 5 portfolios sorted on BE/ME. The extracted latent factors for the cases of regime-independent CAPM are not reported as they are not informative about different states of the market based on their definition. For this sample period, we estimated the model for the portfolio returns sorted by the size, but we did not find a strong size effect across the portfolios.

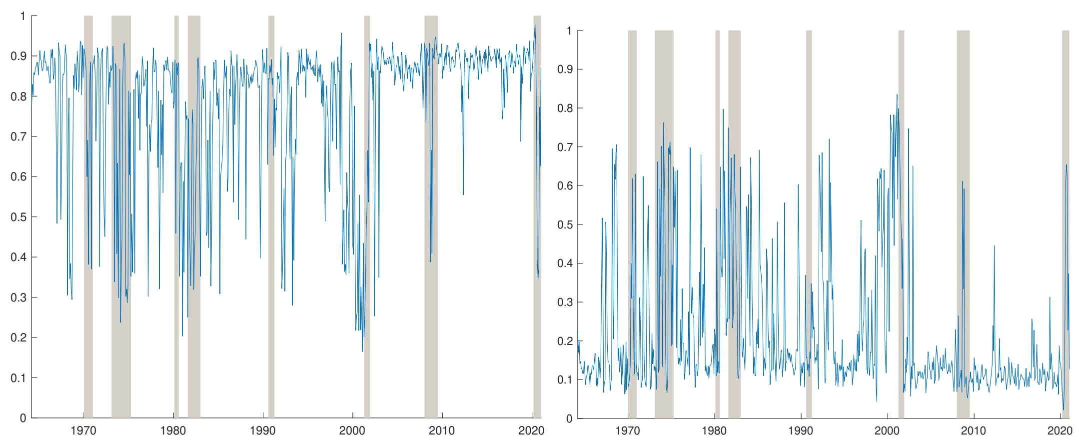
The smoothed state probabilities of being in the low and high volatility regime for the 3 portfolios estimated from our endogenous regime-switching model with the NBER recession periods are displayed in the left- and right-hand side graphs of Figure II, respectively. The information that can be extracted from these graphs is inline with what was discussed before. At the times that our model predicts the market should be in the high volatility state, the state probability of being the high volatility is observed to be high. The opposite is observed for periods of low volatility. Furthermore, the recession periods indicated by the NBER are all experiencing a high probability of being in the high volatility regime (and low probability of being in the low volatility regime) which is consistent with the results from the extracted latent factor.

To further investigate the behavior of the factor loadings under different states of the



(a) 3 portfolios (b) 5 portfolios

Figure I: Extracted latent factor. *Notes*: This figure presents the sample path of the latent factor extracted from the endogenous volatility switching model (solid blue line) and the threshold τ (dashed red line) along with the NBER recession periods (grey shaded area) for 3 and 5 portfolios sorted on the book-to-market ratio for the period 1964–2021, respectively, on the left and right vertical axis.



(a) Low Volatility Regime (b) high Volatility Regime

Figure II: Smoothed high and low State Probabilities. *Notes*: This figure presents the time series of the probabilities of being in the high and low volatility regimes (solid blue line) along with the NBER recession periods (grey shaded area). The left panel plots the low volatility probability series and the right panel plots the high volatility probability series obtained from the endogenous volatility switching model

market, we added another factor, HML (High minus Low), which represents the effect of increasing the book-to-market ratio, keeping everything else constant. The HML is the difference between the returns of the portfolio with the highest BE/ME and the return of the portfolio with the lowest BE/ME. The two components of HML are returns on high and low BE/ME portfolios with about the same weighted average size and other features.

Therefore, HML should be independent of other factors in returns, focusing on the different return behaviors of high and low BE/ME firms.

Table V reports the estimation results for this new specification for 3 portfolios sorted on BE/ME. The results are consistent with the discussion made previously about the expectations one may have with respect to the behavior of the model in different volatility regimes. The market and HML factor coefficients, β and h, are both higher for the portfolios with higher book-to-market ratios when the market is in the low volatility regime. However, this behavior no longer can be observed when we look at the estimated parameters in the high volatility regime.

The pricing errors are behaving similarly to the state-dependent CAPM where the α's in the low volatility regime are smaller than their corresponding value in the high volatility regime (except for the first portfolio). In addition, by comparing the pricing errors in the low volatility regime in this specification with the ones in the regime-dependent CAPM, we can easily see that the magnitude of the pricing errors is getting smaller (except for the first portfolio).

All the evidence provided and the discussions made so far were to show why we need to take the behavior of the market into account when we analyze the performance of an asset pricing model. What we've shown here can be extended to more complex asset pricing models to create a structure on how a portfolio return might behave under different states of the market. Furthermore, our endogenous regime-switching model can be extended to a version that considers more than two volatility states to evaluate the performance of the CAPM or other asset pricing models. As shown before, the performance of an appropriate asset pricing model improves when it is evaluated in the relatively low volatility regime of the market, but there has been some deviation from the general expectations. We think that including a dummy variable that can capture the effects of a few of the extreme outliers (the October of 2008 for instance) can improve the performance of the model. Nonetheless, without including such a dummy variable, we still can see a considerable improvement in how a model can perform when it is applied to our endogenous regime-switching specification.

5 Conclusion

In this article, we proposed a new approach to model a panel regression with regime-switching using a latent autoregressive factor. In this setup, we test the performance of the Capital Asset Pricing Model (CAPM) where we allow for discrete time-variations in the CAPM betas for portfolios that are sorted by the book-to-market ratios based on a two-state endogenous regime-switching process determined by the uncertainty observed in the stock market return

Table V: Estimated parameters for the 2 factor model (3 Portfolios Sorted on BE/ME)

		Low	2	High
High Volatility	α	0.3694***	0.4949***	1.0761
		(1.5415)	(1.7451)	(0.6655)
	β	1.1629***	0.8868***	0.7912***
		(1.0691)	(0.5067)	(0.9238)
	h	0.0251***	0.5892***	0.427***
		(0.7879)	(0.1736)	(0.7845)
	π	0.0651***	0***	1.8824***
		(0.4660)	(0.3184)	(0.4151)
Low Volatility	α	0.4482***	0.3101***	0.3141***
		(0.1892)	(0.1796)	(0.1383)
	β	0.9786***	0.9877***	1.1126***
		(0.8000)	(0.7421)	(1.2156)
	h	-0.3102***	0.2363***	0.7505***
		(0.8583)	(0.2052)	(0.1775)
	π	0.0651***	0***	0.9801***
		(0.5268)	(0.5489)	(3263)
	σ	0.6696***	0.9853***	0.0005***
		(0.2290)	(0.2126)	(0.5488)
	ρ		-0.7450***	
			(0.5872)	
	λ		0.9837***	
			(0.4719)	
	τ		-8.8408***	
			(1.2369)	

Notes: The standard errors are calculated using bootstrap method and reported in parenthesis. Significance is computed using bootstrap confidence intervals. *p<0.1; **p<0.05; ***p<0.01

behaviors. Our method has a couple of advantages by using an endogenous regime-switching setup rather than a Markov-switching process. We found that the behavior of this asset pricing model significantly differs across different volatility regimes and returns behave more closely to the rules of the proposed models. Even though the regime-dependent version of the CAPM can still be rejected, it provides strong evidence on how important it is to consider the occasional shifts observed in the market return when we want to evaluate the performance of an asset pricing model. In addition, based on the information that we can extract from the latent factor, there seems to be a correlation between the periods of high volatility and economic recessions such that the latter is a subset of the former, according to our empirical findings.

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Appendix A

We may find the determinant and inverse of covariance matrix $\Omega(s_t)$ of $\varepsilon_t(s_t)$ analytically, which would be very useful in computing the likelihood function. Define

$$\Sigma = \text{diag}\ (\sigma_1^2, ..., \sigma_N^2)$$

and write

$$\Omega(s_t) = \Sigma + \pi(s_t)\pi(s_t)' = \Sigma^{1/2}(I + \Sigma^{-1/2}\pi(s_t)\pi(s_t)'\Sigma^{-1/2})\Sigma^{1/2} \tag{25}$$

where $\Omega(s_t)$ is defined in (16).

Let $\tau(s_t) = \Sigma^{-1/2}\pi(s_t)$, and note that

$$\begin{aligned} I + \Sigma^{-1/2}\pi(s_t)\pi(s_t)'\Sigma^{-1/2} &= I + \tau(s_t)\tau(s_t)' \\ &= I + ||\tau(s_t)||^2 P_{\tau(s_t)} \\ &= (1 + ||\tau(s_t)||^2)P_{\tau(s_t)} + (I - P_{\tau(s_t)}), \end{aligned}$$

where $P_{\tau(s_t)} = \tau(s_t)\tau(s_t)'/||\tau(s_t)||^2$ is the orthogonal projection on the span of $\tau(s_t)$, from which it follows immediately that

$$\left(I + \Sigma^{-1/2}\pi(s_t)\pi(s_t)'\Sigma^{-1/2}\right)^{-1} = \frac{1}{1 + ||\tau(s_t)||^2}P_{\tau(s_t)} + (I - P_{\tau(s_t)}) = I - \frac{||\tau(s_t)||^2}{1 + ||\tau(s_t)||^2}P_{\tau(s_t)}. \tag{26}$$

Therefore, we may deduce from (25) and (26) that

$$\Omega^{-1}(s_t) = \Sigma^{-1/2}\left(I - \frac{||\tau(s_t)||^2}{1 + ||\tau(s_t)||^2}P_{\tau(s_t)}\right)\Sigma^{-1/2} = \Sigma^{-1} - \frac{1}{1 + \pi(s_t)'\Sigma^{-1}\pi(s_t)}\Sigma^{-1}\pi(s_t)\pi(s_t)'\Sigma^{-1} \tag{27}$$

Moreover, we have

$$\det\Omega(s_t) = (\det\Sigma)(1 + \pi(s_t)'\Sigma^{-1}\pi(s_t)) \tag{28}$$

due to (25).

Finally, we may also easily derive that

$$1 - \rho^2\pi(s_t)'\Omega^{-1}(s_t)\pi(s_t) = \frac{1 + (1 - \rho^2)\pi(s_t)'\Sigma^{-1}\pi(s_t)}{1 + \pi(s_t)'\Sigma^{-1}\pi(s_t)}$$

and

$$\Omega^{-1}(s_t)\pi(s_t) = \frac{\Sigma^{-1}\pi(s_t)}{1 + \pi(s_t)'\Sigma^{-1}\pi(s_t)}$$

from (27).

Appendix B

We provide a brief proof for some of the expressions presented in the main part of the paper.

- **Equation** (8)**:**
  According to how we defined the latent factor in (4) and the assumption of normality for the error term, it follows that

$$\begin{aligned}\mathbb{P}\{w_t < \tau | w_{t-1}\} &= \mathbb{P}\{\lambda w_{t-1} + v_t < \tau | w_{t-1}\} \\ &= \mathbb{P}\{v_t < \tau - \lambda w_{t-1} | w_{t-1}\} \\ &= \Phi\left(\tau - \lambda w_{t-1}\right) \quad \square\end{aligned}$$

- **Equation** (10)&(11)**:**
  From (8), we may easily write

$$\mathbb{P}\{s_t = 0 | w_{t-1}\sqrt{1-\lambda^2} = x\} = \Phi\left(\tau - \frac{\lambda x}{\sqrt{1-\lambda^2}}\right).$$

  It follows that

$$\begin{aligned}\mathbb{P}\{s_t = 0 | s_{t-1} = 0\} &= \mathbb{P}\{s_t = 0 | w_{t-1} < \tau\} \\ &= \mathbb{P}\{s_t = 0 | w_{t-1}\sqrt{1-\lambda^2} < \tau\sqrt{1-\lambda^2}\} \\ &= \frac{\displaystyle\int_{-\infty}^{\tau\sqrt{1-\lambda^2}} \mathbb{P}\{s_t = 0 | w_{t-1}\sqrt{1-\lambda^2} = x\}\varphi(x)dx}{\mathbb{P}\{w_{t-1}\sqrt{1-\lambda^2} < \tau\sqrt{1-\lambda^2}\}} \\ &= \frac{\displaystyle\int_{-\infty}^{\tau\sqrt{1-\lambda^2}} \Phi\left(\tau - \frac{\lambda x}{\sqrt{1-\lambda^2}}\right)\varphi(x)dx}{\Phi(\tau\sqrt{1-\lambda^2})},\end{aligned}$$

  since $w_{t-1}\sqrt{1-\lambda^2} =_d \mathbb{N}(0,1)$. Similarly, we have

$$\mathbb{P}\{s_t = 1 | w_{t-1}\sqrt{1-\lambda^2} = x\} = 1 - \Phi\left(\tau - \frac{\lambda x}{\sqrt{1-\lambda^2}}\right),$$

from which it follows that

$$
\begin{aligned}
\mathbb{P}\{s_t = 1 | s_{t-1} = 1\} &= \mathbb{P}\{s_t = 1 | w_{t-1} \geq \tau\} \\
&= \mathbb{P}\{s_t = 1 | w_{t-1}\sqrt{1-\lambda^2} \geq \tau\sqrt{1-\lambda^2}\} \\
&= \frac{\displaystyle\int_{\tau\sqrt{1-\lambda^2}}^{\infty} \mathbb{P}\{s_t = 1 | w_{t-1}\sqrt{1-\lambda^2} = x\}\varphi(x)dx}{\mathbb{P}\{w_{t-1}\sqrt{1-\lambda^2} \geq \tau\sqrt{1-\lambda^2}\}} \\
&= \frac{\displaystyle\int_{\tau\sqrt{1-\lambda^2}}^{\infty} \left[1 - \Phi\left(\tau - \frac{\lambda x}{\sqrt{1-\lambda^2}}\right)\right]\varphi(x)dx}{1 - \Phi(\tau\sqrt{1-\lambda^2})},
\end{aligned}
$$

The proof for the case of $\lambda = 1$ in equations (13) and (14) is very similar to what we have done here, except that we have $w_{t-1}/\sqrt{t-1} =_d \mathbb{N}(0,1)$ for $t \geq 2$ in this case instead of $w_{t-1}\sqrt{1-\lambda^2} =_d \mathbb{N}(0,1)$ when $|\lambda| < 1$. □

- **Equation (15):**
For any normal random vector $X = (X_1, ..., X_k)'$ with mean μ and covariance matrix Σ, the probability density function can be written as

$$
f_X(x) = \frac{exp\left(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right)}{\sqrt{(2\pi)^k|\Sigma|}}.
$$

In our panel regression model, we have

$$
y_t = \alpha(s_t) + \beta(s_t)x_t + \sigma(s_t)u_t
$$

where

$$
\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}, \ \beta = \begin{pmatrix} \beta_1' \\ \vdots \\ \beta_N' \end{pmatrix}, \ \sigma = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N \end{pmatrix}
$$

We may easily see

$$
\begin{aligned}
\mathbb{E}(y_t|s_t) &= \alpha(s_t) + \beta'(s_t)x_t \\
Var(y_t|s_t) &= \mathbb{E}\Big(\varepsilon_t(s_t)\varepsilon_t(s_t)'|s_t\Big) = \Omega(s_t)
\end{aligned}
$$

since $\mathbb{E}\big(\varepsilon_t(s_t)|s_t\big) = 0_N$. If we apply the above density function to this regression model, we easily derive (15). $\square$

- **Equation** (18)**:**
  We may rewrite (6) as

$$\begin{pmatrix} v_{t+1} \\ u_t \end{pmatrix} =_d \mathbb{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right).$$

It follows that

$$\left.\begin{pmatrix} v_{t+1} \\ \varepsilon_t(s_t) \end{pmatrix}\right|_{s_t} =_d \mathbb{N}\left(0_{N+1}, \begin{pmatrix} 1 & \rho\pi'(s_t) \\ \rho\pi(s_t) & \Omega(s_t) \end{pmatrix}\right).$$

Generally, if we partition a normal random vector $X =_d (\mu, \Sigma)$ as

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

where X_1 and X_2 are n_1- and n_2-dimensional, respectively, we may write

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \ \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

Then, the conditional distribution of X_1 given X_2 is given by

$$p(X_1|X_2) =_d \mathbb{N}(\mu_{1.2}, \Sigma_{11.2})$$

where $\mu_{1.2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(X_2 - \mu_2)$ and $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. If we apply the above formula to the times series and latent factor innovations, we may easily get

$$p(v_t|s_{t-1}, \varepsilon_{t-1}) =_d \mathbb{N}\left(\rho\pi'\Omega^{-1}\varepsilon_{t-1}, 1 - \rho^2\pi'\Omega^{-1}\pi\right). \quad \square$$

- **Equation** (20)**:**
  Note that for the identification of our model in the case of $|\lambda| < 1$, we assumed that $w_{t-1} =_d N(0, 1/(1-\lambda^2))$. Let us define

$$z_t = \frac{w_t - \rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1 - \rho^2\pi'\Omega^{-1}\pi}} - \frac{\lambda w_{t-1}}{\sqrt{1 - \rho^2\pi'\Omega^{-1}\pi}}$$

Based on what we had in (18), we can derive that

$$p(z_t|w_{t-1}, \mathcal{F}_{t-1}) =_d \mathbb{N}(0,1).$$

It follows that

$$\begin{aligned}\mathbb{P}\left\{w_t < \tau | w_{t-1}, \mathcal{F}_{t-1}\right\} &= \mathbb{P}\left\{z_t < \frac{\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}} - \frac{\lambda w_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}} \middle| w_{t-1}, \mathcal{F}_{t-1}\right\} \\ &= \Phi\left(\frac{\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}} - \frac{\lambda w_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}}\right)\end{aligned}$$

Note that the latent factor at time t only depends on its lagged value of w_{t-1} and the error term v_t which is correlated with u_{t-1} and independent of w_{t-1}. This means that $p(w_t|w_{t-1}, \mathcal{F}_{t-1}) = p(w_t|w_{t-1}, \varepsilon_{t-1})$ and we may deduce that

$$\begin{aligned}\mathbb{P}\left\{s_t = 0 | s_{t-1} = 0, \mathcal{F}_{t-1}\right\} &= \mathbb{P}\left\{w_t < \tau | w_{t-1} < \tau, \mathcal{F}_{t-1}\right\} \\ &= \mathbb{P}\left\{w_t < \tau | w_{t-1}\sqrt{1-\lambda^2} < \tau\sqrt{1-\lambda^2}, \mathcal{F}_{t-1}\right\} \\ &= \frac{\int_{-\infty}^{\tau\sqrt{1-\lambda^2}} \Phi\left(\frac{\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}} - \frac{\lambda x}{\sqrt{(1-\lambda^2)(1-\rho^2\pi'\Omega^{-1}\pi)}}\right)\varphi(x)dx}{\Phi\left(\tau\sqrt{1-\lambda^2}\right)}\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathbb{P}\left\{s_t = 0 | s_{t-1} = 1, \mathcal{F}_{t-1}\right\} &= \mathbb{P}\left\{w_t < \tau | w_{t-1} \geq \tau, \mathcal{F}_{t-1}\right\} \\ &= \mathbb{P}\left\{w_t < \tau | w_{t-1}\sqrt{1-\lambda^2} \geq \tau\sqrt{1-\lambda^2}, \mathcal{F}_{t-1}\right\} \\ &= \frac{\int_{\tau\sqrt{1-\lambda^2}}^{\infty} \Phi\left(\frac{\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}} - \frac{\lambda x}{\sqrt{(1-\lambda^2)(1-\rho^2\pi'\Omega^{-1}\pi)}}\right)\varphi(x)dx}{1-\Phi\left(\tau\sqrt{1-\lambda^2}\right)}\end{aligned}$$

By combining the above equations we may easily derive (20). The proof for the case of $\lambda = 1$ in equation (21) is very similar to what we have done here, except that we have $w_{t-1}/\sqrt{t-1} =_d \mathbb{N}(0,1)$ for $t \geq 2$ in this case instead of $w_{t-1}\sqrt{1-\lambda^2} =_d \mathbb{N}(0,1)$ when $|\lambda| < 1$. $\square$

- **ω_ρ of (s_t) when $0 < \lambda < 1$ and $|\rho| = 1$:**

Note that

$$
\begin{aligned}
\mathbb{P}\left\{w_t < \tau | w_{t-1}, \mathcal{F}_{t-1}\right\} &= \mathbb{P}\left\{\lambda w_{t-1} + v_t < \tau | w_{t-1}, \mathcal{F}_{t-1}\right\} \\
&= \mathbb{P}\left\{\lambda w_{t-1} + \rho\pi'\Omega^{-1}\varepsilon_{t-1} < \tau | w_{t-1}, \varepsilon_{t-1}\right\} \\
&= 1\left\{\lambda w_{t-1} + \rho\pi'\Omega^{-1}\varepsilon_{t-1} < \tau\right\}.
\end{aligned}
$$

Consequently, we may write

$$
\begin{aligned}
\mathbb{P}\left\{s_t = 0 | s_{t-1} = 0, \mathcal{F}_{t-1}\right\} &= \mathbb{P}\left\{w_t < \tau | w_{t-1} < \tau, \mathcal{F}_{t-1}\right\} \\
&= \mathbb{P}\left\{\lambda w_{t-1} + \rho\pi'\Omega^{-1}\varepsilon_{t-1} < \tau | w_{t-1} < \tau, \mathcal{F}_{t-1}\right\} \\
&= \mathbb{P}\Bigg\{ w_{t-1}\sqrt{1-\lambda^2} < \frac{\sqrt{1-\lambda^2}}{\lambda}(\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1}) \Bigg| \\
&\times w_{t-1}\sqrt{1-\lambda^2} < \tau\sqrt{1-\lambda^2}, \mathcal{F}_{t-1} \Bigg\} \\
&= \begin{cases} 1, & \text{if } \frac{1}{\lambda}(\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1}) \geq \tau, \\ \frac{\Phi\left(\frac{\sqrt{1-\lambda^2}}{\lambda}(\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1})\right)}{\Phi(\tau\sqrt{1-\lambda^2})}, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\mathbb{P}\left\{s_t = 0 | s_{t-1} = 1, \mathcal{F}_{t-1}\right\} &= \mathbb{P}\left\{w_t < \tau | w_{t-1} \geq \tau, \mathcal{F}_{t-1}\right\} \\
&= \mathbb{P}\left\{\lambda w_{t-1} + \rho\pi'\Omega^{-1}\varepsilon_{t-1} < \tau | w_{t-1} \geq \tau, \mathcal{F}_{t-1}\right\} \\
&= \mathbb{P}\Bigg\{ w_{t-1}\sqrt{1-\lambda^2} < \frac{\sqrt{1-\lambda^2}}{\lambda}(\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1}) \Bigg| \\
&\times w_{t-1}\sqrt{1-\lambda^2} \geq \tau\sqrt{1-\lambda^2}, \mathcal{F}_{t-1} \Bigg\} \\
&= \frac{\Phi\left(\frac{\sqrt{1-\lambda^2}}{\lambda}(\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1})\right) - \Phi(\tau\sqrt{1-\lambda^2})}{1 - \Phi(\tau\sqrt{1-\lambda^2})} \\
&\times 1\left\{\frac{1}{\lambda}(\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1}) \geq \tau\right\}
\end{aligned}
$$

When $-1 < \lambda < 0$, with a similar approach, we may easily see that

$$
\mathbb{P}\left\{s_t = 0 | s_{t-1} = 0, \mathcal{F}_{t-1}\right\} = \begin{cases} 0, & \text{if } \frac{1}{\lambda}(\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1}) \geq \tau, \\ \frac{\Phi(\tau\sqrt{1-\lambda^2}) - \Phi\left(\frac{\sqrt{1-\lambda^2}}{\lambda}(\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1})\right)}{\Phi(\tau\sqrt{1-\lambda^2})}, & \text{otherwise.} \end{cases}
$$

and

$$\mathbb{P}\left\{s_t = 0 | s_{t-1} = 1, \mathcal{F}_{t-1}\right\} = \begin{cases} 1, & \text{if } \frac{1}{\lambda}(\tau - \rho\pi'\Omega^{-1}\varepsilon_{t-1}) < \tau, \\ \frac{1-\Phi\left(\frac{\sqrt{1-\lambda^2}}{\lambda}(\tau-\rho\pi'\Omega^{-1}\varepsilon_{t-1})\right)}{1-\Phi(\tau\sqrt{1-\lambda^2})}, & \text{otherwise.} \end{cases}$$

The proof for the case of $\lambda = 0$ is trivial and the proof for the case of $\lambda = 1$ is very similar to what we did for $0 < \lambda < 1$, except that we have $w_{t-1}/\sqrt{t-1} =_d \mathbb{N}(0,1)$ for $t \geq 2$ in this case instead of $w_{t-1}\sqrt{1-\lambda^2} =_d \mathbb{N}(0,1)$ when $|\lambda| < 1$. □

- **ω_ρ of (w_t) when $|\lambda| < 1$ and $|\rho| < 1$:**
Based on how we define our latent autoregressive process, we may easily see that

$$w_t | w_{t-1}, \varepsilon_{t-1} =_d \mathbb{N}(\lambda w_{t-1} + \rho\pi'\Omega^{-1}\varepsilon_{t-1}, 1 - \rho^2\pi'\Omega^{-1}\pi).$$

It follows that

$$p(w_t | w_{t-1}, \varepsilon_{t-1}) = \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}} \exp\left[-\frac{1}{2}\left(\frac{w_t - \lambda w_{t-1} - \rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}}\right)^2\right].$$

Note that for the conditional transition density of the latent factor, we have

$$\begin{aligned} p(w_t | s_{t-1} = 1, \mathcal{F}_{t-1}) &= p(w_t | s_{t-1} = 1, y_{t-1}, ..., y_1) \\ &= p(w_t | w_{t-1} \geq \tau, \varepsilon_{t-1}) \\ &= \frac{\int_\tau^\infty p(w_t, w_{t-1}, \varepsilon_{t-1}) dw_{t-1}}{\int_\tau^\infty p(w_{t-1}, \varepsilon_{t-1}) dw_{t-1}} \\ &= \frac{\int_\tau^\infty p(w_t | w_{t-1}, \varepsilon_{t-1}) p(w_{t-1}, \varepsilon_{t-1}) dw_{t-1}}{\int_\tau^\infty p(w_{t-1}) p(\varepsilon_{t-1}) dw_{t-1}} \\ &= \frac{\int_\tau^\infty p(w_t | w_{t-1}, \varepsilon_{t-1}) p(w_{t-1}) dw_{t-1}}{\int_\tau^\infty p(w_{t-1}) dw_{t-1}}. \end{aligned}$$

With a similar approach, we may derive

$$p(w_t | s_{t-1} = 0, \mathcal{F}_{t-1}) = \frac{\int_{-\infty}^\tau p(w_t | w_{t-1}, \varepsilon_{t-1}) p(w_{t-1}) dw_{t-1}}{\int_{-\infty}^\tau p(w_{t-1}) dw_{t-1}}$$

Since $w_t =_d w_{t-1} =_d \mathbb{N}(0, 1/(1-\lambda^2))$, it follows that

$$p(w_{t-1}) = \frac{\sqrt{1-\lambda^2}}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}w_{t-1}^2(1-\lambda^2)\right).$$

With a simple multiplication, we may get

$$p(w_t|w_{t-1},\varepsilon_{t-1})p(w_{t-1}) = \frac{\sqrt{1-\lambda^2}}{2\pi\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}} \times \exp\left[\underbrace{-\frac{1}{2}\left(\frac{w_t - \lambda w_{t-1} - \rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}}\right)^2 - \frac{1}{2}w_{t-1}^2(1-\lambda^2)}_{-\frac{1}{2}C}\right].$$

Let us simplify C as follow

$$\begin{aligned}
C &= \left(\frac{w_t - \rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}}\right)^2 + \frac{\lambda^2 w_{t-1}^2}{1-\rho^2\pi'\Omega^{-1}\pi} - \frac{2\lambda(w_t - \rho\pi'\Omega^{-1}\varepsilon_{t-1})w_{t-1}}{1-\rho^2\pi'\Omega^{-1}\pi} + w_{t-1}^2(1-\lambda^2) \\
&= \left(\frac{w_t - \rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}}\right)^2 + \frac{(1-\rho^2\pi'\Omega^{-1}\pi + \lambda^2\rho^2\pi'\Omega^{-1}\pi)w_{t-1}^2}{1-\rho^2\pi'\Omega^{-1}\pi} - \frac{2\lambda(w_t - \rho\pi'\Omega^{-1}\varepsilon_{t-1})w_{t-1}}{1-\rho^2\pi'\Omega^{-1}\pi} \\
&= \left(\sqrt{\frac{1-\rho^2\pi'\Omega^{-1}\pi + \lambda^2\rho^2\pi'\Omega^{-1}\pi}{1-\rho^2\pi'\Omega^{-1}\pi}}w_{t-1} - \frac{\lambda(w_t - \rho\pi'\Omega^{-1}\varepsilon_{t-1})}{\sqrt{(1-\rho^2\pi'\Omega^{-1}\pi)(1-\rho^2\pi'\Omega^{-1}\pi + \lambda^2\rho^2\pi'\Omega^{-1}\pi)}}\right)^2 \\
&\quad + \left(\frac{w_t - \rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}}\right)^2 - \frac{\lambda^2(w_t - \rho\pi'\Omega^{-1}\varepsilon_{t-1})^2}{(1-\rho^2\pi'\Omega^{-1}\pi)(1-\rho^2\pi'\Omega^{-1}\pi + \lambda^2\rho^2\pi'\Omega^{-1}\pi)} \\
&= \left(\sqrt{\frac{1-\rho^2\pi'\Omega^{-1}\pi + \lambda^2\rho^2\pi'\Omega^{-1}\pi}{1-\rho^2\pi'\Omega^{-1}\pi}}w_{t-1} - \frac{\lambda(w_t - \rho\pi'\Omega^{-1}\varepsilon_{t-1})}{\sqrt{(1-\rho^2\pi'\Omega^{-1}\pi)(1-\rho^2\pi'\Omega^{-1}\pi + \lambda^2\rho^2\pi'\Omega^{-1}\pi)}}\right)^2 \\
&\quad + \frac{(1-\lambda^2)(w_t - \rho\pi'\Omega^{-1}\varepsilon_{t-1})^2}{1-\rho^2\pi'\Omega^{-1}\pi + \lambda^2\rho^2\pi'\Omega^{-1}\pi}.
\end{aligned}$$

By substituting the expression we got for C, we can get

$$
\begin{aligned}
p(w_t|w_{t-1},\varepsilon_{t-1})p(w_{t-1}) &= \frac{\sqrt{1-\lambda^2}}{2\pi\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}} \\
&\quad \underbrace{\times \exp\left[-\frac{1}{2}\left(\sqrt{\frac{1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi}{1-\rho^2\pi'\Omega^{-1}\pi}}\left(w_{t-1}-\frac{\lambda(w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1})}{(1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi)}\right)\right)^2\right]}_{K} \\
&\quad \underbrace{\times \exp\left[-\frac{1}{2}\left(\frac{w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{\frac{1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi}{1-\lambda^2}}}\right)^2\right]}_{S}.
\end{aligned}
$$

We may write

$$
\begin{aligned}
p(w_t|w_{t-1},u_{t-1})p(w_{t-1}) &= \frac{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi}}{\sqrt{2\pi}\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}}K \\
&\quad \times \frac{\sqrt{1-\lambda^2}}{\sqrt{2\pi}\sqrt{1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi}}S \\
&= \bar{K}\times\bar{S}.
\end{aligned}
$$

Note that

$$
\bar{S} = \mathbb{N}\left(\rho\pi'\Omega^{-1}\varepsilon_{t-1}, \frac{1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi}{1-\lambda^2}\right).
$$

It follows that

$$
\begin{aligned}
p(w_t|s_{t-1}=1,\mathcal{F}_{t-1}) &= \frac{\int_\tau^\infty \bar{K}\times\bar{S}dw_{t-1}}{\int_\tau^\infty p(w_{t-1})dw_{t-1}} \\
&= \frac{\int_\tau^\infty \bar{K}dw_{t-1}}{\int_\tau^\infty p(w_{t-1})dw_{t-1}}\bar{S} \\
&= \frac{1-\int_{-\infty}^\tau \bar{K}dw_{t-1}}{\int_\tau^\infty p(w_{t-1})dw_{t-1}}\bar{S} \\
&= \frac{1-\Phi\left(\sqrt{\frac{1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi}{1-\rho^2\pi'\Omega^{-1}\pi}}\left(\tau-\frac{\lambda(w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1})}{(1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi)}\right)\right)}{1-\Phi(\tau\sqrt{1-\lambda^2})} \\
&\quad \times \mathbb{N}\left(\rho\pi'\Omega^{-1}\varepsilon_{t-1}, \frac{1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi}{1-\lambda^2}\right).
\end{aligned}
$$

Similarly, we may derive

$$p(w_t|s_{t-1}=0,\mathcal{F}_{t-1}) = \frac{\Phi\left(\sqrt{\frac{1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi}{1-\rho^2\pi'\Omega^{-1}\pi}}\left(\tau - \frac{\lambda(w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1})}{(1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi)}\right)\right)}{\Phi(\tau\sqrt{1-\lambda^2})}$$
$$\times\, \mathbb{N}\left(\rho\pi'\Omega^{-1}\varepsilon_{t-1}, \frac{1-\rho^2\pi'\Omega^{-1}\pi+\lambda^2\rho^2\pi'\Omega^{-1}\pi}{1-\lambda^2}\right).$$

- **ω_ρ of (w_t) when $\lambda = 1$ and $|\rho| < 1$:**
  Based on how we define our latent autoregressive process, we may easily see that

$$w_t|w_{t-1},\varepsilon_{t-1} =_d \mathbb{N}(w_{t-1}+\rho\pi'\Omega^{-1}\varepsilon_{t-1}, 1-\rho^2\pi'\Omega^{-1}\pi).$$

It follows that

$$p(w_t|w_{t-1},\varepsilon_{t-1}) = \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}} \exp\left[-\frac{1}{2}\left(\frac{w_t-w_{t-1}-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}}\right)^2\right].$$

Note that for the conditional transition density of the latent factor, we have

$$\begin{aligned}
p(w_t|s_{t-1}=1,\mathcal{F}_{t-1}) &= p(w_t|s_{t-1}=1,y_{t-1},...,y_1) \\
&= p(w_t|w_{t-1}\geq\tau,\varepsilon_{t-1}) \\
&= \frac{\int_\tau^\infty p(w_t,w_{t-1},\varepsilon_{t-1})dw_{t-1}}{\int_\tau^\infty p(w_{t-1},\varepsilon_{t-1})dw_{t-1}} \\
&= \frac{\int_\tau^\infty p(w_t|w_{t-1},\varepsilon_{t-1})p(w_{t-1},\varepsilon_{t-1})dw_{t-1}}{\int_\tau^\infty p(w_{t-1})p(\varepsilon_{t-1})dw_{t-1}} \\
&= \frac{\int_\tau^\infty p(w_t|w_{t-1},\varepsilon_{t-1})p(w_{t-1})dw_{t-1}}{\int_\tau^\infty p(w_{t-1})dw_{t-1}}
\end{aligned}$$

With a similar approach, we may derive

$$p(w_t|s_{t-1}=0,\mathcal{F}_{t-1}) = \frac{\int_{-\infty}^\tau p(w_t|w_{t-1},\varepsilon_{t-1})p(w_{t-1})dw_{t-1}}{\int_{-\infty}^\tau p(w_{t-1})dw_{t-1}}$$

Since $w_{t-1} =_d \mathbb{N}(0, t-1)$, it follows that

$$p(w_{t-1}) = \frac{1}{\sqrt{2\pi}\sqrt{t-1}} \exp\left(-\frac{w_{t-1}^2}{2(t-1)}\right).$$

With a simple multiplication, we may get

$$p(w_t|w_{t-1},\varepsilon_{t-1})p(w_{t-1}) = \frac{1}{2\pi\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}\sqrt{t-1}} \times \exp\left[\underbrace{-\frac{1}{2}\left(\frac{w_t-w_{t-1}-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}}\right)^2 - \frac{w_{t-1}^2}{2(t-1)}}_{-\frac{1}{2}C}\right].$$

Let us simply C as follow

$$\begin{aligned}
C &= \left(\frac{w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}}\right)^2 + \frac{w_{t-1}^2}{1-\rho^2\pi'\Omega^{-1}\pi} - \frac{2(w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1})w_{t-1}}{1-\rho^2\pi'\Omega^{-1}\pi} + \frac{w_{t-1}^2}{t-1} \\
&= \left(\frac{w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}}\right)^2 + \frac{(t-\rho^2\pi'\Omega^{-1}\pi)w_{t-1}^2}{(t-1)(1-\rho^2\pi'\Omega^{-1}\pi)} - \frac{2(w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1})w_{t-1}}{1-\rho^2\pi'\Omega^{-1}\pi} \\
&= \left(\sqrt{\frac{t-\rho^2\pi'\Omega^{-1}\pi}{(t-1)(1-\rho^2\pi'\Omega^{-1}\pi)}}w_{t-1} - \sqrt{\frac{t-1}{(1-\rho^2\pi'\Omega^{-1}\pi)(t-\rho^2\pi'\Omega^{-1}\pi)}}(w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1})\right)^2 \\
&+ \left(\frac{w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}}\right)^2 - \frac{t-1}{(1-\rho^2\pi'\Omega^{-1}\pi)(t-\rho^2\pi'\Omega^{-1}\pi)}(w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1})^2 \\
&= \left(\sqrt{\frac{t-\rho^2\pi'\Omega^{-1}\pi}{(t-1)(1-\rho^2\pi'\Omega^{-1}\pi)}}w_{t-1} - \sqrt{\frac{t-1}{(1-\rho^2\pi'\Omega^{-1}\pi)(t-\rho^2\pi'\Omega^{-1}\pi)}}(w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1})\right)^2 \\
&+ \frac{(w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1})^2}{t-\rho^2\pi'\Omega^{-1}\pi}.
\end{aligned}$$

By substituting the expression we got for C, we can get

$$\begin{aligned}
p(w_t|w_{t-1},\varepsilon_{t-1})p(w_{t-1}) &= \frac{1}{2\pi\sqrt{1-\rho^2\pi'\Omega^{-1}\pi}\sqrt{t-1}} \\
&\underbrace{\times \exp\left[-\frac{1}{2}\left(\sqrt{\frac{t-\rho^2\pi'\Omega^{-1}\pi}{(t-1)(1-\rho^2\pi'\Omega^{-1}\pi)}}\left(w_{t-1}-\frac{(t-1)(w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1})}{t-\rho^2\pi'\Omega^{-1}\pi}\right)\right)^2\right]}_{K} \\
&\underbrace{\times \exp\left[-\frac{1}{2}\left(\frac{w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\sqrt{t-\rho^2\pi'\Omega^{-1}\pi}}\right)^2\right]}_{S}.
\end{aligned}$$

We may write

$$
\begin{aligned}
p(w_t|w_{t-1},\varepsilon_{t-1})p(w_{t-1}) &= \frac{\sqrt{t-\rho^2\pi'\Omega^{-1}\pi}}{\sqrt{2\pi}\sqrt{(1-\rho^2\pi'\Omega^{-1}\pi)(t-1)}}K \\
&\times \frac{1}{\sqrt{2\pi}\sqrt{t-\rho^2\pi'\Omega^{-1}\pi}}S \\
&= \bar{K}\times\bar{S}.
\end{aligned}
$$

Note that

$$
\bar{S} = \mathbb{N}\left(\rho\pi'\Omega^{-1}\varepsilon_{t-1}, t-\rho^2\pi'\Omega^{-1}\pi\right).
$$

It follows that

$$
\begin{aligned}
p(w_t|s_{t-1}=1,\mathcal{F}_{t-1}) &= \frac{\int_\tau^\infty \bar{K}\times\bar{S}dw_{t-1}}{\int_\tau^\infty p(w_{t-1})dw_{t-1}} \\
&= \frac{\int_\tau^\infty \bar{K}dw_{t-1}}{\int_\tau^\infty p(w_{t-1})dw_{t-1}}\bar{S} \\
&= \frac{1-\int_{-\infty}^\tau \bar{K}dw_{t-1}}{\int_\tau^\infty p(w_{t-1})dw_{t-1}}\bar{S} \\
&= \frac{1-\Phi\left(\sqrt{\frac{t-\rho^2\pi'\Omega^{-1}\pi}{(t-1)(1-\rho^2\pi'\Omega^{-1}\pi)}}\left(\tau-\frac{(t-1)(w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1})}{t-\rho^2\pi'\Omega^{-1}\pi}\right)\right)}{1-\Phi(\tau/\sqrt{t-1})} \\
&\times\mathbb{N}\left(\rho\pi'\Omega^{-1}\varepsilon_{t-1}, t-\rho^2\pi'\Omega^{-1}\pi\right).
\end{aligned}
$$

Similarly, we may derive

$$
\begin{aligned}
p(w_t|s_{t-1}=0,\mathcal{F}_{t-1}) &= \frac{\Phi\left(\sqrt{\frac{t-\rho^2\pi'\Omega^{-1}\pi}{(t-1)(1-\rho^2\pi'\Omega^{-1}\pi)}}\left(\tau-\frac{(t-1)(w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1})}{t-\rho^2\pi'\Omega^{-1}\pi}\right)\right)}{\Phi(\tau/\sqrt{t-1})} \\
&\times\mathbb{N}\left(\rho\pi'\Omega^{-1}\varepsilon_{t-1}, t-\rho^2\pi'\Omega^{-1}\pi\right).
\end{aligned}
$$

- **ω_ρ of (w_t) when $0<\lambda<1$ and $|\rho|=1$:**
  We have

$$
\begin{aligned}
p(w_t|s_{t-1}=1,\mathcal{F}_{t-1}) &= p(w_t|s_{t-1}=1,y_{t-1},...,y_1) \\
&= p(w_t|w_{t-1}\geq\tau,\varepsilon_{t-1}) \\
&= p(\lambda w_{t-1}+v_t|w_{t-1}\geq\tau,\mathcal{F}_{t-1}).
\end{aligned}
$$

Note that

$$p(w_{t-1}|w_{t-1} \geq \tau) = \frac{\sqrt{1-\lambda^2}\varphi(w_{t-1}\sqrt{1-\lambda^2})}{1-\Phi(\tau\sqrt{1-\lambda^2})} 1\{w_{t-1} \geq \tau\}.$$

Lemma: Given probability density function, $p_X(x)$, the probability density function, $p_Y(y)$, for $Y = \alpha + \beta X$ with $\beta \neq 0$ is given by

$$p_Y(y) = \frac{1}{|\beta|} p_X\left(x = \frac{y-\alpha}{\beta}\right).$$

Since $w_t = \lambda w_{t-1} + \rho\pi'\Omega^{-1}\varepsilon_{t-1}$, by choosing $\alpha = \rho\pi'\Omega^{-1}\varepsilon_{t-1}$ and $\beta = \lambda$, we may derive

$$\begin{aligned}
p(w_t|s_{t-1} = 1, \mathcal{F}_{t-1}) &= \frac{\frac{\sqrt{1-\lambda^2}}{\lambda}\varphi(\frac{w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\lambda}\sqrt{1-\lambda^2})}{1-\Phi(\tau\sqrt{1-\lambda^2})} 1\{w_t \geq \lambda\tau + \rho\pi'\Omega^{-1}\varepsilon_{t-1}\} \\
p(w_t|s_{t-1} = 0, \mathcal{F}_{t-1}) &= \frac{\frac{\sqrt{1-\lambda^2}}{\lambda}\varphi(\frac{w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\lambda}\sqrt{1-\lambda^2})}{\Phi(\tau\sqrt{1-\lambda^2})} 1\{w_t < \lambda\tau + \rho\pi'\Omega^{-1}\varepsilon_{t-1}\}.
\end{aligned}$$

Similarly, if we let $-1 < \lambda < 0$, then the conditional transition density of w_t can be obtained using

$$\begin{aligned}
p(w_t|s_{t-1} = 1, \mathcal{F}_{t-1}) &= \frac{\frac{\sqrt{1-\lambda^2}}{\lambda}\varphi(\frac{w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\lambda}\sqrt{1-\lambda^2})}{1-\Phi(\tau\sqrt{1-\lambda^2})} 1\{w_t \leq \lambda\tau + \rho\pi'\Omega^{-1}\varepsilon_{t-1}\} \\
p(w_t|s_{t-1} = 0, \mathcal{F}_{t-1}) &= \frac{\frac{\sqrt{1-\lambda^2}}{\lambda}\varphi(\frac{w_t-\rho\pi'\Omega^{-1}\varepsilon_{t-1}}{\lambda}\sqrt{1-\lambda^2})}{\Phi(\tau\sqrt{1-\lambda^2})} 1\{w_t > \lambda\tau + \rho\pi'\Omega^{-1}\varepsilon_{t-1}\}.
\end{aligned}$$

The proof for the case of $\lambda = 1$ is very similar to what we have done for $0 < \lambda < 1$, except that we have $w_{t-1}/\sqrt{t-1} =_d \mathbb{N}(0,1)$ for $t \geq 2$ in this case instead of $w_{t-1}\sqrt{1-\lambda^2} =_d \mathbb{N}(0,1)$ when $|\lambda| < 1$. □